WEYL GROUP INVARIANTS AND APPLICATION TO SPHERICAL HARMONIC ANALYSIS ON SYMMETRIC SPACES

GESTUR ÓLAFSSON AND JOSEPH A. WOLF

Abstract. Polynomial invariants are fundamental objects in analysis on Lie groups and symmetric spaces. Invariant differential operators on symmetric spaces are described by Weyl group invariant polynomial. In this article we give a simple criterion that ensure that the restriction of invariant polynomials to subspaces is surjective. We apply our criterion to problems in Fourier analysis on projective/injective limits, specifically to theorems of Paley–Wiener type.

Introduction

Invariant polynomials play a fundamental role in several branches of mathematics. A well known example related to the topic of this article comes from the representation theory of semisimple Lie groups and from the related analysis on Riemannian symmetric spaces. Let G be a connected semisimple real Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Then the algebra of G–invariant polynomials on $\mathfrak{g}$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}$. Also, the restriction of invariant polynomials to $\mathfrak{h}$ is an isomorphism onto the algebra of Weyl group invariant polynomials on $\mathfrak{h}$. Replace G by a Riemannian symmetric space $M = G/K$ corresponding to a Cartan involution θ and replace $\mathfrak{h}$ by a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{s} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then the Weyl group invariant polynomials correspond to the invariant differential operators on M. They are therefor closely related to harmonic analysis on M, in particular to the determination of the spherical functions on M.

In general we need $\mathfrak{a} \subset \mathfrak{h}$ and $\theta\mathfrak{h} = \mathfrak{h}$. For this, of course, we need only choose $\mathfrak{h}$ to be a Cartan subalgebra of the centralizer of $\mathfrak{a}$.

Denote by $W(\mathfrak{g}, \mathfrak{h})$ the Weyl group of $\mathfrak{g}$ relative to $\mathfrak{h}$, $W(\mathfrak{g}, \mathfrak{a})$ the "baby" Weyl group of $\mathfrak{g}$ relative to $\mathfrak{a}$, $W_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h}) = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\mathfrak{a}) = \mathfrak{a}\}$, $\mathrm{I}(\mathfrak{g})$ the algebra of $W(\mathfrak{g}, \mathfrak{h})$–invariant polynomials on $\mathfrak{h}$ and finally $\mathrm{I}(\mathfrak{a})$ the algebra of $W(\mathfrak{g}, \mathfrak{a})$–invariant polynomials on $\mathfrak{a}$. It is well known for all semisimple Lie algebras that $W_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h})|_{\mathfrak{a}} = W(\mathfrak{g}, \mathfrak{a})$. In [8] Helgason

2000 *Mathematics Subject Classification.* 43A85, 53C35, 22E46.

Key words and phrases. Invariant polynomials; Injective and projective limits; Spherical Fourier transform; Paley-Wiener theorem.

The research of G. Ólafsson was supported by NSF grants DMS-0402068 and DMS-0801010.

The research of J. A. Wolf was partially supported by NSF grant DMS-0652840.

showed for all classical semisimple Lie algebras that $\mathrm{I}(\mathfrak{h})|_{\mathfrak{a}} = \mathrm{I}(\mathfrak{a})$. As an application, this shows that in most cases the invariant differential operators on M come from elements in the center of the universal enveloping algebra of $\mathfrak{g}$.

In this article we discuss similar restriction problems for the case of pairs of Lie groups $G_n \subset G_k$ and symmetric spaces $M_n \subset M_k$. We use the above notation with indices $_n$ respectively $_k$. The first question is about restriction from $\mathfrak{h}_k$ to $\mathfrak{h}_n$. It is clear that neither does the group $W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)$ restrict to $W(\mathfrak{g}_n, \mathfrak{h}_n)$ in general, nor is $\mathrm{I}(\mathfrak{h}_k)|_{\mathfrak{h}_n} = \mathrm{I}(\mathfrak{h}_n)$. To make this work, we introduce the notion that $\mathfrak{g}_k$ is a prolongation of $\mathfrak{g}_n$ using the Dynkin diagram of simple Lie classical Lie algebras. In terms of restricted roots, that means that either the rank and restricted root system of the large and the small symmetric spaces are the same, or roots are added to the left end of the Dynkin diagram. The result is that both symmetric spaces have the same type of root system but the larger one can have higher rank. In that case the restriction result above holds for all cases except when the restricted root systems are of type D. This includes all the cases of classical Lie groups of the same type. If G_k is a prolongation of G_n, then $W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and $\mathrm{I}(\mathfrak{h}_n)|_{\mathfrak{h}_n} = \mathrm{I}(\mathfrak{h}_n)$, except in the case of simple algebras of type D, where a parity condition is needed, i.e., we have to extend the Weyl group by incorporating odd sign changes for simple factors of type D. The resulting finite group is denoted by $\widetilde{W}(\mathfrak{g}, \mathfrak{h})$. Then, in all classical cases, the $\widetilde{W}(\mathfrak{g}_k, \mathfrak{h}_k)$-invariant polynomials restrict to $\widetilde{W}(\mathfrak{g}_n, \mathfrak{h}_n)$-invariant polynomials. We also show that $\widetilde{W}_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{h})|_{\mathfrak{a}} = \widetilde{W}(\mathfrak{g}, \mathfrak{a})$.

When a compact symmetric space M_k is a prolongation of another, say M_n, we prove surjectivity for restriction of smooth functions supported in a ball of a given radius r on M_k to smooth functions supported in a ball of radius r on M_n, resulting is a corresponding restriction result on their Fourier transform spaces. Using results on conjugate and cut locus of compact symmetric spaces we show that the radius of injectivity in the symmetric spaces forming a direct system, related by prolongation, is constant. If R is that radius then the condition on the support size r is given in terms of R, thus constant for the direct system, and this allows us to carry the finite dimensional Paley–Wiener theorem to the limit.

The classical Paley–Wiener Theorem describes the growth of the Fourier transform of a function $f \in C_c^\infty(\mathbb{R}^n)$ in terms of the size of its support. Helgason and Gangolli generalized it to Riemannian symmetric spaces of noncompact type, Arthur extended it to semisimple Lie groups, van den Ban and Schlichtkrull made the extension to pseudo-Riemannian reductive symmetric spaces, and finally Ólafsson and Schlichtkrull worked out the corresponding result for compact Riemannian symmetric spaces. Here we extend their result to a class of infinite dimensional Riemannian symmetric spaces, the classical direct limits compact symmetric spaces. The main idea is to combine the results of Ólafsson and Schlichtkrull with Wolf's results on direct limits $\varinjlim M_n$ of Riemannian symmetric spaces and limits of the corresponding function spaces on the M_n.

Of course compact support in the Paley–Wiener Theorem is irrelevant for functions on a compact symmetric space, and there one concentrates on the radius of the support. The Fourier transform space is interpreted as the parameter space for spherical functions, the linear dual space of the complex span of the restricted roots. When we pass to direct limits it is crucial that these ingredients be properly normalized. In order to do this we introduce the notion of propagation for pairs of root systems, pairs of groups, and pairs of symmetric spaces.

In Section 1 we recall some basic facts concerning Paley–Wiener theorems, their behavior under finite symmetry groups, and restrictions of Paley–Wiener spaces. In order to apply this to direct systems of symmetric spaces, in Section 2 we introduce the notion of propagation and examine the corresponding invariants explicitly for each type of root system. The main result, Theorem 2.7, summarizes the facts on restriction of Weyl groups for propagation of symmetric spaces. The proof is by case by case consideration of each simple root system.

In Section 3 we prove surjectivity of Weyl group invariant polynomials for propagation of symmetric spaces. As mentioned above, this is analogous to Helgason's result on restriction of invariants from the full Cartan $\mathfrak{h}$ of $\mathfrak{g}$ to the Cartan $\mathfrak{a}$ of $(\mathfrak{g}, \mathfrak{k})$.

In Section 4 we apply our results on Weyl group invariants to Fourier analysis on Riemannian symmetric spaces of noncompact type. The main result is Theorem 4.8, the Paley–Wiener Theorem for classical direct limits of those spaces. As indicated earlier, a $\mathbb{Z}_2$ extension of the Weyl group is needed in case of root systems of type D. The extension can be realized by an automorphism σ of the of the Dynkin diagram. We show that there exists an automorphism $\widetilde{\sigma}$ of G or a double cover such that $d\widetilde{\sigma}|_{\mathfrak{a}} = \sigma$ and the spherical function with spectral parameter λ satisfies $\varphi_\lambda(\widetilde{\sigma}(x)) = \varphi_{\sigma'(\lambda)}(x)$.

In Section 5 we set up the basic surjectivity of the direct limit Paley–Wiener Theorem for the classical sequences $\{SU(n)\}$, $\{SO(2n)\}$, $\{SO(2n+1)\}$ and $\{Sp(2n)\}$. The main tool is Theorem 5.3, the calculation of the injectivity radius; it turns out to be a simple constant ($\sqrt{2}\,\pi$ or 2π) for each of those series. The main result is Theorem 5.9, which sets up the projective systems of functions used in the Paley–Wiener Theorem for $SU(\infty)$, $SO(\infty)$ and $Sp(\infty)$. All this is needed when we go to limits of symmetric spaces. Theorem 6.10, the main result of Section 6, sets up the sequence of function spaces corresponding to a direct system $\{M_n\}$ of compact Riemannian symmetric spaces in which M_k propagates M_n for $k \geqq n$. On the way we show that for compact symmetric spaces the map $Q : C^\infty(G)^G \to C^\infty(G/K)^K$, $Q(f)(xK) = \int_K f(xk)\, dk$, which is surjective, is in fact surjective as a map $C_r^\infty(G)^G \to C_r^\infty(G/K)^K$, where the subscript $_r$ denotes the size of the support.

Finally, in Section 7, we relate the spherical Fourier transforms for the sequence $\{M_n\}$, show how the injectivity radii remain constant on the sequence, and prove the Paley–Wiener Theorem 7.6, and a Paley–Wiener Theorem 7.20, for direct limits $M_\infty = \varinjlim M_n$

of compact Riemannian symmetric spaces in which M_k propagates M_n for $k \geqq n$. Along the way we obtain a stronger form, Theorem 7.8, of one of the key ingredients in the proof of the surjectivity.

Our discussion of direct limit Paley–Wiener Theorems involves function space maps that have a somewhat complicated relation [24] to the L^2 theory of [22]. This is discussed in Section 8, where we compare our maps with the partial isometries of [22].

1. Polynomial Invariants and Restriction of Paley-Wiener spaces

Let $E \cong \mathbb{R}^n$ be a finite dimensional Euclidean space and $E_{\mathbb{C}} \cong \mathbb{C}^n$ its complexification. Denote by $C_r^\infty(E)$ the space of smooth functions on E with support in a closed ball $\overline{B_r(0)}$ of radius $r > 0$. Denote by $\mathrm{PW}_r(E_{\mathbb{C}})$ the space of holomorphic function on $E_{\mathbb{C}}$ with the property that for each $n \in \mathbb{Z}^+$ there exists a constant $C_n > 0$ such that

$$|F(z)| \leqq C_n(1+|z|)^{-n} e^{r|\operatorname{Im} z|}\,. \tag{1.1}$$

Let $\langle x, y\rangle_E = \langle x, y\rangle = x \cdot y$ denote the inner product on E as well as its $\mathbb{C}$–bilinear extension to $E_{\mathbb{C}}$. Denote by $\mathrm{O}(E)$ the orthogonal group of E with respect to this inner product. If $T : E \to E$ is $\mathbb{R}$-linear then we will also view T as a complex linear map on $E_{\mathbb{C}}$. If $w \in \mathrm{O}(E)$ and f is a function on E or $E_{\mathbb{C}}$ then $L_w(f)$ denotes the (left) translate of f by w, $L_w(f)(x) = f(w^{-1}x)$. If G is a subgroup of $\mathrm{O}(E)$ and $L_w f = f$ for all $w \in G$ then we say that f is G–invariant. For a G-module V set

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}\,. \tag{1.2}$$

In particular $\mathrm{PW}_r(E_{\mathbb{C}})^G$ and $C_r^\infty(E)^G$ are well defined.

We normalize the Fourier transform on E as

$$\mathcal{F}_E(f)(\lambda) = \widehat{f}(\lambda) = (2\pi)^{-n/2} \int_E f(x) e^{-i\lambda \cdot x}\, dx. \tag{1.3}$$

The Paley–Wiener Theorem says that $\mathcal{F}_E : C_r^\infty(E)^G \to \mathrm{PW}_r(E_{\mathbb{C}})^G$ is an isomorphism.

Denote by $\mathrm{S}(E)$ the algebra of polynomial functions on E and $\mathrm{I}(E) = \mathrm{I}_G(E) = \mathrm{S}(E)^G$ the algebra of G–invariant invariant polynomial functions on E.

From now on we assume that F is another Euclidean space and that $E \subsetneqq F$. We will always assume that the inner products on E and F are chosen so that $\langle x, y\rangle_E = \langle x, y\rangle_F$ for all $x, y \in E$. Furthermore, if $W(E)$ and $W(F)$ are closed subgroups of the respective orthogonal groups acting on E and F, then

$$W_E(F) = \{w \in W(F) \mid w(E) = E\}$$

is the subgroup of $W(F)$ that maps E into E. We will always assume that $W(E)$ and $W(F)$ are generated by reflections $s_\alpha : v \mapsto v - \frac{2(\alpha,v)}{(\alpha,\alpha)}\alpha$, for α in a root system in E

respectively F. However, it should be pointed out that the Cowling result, see below, holds for arbitrary closed subgroup of $\mathrm{O}(E)$ respectively $\mathrm{O}(F)$.

We recall the following theorems of Cowling [3] and Rais [19] in the form that we need in the sequel. Cowling states his result for finite groups but his argument is valid for compact groups.

Theorem 1.4 (Cowling)**.** *The restriction map* $\mathrm{PW}_r(F_{\mathbb{C}})^{W_E(F)} \to \mathrm{PW}_r(E_{\mathbb{C}})^{W_E(F)|_{E_{\mathbb{C}}}}$, *given by* $F \mapsto F|_{E_{\mathbb{C}}}$, *is surjective.*

Theorem 1.5 (Rais)**.** *Let* $P_1, \ldots, P_n$ *be a basis for* $\mathrm{S}(F)$ *over* $I_{W(F)}(F)$. *If* $F \in \mathrm{PW}_r(F_{\mathbb{C}})$ *there exist* $\Phi_1, \ldots, \Phi_n \in \mathrm{PW}_r(F_{\mathbb{C}})^{W(F)}$ *such that*

$$F = P_1\Phi_1 + \ldots + P_n\Phi_n\,.$$

If $W_E(F)|_E = W(E)$ then Cowling's Theorem implies that the restriction map

$$\mathrm{PW}_r(F_{\mathbb{C}})^{W_E(F)} \to \mathrm{PW}_r(E_{\mathbb{C}})^{W_E(F)|_{E_{\mathbb{C}}}}\,, \quad F \mapsto F|_{E_{\mathbb{C}}}\,,$$

is surjective. If we, on the right hand side, replace $W_E(F)$ by the full group $W(F)$ then the subspace of invariant functions becomes smaller, so one would in general not expect the restriction map to remain surjective. The following theorem gives a sufficient condition for that to happen.

Theorem 1.6. *Let the notation be as above. Assume that*

(1) $W_E(F)|_E = W(E)$
(2) *The restriction map* $\mathrm{I}_{W(F)}(F) \to \mathrm{I}_{W(E)}(E)$ *is surjective.*

Then the restriction map

$$\mathrm{PW}_r(F_{\mathbb{C}})^{W(F)} \to \mathrm{PW}_r(E_{\mathbb{C}})^{W(E)}\,,\ \textit{given by}\ F \mapsto F|_{E_{\mathbb{C}}}\,,$$

is surjective.

Proof. It follows from assumption (1) that if $F \in \mathrm{PW}_r(F_{\mathbb{C}})^{W(F)}$ then $F|_E \in \mathrm{PW}_r(E_{\mathbb{C}})^{W(E)}$.

Now, let $G \in \mathrm{PW}_r(E_{\mathbb{C}})^{W(E)}$. By Theorem 1.4 and assumption (1) there exists a function $\widetilde{G} \in \mathrm{PW}_r(F_{\mathbb{C}})^{W_E(F)}$ such that $\widetilde{G}|_{E_{\mathbb{C}}} = G$. By Theorem 1.5, there exist $\Phi_1, \ldots, \Phi_n \in \mathrm{PW}_r(F_{\mathbb{C}})^{W(F)}$ and polynomials $P_1, \ldots, P_n \in \mathrm{S}(F)$ such that

$$\widetilde{G} = P_1\Phi_1 + \ldots + P_n\Phi_n\,.$$

But then

$$G = \widetilde{G}|_{E_{\mathbb{C}}} = (P_1|_{E_{\mathbb{C}}})(\Phi_1|_{E_{\mathbb{C}}}) + \ldots + (P_n|_{E_{\mathbb{C}}})(\Phi_n|_{E_{\mathbb{C}}})\,.$$

As $W(E) = W_E(F)|_E$, G is $W(E)$–invariant and the functions Φ_j are $W(F)$–invariant, we can assume that $P_j|_{E_{\mathbb{C}}} \in \mathrm{I}_{W(E)}(E)$. By assumption (2) there exists $Q_j \in \mathrm{I}_{W(F)}(F)$ such that $Q_j|_{E_{\mathbb{C}}} = P_j|_{E_{\mathbb{C}}}$. But then

$$\Phi := Q_1\Phi_1 + \ldots + Q_r\Phi_r \in \mathrm{PW}_r(F_{\mathbb{C}})^{W(F)} \text{ satisfies } \Phi|_{E_{\mathbb{C}}} = G\,.$$

Hence the restriction map is surjective. $\square$

Remark 1.7. We note that (1) above does not imply (2) in Theorem 1.6. Let G/K be a semisimple symmetric space of the noncompact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition. Thus, there exists an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{k}$, the Lie algebra of K, is the $+1$-eigenspace of θ and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}$ and let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. We write $W(\mathfrak{g}, \mathfrak{a})$ for the corresponding Weyl group. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{h}_k \oplus \mathfrak{a}$, where $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$. Let $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ be the roots of $\mathfrak{h}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ and let $W(\mathfrak{g}, \mathfrak{h})$ be the corresponding Weyl group. Then by [12, p. 366],

$$W(\mathfrak{g}, \mathfrak{a}) = \{w|_\mathfrak{a} \mid w \in W(\mathfrak{g}, \mathfrak{h}) \text{ such that } w(\mathfrak{a}) = \mathfrak{a}\} = W_\mathfrak{a}(\mathfrak{g}, \mathfrak{h})|_\mathfrak{a}.$$

But for some of the exceptional algebras (2) can fail; see [10] for exact statement. $\diamondsuit$

Let $n = \dim E$ and $m = \dim F$. Denote by $\mathcal{F}_E$ respectively $\mathcal{F}_F$ the Euclidean Fourier transforms on E and F. The following map C was denoted by P in [3].

Corollary 1.8 (Cowling)**.** *Let the assumptions be as above. Then the map*

$$C : C_r^\infty(F)^{W(F)} \to C_r^\infty(E)^{W(E)}\,, \text{ given by } Cf(x) = \int_{E^\perp} f(x, y)\, dy,$$

is surjective.

Proof. We follow [3]. Let $c = (2\pi)^{(n-m)/2}$ Let $g \in C_r^\infty(E)^{W(E)}$ and $G = \mathcal{F}_E(g)$ its Fourier transform in $\mathrm{PW}_r(E_\mathbb{C})^{W(E)}$. Let $F \in \mathrm{PW}_r(F_\mathbb{C})^{W(F)}$ be such that $F|_E = c^{-1}G$ and let f be the inverse Fourier transform of F. We claim that $g = Cf$. For that we note

$$\begin{aligned}
\mathcal{F}_E(g)(\lambda) &= c^{-1}F(\lambda, 0) \\
&= c^{-1}(2\pi)^{-m/2} \int_{E_x} \int_{E_y^\perp} f(x, y) e^{-i\lambda\cdot x}\, dxdy \\
&= (2\pi)^{-n/2} \int_E Cf(x) e^{-i\lambda\cdot x}\, dx \\
&= \mathcal{F}_E(Cf)(\lambda)\,.
\end{aligned}$$

The claim follows now, as obviously Cf has compact support and the Fourier transform is injective on the space of compactly supported functions. $\square$

Theorem 1.9. *Let $\{E_j\}$ be a sequence of Euclidean spaces such that $E_j \subsetneqq E_{j+1}$ and such that hypotheses* (1) *and* (2) *of* Theorem 1.6 *are satisfied for each pair (E_j, E_k), $k \geqq j$. Let $P_{k,j} : \mathrm{PW}_r(E_{k,\mathbb{C}})^{W(E_k)} \to \mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}$ be the restriction map. Then $\{\mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}, P_j\}$ is a projective system and $\varprojlim\{\mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}\} \neq \{0\}$.*

Proof. It is clear that $\{\mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}, P_{n,j}\}$ is a projective system. Fix j and let $F \in \mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}$, $F \neq 0$. Recursively choose $F_k \in \mathrm{PW}_r(E_{k,\mathbb{C}})^{W(E_k)}$, $k \geqq j$ such that $F_{k+1}|_{E_{k,\mathbb{C}}} = F_k$. Then the sequence $\{F_k\}$ is a non-zero element of $\varprojlim \mathrm{PW}_r(E_{j,\mathbb{C}})^{W(E_j)}$. $\square$

Theorem 1.10. *Let $\{E_j\}$ be a sequence of Euclidean spaces such that $E_j \subseteqq E_{j+1}$ and such that hypotheses* (1) *and* (2) *of* Theorem 1.6 *are satisfied for each pair (E_j, E_k), $k \geqq j$. Define $C_{k,j} : C_r^\infty(E_k)^{W(E_k)} \to C_r^\infty(E_j)^{W(E_j)}$ by*

$$[C_{k,j}(f)](x) = \int_{E_j^\perp} f(x,y)\, dy\,.$$

Then the maps $C_{k,j}$ are surjective, $\{C_r^\infty(E_j)^{W(E_j)}, C_{k,j}\}$ is a projective system, and its limit satisfies $\varprojlim C_r^\infty(E_j)^{W(E_j)} \neq \{0\}$.

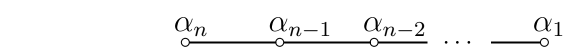
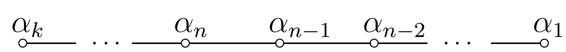
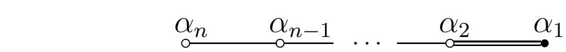
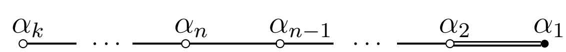
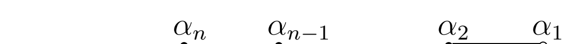
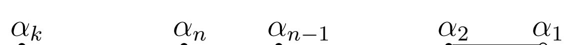
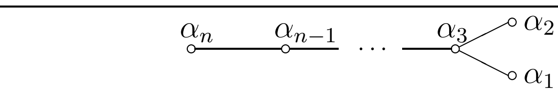
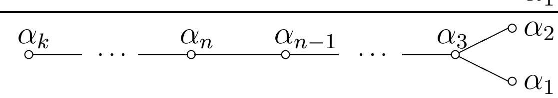
Proof. The proof is the same as that of Theorem 1.9, making use of Corollary 1.8. □

Remark 1.11. *The last two theorems remain valid if the assumptions holds for a cofinite sequence $\{E_j\}_{j\in J}$.* ♢

2. Restriction of Invariants for Classical Simple Lie Algebras

We will now apply this to the classical simple Lie algebras and related symmetric spaces. Let $\mathfrak{g}_n$ be a simple Lie algebra of classical type and let $\mathfrak{h}_n \subset \mathfrak{g}_n$ be a Cartan subalgebra. Let $\Delta_n = \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ be the set of roots of $\mathfrak{h}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$ and $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{h}_n)$ a set of simple roots. We label the corresponding Dynkin diagram so that α_1 is the *right* endpoint. If $\mathfrak{g}_n \subseteqq \mathfrak{g}_k$ then we chose $\mathfrak{h}_n$ and $\mathfrak{h}_k$ so that $\mathfrak{h}_n = \mathfrak{g}_n \cap \mathfrak{h}_k$. We say that $\mathfrak{g}_k$ *propagates* $\mathfrak{g}_n$, if Ψ_k is constructed from Ψ_n by adding simple roots to the *left* end of the Dynkin diagrams. Thus

(2.1)

$\Psi_n = A_n$		$n \geqq 1$
$\Psi_k = A_k$		$k \geqq n$
$\Psi_n = B_n$		$n \geqq 2$
$\Psi_k = B_k$		$k \geqq n$
$\Psi_n = C_n$		$n \geqq 3$
$\Psi_k = C_k$		$k \geqq n$
$\Psi_n = D_n$		$n \geqq 4$
$\Psi_k = D_k$		$k \geqq n$

Let $\mathfrak{g}$ and $'\mathfrak{g} \subset \mathfrak{g}$ be semisimple Lie algebras. Then $\mathfrak{g}$ *propagates* $'\mathfrak{g}$ if we can number the simple ideals $\mathfrak{g}_j$, $j = 1, 2, \ldots, r$, in $\mathfrak{g}$ and the simple ideals $'\mathfrak{g}_i$, $i = 1, 2, \ldots, s$, in $'\mathfrak{g}$, so that $\mathfrak{g}_j$ propagates $'\mathfrak{g}_j$ for $j = 1, \ldots, s$.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$ as above, they have Cartan subalgebra $\mathfrak{h}_k$ and $\mathfrak{h}_n$ such that $\mathfrak{h}_n \subseteqq \mathfrak{h}_k$, and we have choices of root order such that

$$\text{if } \alpha \in \Psi_n \text{ then there is a unique } \alpha' \in \Psi_k \text{ such that } \alpha'|_{\mathfrak{h}_n} = \alpha.$$

It follows that

$$\Delta_n \subseteqq \{\alpha|_{\mathfrak{h}_n} \mid \alpha \in \Delta_k \text{ and } \alpha|_{\mathfrak{h}_n} \neq 0\}\,.$$

For a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ in a simple complex Lie algebra $\mathfrak{g}_\mathbb{C}$ denote by $\mathfrak{h}_\mathbb{R}$ the Euclidean vector space

$$\mathfrak{h}_\mathbb{R} = \{X \in \mathfrak{h}_\mathbb{C} \mid \alpha(X) \in \mathbb{R} \text{ for all } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\}\,.$$

We now discuss case by case the classical simple Lie algebras and how the Weyl group and the invariants behave under propagation. The result will be collected in Theorem 2.7 below. The corresponding result for Riemannian symmetric spaces is Theorem 3.4.

For $s \in \mathbb{N}$ identify $\mathbb{R}^s$ with its dual. Let $f_1 = (0, 0, \ldots, 0, 1)$, $\ldots$, $f_s = (1, 0, 0, \ldots, 0)$ be the standard basis for $\mathbb{R}^s$ numbered in order opposite to the usual one. We write

$$x = x_1 f_1 + \ldots + x_s f_s = (x_s, \ldots, x_1)$$

to indicate that in the following we will be adding zeros to the left to adjust for our numbering in the Dynkin diagrams. We use the discussion in [20, p. 293] as a reference for the realization of the classical Lie algebras.

For a classical simple Lie algebra $\mathfrak{g}$ of rank n denote by π_n the defining representation and

$$F_n(t, X) := \det(t + \pi_n(X))\,.$$

We denote by the same letter the restriction of $F_n(t, \cdot)$ to $\mathfrak{h}_n$. In this section only we use the following simplified notation: $W_k = W(\mathfrak{g}_k, \mathfrak{h}_k)$ denotes the usual Weyl group of the pair $(\mathfrak{g}_k, \mathfrak{h}_k)$ and

$$W_{k,n} = W_{\mathfrak{h}_{n,\mathbb{R}}}(\mathfrak{g}, \mathfrak{h}_k) = \{w \in W_k \mid w(\mathfrak{h}_{n,\mathbb{R}}) = \mathfrak{h}_{n,\mathbb{R}}\}$$

is the subgroup with well defined restriction to $\mathfrak{h}_n$.

The case $\mathbf{A_n}$, where $\mathfrak{g} = \mathfrak{sl}(\mathbf{n+1}, \mathbb{C})$. In this case

$$\mathfrak{h}_{k,\mathbb{R}} = \{(x_{k+1}, \ldots, x_1) \in \mathbb{R}^{k+1} \mid x_1 + \ldots + x_{k+1} = 0\}\,, \tag{2.2}$$

where $x \in \mathbb{R}^{k+1}$ corresponds to the diagonal matrix

$$x \leftrightarrow \operatorname{diag}(x) := \begin{pmatrix} x_{k+1} & 0 & \dots & 0 \\ 0 & x_k & & \\ & & \ddots & \\ & & & x_1 \end{pmatrix}$$

Then $\Delta = \{f_i - f_j \mid 1 \leqq i \neq j \leqq k+1\}$ where f_ℓ maps a diagonal matrix to its ℓ^{th} diagonal element. Here $W(\mathfrak{g}_k, \mathfrak{h}_k)$ is the symmetric group $\mathfrak{S}_{k+1}$, all permutations of $\{1, \dots, k+1\}$, acting on the $\mathfrak{h}_k$ by

$$\sigma \cdot (x_{k+1}, \dots, x_1) = (x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(1)})\,.$$

We will use the simple root system

$$\Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{f_j - f_{j-1} \mid j = 2, \dots, k+1\}\,.$$

The analogous notation will be used for A_n. In particular, denoting the zero vector of length j by 0_j, we have

$$\mathfrak{h}_{n,\mathbb{R}} = \left\{ (0_{k-n}, x_{n+1}, \dots, x_1) \mid x_j \in \mathbb{R} \quad \text{and} \quad \sum_{j=1}^{n+1} x_j = 0 \right\} \subset \mathfrak{h}_{k,\mathbb{R}}\,. \tag{2.3}$$

This corresponds to the embedding

$$\mathfrak{sl}(n, \mathbb{C}) \hookrightarrow \mathfrak{sl}(k, \mathbb{C})\,, \quad X \mapsto \begin{pmatrix} 0_{k-n,k-n} & 0 \\ 0 & X \end{pmatrix}\,.$$

It follows that

$$W_{k,n} = \mathfrak{S}_{k-n} \times \mathfrak{S}_{n+1}\,.$$

Hence $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n, \mathfrak{h}_)$ and the kernel of the restriction map is the first factor $\mathfrak{S}_{k-n}$.

According to [20, Exercise 58, p. 410] we have

$$F_k(t, X) = \prod_{j=1}^{k+1} (t + x_j) = t^{k+1} + \sum_{\nu=1}^{k+1} p_{k,\nu}(X) t^{\nu-1}\,.$$

The polynomials $p_{k,\nu}$ generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. By (2.3), if $X = (0_{k-n}, x) \in \mathfrak{h}_{n,\mathbb{R}}$, then

$$\begin{aligned} F_k(t, (0_{k-n}, x)) &= t^{k+1} + \sum_{\nu=1}^{k+1} p_{k,\nu}(X) t^{\nu-1} \\ &= t^{k-n} \det(t + \pi_n(x)) \\ &= t^{k-n}(t^{n+1} + \sum_{\nu=1}^{n+1} p_{n,\nu}(x) t^{\nu-1}) \\ &= t^{k+1} + \sum_{\nu=k-n+1}^{k+1} p_{n,\nu+n-k}(x) t^{\nu-1}\,. \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k-n\,.$$

In particular the restriction map $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{B_n}$, where $\mathfrak{g} = \mathfrak{so}(\mathbf{2n+1},\mathbb{C})$. In this case $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ where $\mathbb{R}^k$ is embedded into $\mathfrak{so}(2n+1,\mathbb{C})$ by

$$x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{diag}(x) & 0 \\ 0 & 0 & -\mathrm{diag}(x) \end{pmatrix}. \tag{2.4}$$

Here $\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\} \bigcup \{\pm f_1, \dots, \pm f_k\}$ and we have the positive system $\Delta_k^+ = \{f_i \pm f_j \mid 1 \leqq j < i \leqq n\} \cup \{f_1, \dots, f_n\}$. The simple root system is $\Psi = \Psi(\mathfrak{g}_k, \mathfrak{h}_k) = \{\alpha_1, \dots, \alpha_k\}$ where

$$\text{the simple root } \alpha_1 = f_1, \text{ and } \alpha_j = f_j - f_{j-1} \text{ for } 2 \leqq j \leqq k.$$

In this case the Weyl group $W(\mathfrak{g}_k, \mathfrak{h}_k)$ is the semidirect product $\mathfrak{S}_k \rtimes \{1,-1\}^k$, where $\mathfrak{S}_k$ acts as before and

$$\{1,-1\}^k \cong (\mathbb{Z}/2\mathbb{Z})^k = \{\epsilon = (\epsilon_k, \dots, \epsilon_1) \mid \epsilon_j = \pm 1\}$$

acts by sign changes

$$\epsilon \cdot x = (\epsilon_k x_k, \dots, \epsilon_n x_1)\,.$$

Similar notation holds for $\mathfrak{h}_{n,\mathbb{R}}$. Our embedding of $\mathfrak{h}_{n,\mathbb{R}} \hookrightarrow \mathfrak{h}_{k,\mathbb{R}}$ corresponds to the (non-standard) embedding of $\mathfrak{so}(2n+1,\mathbb{C})$ into $\mathfrak{so}(2k+1,\mathbb{C})$ given by

$$\begin{pmatrix} 0 & a & b \\ -b^t & A & B \\ -a^t & C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0_{k-n} & a & 0_{k-n} & b \\ 0_{k-n}^t & 0 & 0 & 0 & 0 \\ -b^t & 0 & A & 0 & B \\ 0_{k-n}^t & 0 & 0 & 0 & 0 \\ -a^t & 0 & C & 0 & -A^t \end{pmatrix}$$

where the zeros stands for the zero matrix of the obvious size and we use the realization from [20, p. 303].

We see that

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}) \times (\mathfrak{S}_n \rtimes \{1,-1\}^n)\,.$$

Thus $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n, \mathfrak{h}_n)$ and the kernel of the restriction map is $\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}$.

For the invariant polynomials we have, again using [20, Exercise 58, p. 410], that

$$F_k(t,X) = \det(t + \pi_k(X)) = t^{2k+1} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2\nu-1}$$

and the polynomials $p_{k,\nu}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. For $X \in \mathfrak{h}_k$, $F_k(t,X)$ is given by $t\prod_{j=1}^n(t+x_j)(t-x_j) = t\prod_{j=1}^n(t^2-x_j^2)$. By the same argument as above we have for $X = (0_{k-n}, x) \in \mathfrak{h}_{n,\mathbb{R}} \subseteqq \mathfrak{h}_{k,\mathbb{R}}$:

$$\begin{aligned} F_k(t,(0_{k-n},x)) &= t^{2k+1} + \sum_{\nu=1}^{k} p_{k,\nu}(X)t^{2\nu-1} \\ &= t^{2(k-n)}\det(t+\pi_n(x)) \\ &= t^{2(k-n)}(t^{2n+1} + \sum_{\nu=1}^{n} p_{n,\nu}(x)t^{2\nu-1}) \\ &= t^{2k+1} + \sum_{\nu=k-n+1}^{k} p_{n,\nu+n-k}(x)t^{2\nu-1}\,. \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k-n\,.$$

In particular, the restriction map $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{C_n}$, where $\mathfrak{g} = \mathfrak{sp}(\mathbf{n},\mathbb{C})$. Again $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ embedded in $\mathfrak{sp}(n,\mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \mathrm{diag}(x) & 0 \\ 0 & -\mathrm{diag}(x) \end{pmatrix}\,. \tag{2.5}$$

In this case

$$\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\} \bigcup \{\pm 2f_1, \ldots, \pm 2f_k\}\,.$$

Take $\Delta_k^+ = \{f_i - f_j \mid 1 \leqq j < i \leqq n\} \cup \{2f_1, \ldots, 2f_n\}$ as a positive system. Then the simple root system $\Psi = \Psi(\mathfrak{g}_k,\mathfrak{h}_k) = \{\alpha_1, \ldots, \alpha_k\}$ is given by

the simple root $\alpha_1 = 2f_1$, and $\alpha_j = f_j - f_{j-1}$ for $2 \leqq j \leqq k$.

The Weyl group $W(\mathfrak{g}_k,\mathfrak{h}_k)$ is again $\mathfrak{S}_k \rtimes \{1,-1\}^k$ and

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}) \times (\mathfrak{S}_n \rtimes \{1,-1\}^n)\,.$$

Thus, $W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = W(\mathfrak{g}_n,\mathfrak{h}_n)$ and the kernel of the restriction map is $\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}$.

For the invariant polynomials we have, again using [20, Exercise 58, p. 410], that

$$F_k(t,X) = t^{2k} + \sum_{\nu=1}^{k} p_{k,\nu}(X)t^{2(\nu-1)} = \prod_{j=1}^{n}(t^2 - x_j^2)$$

and the polynomials $p_{k,\nu}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$. We embed $\mathfrak{sp}(n,\mathbb{C})$ into $\mathfrak{sp}(k,\mathbb{C})$ by

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0_{k-n,k-n} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0_{k-n,k-n} & 0 \\ 0 & C & 0 & -A^t \end{pmatrix}$$

where as usual 0 stands for a zero matrix of the correct size. Then

$$\begin{aligned} F_k(t,(0_{k-n},x)) &= t^{2k} + \sum_{\nu=1}^{k} p_{k,\nu}(X) t^{2(\nu-1)} \\ &= t^{2(k-n)} \det(t+\pi_n(x)) \\ &= t^{2(k-n)}(t^{2n} + \sum_{\nu=1}^{n} p_{n,\nu}(x) t^{2(\nu-1)}) \\ &= t^{2k} + \sum_{\nu=k-n+1}^{k} p_{n,\nu+n-k}(x) t^{2(\nu-1)} \,. \end{aligned}$$

Hence

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+1 \leqq \nu \leqq k$$

and

$$p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 \text{ for } 1 \leqq \nu \leqq k-n \,.$$

In particular, the restriction map $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$ is surjective.

The case $\mathbf{D_n}$, where $\mathfrak{g} = \mathfrak{so}(\mathbf{2n},\mathbb{C})$. We take $\mathfrak{h}_{k,\mathbb{R}} = \mathbb{R}^k$ embedded in $\mathfrak{so}(2n,\mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \mathrm{diag}(x) & 0 \\ 0 & -\mathrm{diag}(x) \end{pmatrix} \,. \tag{2.6}$$

Then $\Delta_k = \{\pm(f_i \pm f_j) \mid 1 \leqq j < i \leqq k\}$ and we use the simple root system $\Psi(\mathfrak{g}_k,\mathfrak{h}_k) = \{\alpha_1,\dots,\alpha_k\}$ given by

$$\alpha_1 = f_1 + f_2, \text{ and } \alpha_i = f_i - f_{i-1} \text{ for } 2 \leqq i \leqq k$$

The Weyl group is

$$W(\mathfrak{g}_k,\mathfrak{h}_k) = \mathfrak{S}_k \rtimes \{\epsilon \in \{1,-1\}^n \mid \epsilon_1 \cdots \epsilon_n = 1\} \,.$$

In other words the elements of $W(\mathfrak{g}_k,\mathfrak{h}_k)$ contain only an *even* number of sign-changes. The invariants are given by

$$F_k(t,X) = t^{2k} + \sum_{\nu=2}^{k} p_{k,\nu}(X) t^{2(\nu-1)} + p_{k,1}(X)^2 = \prod_{\nu=1}^{n} (t^2 - x_j^2)$$

where p_1 is the Pfaffian, $p_1(X) = (-1)^{k/2} x_1 \dots x_k$, so $p_1(X)^2 = \det(X)$. The polynomials $p_{k,1},\dots,p_{k,k}$ freely generate $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$.

We embed $\mathfrak{h}_{n,\mathbb{R}}$ in $\mathfrak{h}_{k,\mathbb{R}}$ in the same manner as before. This corresponds to

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0_{k-n,k-n} & & 0_{k-n,k-n} & \\ 0 & A & 0 & B \\ 0_{k-n,k-n} & 0 & 0 & \\ 0 & C & 0 & -A^t \end{pmatrix} .$$

It is then clear that

$$W_{k,n} = (\mathfrak{S}_{k-n} \rtimes \{1,-1\}^{k-n}) \times_* (\mathfrak{S}_n \rtimes \{1,-1\}^n)$$

where the $_*$ indicates that $\epsilon_1 \cdots \epsilon_n = 1$. Therefore, the restrictions of elements of $W_{k,n}$, $k > n$, contain all sign changes, and

$$\mathfrak{S}_n \rtimes \{1,-1\}^{n-1} = W(\mathfrak{g}_n, \mathfrak{h}_n) \subsetneqq W_{k,n}|_{\mathfrak{h}_{n,\mathbb{R}}} = \mathfrak{S}_n \rtimes \{1,-1\}^n .$$

The Pfaffian $p_{k,1}(0,X) = 0$ and

$$\begin{aligned} F_k(t,(0,x)) &= t^{2k} + \sum_{\nu=2}^{k} p_{k,\nu}(0,x) t^{2(\nu-1)} \\ &= t^{2(k-n)} F_n(t,x) = t^{2(k-n)} (t^{2n} + \sum_{\nu=2}^{n} p_{n,\nu}(x) t^{2(\nu-1)} + p_{n,1}(x)^2) \\ &= t^{2k} + \sum_{\nu=k-n+2}^{k} p_{n,\nu+n-k}(x) t^{2(\nu-1)} + p_{n,1}(x)^2 t^{2(k-n)} . \end{aligned}$$

Hence

$$\begin{aligned} &p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,\nu+n-k} \text{ for } k-n+2 \leqq \nu \leqq k , \\ &p_{k,k-n+1}|_{\mathfrak{h}_{n,\mathbb{R}}} = p_{n,1}(x)^2 , \text{ and} \\ &p_{k,\nu}|_{\mathfrak{h}_{n,\mathbb{R}}} = 0 , \quad \nu = 1, \ldots, k-n . \end{aligned}$$

In particular the elements in $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}}$ are polynomials in even powers of x_j and $p_{n,1}$ is not in the image of the restriction map. Thus

$$\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}} \subsetneqq \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}}) .$$

We put these calculations together in the following theorem.

Theorem 2.7. *Assume $\mathfrak{g}_n$ and $\mathfrak{g}_k$ are simple complex Lie algebras of ranks n and k, respectively, and that $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$.*

(1) *If $\mathfrak{g}_k \neq \mathfrak{so}(2n,\mathbb{C})$ then*

$$W(\mathfrak{g}_n,\mathfrak{h}_n) = W_{\mathfrak{h}_n}(\mathfrak{g}_k,\mathfrak{h}_k)|_{\mathfrak{h}_n} = \{w|_{\mathfrak{h}_n} \mid w \in W(\mathfrak{g}_k,\mathfrak{h}_k) \text{ with } w(\mathfrak{h}_n) = \mathfrak{h}_n\}$$

and the restriction map

$$\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$$

is surjective.

(2) *If* $\mathfrak{g}_k = \mathfrak{so}(2k, \mathbb{C})$ *(so* $\mathfrak{g}_n = \mathfrak{so}(2n, \mathbb{C})$*), then*

$$W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n} = \{w|_{\mathfrak{h}_n} \mid w \in W(\mathfrak{g}_k, \mathfrak{h}_k) \text{ with } w(\mathfrak{h}_n) = \mathfrak{h}_n\} = \mathfrak{S}_n \rtimes \{1, -1\}^n$$

contains all sign changes, while the elements of $W(\mathfrak{g}_n, \mathfrak{h}_n)$ *contain only even numbers of sign changes. In particular* $W(\mathfrak{g}_n, \mathfrak{h}_n) \subsetneqq W_{\mathfrak{h}_n}(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n}$*. The elements of* $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})|_{\mathfrak{h}_{n,\mathbb{R}}}$ *are polynomials in the* x_j^2*, and the Pfaffian (square root of the determinant) is not in the image of the restriction map* $\mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{h}_n)}(\mathfrak{h}_{n,\mathbb{R}})$*. Denote by* $\mathrm{I}^{even}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}})$ *the algebra of invariants that are polynomials in* $x_1^2, \ldots, x_k^2$ *and similarly for* n*. Then the restriction map* $\mathrm{I}^{even}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{k,\mathbb{R}}) \to \mathrm{I}^{even}_{W(\mathfrak{g}_k,\mathfrak{h}_k)}(\mathfrak{h}_{n,\mathbb{R}})$ *is surjective.*

Remark 2.8. If $\mathfrak{g}_k = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g}_n$ is constructed from $\mathfrak{g}_k$ by removing any $n - k$ simple roots from the Dynkin diagram of $\mathfrak{g}_k$, then part 1 of Theorem 2.7 remains valid because all the Weyl groups are permutation groups. On the other hand, if $\mathfrak{g}_k$ is of type B_k, C_k, or D_k ($k \geqq 3$) and if $\mathfrak{g}_n$ is constructed from $\mathfrak{g}_k$ by removing at least one simple root α_i with $k - i \geqq 2$, then $\mathfrak{g}_n$ contains at least one simple factor $\mathfrak{l}$ of type A_ℓ, $\ell \geqq 2$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{l}$. Then the restriction of the Weyl group of $\mathfrak{g}_k$ to $\mathfrak{a}_\mathbb{R}$ will contain $-\mathrm{id}$. But $-\mathrm{id}$ is not in the Weyl group $W(\mathfrak{sl}(\ell+1, \mathbb{C}))$, and the restriction of the invariants will only contain even polynomials. Hence the conclusion of part 1 in the Theorem fails in this case. ♢

We also note the following consequence of the definition of propagation. It is implicit in the diagrams following that definition.

Lemma 2.9. *Assume that* $\mathfrak{g}_k$ *propagates* $\mathfrak{g}_n$*. Let* $\mathfrak{h}_k$ *be a Cartan subalgebra of* $\mathfrak{g}_k$ *such that* $\mathfrak{h}_n = \mathfrak{h}_k \cap \mathfrak{g}_n$ *is a Cartan subalgebra of* $\mathfrak{g}_n$*. Choose positive systems* $\Delta^+(\mathfrak{g}_k, \mathfrak{h}_k) \subset \Delta(\mathfrak{g}_k, \mathfrak{h}_k)$ *and* $\Delta^+(\mathfrak{g}_n, \mathfrak{h}_n) \subset \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ *such that* $\Delta^+(\mathfrak{g}_n, \mathfrak{h}_n) \subseteqq \Delta^+(\mathfrak{g}_k, \mathfrak{h}_k)|_{\mathfrak{h}_n}$*. Then we can number the simple roots such that*

$$\alpha_{n,j} = \alpha_{k,j}|_{\mathfrak{h}_n}$$

for $j = 1, \ldots, \dim \mathfrak{h}_n$*.*

3. Symmetric Spaces

In this section we discuss restriction of invariant polynomials related to Riemannian symmetric spaces. Let $M = G/K$ be a Riemannian symmetric space of compact or noncompact type. Thus G is a connected semisimple Lie group with an involution θ such that

$$(G^\theta)_o \subseteqq K \subseteqq G^\theta$$

where $G^\theta = \{x \in G \mid \theta(x) = x\}$ and the subscript $_o$ denotes the connected component containing the identity element. If G is simply connected then G^θ is connected and $K = G^\theta$. If G is noncompact and with finite center, then $K \subset G$ is a *maximal compact* subgroup of G, K is connected, and G/K is simply connected.

Denote the Lie algebra of G by $\mathfrak{g}$. Then θ defines an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ is the Lie algebra of K and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.

Cartan Duality is a bijection between the classes of simply connected symmetric spaces of noncompact type and of compact type. On the Lie algebra level this isomorphism is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \leftrightarrow \mathfrak{k} \oplus i\mathfrak{s} = \mathfrak{g}^d$. We denote this bijection by $M \leftrightarrow M^d$.

Fix a maximal abelian subset $\mathfrak{a} \subset \mathfrak{s}$. For $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ let

$$\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_{\mathbb{C}}\}.$$

If $\mathfrak{g}_{\mathbb{C},\alpha} \neq \{0\}$ then α is called a (restricted) root. Denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$ the set of roots. If M is of noncompact type, then all the roots are in the real dual space $\mathfrak{a}^*$ and $\mathfrak{g}_{\mathbb{C},\alpha} = \mathfrak{g}_\alpha + i\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g}$. If M is of compact type, then the roots are purely imaginary on $\mathfrak{a}$, $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset i\mathfrak{a}^*$, and $\mathfrak{g}_{\mathbb{C},\alpha} \cap \mathfrak{g} = \{0\}$. The set of roots is preserved under duality, $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma(\mathfrak{g}^d, i\mathfrak{a})$, where we view those roots as $\mathbb{C}$–linear functionals on $\mathfrak{a}_{\mathbb{C}}$.

If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ it can happen that $\frac{1}{2}\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ or $2\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. Define

$$\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \tfrac{1}{2}\alpha \notin \Sigma(\mathfrak{g}, \mathfrak{a})\}.$$

Then $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a root system in the usual sense and the Weyl group corresponding to $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the same as the Weyl group generated by the reflections s_α, $\alpha \in \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Furthermore, M is irreducible if and only if $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$ is irreducible, i.e., can not be decomposed into two mutually orthogonal root systems.

Let $\Sigma^+(\mathfrak{g}, \mathfrak{a}) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$ be a positive system and $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a}) = \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cap \Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a positive root system in $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. Denote by $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ the set of simple roots in $\Sigma^+_{1/2}(\mathfrak{g}, \mathfrak{a})$. Then $\Psi_{1/2}(\mathfrak{g}, \mathfrak{a})$ is a basis for $\Sigma(\mathfrak{g}, \mathfrak{a})$.

The list of irreducible symmetric spaces is given by the following table. The indices j and k are related by $k = 2j + 1$. In the fifth column we list the realization of K as a subgroup of the compact real form. The second column indicates the type of the root system $\Sigma_{1/2}(\mathfrak{g}, \mathfrak{a})$. (More detailed information is given by the Satake–Tits diagram for M; see [1] or [11, pp. 530–534]. In that classification the case $\mathrm{SU}(p, 1)$, $p \geqq 1$, is denoted by AIV, but here it appears in $AIII$. The case $\mathrm{SO}(p, q)$, $p + q$ odd, $p \geq q > 1$, is denoted by BI as in this case the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p + q, \mathbb{C})$ is of type B. The case $\mathrm{SO}(p, q)$, with $p + q$ even, $p \geq q > 1$ is denoted by DI as in this case $\mathfrak{g}_{\mathbb{C}}$ is of type D. Finally, the case $\mathrm{SO}(p, 1)$, p even, is denoted by BII and $\mathrm{SO}(p, 1)$, p odd, is denoted by DII.)

Irreducible Riemannian Symmetric $M = G/K$, K connected

(3.1)

		G noncompact	G compact	K	Rank$M$	Dim$M$
1	A_j	$\mathrm{SL}(j,\mathbb{C})$	$\mathrm{SU}(j)\times\mathrm{SU}(j)$	$\mathrm{diag}\,\mathrm{SU}(j)$	$j-1$	j^2-1
2	B_j	$\mathrm{SO}(k,\mathbb{C})$	$\mathrm{SO}(k)\times\mathrm{SO}(k)$	$\mathrm{diag}\,\mathrm{SO}(k)$	j	$2j^2+j$
3	D_j	$\mathrm{SO}(2j,\mathbb{C})$	$\mathrm{SO}(2j)\times\mathrm{SO}(2j)$	$\mathrm{diag}\,\mathrm{SO}(2j)$	j	$2j^2-j$
4	C_j	$\mathrm{Sp}(j,\mathbb{C})$	$\mathrm{Sp}(j)\times\mathrm{Sp}(j)$	$\mathrm{diag}\,\mathrm{Sp}(j)$	j	$2j^2+j$
5	$AIII$	$\mathrm{SU}(p,q)$	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$\min(p,q)$	$2pq$
6	AI	$\mathrm{SL}(j,\mathbb{R})$	$\mathrm{SU}(j)$	$\mathrm{SO}(j)$	$j-1$	$\frac{(j-1)(j+2)}{2}$
7	AII	$\mathrm{SU}^*(2j)$	$\mathrm{SU}(2j)$	$\mathrm{Sp}(j)$	$j-1$	$2j^2-j-1$
8	BDI	$\mathrm{SO}_o(p,q)$	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\min(p,q)$	pq
9	$DIII$	$\mathrm{SO}^*(2j)$	$\mathrm{SO}(2j)$	$\mathrm{U}(j)$	$[\frac{j}{2}]$	$j(j-1)$
10	CII	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\min(p,q)$	$4pq$
11	CI	$\mathrm{Sp}(j,\mathbb{R})$	$\mathrm{Sp}(j)$	$\mathrm{U}(j)$	j	$j(j+1)$

Only in the following cases do we have $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a}) \neq \Sigma(\mathfrak{g},\mathfrak{a})$:

- $AIII$ for $1 \leqq p < q$,
- CII for $1 \leqq p < q$, and
- $DIII$ for j odd.

In those three cases there is exactly one simple root with $2\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ and this simple root is at the right end of the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$. Also, either $\Psi_{1/2}(\mathfrak{g},\mathfrak{a}) = \{\alpha\}$ contains one simple root or $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type B_r where $r = \dim\mathfrak{a}$ is the rank of M.

Finally, the only two cases where $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D are the case $\mathrm{SO}(2j,\mathbb{C})/\mathrm{SO}(2j)$ or the split case $\mathrm{SO}_o(p,p)/\mathrm{SO}(p)\times\mathrm{SO}(p)$.

Later on we will also need the root system $\Sigma_2(\mathfrak{g},\mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \mid 2\alpha \notin \Sigma(\mathfrak{g},\mathfrak{a})\}$. According to the above discussion, this will only change the simple root at the right end of the Dynkin diagram. If $\Psi_2(\mathfrak{g},\mathfrak{a})$ is of type B the root system $\Sigma_2(\mathfrak{g},\mathfrak{a})$ will be of type C.

Let G/K be an irreducible symmetric space of compact or non-compact type. As before let $\mathfrak{a} \subset \mathfrak{s}$ be maximal abelian. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{k})\oplus\mathfrak{a}$. Let $\Delta(\mathfrak{g},\mathfrak{h})$, $\Sigma(\mathfrak{g},\mathfrak{a})$, and $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$ denote the corresponding root systems and $W(\mathfrak{g},\mathfrak{h})$ respectively $W(\mathfrak{g},\mathfrak{a})$ the Weyl group corresponding to $\Delta(\mathfrak{g},\mathfrak{h})$ respectively $\Sigma(\mathfrak{g},\mathfrak{a})$. We define an extension of those Weyl groups $\widetilde{W}(\mathfrak{g},\mathfrak{h})$ and $\widetilde{W}(\mathfrak{g},\mathfrak{a})$ in the following way: If the root system in question (i.e., $\Delta(\mathfrak{g},\mathfrak{h})$ or $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$) is not of type D then $\widetilde{W}$ is just the Weyl group. If the root system is of type D, so the Weyl group elements involve only even numbers of sign changes, then $\widetilde{W}$ is the $\mathbb{Z}_2$–extension of the Weyl group allowing all sign changes. Denote $\widetilde{W}_{\mathfrak{a}}(\mathfrak{g},\mathfrak{h}) = \{w \in \widetilde{W}(\mathfrak{g},\mathfrak{a}) \mid w(\mathfrak{a}) = \mathfrak{a}\}$.

Note $\widetilde{W}(\mathfrak{g},\mathfrak{a}) \neq W(\mathfrak{g},\mathfrak{a})$ only for M locally isomorphic to $\mathrm{SO}(2j,\mathbb{C})/\mathrm{SO}(2j)$ (where $\mathfrak{h} = \mathfrak{a}_{\mathbb{C}}$) or its compact dual $(SO(2j) \times SO(2j))/\mathrm{diag}\, SO(2j)$ (where $\mathfrak{h} \cong \mathfrak{a} \oplus \mathfrak{a}$), or to $\mathrm{SO}_o(j,j)/\mathrm{SO}(j) \times \mathrm{SO}(j)$ or its compact dual $\mathrm{SO}(2j)/\mathrm{SO}(j) \times \mathrm{SO}(j)$ where $\mathfrak{h} = \mathfrak{a}$.

If G/K is reducible without Euclidean factors then the Weyl groups are direct products of Weyl groups for the irreducible factors. Then $\widetilde{W}(\mathfrak{g},\mathfrak{h})$ and $\widetilde{W}(\mathfrak{g},\mathfrak{a})$ denote the corresponding products of the extended Weyl groups for each irreducible factor.

Theorem 3.2. *Let G/K be a symmetric space of compact or non-compact type (thus no Euclidean factors). In the above notation, $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = \widetilde{W}_{\mathfrak{a}}(\mathfrak{g},\mathfrak{h})|_{\mathfrak{a}}$ and the restriction map $I_{\widetilde{W}(\mathfrak{g},\mathfrak{h})}(\mathfrak{h}_{\mathbb{R}}) \to I_{\widetilde{W}(\mathfrak{g},\mathfrak{a})}(\mathfrak{a})$ is surjective.*

Proof. We can assume that G/K is irreducible. If neither $\Delta(\mathfrak{g},\mathfrak{h})$ nor $\Sigma(\mathfrak{g},\mathfrak{a})$ is of type D this is Theorem 5 from [8]. According to the above discussion, the only cases where $\Sigma(\mathfrak{g},\mathfrak{a})$ is of type D are where $\Delta(\mathfrak{g},\mathfrak{h})$ is also of type D and $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}}$, or $\mathfrak{a}$ is the diagonal in $\mathfrak{h} \cong \mathfrak{a} \oplus \mathfrak{a}$, or $\mathfrak{a} = \mathfrak{h}$. The statement is clear when $\mathfrak{a}$ is $\mathfrak{h}$ or $\mathfrak{h}_{\mathbb{R}}$. If $\mathfrak{a}$ is the diagonal in $\mathfrak{h} \cong \mathfrak{a} \oplus \mathfrak{a}$ then $\widetilde{W}_{\mathfrak{a}}(\mathfrak{g},\mathfrak{h})$ is the diagonal in $\widetilde{W}(\mathfrak{g},\mathfrak{h}) \cong \widetilde{W}(\mathfrak{g},\mathfrak{a}) \times \widetilde{W}(\mathfrak{g},\mathfrak{a})$, hence again is $\widetilde{W}(\mathfrak{g},\mathfrak{a})$.

Now suppose that neither $\Delta(\mathfrak{g},\mathfrak{h})$ nor $\Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D. Then $\widetilde{W}(\mathfrak{g},\mathfrak{a}) = W(\mathfrak{g},\mathfrak{a})$ consists of all permutations with sign changes (with respect to the correct basis). The claim now follows from the explicit calculations in [8, pp. 594, 596]. □

Let $M_k = G_k/K_k$ and $M_n = G_n/K_n$ be irreducible symmetric spaces of compact or noncompact type. We say that M_k *propagates* M_n, if $G_n \subsetneqq G_k$, $K_n = K_k \cap G_n$, and either $\mathfrak{a}_k = \mathfrak{a}_n$ or choosing $\mathfrak{a}_n \subsetneqq \mathfrak{a}_k$ we only add simple roots to the left end of the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ to obtain the Dynkin diagram for $\Psi_{1/2}(\mathfrak{g}_k,\mathfrak{a}_k)$. So, in particular $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ and $\Psi_{1/2}(\mathfrak{g}_k,\mathfrak{a}_k)$ are of the same type. In general, if M_k and M_n are Riemannian symmetric spaces of compact or noncompact type, with universal covering $\widetilde{M}_k$ respectively $\widetilde{M}_n$, then M_k *propagates* M_n if we can enumerate the irreducible factors of $\widetilde{M}_k = M_k^1 \times \ldots \times M_k^j$ and $\widetilde{M}_n = M_n^1 \times \ldots \times M_n^i$, $i \leqq j$ so that M_k^s propagates M_n^s for $s = 1,\ldots,i$. Thus, each M_n is, up to covering, a product of irreducible factors listed in Table 3.1.

In general we can construct infinite sequences of propagations by moving along each row in Table 3.1. But there are also inclusions like $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) \subset \mathrm{SL}(k,\mathbb{C})/\mathrm{SU}(k)$ which satisfy the definition of propagation.

When $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$, and θ_k and θ_n are the corresponding involutions with $\theta_k|_{\mathfrak{g}_n} = \theta_n$, the corresponding eigenspace decompositions $\mathfrak{g}_k = \mathfrak{k}_k \oplus \mathfrak{s}_k$ and $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{s}_n$ give us

$$\mathfrak{k}_n = \mathfrak{k}_k \cap \mathfrak{g}_n\,, \quad \text{and} \quad \mathfrak{s}_n = \mathfrak{g}_n \cap \mathfrak{s}_k\,.$$

We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subset \mathfrak{s}_k$ such that $\mathfrak{a}_n \subseteqq \mathfrak{a}_k$ for $k \geqq n$. Denote by $W(\mathfrak{g}_n,\mathfrak{a}_n)$ and $W(\mathfrak{g}_k,\mathfrak{a}_k)$ the corresponding Weyl groups. The extensions

$\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)$ and $\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)$ are defined as just before Theorem 3.2. Let $\mathrm{I}(\mathfrak{a}_n)=\mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$, $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$, and $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)$ denote the respective sets of Weyl group invariant or $\widetilde{W}$–invariant polynomials on $\mathfrak{a}_n$ and $\mathfrak{a}_k$. As before we let

$$W_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k):=\{w\in W(\mathfrak{g}_k,\mathfrak{a}_k)\mid w(\mathfrak{a}_n)=\mathfrak{a}_n\} \tag{3.3}$$

and define $\widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)$ in the same way.

Theorem 3.4. *Assume that M_k and M_n are symmetric spaces of compact or noncompact type and that M_k propagates M_n.*

(1) *If M_n does not contain any irreducible factor with $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ of type D, then*

$$W_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)|_{\mathfrak{a}_n}=W(\mathfrak{g}_n,\mathfrak{a}_n) \tag{3.5}$$

and the restriction map $\mathrm{I}(\mathfrak{a}_k)\to\mathrm{I}(\mathfrak{a}_n)$ is surjective.

(2) *If $\Psi_{1/2}(\mathfrak{g}_n,\mathfrak{a}_n)$ is of type D then*

$$W(\mathfrak{g}_n,\mathfrak{a}_n)\subsetneqq W_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)|_{\mathfrak{a}_n}\ \text{and}\ \mathrm{I}_{W(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n}\subsetneqq \mathrm{I}_{W(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n).$$

On the other hand $\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)=\widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)|_{\mathfrak{a}_n}$ and $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)=\mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n}$.

(3) *In all cases $\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)=\widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)|_{\mathfrak{a}_n}$ and $\mathrm{I}_{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(\mathfrak{a}_k)|_{\mathfrak{a}_n}=\mathrm{I}_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)$.*

Proof. It suffices to prove this for each irreducible component of M_n. The argument of Theorem 2.7 is valid here as well, and our assertion follows. □

4. Application to Fourier Analysis on Symmetric Spaces of the noncompact Type

In this section we apply the above results to harmonic analysis on symmetric spaces. We start by recalling the main ingredients for the Helgason Fourier transform on a Riemannian symmetric space G/K of the noncompact type. The material is standard and we refer to the books of Helgason, in particular [12], for details. We use the notation from the previous section: $\Sigma(\mathfrak{g},\mathfrak{a})$ is the set of (restricted) roots of $\mathfrak{a}$ in $\mathfrak{g}$ and $\Sigma^+(\mathfrak{g},\mathfrak{a})\subset\Sigma(\mathfrak{g},\mathfrak{a})$ is a positive system. Let

$$\mathfrak{n}=\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\mathfrak{g}_\alpha,\quad \mathfrak{m}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}),\quad\text{and}\quad \mathfrak{p}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n}.$$

Denote by N (respectively A) the analytic subgroup of G with Lie algebra $\mathfrak{n}$ (respectively $\mathfrak{a}$). Let $M=Z_K(\mathfrak{a})$ and $P=MAN$. Then M and P are closed subgroup of G and P is a *minimal parabolic subgroup.* Note, that we are using M in two different ways, once as the symmetric space G/K and also as a subgroup of G. The meaning will always be clear from the context.

We have the Iwasawa decomposition

$$G = NAK : \ C^\omega\text{–diffeomorphic to } N\times A\times K \text{ under } (n,a,k)\mapsto nak\,.$$

For $x\in G$ denote by $a(x)\in A$ the unique element in A such that $x\in Na(x)K$. Then $x\mapsto a(x)$ is analytic. For $\lambda\in\mathfrak{a}_\mathbb{C}^*$ let

$$a^\lambda := e^{\lambda(\log(a))}\,.$$

Then the characters on A are given by

$$a\mapsto \chi_\lambda(a) := a^\lambda$$

for some $\lambda\in\mathfrak{a}_\mathbb{C}^*$. χ_λ is unitary if and only if $\lambda\in i\mathfrak{a}^*$. Let $m_\alpha=\dim\mathfrak{g}_\alpha$ and

$$\rho = \tfrac{1}{2}\sum_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})} m_\alpha\,\alpha\,.$$

For a moment let G be a locally compact topological group and $K\subset G$ a compact subgroup A continuous, non-zero, K–biinvariant function $\varphi : G\to\mathbb{C}$ is K-*spherical* or just *spherical* if for all $x,y\in G$ we have

$$\int_K \varphi(xky)\,dk = \varphi(x)\varphi(y)\,.$$

We will view the spherical functions on G as K–invariant functions on G/K. The importance of the spherical functions comes from the fact that a map $\chi : L^1(G/K)^K\to\mathbb{C}$ is a continuous algebra homomorphism if and only if there exists a bounded spherical function φ such that $\chi(f)=\int_{G/K} f(x)\overline{\varphi(x)}\,dx$. The spherical function φ is called *positive definite* if for all $n\in\mathbb{N}$, $c_j\in\mathbb{C}$, $x_j\in G$, $1\leqq j\leqq n$ we have

$$\sum_{\nu,\mu=1}^{n} c_\nu\overline{c_\mu}\varphi(x_\nu^{-1}x_\mu)\geqq 0\,,$$

in other words, if the matrix $(\varphi(x_\nu^{-1}x_\mu))_{\nu,\mu}$ is positive semidefinite for finite subsets $\{x_1,\dots,x_n\}$ of G. The positive definite spherical functions are particular coefficient functions

$$\varphi(g) = (e_\pi, \pi(g)e_\pi) \tag{4.1}$$

where π is an irreducible unitary representation with nonzero K–fixed vectors and $e_\pi\in V_\pi$ is K–fixed unit vector.

For $\lambda\in\mathfrak{a}_\mathbb{C}^*$ define

$$\varphi_\lambda(x) = \int_K \chi_{\lambda-\rho}(a(kx))\,dk \tag{4.2}$$

where the Haar measure dk on K is normalized by $\int_K dk=1$. Then φ_λ is a *spherical function* on G, $\varphi_\lambda=\varphi_\mu$ if and only if $\mu\in W(\mathfrak{g},\mathfrak{a})'\cdot\lambda$, and every spherical function is equal to some φ_λ. Here $'$ stands for the transpose of the elements in $W(\mathfrak{g},\mathfrak{a})$ acting on $\mathfrak{a}^*$ and $\mathfrak{a}^*_\mathbb{C}$ The function φ_λ is positive definite when $\lambda \in i\mathfrak{a}^*$.

The *spherical Fourier transform* on M is given by

$$\mathcal{F}(f)(\lambda) = \widehat{f}(\lambda) := \int_M f(x)\varphi_{-\lambda}(x)\,dx \qquad f \in C_c^\infty(M)^K\,.$$

The invariant measure dx on M can be normalized so that the spherical Fourier transform extends to an unitary isomorphism

$$f \mapsto \widehat{f}\,, \quad L^2(M)^K \cong L^2(i\mathfrak{a}^*_+, |c(\lambda)|^{-2}d\lambda) \cong L^2\left(i\mathfrak{a}^*, \tfrac{d\lambda}{\#W(\mathfrak{g},\mathfrak{a})|c(\lambda)|^2}\right)^{W(\mathfrak{g},\mathfrak{a})}$$

where $\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})\}$ and $c(\lambda)$ denotes the Harish-Chandra c–function. For $f \in C_c^\infty(M)^K$ the inversion is given by

$$f(x) = \frac{1}{\#W(\mathfrak{g},\mathfrak{a})}\int_{i\mathfrak{a}^*} \widehat{f}(\lambda)\varphi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}\,.$$

If $\sigma : \mathfrak{a} \to \mathfrak{a}$ is linear, then $\sigma' : \mathfrak{a}^*_\mathbb{C} \to \mathfrak{a}^*_\mathbb{C}$ is its transpose, $\sigma'(\lambda)(H) = \lambda(\sigma(H))$.

Recall the notation from the last section. If $\Sigma(\mathfrak{g},\mathfrak{a}) = \Sigma_{1/2}(\mathfrak{g},\mathfrak{a})$ then $\Psi(\mathfrak{g},\mathfrak{a})$ is the simple root system.

A connected semisimple Lie group G is *algebraically simply connected* if it is an analytic subgroup of the connected simply connected group $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{C}$. Then the analytic subgroup K of G for $\mathfrak{k}$ is compact, and every automorphism of $\mathfrak{g}$ integrates to an automorphism of G.

Lemma 4.3. *Let G/K be a Riemannian symmetric space of noncompact type with G simple and algebraically simply connected. Suppose that $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e., that $\mathfrak{g}$ is a split real form of $\mathfrak{g}_\mathbb{C}$. If $\sigma : \mathfrak{a} \to \mathfrak{a}$ is a linear isomorphism such that σ' defines an automorphism of the Dynkin diagram of $\Psi(\mathfrak{g},\mathfrak{a})$, then there exists a automorphism $\widetilde{\sigma} : G \to G$ such that*

(1) *$\widetilde{\sigma}|_\mathfrak{a} = \sigma$ where by abuse of notation we write $\widetilde{\sigma}$ for $d\widetilde{\sigma}$,*
(2) *$\widetilde{\sigma}$ commutes with the the Cartan involution θ, and in particular $\widetilde{\sigma}(K) = K$,*
(3) *$\widetilde{\sigma}(N) = N$.*

Proof. The complexification of $\mathfrak{a}$ is a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_\mathbb{C}$ such that $\mathfrak{h}_\mathbb{R} = \mathfrak{a}$. Let $\{Z_\alpha\}_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})}$ be a Weyl basis for $\mathfrak{g}_\mathbb{C}$ (see, for example, [20, page 285]). Then (see, for example, [20, Theorem 4.3.26]),

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathbb{R}Z_\alpha$$

is a real form of $\mathfrak{g}_\mathbb{C}$. Denote by B the Killing form of $\mathfrak{g}_\mathbb{C}$. Then $B(Z_\alpha, Z_{-\alpha}) = -1$ and it follows that B is positive definite on $\mathfrak{a}$ and on $\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\mathbb{R}(Z_\alpha - Z_{-\alpha})$, and negative definite on $\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\mathbb{R}(Z_\alpha+Z_{-\alpha})$. Hence, the map

$$\theta|_{\mathfrak{a}}=-\mathrm{id} \text{ and } \theta(Z_\alpha)=Z_{-\alpha}$$

defines a Cartan involution on $\mathfrak{g}_0$ such that the Cartan subalgebra $\mathfrak{a}$ is contained in the corresponding -1 eigenspace $\mathfrak{s}$. As there is (up to isomorphism) only one real form of $\mathfrak{g}_{\mathbb{C}}$ with Cartan involution such that $\mathfrak{a}\subset\mathfrak{s}$ we can assume that $\mathfrak{g}=\mathfrak{g}_0$ and the above Cartan involution θ is the the one we started with.

Going back to the proof of [20, Lemma 4.3.24] the map defined by

$$\widetilde{\sigma}|_{\mathfrak{a}}=\sigma \quad \text{and} \quad \widetilde{\sigma}(Z_\alpha)=Z_{\sigma\alpha}$$

is a Lie algebra isomorphism $\widetilde{\sigma}:\mathfrak{g}\to\mathfrak{g}$. But then

$$\widetilde{\sigma}(\theta(Z_\alpha))=\widetilde{\sigma}(Z_{-\alpha})=Z_{\sigma(-\alpha)}=Z_{-\sigma(\alpha)}=\theta(\widetilde{\sigma}(Z_\alpha)).$$

Finally, $\theta|_{\mathfrak{a}}=-\mathrm{id}$ and it follows that $\widetilde{\sigma}$ and θ commute. As

$$\mathfrak{k}=\bigoplus_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\mathbb{R}(Z_\alpha+\theta(Z_\alpha))$$

and $\sigma(\Sigma^+(\mathfrak{g},\mathfrak{a}))=\Sigma^+(\mathfrak{g},\mathfrak{a})$ it follows that $\widetilde{\sigma}(\mathfrak{k})=\mathfrak{k}$.

As $\sigma'(\Sigma^+(\mathfrak{g},\mathfrak{a}))=\Sigma^+(\mathfrak{g},\mathfrak{a})$ it follows that $\widetilde{\sigma}(\mathfrak{n})=\mathfrak{n}$.

As G is assumed to be algebraically simply connected, there is an automorphism of G with differential $\widetilde{\sigma}$. Denote this automorphism also by $\widetilde{\sigma}$. It is clear that $\widetilde{\sigma}$ satisfies the assertions of the Lemma. □

Theorem 4.4. *Let G/K be a Riemannian symmetric space of noncompact type with G simple and algebraically simply connected. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\sigma:\mathfrak{a}\to\mathfrak{a}$ a linear isomorphism such that σ' defines an automorphism of the Dynkin diagram of $\Psi(\mathfrak{g},\mathfrak{a})$. Then for $x\in G$*

$$\varphi_\lambda(\widetilde{\sigma}(x))=\varphi_{\sigma'(\lambda)}(x)\,.$$

If $f\in L^2(G/K)^K$ is such that f is $\widetilde{\sigma}$-invariant, then $\mathcal{F}(f)$ is σ'-invariant.

Proof. Let $\widetilde{\sigma}:G\to G$ be the automorphism of from Lemma 4.3. As $\widetilde{\sigma}(K)=K$, $\widetilde{\sigma}(A)=A$, and $\widetilde{\sigma}(N)=N$, it follows that if $x=n(x)a(x)k(x)$ is the Iwasawa decomposition of $x\in G$, then

$$\widetilde{\sigma}(x)=\widetilde{\sigma}(n(x))\widetilde{\sigma}(a(x))\widetilde{\sigma}(k(x))$$

is the corresponding decomposition of $\widetilde{\sigma}(x)$. By (4.2) and the fact that $\sigma'(\rho) = \rho$ we get for $x \in G$:

$$\begin{aligned}\varphi_\lambda(\widetilde{\sigma}(x)) &= \int_K \chi_{\lambda-\rho}(a(k\widetilde{\sigma}(x)))\,dk \\ &= \int_K \chi_{\lambda-\rho}(\widetilde{\sigma}(a(kx)))\,dk \\ &= \int_K \chi_{\sigma'(\lambda-\rho)}(a(kx))\,dk = \varphi_{\sigma'(\lambda)}(x)\end{aligned}$$

where we have used that the Haar measure on K is invariant under $\widetilde{\sigma}|_K$.

The first statement follows now by applying this to $x = \exp(H)$, $H \in \mathfrak{a}$. Using $f \circ \widetilde{\sigma} = f$, the second statement follows because G-invariant measure on G/K is is $\widetilde{\sigma}$-invariant and we can assume that $f \in C_c^\infty(G/K)$. □

Fix a positive definite K–invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$. It defines an invariant Riemannian structure on M and hence also an invariant metric $d(x, y)$. Let $x_o = eK \in M$ and for $r > 0$ denote by $B_r = B_r(x_o)$ the closed ball

$$B_r = \{x \in M \mid d(x, x_o) \leqq r\}\,.$$

Note that B_r is K–invariant. Denote by $C_r^\infty(M)^K$ the space of smooth K–invariant functions on M with support in B_r. The restriction map $f \mapsto f|_A$ is a bijection from $C_r^\infty(M)^K$ onto $C_r^\infty(A)^{W(\mathfrak{g},\mathfrak{a})}$ (using the obvious notation). Define $\widetilde{W}(\mathfrak{g},\mathfrak{a})$ as just before Theorem 3.4. Let $C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(M)^K$ be the preimage of $C_r^\infty(A)^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ in $C_r^\infty(M)^K$. In particular $C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(M)^K$ is just $C_r^\infty(M)^K$ if there is no irreducible factor for which $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D. In the case $\Psi_{1/2}(\mathfrak{g},\mathfrak{a})$ is of type D we can replace G by $G_{\widetilde{\sigma}} = G \ltimes \{1, \widetilde{\sigma}\}$ and K by $K_{\widetilde{\sigma}} = K \ltimes \{1, \widetilde{\sigma}\}$. Then $M = G_{\widetilde{\sigma}}/K_{\widetilde{\sigma}}$ and $C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(M)^K = C_r^\infty(M)^{K_{\widetilde{\sigma}}}$. This corresponds to replacing $\mathrm{SO}(2j, \mathbb{C})$ by $\mathrm{O}(2j, \mathbb{C})$. In that case we choose

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 X & & \\ & \ddots & \\ & & t_n X \end{pmatrix} \middle|\, t_1, \dots, t_n \in \mathbb{R} \right\} \text{ where } X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and then then $\widetilde{\sigma}$ is conjugation by $\mathrm{diag}(1, \dots, 1, -1)$.

Denote by $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)$ the Paley-Wiener space on $\mathfrak{a}_\mathbb{C}^*$ and by $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ the space of $\widetilde{W}(\mathfrak{g},\mathfrak{a})$–invariant functions in $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)$.

The following is a simple modification of the Paley-Wiener theorem of Helgason [9, 12] and Gangolli [5]; see [15] for a short overview.

Theorem 4.5 (The Paley-Wiener Theorem)**.** *The Fourier transform defines bijections*

$$C_r^\infty(M)^K \cong \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{W(\mathfrak{g},\mathfrak{a})} \text{ and } C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(M)^K \cong \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}\,.$$

Proof. For the proof we need only consider factors of type $\mathrm{SO}(2j,\mathbb{C})/\mathrm{SO}(2j)$ and of type $\mathrm{SO}(j,j)/\mathrm{SO}(j)\times\mathrm{SO}(j)$. It is enough to show that in both cases we have

$$\varphi_\lambda(\exp(w(H))) = \varphi_{w'\lambda}(\exp H)$$

for all $w\in\widetilde{W}(\mathfrak{g},\mathfrak{a})$. This is well known for $w\in W(\mathfrak{g},\mathfrak{a})$. Note that in both cases the root system $\Sigma(\mathfrak{g},\mathfrak{a})=\Delta(\mathfrak{g},\mathfrak{h})$ is of type D.

In the first case, the spherical function is given (see [9], p. 432) by

$$\varphi_\lambda(a) = \frac{\varpi(\rho)}{\varpi(\lambda)}\frac{\sum_{w\in W(\mathfrak{g},\mathfrak{a})}(\det w)a^{w'\lambda}}{\prod_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}(a^{\alpha/2}-a^{-\alpha/2})} \quad\text{where}\quad \varpi(\lambda)=\prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle\,,$$

and the claim follows by direct calculation.

For the second case we recall first that the Weyl group contains all *even* sign changes and permutations, whereas $\widetilde{W}(\mathfrak{g},\mathfrak{a})$ contains all sign changes and the permutations. Thus, fixing one element $\sigma\in\widetilde{W}(\mathfrak{g},\mathfrak{a})\setminus W(\mathfrak{g},\mathfrak{h})$ we have

$$\widetilde{W}(\mathfrak{g},\mathfrak{a}) = W(\mathfrak{g},\mathfrak{a})\cup\sigma W(\mathfrak{g},\mathfrak{a})$$

Using the notation for the root system D_j in Section 2 we take

$$\sigma(f_1)=-f_1 \quad\text{and}\quad \sigma(f_k)=f_k\,,\quad k=2,\ldots,j\,.$$

Then $\sigma(\alpha_1)=\alpha_2$, $\sigma(\alpha_2)=\alpha_1$ and $\sigma(\alpha_i)=\alpha_i$ for all $i\geq 3$. Thus σ defines an automorphism of the Dynkin diagram for $\Psi(\mathfrak{g},\mathfrak{a})$. The statement follows now from Theorem 4.4. □

We assume now that M_k propagates M_n, $k\geqq n$. The index j refers to the symmetric space M_j, for a function F on $\mathfrak{a}^*_{k,\mathbb{C}}$ let $P_{k,n}(F):=F|_{\mathfrak{a}_{n,\mathbb{C}}}$. We fix a compatible K–invariant inner products on $\mathfrak{s}_n$ and $\mathfrak{s}_k$, i.e., for all $X,Y\in\mathfrak{s}_n\subseteqq\mathfrak{s}_k$ we have

$$\langle X,Y\rangle_k = \langle X,Y\rangle_n\,.$$

Theorem 4.6 (Paley-Wiener Isomorphisms)**.** *Assume that M_k propagates M_n. Let $r>0$. Then the following holds:*

1. *The map $P_{k,n}:\mathrm{PW}_r(\mathfrak{a}^*_{k,\mathbb{C}})^{\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}\to\mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}})^{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}$ is surjective.*
2. *The map $C_{k,n}=\mathcal{F}_n^{-1}\circ P_{k,n}\circ\mathcal{F}_k : C^\infty_{r,\widetilde{W}(\mathfrak{g}_k,\mathfrak{a}_k)}(M_k)^{K_k}\to C^\infty_{r,\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(M_n)^{K_n}$ is surjective.*

Proof. This follows from Theorem 1.6 and Theorem 4.5. □

We assume now that $\{M_n,\iota_{k,n}\}$ is a injective system of symmetric spaces such that M_k is a propagation of M_n. Here $\iota_{k,n}:M_n\to M_k$ is the injection. Let

$$M_\infty = \varinjlim M_n\,.$$

We have also, in a natural way, injective systems $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_k$, $\mathfrak{k}_n \hookrightarrow \mathfrak{k}_k$, $\mathfrak{s}_n \hookrightarrow \mathfrak{s}_k$, and $\mathfrak{a}_n \hookrightarrow \mathfrak{a}_k$ giving rise to corresponding injective systems. Let

$$\mathfrak{g}_\infty := \varinjlim \mathfrak{g}_n\,, \quad \mathfrak{k}_\infty := \varinjlim \mathfrak{k}_n\,, \quad \mathfrak{s}_\infty := \varinjlim \mathfrak{s}_n\,, \quad \mathfrak{a}_\infty := \varinjlim \mathfrak{a}_n\,, \quad \text{and} \quad \mathfrak{h}_\infty := \varinjlim \mathfrak{h}_n\,.$$

Then $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{s}_\infty$ is the eigenspace decomposition of $\mathfrak{g}_\infty$ with respect to the involution $\theta_\infty := \varinjlim \theta_n$, $\mathfrak{a}_\infty$ is a maximal abelian subspace of $\mathfrak{s}_\infty$, and $\mathfrak{h}_\infty$ is a Cartan subalgebra of $\mathfrak{g}_\infty$.

The restriction maps $\mathrm{res}_{k,n}$ and the maps from Theorem 4.6 define projective systems $\{I_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)\}_n$, $\{\mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}})^{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}\}_n$, and $\{C_{r,\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(M_n)^{K_n}\}_n$. Let

$$\begin{aligned}
I_\infty(\mathfrak{a}_\infty) &:= \varprojlim I_{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(\mathfrak{a}_n)\\
\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}}) &:= \varprojlim \mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}})^{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}\\
C^\infty_{r,\infty}(M_\infty)^{K_\infty} &:= \varprojlim C^\infty_{r,\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)}(M_n)^{K_n}\,.
\end{aligned}$$

We note, that by the explicit construction in Section 2, there is a canonical inclusion $\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n) \overset{\iota_{k,n}}{\hookrightarrow} \widetilde{W}_{\mathfrak{a}_n}(\mathfrak{g}_k,\mathfrak{a}_k)$ such that $\iota_{k,n}(s)|_{\mathfrak{a}_n} = s$. In this way, we get an injective system $\{\widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)\}_n$. Let

$$\widetilde{W}_\infty = \varinjlim \widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)\,.$$

Then we can view $I_{\widetilde{W}_\infty}(\mathfrak{a}_\infty)$ as $\widetilde{W}_\infty$–invariant polynomials on $\mathfrak{a}^{\infty *}_{\mathbb{C}}$ and $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})$ as $\widetilde{W}_\infty$–invariant functions on $\mathfrak{a}^{\infty *}_{\mathbb{C}}$. The projective limit $C^\infty_{r,\infty}(M_\infty)^{K_\infty}$ consists of functions on on $A_\infty = \varinjlim A_n$, where $A_n = \exp \mathfrak{a}_n$. In Section 8 we discuss a direct limit function space on M_∞ that is more closely related to the representation theory of G_∞.

For $\mathbf{f} = (f_n)_n \in C^\infty_{r,\infty}(M_\infty)^{K_\infty}$ define $\mathcal{F}_\infty(\mathbf{f}) \in \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})$ by

$$\mathcal{F}_\infty(\mathbf{f}) := \{\mathcal{F}_n(f_n)\}\,. \tag{4.7}$$

Simplify the notation by setting $\widetilde{W}_n = \widetilde{W}(\mathfrak{g}_n,\mathfrak{a}_n)$. Then $\mathcal{F}_\infty(\mathbf{f})$ is well defined by Theorem 4.6 and we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots & C^\infty_{r,\widetilde{W}_n}(M_n)^{K_n} & \xleftarrow{C_{n+1,n}} & C^\infty_{r,\widetilde{W}_{n+1}}(M_{n+1})^{K_{n+1}} & \xleftarrow{C_{n+2,n+1}} & \cdots & C^\infty_{r,\infty}(M_\infty)^{K_\infty}\\
 & \downarrow{\scriptstyle \mathcal{F}_n} & & \downarrow{\scriptstyle \mathcal{F}_{n+1}} & & & \downarrow{\scriptstyle \mathcal{F}_\infty}\\
\cdots & \mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}})^{\widetilde{W}_n} & \xleftarrow[P_{n+1,n}]{} & \mathrm{PW}_r(\mathfrak{a}^*_{n+1,\mathbb{C}})^{\widetilde{W}_{n+1}} & \xleftarrow[P_{n+2,n+1}]{} & \cdots & \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})
\end{array}$$

Theorem 4.8 (Infinite dimensional Paley-Wiener Theorem)**.** *Let the notation be as above. Then* $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}}) \neq \{0\}$, $C^\infty_{r,\infty}(M_\infty)^{K_\infty} \neq \{0\}$ *and*

$$\mathcal{F}_\infty : C^\infty_{r,\infty}(M_\infty)^{K_\infty} \to \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})$$

is a linear isomorphism.

5. Central Functions on Compact Lie Groups

The following results on compact Lie groups are a special case of the more general statements on compact symmetric spaces discussed in the next section, as every group can be viewed as a symmetric space $G \times G/\mathrm{diag}(G)$ via the map

$$(g,1)\mathrm{diag}(G) \mapsto g, \text{ in other words } (a,b)\mathrm{diag}(G) \mapsto ab^{-1}$$

corresponding to the involution $\tau(a,b) = (b,a)$. The action of $G\times G$ is the left-right action $(L \times R)(a,b) \cdot x = axb^{-1}$ and the $\mathrm{diag}(G)$–invariant functions are the *central* functions $f(axa^{-1}) = f(x)$ for all $a, x \in G$. Thus f is central if and only if $f \circ \mathrm{Ad}(a) = f$ for all $a \in G$, where as usual $\mathrm{Ad}(a)(x) = axa^{-1}$. From now on, if E is a function space on G, then E^G denotes the space of central functions in E. But, because of the special role played by the group and the central functions, it is worthwhile to discuss this case separately.

In this section G, G_n and G_k will denote a compact connected semisimple Lie group. For simplicity, we will assume that those groups are simply connected. The only change that need to be made for the general case is to change the semi-lattice of highest weights of irreducible representations and the injectivity radius, whose numerical value does not play an important rule in the following. We say that G_k propagates G_n if $\mathfrak{g}_k$ propagates $\mathfrak{g}_n$. This is the same as saying that G_k propagates G_n as a symmetric space. We fix a Cartan subalgebra $\mathfrak{h}_k$ of $\mathfrak{g}_k$ such that $\mathfrak{h}_n := \mathfrak{h}_k \cap \mathfrak{g}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$. We use the notation from the previous section. In the following we will introduce notations for G. The index n respectively k will then denote the corresponding object for G_n respectively G_k. We fix an inner product $\langle\cdot,\cdot\rangle_n$ respectively $\langle\cdot,\cdot\rangle_k$ on $\mathfrak{g}_n$ respectively $\mathfrak{g}_k$ such that $\langle X, Y\rangle_n = \langle X, Y\rangle_k$ for $X, Y \in \mathfrak{g}_n \subsetneqq \mathfrak{g}_k$. This can be done by viewing $G_n \subset G_k$ as locally isomorphic to linear groups and use the trace form $X, Y \mapsto -\mathrm{Tr}\,(XY)$. We denote by R the injectivity radius; Theorem 5.4 below show that the injectivity radius is the same for G_n and G_k.

The following is a reformulation of results of Crittenden [4].

Theorem 5.1. *The minimum locus and the first conjugate locus of G coincide and are given by $\mathrm{Ad}(G)\mathfrak{f}$ where $\mathfrak{f} = \{X \in \mathfrak{h} \mid \max_{\alpha\in\Delta} |\alpha(X)| = 2\pi\}$, and the injectivity radius $R = \min_{X\in\mathfrak{f}} \|X\|$.*

Remark 5.2. Crittenden actually proves the analogous result for symmetric spaces of compact type. That will be used in Section 7. ♢

Proof. The "roots" of [4] are the "angular parameters" of Hopf and Stiefel: the formulae of [4, Section 2] shows that they are $2\pi i$ times what we now call roots. Thus [4, Theorem 3] says that $X \in \mathfrak{h}$ belongs to the conjugate locus just when there is a root $\alpha \in \Delta$ such that $|\alpha(X)|$ is a nonzero multiple of 2π. So $X \in \mathfrak{h}$ belongs to the first conjugate locus just when there is a root $\alpha \in \Delta$ such that $|\alpha(X)| = 2\pi$ but there is no root $\beta \in \Delta$ such that $|\beta(tX)| = 2\pi$ with $|t| < 1$. In other words the first conjugate locus is $\mathrm{Ad}(G)\mathfrak{f}$ as asserted.

The statement of [4, Theorem 5] is that the minimum locus is equal to the first conjugate locus. Now the injectivity radius R is the minimal length $\|X\| = \langle X, X\rangle^{1/2}$ of an element of the first conjugate locus. In other words $R = \min_{X\in\mathfrak{f}} \|X\|$. $\square$

For $\alpha \in \Delta$ let $t_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ be so that $\alpha(t_\alpha) = 2$. Let $t_1, \ldots, t_r$ be the t_{α_i} for the simple roots $\alpha_1, \ldots, \alpha_r$. Let $\Gamma = \{H \in \mathfrak{h} \mid \exp H = e\}$. Then $\Gamma = \bigoplus_{j=1}^r 2\pi\mathbb{Z}t_j$. It follows that the injectivity radius is given by

$$R = \min_{j=1,\ldots,r} \pi\|t_j\| \,. \tag{5.3}$$

Now we use (5.3) to run through the four cases, making Theorem 5.1 explicit for our setting.

Here we use the matrix realization notation of (2.2), (2.4), (2.5) and (2.6), we use the realizations of roots as matrices as introduced in Section 2, and we use the Riemannian metric on G defined by the positive definite inner product $\langle X, Y\rangle = -\mathrm{Tr}\,(XY)$.

A_n: We have $(f_i - f_j, f_i - f_j) = 2$ hence $t_i = f_i - f_{i+1}$. It follows that $R = \sqrt{2}\pi$.

B_n: The simple roots are the f_1 and the $f_i - f_j$. Hence $t_1 = 2f_1$, and $t_i = f_i - f_{i+1}$ for $i > 1$. The realization of $x \in \mathfrak{h}$ as a matrix is

$$x \mapsto \begin{pmatrix} 0 & & \\ & \mathrm{diag}(x) & \\ & & -\mathrm{diag}(x) \end{pmatrix}.$$

Hence $\|t\| = 2\sqrt{2}$ and $\|t_j\| = 2$ and so $R = 2\pi$.

C_n: The simple roots are $2f_1$ and the $f_j - f_{j-1}$ for $j > 1$. The realization of $x \in \mathfrak{h}$ as a matrix is

$$x \mapsto \begin{pmatrix} \mathrm{diag}(x) & \\ & -\mathrm{diag}(x) \end{pmatrix}.$$

That gives us $t_1 = f_1$ and $t_j = f_j - f_{j-1}$ for $j > 1$. Thus $\|t_1\| = \sqrt{2}$ and $\|t_j\| = 2$ for $j > 1$, so $R = \sqrt{2}\pi$.

D_n: The realization of $x \in \mathfrak{h}$ as a matrix is the same as for C_n. The simple roots are $f_1 + f_2 = t_1$ and the $f_j - f_{j-1} = t_j$. Hence

Theorem 5.4. *The injectivity radius of the classical compact simply connected Lie groups G, in the Riemannian metric given by the inner product $\langle X, Y\rangle = -\mathrm{Tr}\,(XY)$ on $\mathfrak{g}$, is $\sqrt{2}\,\pi$ for $SU(m+1)$ and $Sp(m)$, 2π for $SO(2m)$ and $SO(2m+1)$. In particular for each of the four series the injectivity radius R is independent of m.*

The invariant measures on G, G_n and G_k all are normalized to total mass 1.

We start by recalling Gonzalez' Paley-Wiener theorem [6] (also see [16]). Denote by $\Lambda^+(G) \subset i\mathfrak{h}^*$ the set of dominant integral weights,

$$\Lambda^+(G) = \left\{ \mu \in i\mathfrak{h}^* \,\middle|\, \tfrac{2(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \right\} .$$

For $\mu \in \Lambda^+(G)$ denote by π_μ the corresponding representation with highest weight μ. As G is assumed simply connected $\mu \mapsto \pi_\mu$, is a bijection from $\Lambda^+(G)$ onto $\widehat{G}$. The representation space for π_μ is denoted by V_μ. Let $\chi_\mu = \mathrm{Tr} \circ \pi_\mu$ be the character of π_μ and $\deg(\mu) = \dim V_\mu$ its dimension. Note that $\deg(\mu)$ extends to a polynomial function on $\mathfrak{h}_{\mathbb{C}}^*$. As Haar measure is normalized to total mass 1, the characters $\{\chi_\mu\}_{\mu \in \Lambda^+(G)}$ form a complete orthonormal set for $L^2(G)^G := \{f \in L^2(G) \mid f \circ \mathrm{Ad}(g) = f \text{ for all } g \in G\}$.

For $f \in C(G)^G$ define the Fourier transform $\mathcal{F}(f) = \widehat{f} : \Lambda^+(G) \to \mathbb{C}$ by

$$\widehat{f}(\mu) = (f, \chi_\mu) = \int_G f(x)\overline{\chi_\mu(x)}\, dx, \quad \mu \in \Lambda^+(G)\,,$$

where (f, χ_μ) is the inner product in $L^2(G)$. The Fourier transform extends to an unitary isomorphism $\mathcal{F} : L^2(G)^G \to \ell^2(\Lambda^+(G))$ and

$$f = \sum_{\mu \in \Lambda^+(G)} \widehat{f}(\mu)\chi_\mu$$

in $L^2(G)^G$. If f is smooth the Fourier series converges absolutely and uniformly.

If not otherwise stated we will assume that G does not contain any simple factor of exceptional type. As before $W(\mathfrak{g}, \mathfrak{h})$ denotes the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\widetilde{W} = \widetilde{W}(\mathfrak{g}, \mathfrak{h})$ also denotes that Weyl group when G is of type A_n, B_n or C_n, and $\widetilde{W} = \widetilde{W}(\mathfrak{g}, \mathfrak{h})$ is the Weyl group extended by including odd sign changes in the D_n cases. For $r > 0$ let $\mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$ denote the space of holomorphic functions Φ on $\mathfrak{h}_{\mathbb{C}}^*$ such that

(1) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that
$$|\Phi(\lambda)| \leqq C_k(1 + |\lambda|)^{-k} e^{r|\mathrm{Re}\lambda|} \text{ for all } \lambda \in \mathfrak{h}_{\mathbb{C}}^*,$$
(2) $\Phi(w(\lambda + \rho) - \rho) = \det(w)\Phi(\lambda)$ for all $w \in \widetilde{W}$, $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$.

Let $H = \exp(\mathfrak{h})$. For $0 < r < R$ denote by $C^\infty_{r,\widetilde{W}}(G)^G$ the space of smooth central functions with support in a closed geodesic ball $B_r(e)$ of radius r such that the restriction to H is $\widetilde{W}$–invariant. We refer to a much more detailed discussion in Section 3. In this terminology the theorem of Gonzalez [6] reads as follows.

Theorem 5.5. *Let G be an arbitrary connected simply connected compact Lie group. Let $0 < r < R$ and let $f \in C^\infty(G)^G$ be given. Then f belongs to $C^\infty_{r,\widetilde{W}}(G)^G$ if and only if the Fourier transform $\mu \mapsto \widehat{f}(\mu)$ extends to a holomorphic function Φ_f on $\mathfrak{h}_{\mathbb{C}}^*$ such that $\Phi_f \in \mathrm{PW}_r^\rho(\mathfrak{h}_{\mathbb{C}}^*)^{\widetilde{W}}$.*

Proof. We only have to check that $f \in C^\infty_{r,\widetilde{W}}(G)^G$ if and only if $\widehat{f}(w(\mu+\rho)-\rho) = \widehat{f}(\mu)$. For factors not of type D_n that follows from Gonzalez's theorem. For factors of type D_n it follows Weyl's character formula. $\square$

In [16] it is shown that the extension Φ_f is unique whenever r is sufficiently small. In that case Fourier transform, followed by holomorphic extension, is a bijection $C^\infty_{r,\widetilde{W}}(G)^G \cong \mathrm{PW}^\rho_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$.

We will now extend these results to projective limits. We start with two simple lemmas.

Lemma 5.6. *Let $\Phi \in \mathrm{PW}^\rho_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$. Assume that $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ is such that $\langle \lambda, \alpha\rangle = 0$ for some $\alpha \in \Delta$. Then $\Phi(\lambda - \rho) = 0$.*

Proof. Let s_α be the reflection in the hyper plane perpendicular to α. Then

$$\begin{aligned} \Phi(\lambda-\rho) &= \Phi(s_\alpha(\lambda)-\rho) \\ &= \Phi(s_\alpha(\lambda-\rho+\rho)-\rho) = \det(s_\alpha)\Phi(\lambda-\rho)\,. \end{aligned}$$

The claim now follows as $\det(s_\alpha) = -1$. $\square$

Lemma 5.7. *Let $r > 0$ and let $\widetilde{W}$ be as before. For $\Phi \in \mathrm{PW}^\rho_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$ define*

$$T(\Phi)(\lambda) = F_\Phi(\lambda) := \tfrac{\varpi(\rho)}{\varpi(\lambda)}\Phi(\lambda-\rho) \text{ where } \varpi(\lambda) = \prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle\,.$$

Then $T(\Phi) \in \mathrm{PW}_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$ and the map $\mathrm{PW}^\rho_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}} \to \mathrm{PW}_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$, $\Phi \mapsto F_\Phi$, is a linear isomorphism.

Proof. Let $\alpha \in \Delta^+$. Then

$$\lambda \mapsto \frac{1}{(\lambda,\alpha)}\Phi(\lambda)$$

is holomorphic by Lemma 5.6. According to [13], Lemma 5. 13, p. 288, it follows that this function is also of exponential type r. Iterating this for each root it follows that F_Φ is holomorphic of exponential type r. As $\varpi(w(\lambda)) = \det(w)\varpi(\lambda)$ it follows using the same arguments as in the proof of Lemma 5.6 that F_Φ is $\widetilde{W}$–invariant. The surjectivity follow as $F \mapsto \varpi(\lambda)F(\cdot+\rho)$ maps $\mathrm{PW}_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$ into $\mathrm{PW}^\rho_r(\mathfrak{h}^*_{\mathbb{C}})^{\widetilde{W}}$. $\square$

Theorem 5.8. *Let $r > 0$ and assume that G_k propagates G_n. Then the map*

$$\Phi \mapsto P_{k,n}(\Phi) := T_n^{-1}(T_k(\Phi)|_{\mathfrak{h}^*_{n,\mathbb{C}}}) = \frac{\varpi_n(\bullet)}{\varpi_n(\rho_n)}\left(\frac{\varpi_k(\rho_k)}{\varpi_k(\bullet)}\Phi(\bullet-\rho_k)|_{\mathfrak{h}^*_{n,\mathbb{C}}}\right)(\bullet+\rho_n)$$

from $\mathrm{PW}^{\rho_k}_r(\mathfrak{h}^*_{k,\mathbb{C}})^{\widetilde{W}_k} \to \mathrm{PW}^{\rho_n}_r(\mathfrak{h}^*_{n,\mathbb{C}})^{\widetilde{W}_n}$ is surjective.

Proof. This follows from Lemma 5.7 and Theorem 1.6. $\square$

Recall from Theorem 5.4 that the injectivity radii R are the same for G_k and G_n. For $0 < r < R$ we now define a map $C_{k,n} : C^\infty_{r,\widetilde{W}_k}(G_k)^{G_k} \to C^\infty_{r,\widetilde{W}_n}(G_n)^{G_n}$ by the commutative diagram using Gonzalez' theorem:

$$\begin{tikzcd} C^\infty_{r,\widetilde{W}_k}(G_k)^{G_k} \arrow[r, "C_{k,n}"] \arrow[d, "\mathcal{F}_k"'] & C^\infty_{r,\widetilde{W}_n}(G_n)^{G_n} \arrow[d, "\mathcal{F}_n"] \\ \mathrm{PW}_r^{\rho_k}(\mathfrak{h}_{k,\mathbb{C}})^{\widetilde{W}_k} \arrow[r, "P_{k,n}"'] & \mathrm{PW}_r^{\rho_n}(\mathfrak{h}_{n,\mathbb{C}})^{\widetilde{W}_n} \end{tikzcd} .$$

Theorem 5.9. *If G_k propagates G_n and $0 < r < R$ then*

$$C_{k,n} : C^\infty_{r,\widetilde{W}_k}(G_k)^{G_k} \to C^\infty_{r,\widetilde{W}_n}(G_n)^{G_n}$$

is surjective.

Proof. This follows from Theorem 5.5 and Theorem 5.8. □

Theorem 5.10. *Let $r > 0$ and assume that G_k propagates G_n. Then the sequences $(\mathrm{PW}_r^{\rho_n}(\mathfrak{h}^*_{n,\mathbb{C}})^{\widetilde{W}_n}, P_{k,n})$ and $(C^\infty_{r,\widetilde{W}_n}(G_n)^{G_n}, C_{k,n})$ form projective systems and*

$$\begin{aligned} \mathrm{PW}_r^{\rho_\infty}(\mathfrak{h}_{\infty,\mathbb{C}}) &:= \varprojlim \mathrm{PW}_r^{\rho_n}(\mathfrak{h}^*_{n,\mathbb{C}})^{\widetilde{W}_n} \\ C^\infty_{r,\infty}(G_\infty)^{G_\infty} &:= \varprojlim C^\infty_{r,\widetilde{W}_n}(G_n)^{G_n} \end{aligned}$$

are nonzero.

Proof. This follows from Theorem 5.8 and Theorem 5.9. □

Remark 5.11. We can view elements $\Phi \in \mathrm{PW}_r^{\rho_\infty}(\mathfrak{h}_{\infty,\mathbb{C}})$ as holomorphic functions on $\mathfrak{h}_{\infty,\mathbb{C}}$ when we view $\mathfrak{h}_{\infty,\mathbb{C}}$ as the spectrum of $\varprojlim \mathrm{PW}_r^{\rho_n}(\mathfrak{h}^*_{n,\mathbb{C}})$. ♢

6. Spherical Representations of Compact Groups

We will now apply the results from Section 1 and Section 2 to the Fourier transform on compact symmetric spaces. We start by an overview over spherical representations, spherical functions and the spherical Fourier transform. Most of the material can be found in [22] and [23] but partially with different proofs. The notation will be as in Section 3 and G or G_n will always stand for a compact group. In particular, $M_n = G_n/K_n$ where G_n is a connected compact semisimple Lie group with Lie algebra $\mathfrak{g}_n$, which we will for simplicity assume is simply connected. The result can easily be formulated for arbitrary compact symmetric spaces by following the arguments in [16]. We will assume that M_k propagates M_n. We denote by r_k respectively r_n the real rank of M_k respectively M_n. As always we fix compatible K_k– and K_n–invariant inner products on $\mathfrak{s}_k$ respectively $\mathfrak{s}_n$.

As in Section 3 let $\Sigma_n = \Sigma_n(\mathfrak{g}_n, \mathfrak{a}_n)$ denote the system of restricted roots of $\mathfrak{a}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$. Let $\mathfrak{h}_n$ be a θ_n-stable Cartan subalgebra such that $\mathfrak{h}_n \cap \mathfrak{s}_n = \mathfrak{a}_n$. Let $\Delta_n = \Delta(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$. Recall that $\Sigma_n \subset i\mathfrak{a}_n^*$. We choose positive subsystems Δ_n^+ and Σ_n^+ so that $\Sigma_n^+ \subseteqq \Delta_n^+|_{\mathfrak{a}_n}$, $\Delta_n^+ \subseteqq \Delta_k^+|_{\mathfrak{h}_{n,\mathbb{C}}}$, and $\Sigma_n^+ \subset \Sigma_k^+|_{\mathfrak{a}_n}$. Consider the reduced root system

$$\Sigma_{2,n} = \{\alpha \in \Sigma_n \mid 2\alpha \notin \Sigma_n\}$$

and its positive subsystem $\Sigma_{2,n}^+ := \Sigma_{2,n} \cap \Sigma_n^+$. Let

$$\Psi_{2,n} = \Psi_{2,n}(\mathfrak{g}_n, \mathfrak{a}_n) = \{\alpha_{n,1}, \dots, \alpha_{n,r_n}\}$$

denote the set of simple roots for $\Sigma_{2,n}^+$. We note the following simple facts; they follow from the explicit realization (2.1) of the root systems discussed in Section 2.

Lemma 6.1. *Suppose that the M_n are irreducible. Let $r_n = \dim \mathfrak{a}_n$, the rank of M_n. Number the simple root systems $\Psi_{2,n}$ as in (2.1). Suppose that M_k propagates M_n. If $j \leqq r_n$ then $\alpha_{k,j}$ is the unique element of $\Psi_{2,k}$ whose restriction to $\mathfrak{a}_n$ is $\alpha_{n,j}$.*

Since M_k propagates M_n each irreducible factor of M_k contains at most one simple factor of M_n. In particular if M_n is not irreducible then M_k is not irreducible, but we still can number the simple roots so that Lemma 6.1 applies.

We denote the positive Weyl chamber in $\mathfrak{a}_n$ by $\mathfrak{a}_n^+$ and similarly for $\mathfrak{a}_k$.

For $\mu \in \Lambda^+(G_n)$ let

$$V_\mu^{K_n} = \{v \in V_\mu \mid \pi_\mu(k)v = v \text{ for all } k \in K_n\}.$$

We identify $i\mathfrak{a}_n^*$ with $\{\mu \in i\mathfrak{h}_n^* \mid \mu|_{\mathfrak{h}_n \cap \mathfrak{k}_n} = 0\}$ and similar for $\mathfrak{a}_n^*$ and $\mathfrak{a}_{n,\mathbb{C}}^*$. With this identification in mind set

$$\Lambda^+(G_n, K_n) = \left\{ \mu \in i\mathfrak{a}_n^* \;\middle|\; \tfrac{(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+ \right\}.$$

Since G_n is connected and M_n is simply connected it follows that K_n is connected. As K_n is compact there exists a unique (up to multiplication by a positive scalar) G_n–invariant measure μ_{M_n} on M_n. For brevity we sometimes write dx instead of $d\mu_{M_n}$. If G_n is compact, in other words if M_n is compact, then we normalize μ_{M_n} so that $\mu_{M_n}(M_n) = 1$, i.e., μ_{M_n} is a probability measure on M_n.

Theorem 6.2 (Cartan-Helgason)**.** *Assume that G_n is compact and simply connected. Then the following are equivalent.*

(1) $\mu \in \Lambda^+(G_n, K_n)$,
(2) $V_\mu^{K_n} \neq 0$,
(3) π_μ *is a subrepresentation of the representation of* G_n *on* $L^2(M_n)$.

When those conditions hold, $\dim V_\mu^{K_n} = 1$ *and* π_μ *occurs with multiplicity* 1 *in the representation of* G_n *on* $L^2(G_n/K_n)$.

Proof. See [12, Theorem 4.1, p. 535]. □

Remark 6.3. If G_n is compact but not simply connected, then one has to replace Λ_n^+ and $\Lambda^+(G_n, K_n)$ by sub semi–lattices of weights μ such that the group homomorphism $\exp(X) \mapsto e^{\mu(X)}$ is well defined on the maximal torus H_n, and then the proof of Theorem 6.2 goes through without change. See, for example, [16]. ♢

Define linear functionals $\xi_{n,j} \in i\mathfrak{a}_n^*$ by

$$\frac{\langle \xi_{n,i}, \alpha_{n,j}\rangle}{\langle \alpha_{n,j}, \alpha_{n,j}\rangle} = \delta_{i,j} \text{ for } 1 \leqq j \leqq r_n \quad . \tag{6.4}$$

Then for $\alpha \in \Sigma_{2,n}^+$

$$\frac{\langle \xi_{n,i}, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^+ .$$

If $\alpha \in \Sigma^+ \setminus \Sigma_{2,n}^+$, then $2\alpha \in \Sigma_{2,n}^+$ and

$$\frac{\langle \xi_{n,i}, \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\langle \xi_{n,i}, 2\alpha\rangle}{\langle 2\alpha, 2\alpha\rangle} \in \mathbb{Z}^+ .$$

Hence $\xi_{n,i} \in \Lambda_n^+$. The weights $\xi_{n,j}$ are the *class 1 fundamental weights for* $(\mathfrak{g}_n, \mathfrak{k}_n)$. We set

$$\Xi_n = \{\xi_{n,1}, \ldots, \xi_{n,r_n}\} .$$

For $I = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ define $\mu_I := \mu(I) = k_1\xi_{n,1} + \ldots + k_{r_n}\xi_{n,r_n}$.

Lemma 6.5. *If $\mu \in i\mathfrak{a}_n^*$ then $\mu \in \Lambda^+(G_n, K_n)$ if and only if $\mu = \mu_I$ for some $I \in (\mathbb{Z}^+)^{r_n}$.*

Proof. This follows directly from the definition of $\xi_{n,j}$. □

Lemma 6.6. *Suppose that M_k is a propagation of M_n. Let $I_k = (m_1, \ldots, m_k) \in (\mathbb{Z}^+)^{r_k}$ and $\mu = \mu_{I_k}$. Then $\mu|_{\mathfrak{a}_n} \in \Lambda^+(G_n, K_n)$. In particular $\xi_{k,j}|_{\mathfrak{a}_n} \in \Lambda^+(G_n, K_n)$ for $1 \leqq j \leqq r_k$.*

Proof. Let $v_\mu \in V_\mu$ be a nonzero highest weight vector and $e_\mu \in V_\mu$ a K_k–fixed unit vector. Denote by $W = \langle \pi_\mu(G_n)v_\mu\rangle$ the cyclic G_n-module generated by v_μ and let $\mu_n = \mu|_{\mathfrak{a}_n}$.

Write $W = \bigoplus_{j=1}^s W_j$ with W_j irreducible. If W_j has highest weight $\nu_j \neq \mu$ then $v_\mu \perp W_j$ so $\langle \pi_\mu(G_n)v_\mu\rangle \perp W_j$, contradicting $W_j \subset W = \bigoplus W_i$. Now each W_j has highest weight μ. Write $v_\mu = v_1 + \ldots + v_s$ with $0 \neq v_j \in W_j$. As $(v_\mu, e_\mu) \neq 0$ it follows that $(v_j, e_\mu) \neq 0$ for some j. But then the projection of e_μ onto W_j is a non-zero K_n fixed vector in $W_j^{K_n} \neq 0$ and hence $\mu|_n \in \Lambda^+(G_n, K_n)$. □

Lemma 6.7 ([22], Lemma 6)**.** *Assume that M_k is a propagation of M_n. Recall the root ordering of* (2.1)*. If $1 \leqq j \leqq r_n$ then $\xi_{k,j}$ is the unique element of Ξ_k whose restriction of $\mathfrak{a}_n$ is $\xi_{n,j}$.*

Proof. This is clear when $\mathfrak{a}_k = \mathfrak{a}_n$. If $r_n < r_k$ it follows from the explicit construction of the fundamental weights for classical root system; see [7, p. 102]. □

Lemma 6.8. *Assume that $\mu_k \in \Lambda^+(G_k, K_k)$ is a combination of the first r_n fundamental weights, $\mu = \sum_{j=1}^{r_n} k_j \xi_{k,j}$. Let $\mu_n := \mu|_{\mathfrak{a}_n} = \sum_{j=1}^{r_n} k_j \xi_{n,j}$. If v is a nonzero highest weight vector in V_{μ_k} then $\langle \pi_{\mu_k}(G_n)v\rangle$ is irreducible and isomorphic to V_{μ_n}. Furthermore, π_{μ_n} occurs with multiplicity one in $\pi_{\mu_k}|_{G_n}$.*

Proof. Each G_n–irreducible summand W in $\langle \pi_{\mu_k}(G_n)v\rangle$ has highest weight μ_n. Fix one such G_n–submodule W and let $w \in W$ be a nonzero highest weight vector. Write $w = w_1 + \ldots + w_s$ where each w_j is of some $\mathfrak{h}_k$–weight $\mu_k - \sum_i k_{j,i}\beta_i$ and where each β_i is a simple root in $\Sigma^+(\mathfrak{g}_k, \mathfrak{h}_k)$ and each $k_{j,i} \in \mathbb{Z}^+$. As $\mu_k|_{\mathfrak{h}_n} = \mu_n$ it follows that $\langle \sum_i k_{j,i}\beta_i|_{\mathfrak{h}_n}, \alpha\rangle = 0$ for all $\alpha \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$. Thus $\sum_i k_{j,i}\beta_i|_{\mathfrak{h}_n} = 0$. In view of (2.1) each $\langle \beta_i, \alpha_j\rangle \leqq 0$ for $\alpha_j \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ simple (specifically $\langle \beta_i, \alpha_j\rangle = 0$ unless $\beta_i = f_{c+1} - f_c$ and $\alpha_j = f_c - f_{c-1}$, for some c, in which case $\langle \beta_i, \alpha_j\rangle = -1$). Since every $k_{j,i} \in \mathbb{Z}^+$ now $\langle \beta_i, \alpha_j\rangle = 0$ for each $\alpha_j \in \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ simple. Thus $\beta_i|_{\mathfrak{h}_n} = 0$.

Because of the compatibility of the positive systems $\Delta^+(\mathfrak{g}_{k,\mathbb{C}}, \mathfrak{h}_{k,\mathbb{C}})$ and $\Delta^+(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ there exists a $\beta \in \Delta^+(\mathfrak{g}_{k,\mathbb{C}}, \mathfrak{h}_{k,\mathbb{C}})$, $\beta|_{\mathfrak{h}_n} = 0$, such that $\mu_k - \beta$ is a weight in V_{μ_n}. Writing β as a sum of simple roots, we see that each of the simple roots has to vanish on $\mathfrak{a}_n$ and hence the restriction to $\mathfrak{a}_k$ can not contain any of the simple roots $\alpha_{k,j}$, $j = 1, \ldots, r_n$. But then β is perpendicular to the fundamental weights $\xi_{k,j}$, $j = 1, \ldots, r_n$. Hence $s_\beta(\mu_n - \beta) = \mu_n + \beta$ is also a weight, contradicting the fact that μ_n is the highest weight. (Here s_β is the reflection in the hyperplane $\beta = 0$.) This shows that π_{μ_n} can only occur once in $\langle \pi_{\mu_k}(G_n)v\rangle$. In particular, $\langle \pi_{\mu_k}(G_n)v\rangle$ is irreducible. □

Lemma 6.8 allows us to form direct system of representations, as follows. For $\ell \in \mathbb{N}$ denote by $0_\ell = (0, \ldots, 0)$ the zero vector in $\mathbb{R}^\ell$. For $I_n = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$ let

(6.9)

- $\mu_{I,n} = \sum_{j=1}^{r_n} k_j \xi_{n,j} \in \Lambda_n^+$;
- $\pi_{I,n} = \pi_{\mu_{I,n}}$ the corresponding spherical representation;
- $V_{I,n} = V_{\mu_{I,n}}$ a fixed Hilbert space for the representation $\pi_{I,n}$;
- $v_{I,n} = v_{\mu_{I,n}}$ a highest weight unit vector in $V_{I,n}$;
- $e_{I,n} = e_{\mu_{I,n}}$ a K_n–fixed unit vector in $V_{I,n}$.

We collect our results in the following Theorem. Compare [22, Section 3].

Theorem 6.10. *Let M_k propagate M_n and let $\pi_{I,n}$ be an irreducible representation of G_n with highest weight $\mu_{I,n} \in \Lambda^+(G_n, K_n)$. Let $I_k = (I_n, 0_{r_k - r_n})$. Then the following hold.*

(1) *$\mu_{I,k} \in \Lambda^+(G_k, K_k)$ and $\mu_{I,k}|_{\mathfrak{a}_n} = \mu_{I,n}$.*
(2) *The G_n-submodule of $V_{I,k}$ generated by $v_{I,k}$ is irreducible.*
(3) *The multiplicity of $\pi_{I,n}$ in $\pi_{I,k}|_{G_n}$ is 1, in other words there is an unique G_n–intertwining operator $T_{k,n} : V_{I,n} \to V_{I,k}$ such that*

$$T_{k,n}(\pi(g)v_{I,n}) = \pi_{I,k}(g)v_{I,k} \, .$$

Remark 6.11. From this point on, when $m \leqq q$ we will always assume that the Hilbert space $V_{I,m}$ is realized inside $V_{I,q}$ as $\langle \pi_{I,q}(G_m)v_{I,q}\rangle$. $\diamondsuit$

7. Spherical Fourier Analysis and the Paley-Wiener Theorem

In this section we give a short description of the spherical functions and Fourier analysis on compact symmetric spaces. Then we state and prove results for limits of compact symmetric spaces analogous to those in Section 4.

For the moment let $M = G/K$ be a compact symmetric space. We use the same notation as in the last section but without the index n. As usual we view functions on M as right K–invariant functions on G via $f(g) = f(g \cdot x_o)$, $x_o = eK$. For $\mu \in \Lambda(G, K)$ denote by $\deg(\mu)$ the dimension of the irreducible representation π_μ. Fix a unit K-fixed vector e_μ and define

$$\psi_\mu(g) = (e_\mu, \pi_\mu(g)e_\mu)\,.$$

Then ψ_μ is positive definite spherical function on G, and every positive definite spherical function is obtained in this way for a suitable representation π. Define

$$\ell^2_d(\Lambda^+(G,K)) = \left\{ \{a_\mu\}_{\mu\in\Lambda^+(G,K)} \,|a_\mu \in \mathbb{C} \text{ and } \sum_{\mu\in\Lambda^+(G,K)} \deg(\mu)|a_\mu|^2 < \infty \right\}. \tag{7.1}$$

Then $\ell^2_d(\Lambda^+(G,K))$ is a Hilbert space with inner product

$$((a(\mu))_\mu, (b(\mu))_\mu) = \sum_{\mu\in\Lambda^+(G,K)} \deg(\mu)a(\mu)\overline{b(\mu)}\,.$$

For $f \in C^\infty(M)$ define the spherical Fourier transform of f, $\mathcal{S}(f) = \widehat{f} : \Lambda^+(G,K) \to \mathbb{C}$ by

$$\widehat{f}(\mu) = (f, \psi_\mu) = \int_M f(g)(\pi_\mu(g)e_\mu, e_\mu)\, dg = (\pi_\mu(f)e_\mu, e_\mu)$$

where $\pi_\mu(f)$ denotes the operator valued Fourier transform of f, $\pi_\mu(f) = \int_G f(g)\pi_\mu(g)\, dg$. Then the sequence $\mathcal{S}(f) = (\mathcal{S}(f)(\mu))_\mu$ is in $\ell^2_d(\Lambda^+(G,K))$ and $\|f\|^2 = \|\mathcal{S}(f)\|^2$. Finally, $\mathcal{S}$ extends by continuity to an unitary isomorphism

$$\mathcal{S} : L^2(M)^K \to \ell^2_d(\Lambda^+(G,K))\,.$$

We denote by $\mathcal{S}_\rho$ the map

$$\mathcal{S}_\rho(f)(\mu) = \mathcal{S}(f)(\mu - \rho)\,, \quad \mu \in \Lambda^+(G,K) + \rho\,. \tag{7.2}$$

If f is smooth, then f is given by

$$f(x) = \sum_{\mu\in\Lambda^+(G,K)} \deg(\mu)\mathcal{S}(f)(\mu)\psi_\mu(x) = \sum_{\mu\in\Lambda^+(G,K)} \deg(\mu)\mathcal{S}_\rho(f)(\mu+\rho)\psi_\mu(x)\,.$$

and the series converges in the usual Fréchet topology on $C^\infty(M)^K$. In general, the sum has to be interpreted as an L^2 limit.

Let

$$\Omega := \{X \in \mathfrak{a} \mid |\alpha(X)| < \pi/2 \text{ for all } \alpha \in \Sigma\}\,.$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let φ_λ denote the spherical function on the dual symmetric space of noncompact type G^d/K, where the Lie algebra of G^d is given by $\mathfrak{k} + i\mathfrak{s}$. Then φ_λ has a holomorphic extension as $K_{\mathbb{C}}$–invariant function to $K_{\mathbb{C}} \exp(2\Omega) \cdot x_o \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, cf. [18, Theorem 3.15], [2] and [14]. Furthermore

$$\overline{\psi_\mu(x)} = \varphi_{\mu+\rho}(x^{-1})$$

for $x \in K_{\mathbb{C}} \exp(2\Omega) \cdot x_o$. We can therefore define a holomorphic function $\lambda \mapsto \mathcal{S}_\rho(f)(\lambda)$ by

$$\mathcal{S}_\rho(f)(\lambda) = \int_M f(x)\varphi_\lambda(x^{-1})\,dx$$

as long as f has support in $K_{\mathbb{C}} \exp(2\Omega) \cdot x_o$. $\mathcal{S}_\rho(f)$ is $W(\mathfrak{g}, \mathfrak{a})$ invariant and $\mathcal{S}_\rho(f)(\mu) = \mathcal{S}(f)(\mu - \rho)$ for all $\mu \in \Lambda^+(G, K) + \rho$.

Denote by R the injectivity radius of the Riemannian exponential map $\mathrm{Exp} : \mathfrak{s} \to M$. As noted in Remark 5.2, Theorem 5.1 holds for compact simply connected Riemannian symmetric spaces [4] generally, leading to the following extension of Theorem 5.4.

Theorem 7.3. *The injectivity radius R of the classical compact simply connected Riemannian symmetric spaces $M = G/K$, in the Riemannian metric given by the inner product $\langle X, Y\rangle = -\mathrm{Tr}\,(XY)$ on $\mathfrak{s}$, depends only on the type of the restricted reduced root system $\Sigma_2(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$. It is $\sqrt{2}\,\pi$ for $\Sigma_2(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ of type A or C and is 2π for $\Sigma_2(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ of type B or D.*

Remark 7.4. Since Ω is given by $|\alpha(X)| < \pi/2$ and the interior of the injectivity radius disk is given by $|\alpha(X)| < 2\pi$ the set Ω is contained in the open disk in $\mathfrak{s}$ of center 0 and radius $R/4$. $\diamondsuit$

Essentially as before, B_r denotes the closed metric ball in M with center x_o and radius r, and $C_r^\infty(M)^K$ denotes the space of K–invariant smooth functions on M supported in B_r.

Remark 7.5. Theorem 7.6 below is, modulo a ρ-shift and W-invariance, Theorem 4.2 and Remark 4.3 of [16]. As pointed out in [16, Remark 4.3], the known value for the constant S can be different in each part of the theorem. In Theorem7.6(1) we need that $S < R$ and the closed ball in $\mathfrak{s}$ with center zero and radius S has to be contained in $K_{\mathbb{C}} \exp(i\Omega) \cdot x_o$ to be able to use the estimates from [18] for the spherical functions to show that we actually end up in the Paley-Wiener space.

In Theorem 7.6(2) we need only that $S < R$. Thus the constant in (1) is smaller than the one in (2). That is used in part (3). For Theorem 7.6(4) we also need $\|X\| \leqq \pi/\|\xi_j\|$ for $j = 1, \ldots, r$. $\diamondsuit$

The group $\widetilde{W} = \widetilde{W}(\mathfrak{g}, \mathfrak{h})$ is defined as before and $C^\infty_{r,\widetilde{W}}(M)^K$ denotes the space of smooth K–invariant functions with support in B_r such that $f|_A$ is $\widetilde{W}$–invariant.

Theorem 7.6 (Paley-Wiener Theorem for Compact Symmetric Spaces)**.** *Let the notation be as above. Then the following hold.*

1. *There exists a constant $S > 0$ such that, for each $0 < r < S$ and $f \in C^\infty_{r,\widetilde{W}}(M)^K$, the ρ-shifted spherical Fourier transform $\mathcal{S}_\rho(f) : \Lambda^+_n + \rho \to \mathbb{C}$ extends to a function in $\mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}}$.*
2. *There exists a constant $S > 0$ such that if $F \in \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}}$, $0 < r < S$, the function*

$$f(x) := \sum_{\mu \in \Lambda^+(G,K)} \deg(\mu) F(\mu + \rho) \psi_\mu(x) \tag{7.7}$$

   is in $C^\infty_{r,\widetilde{W}}(M)^K$ and $\mathcal{S}_\rho f(\mu) = F(\mu)$.
3. *For S as in (1.) define $\mathcal{I}_\rho : \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}} \to C^\infty_{r,\widetilde{W}}(M)^K$ by (7.7). Then $\mathcal{I}_\rho$ is surjective for all $0 < r < S$.*
4. *There exists a constant $S > 0$ such that for all $0 < r < S$ the map $\mathcal{S}_\rho$ followed by holomorphic extension defines a bijection $C_{r,\widetilde{W}}(M)^K \cong \mathrm{PW}_r(\mathfrak{a}_\mathbb{C})^{\widetilde{W}}$.*

Proof. As mentioned above this is Theorem 4.2 and Remark 4.3 in [16] except for the $\widetilde{W}$-invariance. But that has only be checked for factors of type D_n, where it follows as in the proof of Theorem 5.5 by Weyl's character formula. □

A weaker version of the following theorem was used in [16, Section 11]. It used an operator Q which we will define shortly, and some differentiation, to prove the surjectivity part of local Paley–Wiener Theorem. Denote the Fourier transform of $f \in C(G)^G$ by $\mathcal{F}(f)$. Recall the operator $T : \mathrm{PW}^\rho_r(\mathfrak{h}^*_\mathbb{C})^{\widetilde{W}(\mathfrak{g},\mathfrak{h})} \to \mathrm{PW}_r(\mathfrak{h}^*_\mathbb{C})^{\widetilde{W}(\mathfrak{g},\mathfrak{h})}$ from Theorem 5.7. Finally, for $f \in C(G)$ let $f^\vee(x) = f(x^{-1})$. Then $^\vee : C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{h})}(G)^G \to C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{h})}(G)^G$ is a bijection. We will identify $\mathfrak{a}^*_\mathbb{C}$ with the subspace $\{\lambda \in \mathfrak{h}^*_\mathbb{C} \mid \lambda|_{\mathfrak{h}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}} = 0\}$ without comment in the following.

Theorem 7.8. *Let $S > 0$ be as in* Theorem 7.6(1) *and let $0 < r < S$. Then the the restriction map $\mathrm{PW}_r(\mathfrak{h}^*_\mathbb{C})^{\widetilde{W}(\mathfrak{g},\mathfrak{h})} \to \mathrm{PW}_r(\mathfrak{a}^*_\mathbb{C})^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ is surjective. Furthermore, the map $C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{h})}(G)^G \to C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(M)^K$, given by*

$$Q(\varphi)(g \cdot x_o) = \int_K \varphi(gk)\, dk,$$

is surjective, and $\mathcal{S}_\rho \circ Q(f^\vee) = T \circ \mathcal{F}(f)$ on $\Lambda^+(G,K) + \rho$.

Proof. Surjectivity of the restriction map follows from Theorem 1.6 and Theorem 3.2. Next, we have $Q(\chi_\mu^\vee)(x) = \int_K \chi_\mu(x^{-1}k)\,dk$. As $\int_K \pi_\mu(k)\,dk$ is the orthogonal projection onto V_μ^K it follows that $Q(\chi_\mu^\vee) = 0$ if $\mu \notin \Lambda^+(G,K)$ and

$$Q(\chi_\mu^\vee)(x) = (\pi_\mu(x^{-1})e_\mu, e_\mu) = (e_\mu, \pi_\mu(x)e_\mu) = \psi_\mu(x)$$

for $\mu \in \Lambda^+(G,K)$. Thus, if $f = \sum_\mu \mathcal{F}(f)(\mu)\chi_\mu$ we have

$$Q(f^\vee)(x) = \sum_{\mu\in\Lambda^+(G,K)} \mathcal{F}(f)(\mu)\psi_\mu(x) = \sum_{\mu\in\Lambda^+(G,K)} \deg(\mu)\frac{\mathcal{F}(f)(\mu)}{\deg(\mu)}\psi_\mu(x).$$

Using the Weyl dimension formula for finite dimensional representations, $\deg(\mu) = \frac{\varpi(\mu+\rho)}{\varpi(\rho)}$, we get

$$\mathcal{S}_\rho(Q(f^\vee))(\mu+\rho) = \tfrac{\varpi(\mu+\rho)}{\varpi(\rho)}\,\mathcal{F}(f)(\mu) = T(\mathcal{F}(f))|_{\mathfrak{a}}(\mu+\rho)$$

for $\mu \in \Lambda^+(G,K)$. Hence $\mathcal{S}_\rho \circ Q(f^\vee)|_{\Lambda^+(G,K)} = (T\circ\mathcal{F}(f)|_{\mathfrak{a}_\mathbb{C}})|_{\Lambda^+(G,K)}$.

Assume that $f \in C^\infty_{r,\widetilde{W}(\mathfrak{g},\mathfrak{a})}(G/K)^K$. Then, by the Paley-Wiener Theorem, Theorem 7.6, there exists a $\Phi \in \mathrm{PW}_r(\mathfrak{a}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{a})}$ such that $\Phi = \mathcal{S}_\rho(f)$ on $\Lambda^+(G,K)$. Then, by what we just proved, there exists $\Psi \in \mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{\widetilde{W}(\mathfrak{g},\mathfrak{h})}$ such that $\Psi|_{\mathfrak{a}_\mathbb{C}} = \Phi$. By Theorem 5.5 there exists $F \in C_{r,\widetilde{W}(\mathfrak{g},\mathfrak{h})}(G)^G$ such that $T\circ\mathcal{F}(F) = \Psi$. By the above calculation we have

$$\mathcal{S}(f)(\mu) = \mathcal{S}(Q(F^\vee))(\mu) \quad \text{for all} \quad \mu \in \Lambda^+(G,K)\,.$$

As clearly $Q(F^\vee)$ is smooth, it follows that $Q(F^\vee) = f$ and hence Q is surjective. □

Let $\sigma = 2(\alpha_1 + \ldots + \alpha_\ell)$ where the $\alpha_j \in \Sigma_2^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})$ are the simple roots. For M irreducible let

$$\begin{aligned} &\Omega^* := \Omega \text{ if } \Sigma_2(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C}) \text{ is of type } A_\ell \text{ or } C_\ell,\\ &\Omega^* := \bigcap_{w\in W(\mathfrak{g},\mathfrak{a})} \{X\in\mathfrak{a} \mid |\sigma(w(X))| < \pi/2\} \text{ if } \Sigma_2(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C}) \text{ is of type } B_\ell \text{ or } D_\ell. \end{aligned} \tag{7.9}$$

In general, we define Ω^* to be the product of the Ω^*'s for all the irreducible factors. Then Ω^* is a convex Weyl group invariant polygon in $\mathfrak{a}$. We also have $\Omega^* = -\Omega^*$. This is easy to check and in any case will follow from our explicit description of Ω^*.

Using the explicit realization of the irreducible root systems in Section 2 we describe the domains Ω^* in the following way:

$\mathbf{A_n}$**:** We have $\mathfrak{a} = \{x\in\mathbb{R}^{n+1} \mid \sum x_j = 0\}$, $n \geqq 1$, and the roots are the $f_i - f_j : x \mapsto x_i - x_j$ for $i \neq j$. Hence

$$\Omega^* = \Omega = \left\{x\in\mathbb{R}^{n+1} \,\middle|\, \sum x_j = 0 \quad \text{and } |x_i - x_j| < \tfrac{\pi}{2} \text{ for } 1 \leqq i \neq j \leqq n+1\right\}. \tag{7.10}$$

$\mathbf{B_n}$: We have $\mathfrak{a} = \mathbb{R}^n$, $n \geqq 2$ and $\sigma = 2(f_1 + (f_2 - f_1) + \ldots + (f_n - f_{n-1})) = 2f_n$. The Weyl group consists of all permutations and sign changes on the f_i. Hence

$$\Omega^* = \{x \in \mathbb{R}^n \mid |x_j| < \tfrac{\pi}{4} \text{ for } j = 1, \ldots, n\}\,. \tag{7.11}$$

$\mathbf{C_n}$: Again $\mathfrak{a} = \mathbb{R}^n$, $n \geqq 3$, and the roots are the $\pm(f_i \pm f_j)$ and $\pm 2f_j$. If $|x_i|, |x_j| < \pi/4$ then $|x_i \pm x_j| < \pi/2$. Hence

$$\Omega^* = \Omega = \{x \in \mathbb{R}^n \mid |x_j| < \tfrac{\pi}{4} \text{ for } j = 1, \ldots, n\}\,. \tag{7.12}$$

$\mathbf{D_n}$: Also in this case $\mathfrak{a} = \mathbb{R}^n$ with $n \geqq 4$. We have $\sigma = 2(f_1 + f_2 + (f_2 - f_1) + \ldots + (f_n - f_{n-1})) = 2(f_2 + f_n)$. As the Weyl group is given by all permutations and even sign changes on the f_i, we get

$$\Omega^* = \{x \in \mathbb{R}^n \mid |x_i \pm x_j| < \tfrac{\pi}{4} \text{ for } i, j = 1, \ldots, n\,,\ i \neq j\}. \tag{7.13}$$

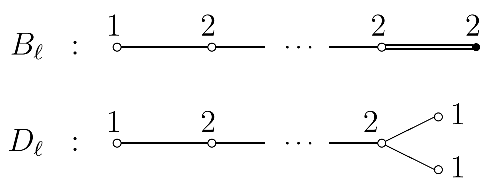
Lemma 7.14. *We have $\Omega^* \subseteqq \Omega$.*

Proof. Let δ be the highest root in Σ^+. Then

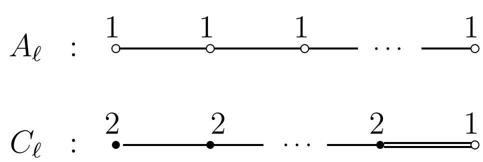
$$\Omega = W(\mathfrak{a})(\{X \in \overline{\mathfrak{a}^+} \mid \delta(X) < \pi/2\})\,.$$

For the classical Lie algebras, the coefficients of the simple roots in the highest root are all 1 or 2. Hence $\Omega^* \subseteqq \Omega$ and the claim follows. $\square$

Remark 7.15. The distinction between Ω and Ω^* is caused by change in the coefficient in the highest root of the simple root on the left. Thus in cases B_n and D_n it goes from 1 to 2 as we move up in the rank of M:



while in cases A_n and C_n it doesn't change:



$\diamondsuit$

Lemma 7.16. *If $S > 0$ such that $\{X \in \mathfrak{s} \mid \|X\| \leqq S\} \subset \mathrm{Ad}(K)\Omega^*\}$, then we can use S as the constant in* Theorem 7.6(1)*.*

Proof. Recall from [16, Remark 4.3] that Theorem 7.6(1) holds when $0 < S < R$ and

$$\{X \in \mathfrak{s} \mid \|X\| \leqq S\} \subseteqq \mathrm{Ad}(K)\Omega\,. \tag{7.17}$$

But $\mathrm{Ad}(K)\Omega$ is open in $\mathfrak{s}$, and $\mathrm{Exp} : \mathrm{Ad}(K)\Omega \to M$ is injective by Theorem 7.3. Hence, if (7.17) holds then $S < R$, and the claim follows from the first part of Remark 7.5. $\square$

We will now apply this to sequences $\{M_n\}$ where M_k is a propagation of M_n for $k \geqq n$. We use the same notation as before and add the index n (or k) to indicate the dependence of the space M_n (or M_k). We start with the following lemma.

Lemma 7.18. *If $k \geqq n$ then $\Omega_n^* = \Omega_k^* \cap \mathfrak{a}_n$.*

Proof. We can assume that M is irreducible. As M_k propagates M_n it follows that we are only adding simple roots to the left on the Dynkin diagram for Σ_2. Let r_n denote the rank of M_n and r_k the rank of M_k. We can assume that $r_n < r_k$, as the claim is obvious for $r_n = r_k$. We use the above explicit description Ω^* given above and case by case inspection of each of the irreducible root system in Section 2.

Assume that $\Sigma_{n,2}$ is of type A_{r_n} and $\Sigma_{k,2}$ is of type A_{r_k} with $r_n < r_k$. It follows from (7.10) that $\Omega_n^* \subseteqq \Omega_k^* \cap \mathfrak{a}_n$. Let $(0,x) \in \Omega_n^*$. For $j > i$ we have

$$\pm (f_j - f_i)((0,x)) = \begin{cases} \pm(x_j - x_i) & \text{for} \quad j \leqq r_n + 1 \\ \mp(-x_i) & \text{for} \quad j > r_n + 1 \geqq i \\ 0 & \text{for} \quad j, i > r_n + 1 \end{cases} \tag{7.19}$$

Let $i \leqq r_n + 1$. Then, using that $x_i = -\sum_{j \neq i} x_j$ and $|x_i - x_j| < \pi/2$, we get

$$-r_k \tfrac{\pi}{2} < \sum_{i \neq j} (x_i - x_j) = r_k x_i - \sum_{j \neq i} x_j = (r_k + 1) x_i < r_k \tfrac{\pi}{2}\,.$$

Hence

$$-\tfrac{\pi}{2} < -\tfrac{r_k}{r_k+1}\,\tfrac{\pi}{2} < x_i < \tfrac{r_k}{r_k+1}\,\tfrac{\pi}{2} < \tfrac{\pi}{2}\,.$$

It follows now from (7.19) that $(0,x) \in \Omega_k^* \cap \mathfrak{a}_n$.

The cases of types B and C are obvious from (7.11) and (7.12). For the case of type D we note that $|x_i \pm x_j| < \frac{\pi}{4}$ implies both -$\frac{\pi}{4} < x_i - x_j < \frac{\pi}{4}$ and -$\frac{\pi}{4} < x_i + x_j < \frac{\pi}{4}$. Adding, -$\frac{\pi}{2} < 2x_i < \frac{\pi}{2}$, so $|x_i| < \frac{\pi}{4}$. Hence $(0,x) \in \Omega_k^* \cap \mathfrak{a}_n$ if and only if $x \in \Omega_n^*$ by (7.13). $\square$

We can now proceed as in Section 4. We will always assume that $S > 0$ is so that the closed ball in $\mathfrak{s}$ of radius S is contained in Ω^*. The group W is defined as before. Define $C_{k,n} : C^\infty_{r,W_k}(M_k)^{K_k} \to C^\infty_{r,W_n}(M_n)^{K_n}$ by $C_{k,n} := \mathcal{I}_{n,\rho_n} \circ P_{k,n} \circ \mathcal{S}_{k,\rho_k}$, in other words

$$C_{k,n}(f)(x) = \sum_{I \in (\mathbb{Z}^+)^{r_n}} \deg(\mu_{I,n}) \widehat{f}(\mu_{I,k} - \rho_k + \rho_n) \psi_{\mu_{I,n}}(x)\,.$$

Theorem 7.20 (Paley-Wiener Isomorphisms-II)**.** *Assume that M_k propagates M_n and $0 < r < S$. Then the following holds:*

1. *The map $P_{k,n} : \mathrm{PW}_r(\mathfrak{a}_{k,\mathbb{C}}^*)^{W_k} \to \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{W_n}$ is surjective.*
2. *The map $C_{k,n} : C^\infty_{r,W_k}(M_k)^{K_k} \to C^\infty_{r,W_n}(M_n)^{K_n}$ is surjective.*

Proof. This follows from Theorem 1.6, Lemma 7.16, and Lemma 7.18. □

We assume now that $\{M_n, \iota_{k,n}\}$ is a injective system of Riemannian symmetric spaces of compact type such that M_k is a propagation of M_n along a cofinite subsequence. Here the direct system maps $\iota_{k,n} : M_n \to M_k$ are injections. We pass to that cofinite subsequence and now assume that M_k is a propagation of M_n whenever $k \geqq n$. Let

$$M_\infty = \varinjlim M_n \, .$$

The compact symmetric spaces in Table 3.1 give rise to the following injective limits of symmetric spaces.

(7.21)

1. $\big(\mathrm{SU}(\infty) \times \mathrm{SU}(\infty)\big)/\mathrm{diag}\,\mathrm{SU}(\infty)$, group manifold $\mathrm{SU}(\infty)$,
2. $\big(\mathrm{Spin}(\infty) \times \mathrm{Spin}(\infty)\big)/\mathrm{diag}\,\mathrm{Spin}(\infty)$, group manifold $\mathrm{Spin}(\infty)$,
3. $\big(\mathrm{Sp}(\infty) \times \mathrm{Sp}(\infty)\big)/\mathrm{diag}\,\mathrm{Sp}(\infty)$, group manifold $\mathrm{Sp}(\infty)$,
4. $\mathrm{SU}(p+\infty)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(\infty))$, $\mathbb{C}^p$ subspaces of $\mathbb{C}^\infty$,
5. $\mathrm{SU}(2\infty)/[\mathrm{S}(\mathrm{U}(\infty) \times \mathrm{U}(\infty))]$, $\mathbb{C}^\infty$ subspaces of infinite codim in $\mathbb{C}^\infty$,
6. $\mathrm{SU}(\infty)/\mathrm{SO}(\infty)$, real forms of $\mathbb{C}^\infty$
7. $\mathrm{SU}(2\infty)/\mathrm{Sp}(\infty)$, quaternion vector space structures on $\mathbb{C}^\infty$,
8. $\mathrm{SO}(p+\infty)/[\mathrm{SO}(p) \times \mathrm{SO}(\infty)]$, oriented $\mathbb{R}^p$ subspaces of $\mathbb{R}^\infty$,
9. $\mathrm{SO}(2\infty)/[\mathrm{SO}(\infty) \times \mathrm{SO}(\infty)]$, $\mathbb{R}^\infty$ subspaces of infinite codim in $\mathbb{R}^\infty$,
10. $\mathrm{SO}(2\infty)/\mathrm{U}(\infty)$, complex vector space structures on $\mathbb{R}^\infty$,
11. $\mathrm{Sp}(p+\infty)/[\mathrm{Sp}(p) \times \mathrm{Sp}(\infty)]$, $\mathbb{H}^p$ subspaces of $\mathbb{H}^\infty$,
12. $\mathrm{Sp}(2\infty)/[\mathrm{Sp}(\infty) \times \mathrm{Sp}(\infty)]$, $\mathbb{H}^\infty$ subspaces of infinite codim in $\mathbb{H}^\infty$,
13. $\mathrm{Sp}(\infty)/\mathrm{U}(\infty)$, complex forms of $\mathbb{H}^\infty$.

We also have as before injective systems $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_k$, $\mathfrak{k}_n \hookrightarrow \mathfrak{k}_k$, $\mathfrak{s}_n \hookrightarrow \mathfrak{s}_k$, and $\mathfrak{a}_n \hookrightarrow \mathfrak{a}_k$ giving rise to corresponding injective systems. Let

$$\mathfrak{g}_\infty := \varinjlim \mathfrak{g}_n \, , \quad \mathfrak{k}_\infty := \varinjlim \mathfrak{k}_n \, , \quad \mathfrak{s}_\infty := \varinjlim \mathfrak{s}_n \, , \quad \mathfrak{a}_\infty := \varinjlim \mathfrak{a}_n \, , \quad \text{and,} \quad \mathfrak{h}_\infty := \varinjlim \mathfrak{h}_n \, .$$

We have that $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{s}_\infty$ is the eigenspace decomposition of $\mathfrak{g}_\infty$ with respect to the involution $\theta_\infty := \varinjlim \theta_n$, $\mathfrak{a}_\infty$ is a maximal abelian subspace of $\mathfrak{s}_\infty$.

We have also projective systems $\{\mathrm{PW}_r(i\mathfrak{a}_n)^{W_n}\}$, and $\{C_{r,W_n}(M_n)^{K_n}\}$ with surjective projections. Let

$$\begin{aligned} \mathrm{PW}_r(\mathfrak{a}_{\mathbb{C}}^{\infty *}) &:= \varprojlim \mathrm{PW}_r(\mathfrak{a}_{n,\mathbb{C}}^*)^{W_n} \\ C_{r,W_\infty}(M_\infty)^{K_\infty} &:= \varprojlim C_{r,W_n}(M_n)^{K_n} \, . \end{aligned}$$

As before we view the elements of $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})$ as W_∞–invariant functions on $\mathfrak{a}^*_{\infty,\mathbb{C}}$, and the elements of of $C_{r,W_\infty}(M_\infty)^{K_\infty}$ as K_∞–invariant functions on M_∞; see Remark 5.11. For $\mathbf{f} = (f_n)_n \in C_{r,W_\infty}(M_\infty)^{K_\infty}$ define $\mathcal{S}_{\rho,\infty}(\mathbf{f}) \in \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})$ by

$$\mathcal{S}_{\rho,\infty}(\mathbf{f}) := \{\mathcal{S}_{\rho,n}(f_n)\}\,. \tag{7.22}$$

Then $\mathcal{S}_{\rho,\infty}(\mathbf{f})$ is well defined by Theorem 7.20 and we have a commutative diagram

$$\begin{tikzcd}
\cdots & C^\infty_{r,W_n}(M_n)^{K_n} \arrow[d, "\mathcal{S}_{\rho,n}"'] & C^\infty_{r,W_{n+1}}(M_{n+1})^{K_{n+1}} \arrow[l, "C_{n+1,n}"'] \arrow[d, "\mathcal{S}_{\rho,n+1}"'] & \cdots \arrow[l, "C_{n+2,n+1}"'] & C_{r,W_\infty}(M_\infty)^{K_\infty} \arrow[d, "\mathcal{S}_{\rho,\infty}"'] \\
\cdots & \mathrm{PW}_r(\mathfrak{a}^*_{n,\mathbb{C}}) & \mathrm{PW}_r(\mathfrak{a}^*_{n+1\mathbb{C}}) \arrow[l, "P_{n+1,n}"] & \cdots \arrow[l, "P_{n+2,n+1}"] & \mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}})
\end{tikzcd},$$

see also [17, 21] for the spherical Fourier transform and direct limits.

Theorem 7.23 (Infinite dimensional Paley-Wiener Theorem-II)**.** *Let the notation be as above. Then* $\mathrm{PW}_r(\mathfrak{a}^*_{\infty,\mathbb{C}}) \neq \{0\}$*,* $C_{r,W_\infty}(M_\infty)^{K_\infty} \neq \{0\}$*, and the spherical Fourier transform*

$$\mathcal{F}_\infty : C_{r,W_\infty}(M_\infty)^{K_\infty} \to \mathrm{PW}_r(\mathfrak{a}^{\infty *}_{\mathbb{C}})$$

is injective.

8. Comparison with the L^2 Theory

The maps considered up to this point are based on C^∞ and C^∞_c spaces rather than L^2 spaces and unitary representation theory. It is standard that L^2 for a compact symmetric space is just a Hilbert space completion of the corresponding C^∞ space, and it turns out [24, Proposition 3.27] that the same is true for inductive limits of compact symmetric spaces. Here we discuss those inductive limits; any consideration of the projective limit of L^2 spaces follows similar lines by replacing the the maps of the inductive limit by the corresponding orthogonal projections, because inductive and projective limits are the same in the Hilbert space category.

The material of this section is taken from [22, Section 3] and [24, Section 3] and adapted to our setting. We always assume without further comments that all extensions are propagations.

There are three steps to the comparison. First, we describe the construction of a direct limit Hilbert space $L^2(M_\infty) := \varinjlim\{L^2(M_n), L_{m,n}\}$ that carries a natural multiplicity–free unitary action of G_∞. Then we describe the ring $\mathcal{A}(M_\infty) := \varinjlim\{\mathcal{A}(M_n), \nu_{m,n}\}$ of regular functions on M_∞ where $\mathcal{A}(M_n)$ consists of the finite linear combinations of the matrix coefficients of the π_μ with $\mu \in \Lambda^+_n(G_n, K_n)$ and such that $\nu_{m,n}(f)|_{M_n} = f$. Thus $\mathcal{A}(M_\infty)$ is a (rather small) G_∞–submodule of the projective limit $\varprojlim\{\mathcal{A}(M_n),\ \text{restriction}\}$. Third, we describe a $\{G_n\}$–equivariant morphism $\{\mathcal{A}(M_n), \nu_{m,n}\} \rightsquigarrow \{L^2(M_n), L_{m,n}\}$ of direct systems that embeds $\mathcal{A}(M_\infty)$ as a dense G–submodule of $L^2(M_\infty)$, so that $L^2(M_\infty)$ is G_∞–isomorphic to a Hilbert space completion of the function space $\mathcal{A}(M_\infty)$.

We recall first some basic facts about the vector valued Fourier transform on M_n as well as the decomposition of $L^2(M_n)$ into irreducible summands. To simplify notation write Λ_n^+ for $\Lambda^+(G_n, K_n)$. Let $\mu \in \Lambda_n^+$ and let $V_{n,\mu}$ denote the irreducible G_n–module of highest weight μ. Recursively in n, we choose a highest weight vector $v_{n,\mu} \in V_{n,\mu}$ and and a K_n–invariant unit vector $e_{n,\mu} \in V_\mu^{K_n}$ such that (i) $V_{n-1,\mu} \hookrightarrow V_{n,\mu}$ is isometric and G_{n-1}–equivariant and sends $v_{n-1,\mu}$ to a multiple of $v_{n,\mu}$, (ii) orthogonal projection $V_{n,\mu} \to V_{n-1,\mu}$ sends $e_{n,\mu}$ to a non–negative real multiple $c_{n,n-1,\mu}e_{n-1,\mu}$ of $e_{n-1,\mu}$, and (iii) $\langle v_{n,\mu}, e_{n,\mu}\rangle = 1$. (Then $0 < c_{n,n-1,\mu} \leqq 1$.) Note that orthogonal projection $V_{m,\mu} \to V_{n,\mu}, m \geqq n$, sends $e_{m,\mu}$ to $c_{m,n,\mu}e_{n,\mu}$ where $c_{m,n,\mu} = c_{m,m-1,\mu} \cdots c_{n+1,n,\mu}$.

The Hermann Weyl degree formula provides polynomial functions on $\mathfrak{a}_\mathbb{C}^*$ that map μ to $\deg(\pi_{n,\mu}) = \dim V_{n,\mu}$. Earlier in this paper we had written $\deg(\mu)$ for that degree when n was fixed, but here it is crucial to track the variation of $\deg(\pi_{n,\mu})$ as n increases. Define a map $v \mapsto f_{n,\mu,v}$ from $V_{n,\mu}$ into $L^2(M_n)$ by

$$f_{n,\mu,v}(x) = \langle v, \pi_{n,\mu}(x)e_\mu\rangle\,. \tag{8.1}$$

It follows by the Frobenius–Schur orthogonality relations that $v \mapsto \deg(\pi_{n,\mu})^{1/2}f_{\mu,v}$ is a unitary G_n map from V_μ onto its image in $L^2(M_n)$.

The operator valued Fourier transform

$$L^2(G_n) \to \bigoplus_{\mu\in\Lambda_n^+} \mathrm{Hom}(V_{n,\mu}, V_{n,\mu}) \cong \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu} \otimes V_{n,\mu}^*$$

is defined by $f \mapsto \bigoplus_{\mu\in\Lambda_n^+} \pi_{n,\mu}(f)$ where $\pi_{n,\mu}(f) \in \mathrm{Hom}(V_{n,\mu}, V_{n,\mu})$ is given by

$$\pi_{n,\mu}(f)v := \int_{G_n} f(x)\pi_{n,\mu}(x)v \ \text{ for } f \in L^2(G_n)\,. \tag{8.2}$$

Denote by $P_\mu^{K_n}$ the orthogonal projection $V_{n,\mu} \to V_{n,\mu}^{K_n}$. Then $P_\mu^{K_n}(v) = \int_{K_n} \pi_{n,\mu}(k)v\,dk$, and if f is right K_n–invariant, then

$$\pi_{n,\mu}(f) = \pi_{n,\mu}(f)P_\mu^{K_n}\,.$$

That gives us the vector valued Fourier transform $f \mapsto \widehat{f} : \Lambda_n^+ \to \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu}$,

$$L^2(M_n) \to \bigoplus_{\mu\in\Lambda_n^+} V_{n,\mu} \text{ defined by } f \mapsto \widehat{f}(\mu) := \pi_{n,\mu}(f)e_{n,\mu}\,. \tag{8.3}$$

Then the Plancherel formula for $L^2(M_n)$ states that

$$f = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu})f_{\mu,\widehat{f}(\mu)} = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu})\langle\widehat{f}(\mu), \pi_{n,\mu}(\,\cdot\,)e_{n,\mu}\rangle \tag{8.4}$$

in $L^2(M_n)$ and

$$\|f\|_{L^2}^2 = \sum_{\mu\in\Lambda_n^+} \deg(\pi_{n,\mu})\|\widehat{f}(\mu)\|_{HS}^2 . \tag{8.5}$$

If f is smooth, then the series in (8.4) converges in the C^∞ topology of $C^\infty(M_n)$.

For $n \leqq m$ and $\mu = \mu_{I,n} \in \Lambda_n^+$ consider the following diagram of unitary G_n-maps, adapted from [24, Equation 3.21]:

$$\begin{tikzcd} V_{\mu_{I,n}} \arrow[r, "v\mapsto v"] \arrow[d, "v\mapsto\deg(\pi_{n,\mu})^{1/2} f_{\mu_{I,n},v}"'] & V_{\mu_{I,m}} \arrow[d, "v\mapsto\deg(\pi_{m,\mu})^{1/2} f_{\mu_{I,m},v}"] \\ L^2(M_n) \arrow[r, "L_{m,n}"'] & L^2(M_m) \end{tikzcd}$$

where $L_{m,n} : L^2(M_n) \to L^2(M_k)$ is the G_n–equivariant partial isometry defined by

$$L_{k,n} : \sum_{I_n} f_{\mu_{I,n},w_I} \mapsto \sum_{I_m} c_{m,n,\mu}\sqrt{\tfrac{\deg(\pi_{m,\mu})}{\deg(\pi_{n,\mu})}}\, f_{\mu_{I,m},w_I}\, , \quad w_I \in V_{n,\mu}\, . \tag{8.6}$$

As in [24, Section 4] we have

Theorem 8.7. *The map $L_{k,n}$ of (8.6) is a G_n–equivariant partial isometry with image*

$$\mathrm{Im}(L_{m,n}) \cong \bigoplus\nolimits_{I\in(\mathbb{Z}^+)^{r_k},\ k_{r_n+1}=\ldots=k_{r_k}=0} V_{\mu_I}\, .$$

If $n \leqq m \leqq k$ then

$$L_{k,n} = L_{m,n}\circ L_{k,m}$$

making $\{L^2(M_n), L_{k,n}\}$ into a direct system of Hilbert spaces.

Define

$$L^2(M_\infty) := \varinjlim L^2(M_n), \tag{8.8}$$

direct limit in the category of Hilbert spaces and unitary injections.

From construction of the $L_{m,n}$ we now have

Theorem 8.9 ([22], Theorem 13)**.** *The left regular representation of G_∞ on $L^2(M_\infty)$ is a multiplicity free discrete direct sum of irreducible representations. Specifically, that left regular representation is $\sum_{I\in\mathcal{I}}\pi_I$ where $\pi_I = \varinjlim \pi_{I,n}$ is the irreducible representation of G_∞ with highest weight $\xi_I := \sum k_r\xi_r$. This applies to all the direct systems of* (7.21)*.*

The problem with the partial isometries $L_{m,n}$ is that they do not work well with restriction of functions, because of the rescalings and because $L_{m,n}(L^2(M_n)^{K_n}) \not\subset L^2(M_m)^{K_m}$ for $n < m$. In particular the spherical functions $\psi_{I,n}(g) := \langle e_{I,n}, \pi_{I,n}(g)e_{I,n})\rangle$ do not map forward, in other words $L_{m,n}(\psi_{I,n}) \neq \psi_{I,m}$.

We deal with this by viewing $L^2(M_\infty)$ as a Hilbert space completion of the ring $\mathcal{A}(M_\infty) := \varinjlim \mathcal{A}(M_n)$ of regular functions on M_∞. Adapting [24, Section 3] to our notation, we define

$$\begin{aligned} &\mathcal{A}(\pi_{n,\mu})^{K_n} = \{\text{finite linear combinations of the } f_{\mu,I_n,w_I} \text{ where } w_I \in V_{n,\mu}\}, \\ &\nu_{m,n,\mu} : \mathcal{A}(\pi_{n,\mu})^{K_n} \hookrightarrow \mathcal{A}(\pi_{m,\mu})^{K_m} \text{ by } f_{\mu,I_n,w_I} \mapsto f_{\mu,I_m,w_I}\ . \end{aligned} \tag{8.10}$$

Thus [24, Lemma 2.30] says that if $f \in \mathcal{A}(\pi_{n,\mu})^{K_n}$ then $\nu_{m,n,\mu}(f)|_{M_n} = f$.

The ring of regular functions on M_n is $\mathcal{A}(M_n) := \mathcal{A}(G_n)^{K_n} = \sum_\mu \mathcal{A}(\pi_{n,\mu})$, and the $\nu_{m,n,\mu}$ sum to define a direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$. Its limit is

$$\mathcal{A}(M_\infty) := \mathcal{A}(G_\infty)^{K_\infty} = \varinjlim \{\mathcal{A}(M_n), \nu_{m,n}\}. \tag{8.11}$$

As just noted, the maps of the direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$ are inverse to restriction of functions, so $\mathcal{A}(M_\infty)$ is a G_∞–submodule of the inverse limit $\varprojlim\{\mathcal{A}(M_n),\ \text{restriction}\}$.

For each n, $\mathcal{A}(M_n)$ is a dense subspace of $L^2(M_n)$ but, because the $\nu_{m,n}$ distort the Hilbert space structure, $\mathcal{A}(M_\infty)$ does not sit naturally as a subspace of $L^2(M_\infty)$. Thus we use the G_n–equivariant maps

$$\eta_{n,\mu} : \mathcal{A}(\pi_{n,\mu})^{K_n} \to \mathcal{H}_{\pi_n} \widehat{\otimes} (w_{n,\mu^*}\mathbb{C}) \text{ by } f_{\mu,I_n,w_I} \mapsto c_{n,1,\mu}\sqrt{\deg \pi_{n,\mu}}\, f_{\mu,I_n,w_I}. \tag{8.12}$$

where $c_{m,n,\mu}$ is the length of the projection of $e_{m,\mu}$ to $V_{n,\mu}$. Now [24, Proposition 3.27] says

Proposition 8.13. *The maps $L_{m,n,\mu}$ of* (8.6)*, $\nu_{m,n,\mu}$ of* (8.10) *and $\eta_{n,\mu}$ of* (8.12) *satisfy*

$$(\eta_{m,\mu} \circ \nu_{m,n,\mu})(f_{\mu,I_n,w_I}) = (L_{m,n,\mu} \circ \eta_{n,\mu})(f_{\mu,I_n,w_I})$$

for $f_{u,v,n} \in \mathcal{A}(\pi_{n,\mu})^{K_n}$. Thus they inject the direct system $\{\mathcal{A}(M_n), \nu_{m,n}\}$ into the direct system $\{L^2(M_n), L_{m,n}\}$. That map of direct systems defines a G_∞–equivariant injection

$$\widetilde{\eta} : \mathcal{A}(M_\infty) \to L^2(M_\infty)$$

with dense image. In particular η defines a pre Hilbert space structure on $\mathcal{A}(M_\infty)$ with completion isometric to $L^2(M_\infty)$.

This describes $L^2(M_\infty)$ as an ordinary Hilbert space completion of a natural function space on M_∞.

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