Geometry of the Borel – de Siebenthal Discrete Series

Bent Ørsted* & Joseph A. Wolf†

27 January 2009

Abstract

Let G_0 be a connected, simply connected real simple Lie group. Suppose that G_0 has a compact Cartan subgroup T_0, so it has discrete series representations. Relative to T_0 there is a distinguished positive root system Δ^+ for which there is a unique noncompact simple root ν, the "Borel – de Siebenthal system". There is a lot of fascinating geometry associated to the corresponding "Borel – de Siebenthal discrete series" representations of G_0. In this paper we explore some of those geometric aspects and we work out the K_0–spectra of the Borel – de Siebenthal discrete series representations. This has already been carried out in detail for the case where the associated symmetric space G_0/K_0 is of hermitian type, i.e. where ν has coefficient 1 in the maximal root μ, so we assume that the group G_0 is not of hermitian type, in other words that ν has coefficient 2 in μ.

Several authors have studied the case where G_0/K_0 is a quaternionic symmetric space and the inducing holomorphic vector bundle is a line bundle. That is the case where μ is orthogonal to the compact simple roots and the inducing representation is 1–dimensional.

1 Introduction

One of Harish–Chandra's great achievements was the existence theorem for discrete series representations of a semisimple Lie group. He characterized the groups with discrete series representations by the equal rank condition, he found the explicit formulae on the regular elliptic set for the characters of the discrete series, and he showed that those formulae specify the characters. At the same time (and as a main motivation) he was able to explicitly construct a particularly simple series, the holomorphic discrete series, for those groups where the corresponding Riemannian symmetric space is a bounded symmetric domain in a complex Euclidean space. For the other discrete series representations however, the actual construction has remained less explicit, although there are several beautiful realizations.

In this paper we initiate the study of a certain family, the so-called Borel – de Siebenthal discrete series, from a point of view as close as possible to that of Harish–Chandra for the holomorphic discrete series. This is motivated in part by the work of Gross and Wallach for the scalar case of the quaternionic discrete series [GW]. As in that case we obtain in particular the admissibility of the series for a small subgroup of the maximal compact subgroup. At the same time we discover a rather appealing geometry for the coadjoint orbit that one wants

*Research partially supported by the Danish Research Council.
†Research partially supported by the NSF, the Danish Research Council, and IMF, Aarhus University.
2000 AMS Subject Classification: Primary 22E46; secondary 22E30, 32L10, 32M10.
Key Words: discrete series, cohomology, compact subvarieties, relative invariants.

to attach to the discrete series in question. In particular we give a detailed classification of the possible structures of such orbits in terms of explicit prehomogeneous vector spaces with relative invariants. We feel these deserve attention in their own right; and while we do give the construction for the Borel – de Siebenthal discrete series here, including the explicit K_0-types and the (important) admissibility for a small subgroup K_1 of K_0, we defer further analysis of continuations of the series to a sequel to this paper. In particular, we shall then elucidate the role of the relative invariants in constructing rather singular representations in the continuation of the series. We mention that much of this has been carried out in the quaternion line bundle case in [GW]. We also mention the papers [Kn2] and [Kn3] treating such questions for the indefinite orthogonal and symplectic groups; here methods from [GW] are used, and the connection to the continuation of unitary modules in the sense of Vogan (with criteria for good and fair range of unitarity of cohomologically induced modules) is made clear. Our approach seeks to employ analytic methods and the geometry of the orbits, and in particular to use reproducing kernels, see e.g. [WaW].

Several questions concerning discrete series representations may be resolved by our methods, for example the question of finding admissible branching laws, where one obtains direct sum decompositions with finite multiplicities. By applying admissibility of K_1 such results may be obtained in complete analogy with what happens for holomorphic discrete series representations.

In this paper we give a complete description of the geometry of the elliptic coadjoint orbits corresponding to the Borel – de Siebenthal discrete series. They are open G_0–orbits in certain complex flag manifolds and we give precise results on their maximal compact subvarieties (which are compact hermitian symmetric spaces) and the holomorphic normal bundles to those subvarieties. We use this structural information to give a concrete geometric construction of the representations in this series, including the structure of the K_0–types. Our construction of the K_0–types provides an analogue of the K_0–type decomposition of holomorphic discrete series representations.

The quaternionic discrete series, studied by Gross and Wallach [GW] in the line bundle case, is the special case of the Borel – de Siebenthal discrete series, where the maximal root is compact and is orthogonal to all the other compact positive roots, or equivalently (see [W0]) where K_0 has a local direct factor isomorphic to $Sp(1)$. While each complex simple Lie group G has exactly one noncompact real form G_0 that has quaternionic discrete series representations, every real simple Lie group G_0 with $\operatorname{rank} G_0 = \operatorname{rank} K_0$ has either holomorphic discrete series representations or Borel – de Siebenthal discrete series representations. Thus the Borel – de Siebenthal discrete series is the natural extension of the holomorphic discrete series.

Our geometric approach allows us to extend several results from [GW]. While Gross and Wallach constructed quaternionic discrete series representations on spaces of holomorphic forms with values in a line bundle, we also allow vector bundles – which is natural in the more general setting considered here. Our construction also provides good concrete examples of minimal cohomology degree realizations of discrete series representations in the sense of Kostant [Ko3].

Our basic tool is complex differential geometry and the associated cohomology groups. An important component of this is a collection of basic spectral sequence arguments, already implicit in the paper [S1]. See also [S2], [S3] and [W4]. Here we make use of some technical results of M. Eastwood and the second named author from [EW] for some crucial identifications of duals of finite dimensional representations of reductive Lie groups, in particular for keeping track of the action of the center in terms of the highest weights and the Dynkin diagrams.

Our results include a careful collection of the data attached to the orbits in question, and an explicit formula for the K_0–types in the Harish–Chandra module corresponding to the discrete series in question. As a by–product we find two natural sets of strongly orthogonal roots, one corresponding to the hermitian symmetric space K_0/L_0 and the other corresponding to the riemannian symmetric space G_0/K_0. They fit together to realize the orbit $G_0(z_0) = G_0/L_0$ as a kind of Siegel domain of Type II. This should provide useful coordinates for explicit calculations of the elements in the Harish–Chandra module.

In Section 2 we work out the general structure of the complex manifold $D = G_0(z_0) \cong G_0/L_0$. We describe the action of $\mathfrak{l}_0$ on the tangent and normal spaces to the maximal compact subvariety $Y = K_0(z_0) \cong K_0/L_0$, and on their duals. then we discuss a negativity condition that is crucial to the realization of our discrete series representations.

In Section 3 we list all instances of simple Lie algebras $\mathfrak{g}_0$ corresponding to Borel – de Siebenthal root orders. Setting aside the well–understood hermitian symmetric cases, we work out the precise structure of the algebras $\mathfrak{k}_0$ and $\mathfrak{l}_0$ and the parts of the complexified Lie algebra $\mathfrak{g}$ that correspond to the holomorphic tangent space of Y and the holomorphic normal space of Y in D, including the representations of $\mathfrak{l}_0$ on those two spaces. In each case this allows explicit parameterization of the Borel – de Siebenthal discrete series.

In Section 4 we consider the prehomogeneous space $(L, \mathfrak{u}_1)$ where $\mathfrak{u}_1 \subset \mathfrak{g}$ represents the holomorphic normal space to Y in D. There we describe the algebra of relative invariants, using our knowledge of the representation of L on the symmetric algebra $S(\mathfrak{u}_1)$. In most cases we can be explicit, but in some we must rely on general results of Sato and Kimura [SK]. These invariants are (in addition to being interesting in themselves) relevant for the next step of understanding the analytic continuation of the discrete series; here the ring of regular functions on the zero set of an invariant will correspond to a module in this continuation. We intend to follow this idea in a sequel to this paper.

In Section 5 we assemble our preparations and work out the exact K–spectrum of the Borel – de Siebenthal discrete series representations. Our main result here, which is the main result of the paper, is Theorem 5.23. In a final example we look a the "sufficient negativity" condition that ensures the non-vanishing of exactly the right analytic cohomology group, and we compare it to the corresponding condition for individual K-types - this will indicate a possibility of continuing the discrete series family.

2 Notation and the Basic Fibration

In general we use capital Latin letters for Lie groups with subscript $_0$ for real groups and no subscript for complexifications. We use the corresponding small Gothic letters for Lie algebras, again with subscript $_0$ for real Lie algebras and no subscript for complexifications. Our basic objects are a connected simply connected simple real Lie group G_0, its Lie algebra $\mathfrak{g}_0$, the complexification G of G_0, and the Lie algebra $\mathfrak{g}$ of G. Here G is a connected simply connected complex Lie group and the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a homomorphism $G_0 \to G$ with discrete central kernel.

When we omit a subscript $_0$ where there had been one before, we mean complexification.

Fix a Cartan involution θ of G_0 and $\mathfrak{g}_0$. The fixed point set $K_0 = G_0^\theta$ is a maximal compactly embedded subgroup of G_0. As usual, we decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ into (± 1)–eigenspaces of θ, where $\mathfrak{k}_0$ (resp. $\mathfrak{k}$) is the Lie algebra of K_0 (resp. K).

We now make two assumptions:

(2.1) $\quad \operatorname{rank} G_0 = \operatorname{rank} K_0$, and the symmetric space $S_0 := G_0/K_0$ is not of hermitian type.

In particular K_0 is a maximal compact subgroup of G_0. Both G_0 and K_0 are simply connected semisimple groups with finite center.

Fix a maximal torus $T_0 \subset K_0$. Then T_0 is a compact Cartan subgroup of G_0, and a celebrated theorem of Harish–Chandra says that G_0 has discrete series representations.

The construction of Borel and de Siebenthal [BoS] provides a positive root system $\Delta^+ = \Delta^+_G$ for $(\mathfrak{g}, \mathfrak{t})$ such that the associated simple root system $\Psi = \Psi_G$ contains just one noncompact root. We denote

$$\Psi = \{\psi_1, \dots, \psi_\ell\} \text{ (Bourbaki root order) and } \nu \in \Psi \text{ is the noncompact simple root.} \tag{2.2}$$

Every root $\alpha \in \Delta^+$ has expression $\alpha = \sum n_i(\alpha)\psi_i$. Since we have excluded the hermitian case, the coefficient of ν in the maximal root μ is 2. Further, a root is compact just when the coefficient of ν in its expansion is 0 or ± 2, noncompact just when that coefficient is ± 1. Also, $(\Psi \setminus \{\nu\}) \cup \{-\mu\}$ is a simple root system for $(\mathfrak{k}, \mathfrak{t})$. Grading by the coefficient n_ν of ν we have a parabolic subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}_-, \text{ reductive part } \mathfrak{l} = \mathfrak{t} + \sum_{n_\nu = 0} \mathfrak{g}_\alpha \text{ and nilradical } \mathfrak{u}_- = \mathfrak{u}_{-2} + \mathfrak{u}_{-1} \tag{2.3}$$

where $\mathfrak{u}_i = \sum_{n_\nu = i} \mathfrak{g}_\alpha$. The opposite parabolic is $\mathfrak{q}^{opp} = \mathfrak{l} + \mathfrak{u}_+$ where $\mathfrak{u}_+ = \mathfrak{u}_1 + \mathfrak{u}_2$. Note that

$$\mathfrak{s} = \mathfrak{u}_{-1} + \mathfrak{u}_1, \text{ so } \mathfrak{q} \cap \mathfrak{s} = \mathfrak{u}_{-1}, \text{ and } \mathfrak{k} = \mathfrak{u}_{-2} + \mathfrak{l} + \mathfrak{u}_2. \tag{2.4}$$

On the group level, we have the parabolic subgroup $Q \subset G$ where Q has Lie algebra $\mathfrak{q}$. The group Q has Chevalley semidirect product decomposition LU_- where L is the reductive component and U_- is the unipotent radical. Note that $G_0 \cap Q$ is a real form L_0 of L and that L_0 is the centralizer in K_0 of a circle subgroup of T_0. The parabolic Q defines a complex flag manifold $Z = G/Q$, say with base point $z_0 = 1Q$, and and open orbit $D = G_0(z_0) \cong G_0/L_0$. The complex manifold D has maximal compact subvariety $Y = K_0(z_0) \cong K_0/L_0$, which is a smaller complex flag manifold $K/(K \cap Q)$.

Our choice of signs in (2.3) is such that

$$\begin{aligned} &\mathfrak{u}_+ \text{ is the holomorphic tangent space of } D \text{ at } z_0\,, \\ &\mathfrak{u}_2 \text{ is the holomorphic tangent space of } Y \text{ at } z_0\,, \text{ and} \\ &\mathfrak{u}_1 \text{ is the holomorphic normal space of } Y \text{ in } D \text{ at } z_0\,. \end{aligned} \tag{2.5}$$

Since G_0/K_0 is irreducible but not hermitian we know that the action of $\mathfrak{k}_0$ on $\mathfrak{s}_0$ is absolutely irreducible. Thus the action of $\mathfrak{k}$ on $\mathfrak{s}$ is irreducible. From [W2, Theorem 8.13.3] we know that the action of $\mathfrak{l}$ on each $\mathfrak{u}_i$ is irreducible. It will be convenient to have the notation

$$\tau_i: \text{ representation of } L \text{ on the vector space } \mathfrak{u}_i. \tag{2.6}$$

The contragredient (dual) of τ_i is $\tau_i^* = \tau_{-i}$. Some obvious highest or lowest weight spaces of the τ_i are given by

$$\begin{aligned} &\tau_2 \text{ has highest weight space } \mathfrak{g}_\mu \text{ and } \tau_{-2} \text{ has lowest weight space } \mathfrak{g}_{-\mu}, \\ &\tau_1 \text{ has lowest weight space } \mathfrak{g}_\nu \text{ and } \tau_{-1} \text{ has highest weight space } \mathfrak{g}_{-\nu}\,. \end{aligned} \tag{2.7}$$

Note that the degree $\deg \tau_i = \dim_{\mathbb{C}} \mathfrak{u}_i$. If $i \neq 0$ it is the number of roots α such that $n_\nu(\alpha) = i$.

The basic tool in this paper is the real analytic fibration

$$D \to S_0 \text{ with fiber } Y, \text{ in other words } G_0/L_0 \to G_0/K_0 \text{ with fiber } K_0/L_0. \tag{2.8}$$

The structure of the holomorphic tangent bundle and the holomorphic normal bundle to Y in D is given by (2.5), (2.6) and (2.7). In the next section we will make this explicit. The fibration (2.8) was first considered by W. Schmid in [S1] and [S2] for a related situation in which $L_0 = T_0$, and then somewhat later by R. O. Wells and one of us [WeW] without that restriction. A much more general setting, which drops the compactness assumption on L_0, is that of the double fibration transform (see [FHW] and the references there), where S_0 is replaced by a complexification $S \subset G/K$. Specialization of the double fibration transform to the Borel – de Siebenthal setting is carried out in [EW].

The simple root system $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ of $\mathfrak{g}$ defines the system

$$\Xi = \{\xi_1, \ldots, \xi_\ell\} \text{ where } \tfrac{2\langle \xi_i, \psi_j\rangle}{\langle \psi_j, \psi_j\rangle} = \delta_{i,j} \tag{2.9}$$

of fundamental simple weights. Let γ be the highest weight of an irreducible representation of L_0. For our discussion of the Borel – de Siebenthal discrete series in Section 5 we will need to know exactly when $\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0$ for all positive complementary roots α (roots α that are not roots of $\mathfrak{l}$) where $\rho_\mathfrak{g}$ denotes half the sum of the positive roots of $\mathfrak{g}$. The condition is Theorem 2.12 below.

Define ν^* by $\langle \nu^*, \psi_j\rangle = 0$ for $\psi_j \neq \nu$ and $2\langle \nu^*, \nu\rangle = \langle \nu, \nu\rangle$ (i.e. the fundamental weight dual to ν). Then $\gamma \in i\mathfrak{t}_0^*$ decomposes as

$$\gamma = \gamma_0 + t\nu^* \text{ where } \langle \gamma_0, \nu\rangle = 0 \text{ and } t \in \mathbb{R}. \tag{2.10}$$

Define $\Delta_i = \{\alpha \in \Delta_G \mid n_\nu(\alpha) = i\}$, so $\mathfrak{u}_i = \sum_{\alpha \in \Delta_i} \mathfrak{g}_\alpha$ for $i \in \{\pm 1, \pm 2\}$. Thus the positive root system decomposes as $\Delta^+ = (\Delta_0 \cap \Delta^+) \cup \Delta_1 \cup \Delta_2$. The highest weight of τ_2, representation of $\mathfrak{l}$ on $\mathfrak{u}_2 = \sum_{\Delta_2} \mathfrak{g}_\alpha$ is μ. If we subtract a positive combination of roots of $\Psi \setminus \{\nu\}$ from μ we decrease the inner product with $\gamma + \rho_\mathfrak{g}$. Thus

$$\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0 \text{ for all } \alpha \in \Delta_2 \text{ if and only if } \langle \gamma + \rho_\mathfrak{g}, \mu\rangle < 0. \tag{2.11a}$$

The highest weight of τ_{-1} is $-\nu$, so τ_1 has highest weight $w^0_\mathfrak{l}(\nu)$ where $w^0_\mathfrak{l}$ is the longest element of the Weyl group of $\mathfrak{l}$. Thus

$$\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0 \text{ for all } \alpha \in \Delta_1 \text{ if and only if } \langle \gamma + \rho_\mathfrak{g}, w^0_\mathfrak{l}(\nu)\rangle < 0. \tag{2.11b}$$

As ν^* is orthogonal to the roots of $\mathfrak{l}$ it is fixed by the inverse of $w^0_\mathfrak{l}$, so $\langle \nu^*, w^0_\mathfrak{l}(\nu)\rangle = \langle \nu^*, \nu\rangle = 1$. Using the decomposition (2.10), and combining (2.11a) and (2.11b), we have

Theorem 2.12 *The following conditions are equivalent.*

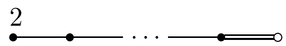
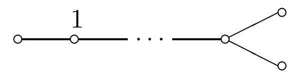
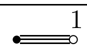
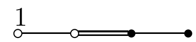
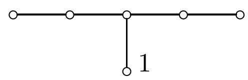
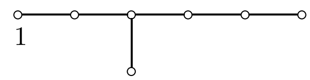
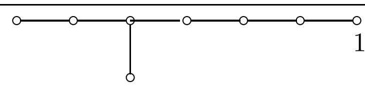
1. *The inequality $\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0$ holds for every root $\alpha \in \Delta_1 \cup \Delta_2$ (i.e. every positive complementary root)*
2. *Both $t < -\frac{1}{2}\langle \gamma_0 + \rho_\mathfrak{g}, \mu\rangle$ and $t < -\langle \gamma_0 + \rho_\mathfrak{g}, w^0_\mathfrak{l}(\nu)\rangle$.*

Remark 2.13 In Theorem 2.12 it is automatic that $\langle \gamma + \rho_\mathfrak{g}, \beta\rangle > 0$ for every positive root of $\mathfrak{l}$, so the conditions of Theorem 2.12 ensure that $\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle \neq 0$ for every root α, in other words that $\gamma + \rho_\mathfrak{g}$ is the Harish–Chandra parameter of a discrete series representation of G_0. Specifically, in our setting, the conditions of Theorem 2.12 will characterize the Borel – de Siebenthal discrete series. $\diamondsuit$

3 Classification

In this section we give a complete list the simple Lie algebras $\mathfrak{g}_0$ for which the hypotheses (2.1) hold. We then specify the complex parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ and the real subalgebras $\mathfrak{k}_0$ and $\mathfrak{l}_0$ of $\mathfrak{g}_0$. Next, we give precise descriptions of the representations τ_i and their representation spaces $\mathfrak{u}_i$. Much of this is done using the Dynkin diagrams to indicate highest weights of representations. There the special cases, where G_0/K_0 is a quaternionic symmetric space, are visible at a glance: they are the ones where $-\mu$ connects directly to ν in the extended Dynkin diagram of $\mathfrak{g}$.

We will denote highest weights of representations as follows. In Dynkin diagrams with two root lengths we denote short root nodes by black dots $\bullet$ and long roots by the usual circles $\circ$. Extended diagrams are those with the negative of the maximal root μ attached by the usual rules. Recall the system Ξ of fundamental simple weights from (2.9). The (irreducible finite dimensional) representation of G and $\mathfrak{g}$ of highest weight $\sum n_i\xi_i$ is indicated by the Dynkin diagram of $\mathfrak{g}$ with n_i written next to the i^{th} node, except that we omit writing zeroes. So for example the adjoint representations are indicated by

$A_\ell\,,\ell \geqq 1$		$B_\ell\,,\ell \geqq 3$	
$C_\ell\,,\ell \geqq 2$		$D_\ell\,,\ell \geqq 4$	
G_2		F_4	
E_6		E_7	
E_8			

We will use this notation for $\mathfrak{k}$ as well. It can be identified from its simple root system $\Psi_{\mathfrak{k}} = (\Psi \setminus \{\nu\}) \cup \{-\mu\}$. We'll follow the notation of [BE], except that we won't darken the dots. Thus the diagram of $\mathfrak{l}$ consists of the diagram of $\mathfrak{g}$, except that the o (resp. $\bullet$) at the node for the noncompact simple root ν is replaced by an $\times$ (resp. $\boxtimes$)[1]. In the diagram of $\mathfrak{l}$, a symbol $\times$ or $\boxtimes$ indicates the 1–dimensional center of L. The irreducible representation of L with highest weight $\sum n_i\xi_i$ now indicated by the Dynkin diagram of $\mathfrak{l}$ with n_i written next to the i^{th} node, for $n_i \neq 0$, whether that node is o, $\bullet$, $\times$ or $\boxtimes$.

If ν^* is the fundamental simple weight corresponding to the noncompact simple root ν, and $x \in \mathfrak{t}$ by $\alpha(x) = \langle \xi, \alpha\rangle$ for $\alpha \in \mathfrak{t}^*$, then $\mathfrak{l}_0$ has center $i\mathbb{R}\nu^*$.

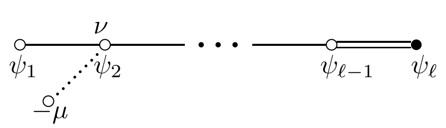
Now we use the fact that K_0 is connected, simply connected and semisimple. The simple root system $\Psi_{\mathfrak{k}}$ decomposes into

$$\begin{aligned} \Psi_{\mathfrak{k}_1} &: \text{ connected component that contains } -\mu, \text{ and} \\ \Psi_{\mathfrak{k}_2} &: \text{ the complement of } \Psi_{\mathfrak{k}_1} \text{ in } \Psi_{\mathfrak{k}} \end{aligned}$$

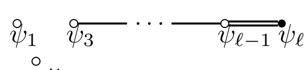
This results in decompositions $K_0 = K_1 \times K_2$ and $L_0 = L_1 \times L_2$, which we make explicit in each case.

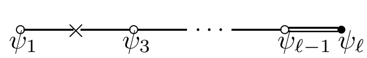
In the following we list the Dynkin diagrams, with the possibilities of the noncompact simple root ν among the simple roots ψ_i. Also in the picture one finds the extended Dynkin diagram node for $-\mu$ where μ is the maximal root. Diagrams of Type A do not occur because ν must have coefficient 2 in the expression of μ as a linear combination of simple roots. We now consider the cases where $\mathfrak{g}$ is of type B.

3.1 Case $Spin(4, 2\ell-3)$. Here G_0 is the 2–sheeted cover of the group $SO(4, 2\ell-3)$ which is a real analytic subgroup of the complex simply connected group $Spin(2\ell+1;\mathbb{C})$. Its extended Dynkin diagram is

(3.1a) 

(type $B_\ell\,,\,\ell > 2$)

Thus $\mathfrak{k}$ is 

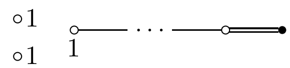
and $\mathfrak{l}$ is 

.

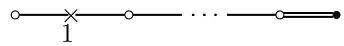
[1]Of course one can also look at L_0 as a subgroup of K_0, and from that viewpoint the diagram of $\mathfrak{l}$ is obtained from that of $\mathfrak{k}$ on replacing the o at the node for $-\mu$ with a $\times$. However it is more convenient to look at L_0 as a subgroup of G_0, and the diagram of $\mathfrak{l}$ from that viewpoint, when we consider the action of L on the subspaces $\mathfrak{u}_i$.

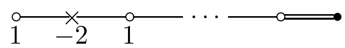
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.1b) $\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell-3))$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \times \mathfrak{so}(2\ell-3))$

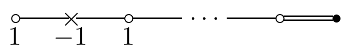
where ν^* the fundamental simple weight corresponding to ν. The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is 

. Using (2.7), the representation

(3.1c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

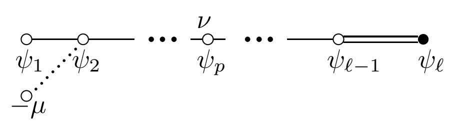
, so the dualizing diagram method of [EW] shows that the representation

(3.1d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

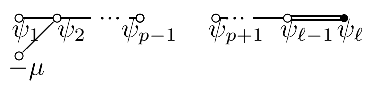
.

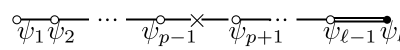
Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = (2\ell-3)(2\ell-4)$.

3.2 Case $Spin(2p, 2\ell-2p+1), 2 < p < \ell$. Here G_0 is the 2–sheeted cover of $SO(2p, 2\ell-2p+1)$, $2 < p < \ell$, contained in $Spin(2\ell+1;\mathbb{C})$. Its extended Dynkin diagram is

(3.2a) 

(type B_ℓ, $\ell > 3$)

Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

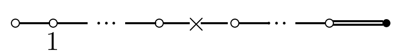
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

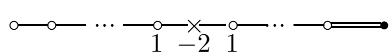
(3.2b) $\mathfrak{k}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2\ell-2p+1)$ and $\mathfrak{l}_0 = \mathfrak{u}(p) \oplus \mathfrak{so}(2\ell-2p+1) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{so}(2\ell-2p+1)$.

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_p$: 

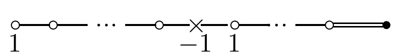
. Using (2.7), the representation

(3.2c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

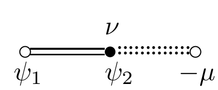
so the dualizing diagram method of [EW] shows that the representation

(3.2d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

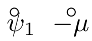
.

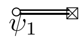
Here $\dim \mathfrak{u}_2 = p(p-1)/2$ and $\dim \mathfrak{u}_1 = p(2\ell-2p+1)$.

3.3 Case $Spin(4,1)$. Here G_0 is the (universal) double cover of the group $SO(4,1)$. Its extended Dynkin diagram is

(3.3a) 

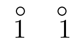
(type B_2)

Thus $\mathfrak{k}$ is 

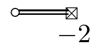
and $\mathfrak{l}$ is 

. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

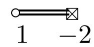
(3.3*b*) $$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(1).$$

Here $\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_2$: 

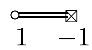
. Using (2.7), the representation

(3.3*c*) $\tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

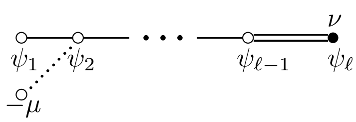
so the dualizing diagram method of [EW] shows that the representation

(3.3*d*) $\tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

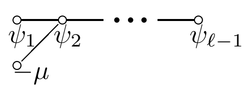
.

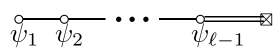
$\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 2$.

3.4 Case $Spin(2\ell, 1)$, $\ell > 1$. Here G_0 is the universal (2–sheeted) cover of the group $SO(2\ell, 1)$ with $\ell > 1$. Its extended Dynkin diagram is

(3.4*a*) 

(type B_ℓ , $\ell > 2$)

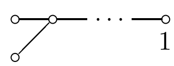
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

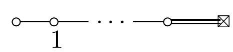
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

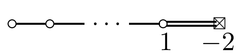
(3.4*b*) $$\mathfrak{k}_0 = \mathfrak{so}(2\ell) \text{ and } \mathfrak{l}_0 = \mathfrak{u}(\ell) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(\ell).$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_\ell$: 

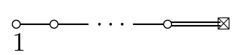
. Using (2.7), the representation

(3.4*c*) $\tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

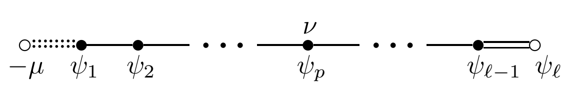
so the dualizing diagram method of [EW] shows that the representation

(3.4*d*) $\tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

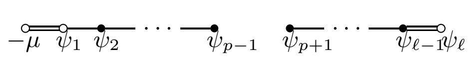
.

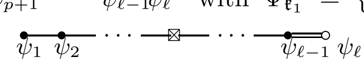
Here $\dim \mathfrak{u}_2 = \ell(\ell-1)/2$ and $\dim \mathfrak{u}_1 = \ell$. This exhausts the cases where $\mathfrak{g}$ is of type B, and we go on to consider the cases where $\mathfrak{g}$ is of type C.

3.5 Case $Sp(p, \ell - p), 1 < p < \ell$. Here G_0 is simply connected, and its extended Dynkin diagram is

(3.5*a*) 

(type C_ℓ , $\ell > 1$)

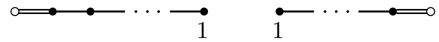
Thus $\mathfrak{k}$ is 

with $\Psi_{\mathfrak{k}_1} = \{-\mu, \psi_1, \psi_2, \ldots, \psi_{p-1}\}$ and $\Psi_{\mathfrak{k}_2} = \{\psi_{p+1}, \psi_{p+2}, \ldots, \psi_\ell\}$, and $\mathfrak{l}$ is 

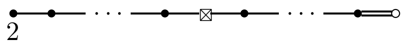
.

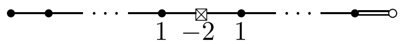
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.5b) $\qquad \mathfrak{k}_0 = \mathfrak{sp}(p) \oplus \mathfrak{sp}(\ell - p)$ and $\mathfrak{l}_0 = \mathfrak{u}(p) \oplus \mathfrak{sp}(\ell - p) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{sp}(\ell - p)$.

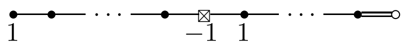
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_p$: 

. Using (2.7), the representation

(3.5c) $\qquad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

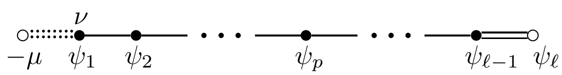
The action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

, so the dualizing diagram method of [EW] shows that the representation

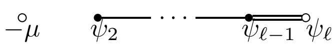
(3.5d) $\qquad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

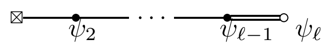
Here $\dim \mathfrak{u}_2 = (p-1)(p+2)/2$ and $\dim \mathfrak{u}_1 = 2p(\ell - p)$.

3.6 Case $Sp(1, \ell - 1)$. Here G_0 is simply connected, and its extended Dynkin diagram is

(3.6a) 

(type C_ℓ , $\ell > 1$)

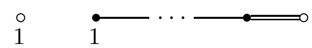
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

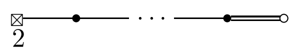
.

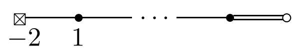
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.6b) $\qquad \mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(\ell - 1)$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(\ell - 1)$.

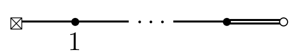
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$: 

. Using (2.7), the representation

(3.6c) $\qquad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

The action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

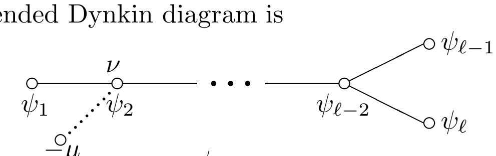
, so the dualizing diagram method of [EW] shows that the representation

(3.6d) $\qquad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

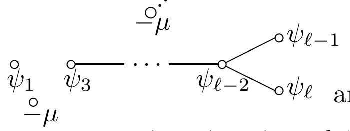
.

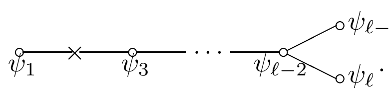
Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 2\ell$. This exhausts the cases where $\mathfrak{g}$ is of type C, and we go on to consider the cases where $\mathfrak{g}$ is of type D.

3.7 Case $Spin(4, 2\ell - 4), \ell > 4$. Here G_0 is the 2–sheeted cover of $SO(4, 2\ell - 4)$ contained in $Spin(2\ell; \mathbb{C})$. Its extended Dynkin diagram is

(3.7a) 

(type D_ℓ , $\ell > 4$)

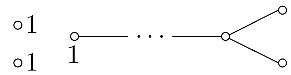
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

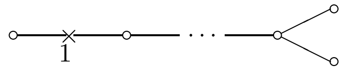
.

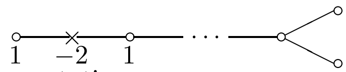
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.7b) $\qquad \mathfrak{k}_0 = \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell - 4))$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell - 4))$.

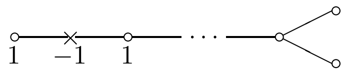
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is 

.
Using (2.7), the representation

(3.7c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

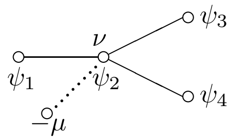
, so the dualizing diagram method of [EW] shows that the representation

(3.7d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

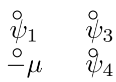
.

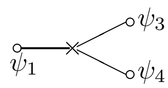
Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 4(\ell - 2)$.

3.8 Case $SO(4,4)$. Here G_0 is the 2–sheeted cover of the group $SO(4,4)$ that is contained in $Spin(8;\mathbb{C})$. Its extended Dynkin diagram is

(3.8a) 

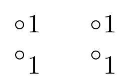
(type D_4)

Thus $\mathfrak{k}$ is 

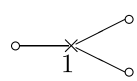
and $\mathfrak{l}$ is 

.
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

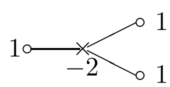
(3.8b) $\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1))$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1))$.

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is 

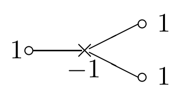
.
Using (2.7), the representation

(3.8c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

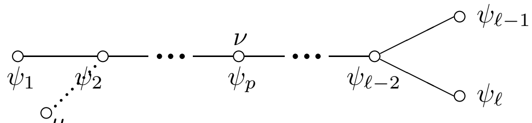
Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

so the dualizing diagram method of [EW] shows that the representation

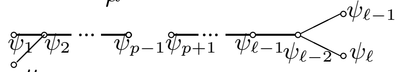
(3.8d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

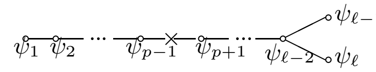
Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 8$.

3.9 Case $Spin(2p, 2\ell - 2p), 2 < p < \ell - 2$. Here G_0 is the 2–sheeted cover of the group $SO(2p, 2\ell - 2p)$ that is contained in $Spin(2\ell;\mathbb{C})$, with $2 < p < \ell - 2$. Its extended Dynkin diagram is

(3.9a) 

(type D_ℓ, $\ell > 5$)

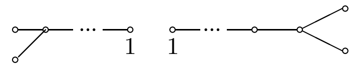
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

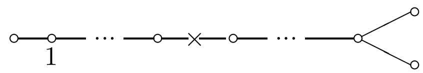
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

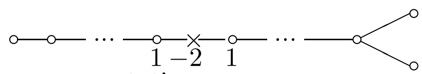
(3.9b) $\quad \mathfrak{k}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2\ell - 2p)$ and $\mathfrak{l}_0 = \mathfrak{u}(p) \oplus \mathfrak{so}(2\ell - 2p) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{so}(2\ell - 2p)$.

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_p$: 

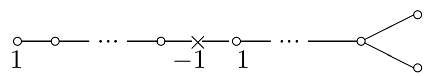
. Using (2.7), the representation

(3.9c) $\qquad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

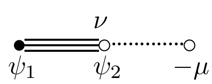
so the dualizing diagram method of [EW] shows that the representation

(3.9d) $\qquad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

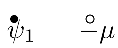
.

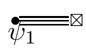
Here $\dim \mathfrak{u}_2 = p(p-1)/2$ and $\dim \mathfrak{u}_1 = 2p(\ell - p)$. This exhausts the cases where $\mathfrak{g}$ is of type D, and thus exhausts the classical cases. We go on the the exceptional cases.

3.10 Case G_{2,A_1A_1}. Here G_0 is the split real Lie group of type G_2. It has maximal compact subgroup $SO(4)$. Its extended Dynkin diagram is

(3.10a) 

(type G_2).

Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

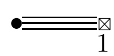
.

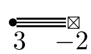
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.10b) $\qquad \mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(1)$.

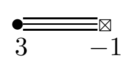
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$: 

. Using (2.7), the representation

(3.10c) $\qquad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

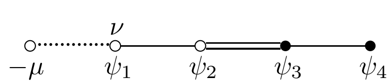
so the representation

(3.10d) $\qquad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

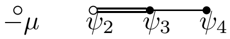
.

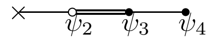
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 4, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$.

3.11 Case F_{4,A_1C_3}. Here G_0 is the simply connected real Lie group of type F_4 whose maximal compact subgroup has 2–sheeted cover $Sp(1) \times Sp(3)$. Its extended Dynkin diagram is

(3.11a) 

(Type F_4)

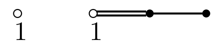
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

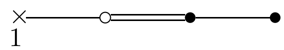
.

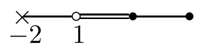
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.11b) $$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(3) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(3).$$

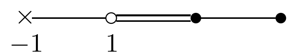
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$: 

. Using (2.7), the representation

(3.11c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

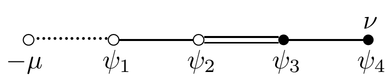
so the representation

(3.11d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

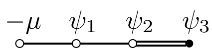
.

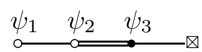
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 14, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$.

3.12 Case F_{4,B_4}. Here G_0 is the simply connected real Lie group of type F_4 with maximal compact subgroup $Spin(9)$. Its extended Dynkin diagram is

(3.12a) 

(Type F_4)

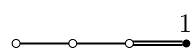
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

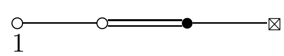
.

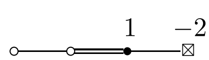
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.12b) $$\mathfrak{k}_0 = \mathfrak{so}(9) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(7).$$

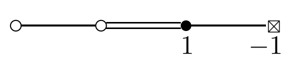
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_4$: 

. Using (2.7), the representation

(3.12c) τ_2 : $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

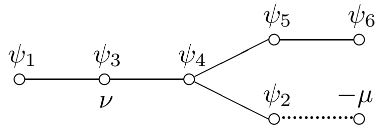
so the representation

(3.12d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

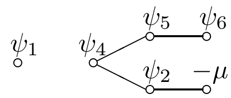
.

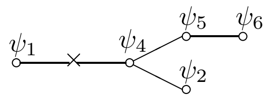
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 8, is self–dual, and has a symmetric bilinear invariant. In effect τ_1 is the action of $Spin(7)$ on the Cayley numbers, and τ_2 is its action (factored through $SO(7)$ on the pure imaginary Cayley numbers. thus $\dim \mathfrak{u}_2 = 7$ and $\dim \mathfrak{u}_1 = 8$.

3.13 Case $E_{6,A_1A_5,1}$. Here G_0 is the group of type E_6 whose maximal compact subgroup is the 2–sheeted cover of $SU(2) \times SU(6)$. The noncompact simple root $\nu = \psi_3$, so L is of type $T_1A_1A_4$. We do not consider the case $\nu = \psi_5$ separately because the two differ only by an outer automorphism of E_6. Here the extended Dynkin diagram (Bourbaki root order) is

(3.13a) 

(Type E_6)

Thus $\mathfrak{k}$ is 

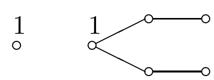
and $\mathfrak{l}$ is 

.

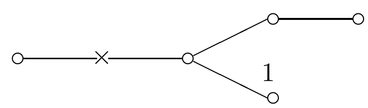
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.13b) $$\mathfrak{k}_0 = \mathfrak{su}(6) \oplus \mathfrak{su}(2) \text{ and } \mathfrak{l}_0 = \big(\mathfrak{su}(5) \oplus i\mathbb{R}\nu^*\big) \oplus \mathfrak{su}(2).$$

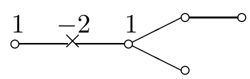
The representation of $\mathfrak{k}$ on $\mathfrak{s}$

has highest weight $-\nu = -\psi_3$: 

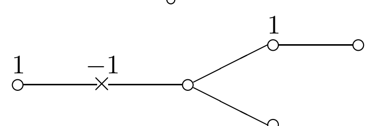
. Using (2.7), the representation

(3.13c) $\tau_2 :$ $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

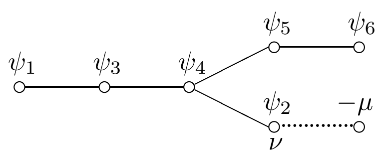
so the representation

(3.13d) $\tau_1 :$ $\mathfrak{l}$ on $\mathfrak{u}_1$ is 

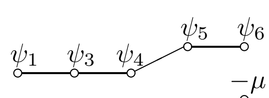
.

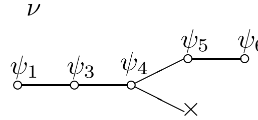
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 20 and is not self–dual. We have $\dim \mathfrak{u}_2 = 5$ and $\dim \mathfrak{u}_1 = 20$.

3.14 Case $E_{6,A_1A_5,2}$. Here G_0 is the group of type E_6 with maximal compact subgroup $SU(2) \times SU(6)$. The noncompact simple root $\nu = \psi_2$, so L is of type T_1A_5. The extended Dynkin diagram (Bourbaki root order) is

(3.14a) 

(Type E_6)

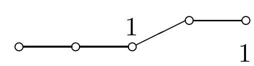
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

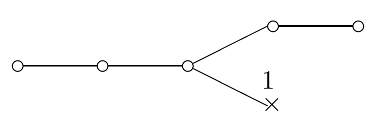
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

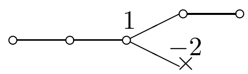
(3.14b) $$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{su}(6) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{su}(6).$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$: 

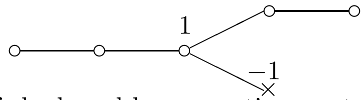
. Using (2.7), the representation

(3.14c) $\tau_2 :$ $\mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

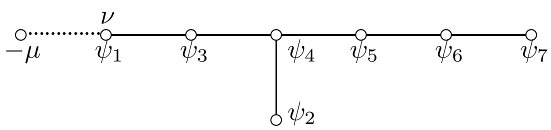
so the representation

(3.14d) $\tau_1:\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

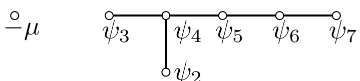
.

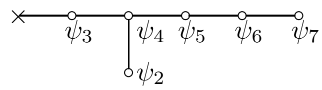
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 20, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$. In brief, $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 20$.

3.15 Case $E_{7,A_1D_6,1}$. Here G_0 is the group of type E_7 with maximal compact subgroup that is the 2–sheeted cover of $SU(2) \times Spin(12)$. The noncompact simple root $\nu = \psi_1$, so L is of type T_1D_6 and the extended Dynkin diagram is

(3.15a) 

(Type E_7)

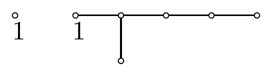
Thus $\mathfrak{k}$ is 

and $\mathfrak{l}$ is 

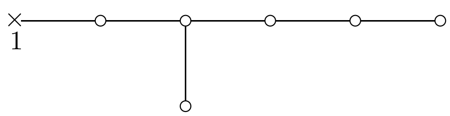
.

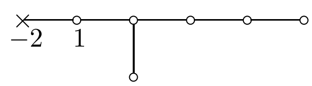
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

(3.15b) $\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{so}(12)$ and $\mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(12)$.

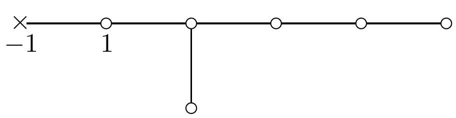
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$: 

. Using (2.7), the representation

(3.15c) $\tau_2:\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

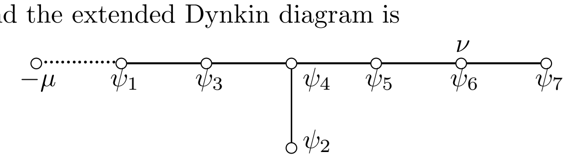
so the representation

(3.15d) $\tau_1:\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

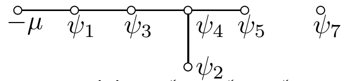
.

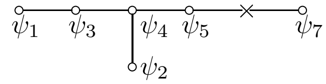
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 32, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$. In brief, $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 32$.

3.16 Case $E_{7,A_1D_6,2}$. Again G_0 is the group of type E_7 with maximal compact subgroup that is a 2–sheeted cover of $SU(2) \times Spin(12)$, but now noncompact simple root $\nu = \psi_6$, so L is of type $T_1A_1D_5$ and the extended Dynkin diagram is

(3.16a) 

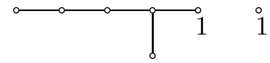
(Type E_7)

Thus $\mathfrak{k}$ is 

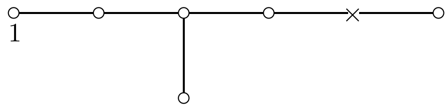
and $\mathfrak{l}$ is 

.
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

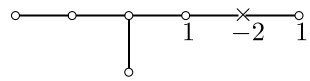
(3.16b) $\quad \mathfrak{k}_0 = \mathfrak{so}(12) \oplus \mathfrak{sp}(1)$ and $\mathfrak{l}_0 = (\mathfrak{so}(2) \oplus \mathfrak{so}(10)) \oplus \mathfrak{sp}(1) = (i\mathbb{R}\nu^* \oplus \mathfrak{so}(10)) \oplus \mathfrak{sp}(1)$.

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_6$: 

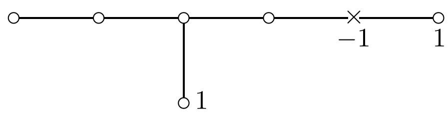
. Using (2.7), the representation

(3.16c) $\quad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

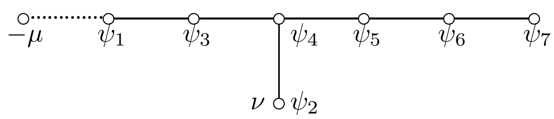
so the representation

(3.16d) $\quad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

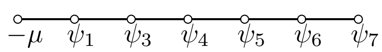
.

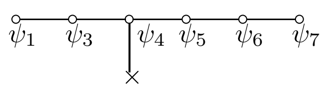
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 16 and is not self–dual. In brief, $\dim \mathfrak{u}_2 = 10$ and $\dim \mathfrak{u}_1 = 16$.

3.17 Case E_{7,A_7}. Here G_0 is the group of type E_7 with maximal compact subgroup $SU(8)/\{\pm 1\}$. The noncompact simple root $\nu = \psi_2$, so L is of type T_1E_6 and the extended Dynkin diagram is

(3.17a) 

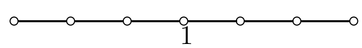
(Type E_7)

Thus $\mathfrak{k}$ is 

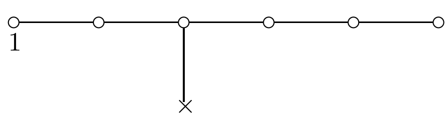
and $\mathfrak{l}$ is 

.
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

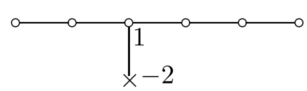
(3.17b) $\quad \mathfrak{k}_0 = \mathfrak{su}(8)$ and $\mathfrak{l}_0 = (\mathfrak{u}(1) \oplus \mathfrak{u}(7)) \cap \mathfrak{su}(8) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(7)$.

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$: 

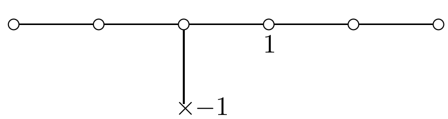
. Using (2.7), the representation

(3.17c) $\quad \tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is 

.

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is 

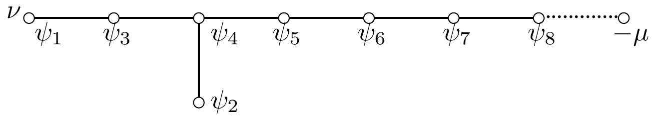
so the representation

(3.17d) $\quad \tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is 

.

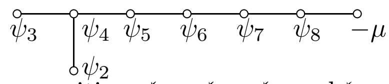
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 35 and is not self–dual. In brief, $\dim \mathfrak{u}_2 = 7$ and $\dim \mathfrak{u}_1 = 35$

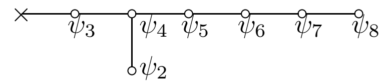
3.18 CASE E_{8,D_8}. Here G_0 is the group of type E_8 with maximal compact subgroup locally isomorphic to $Spin(16)$. The noncompact simple root $\nu = \psi_1$, so L is of type $T_1 D_7$ and the extended Dynkin diagram is



(Type E_8)

(3.18a)

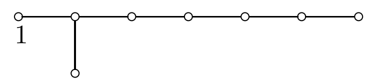
Thus $\mathfrak{k}$ is


and $\mathfrak{l}$ is


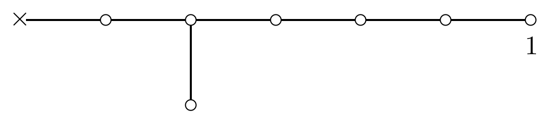
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are'

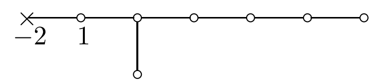
(3.18b) $$\mathfrak{k}_0 = \mathfrak{so}(16) \text{ and } \mathfrak{l}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(14) = i\mathbb{R}\nu^* \oplus \mathfrak{so}(14).$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$:


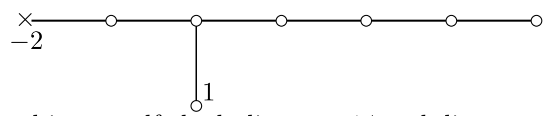
.
Using (2.7), the representation

(3.18c) $\tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is


.

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is


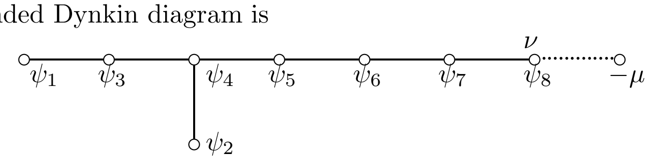
so the representation

(3.18d) $\tau_1 :\ \mathfrak{l}$ on $\mathfrak{u}_1$ is


.

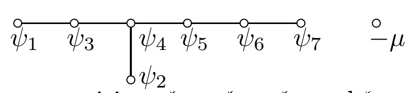
Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 64 and is not self–dual; $\dim \mathfrak{u}_2 = 14$ and $\dim \mathfrak{u}_1 = 64$.

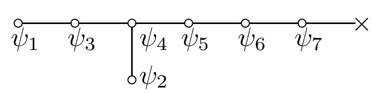
3.19 CASE E_{8,A_1E_7}. Here G_0 is the group of type E_8 with maximal compact subgroup that has $SU(2) \times E_7$ as a double cover.. The noncompact simple root $\nu = \psi_8$, so L is of type $T_1 E_7$ and the extended Dynkin diagram is



(Type E_8)

(3.19a)

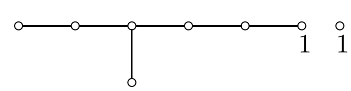
Thus $\mathfrak{k}$ is


and $\mathfrak{l}$ is


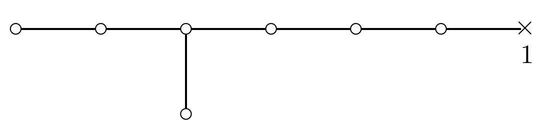
.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

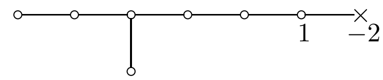
(3.19b) $$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{e}_7 \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{e}7.$$

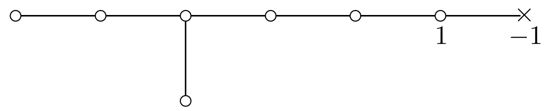
The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_8$:


.
Using (2.7), the representation

(3.19c) $\tau_2 :\ \mathfrak{l}$ on $\mathfrak{u}_2$ is


.

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is  so the representation

(3.19d) τ_1 : $\mathfrak{l}$ on $\mathfrak{u}_1$ is  .

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$, has degree 56, is self-dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$.

This completes our run through the exceptional cases.

4 Prehomogeneity and Relative Invariants for $(L, \mathfrak{u}_1)$

Consider a connected linear algebraic group with a rational representation on a complex vector space. We say that the triple consisting of the group, the representation and the vector space is *prehomogeneous* if there is a Zariski–dense orbit. When no confusion is possible we omit the representation. A general theorem of Vinberg on graded Lie algebras (see [Kn1, Theorem 10.19]) shows that $(L, \mathfrak{u}_1)$ is prehomogeneous. Or one can verify that fact by running through the lists of Section 3 and the classification of [SK]. In fact we will do the latter in order to describe the algebra of relative–invariant polynomials on $\mathfrak{u}_1$ and the L–orbit structure of $\mathfrak{u}_1$ for each instance of $(L, \mathfrak{u}_1)$. We shall also use the notation V for the fundamental space $\mathfrak{u}_1$ and $V^* = \mathfrak{u}_{-1}$ for its dual space.

We recall some material on prehomogeneous spaces as it applies to $(L, \mathfrak{u}_1)$. There is no nonconstant L–invariant rational function $f : \mathfrak{u}_1 \to \mathbb{C}$ because the L–invariance would force it to be constant on the Zariski–dense L–orbit [SK, Proposition 3 in §2]. By *relative invariant* for $(L, \mathfrak{u}_1)$ we mean a nonconstant polynomial function $f : \mathfrak{u}_1 \to \mathbb{C}$ such that $f(\ell\xi) = \chi(\ell)f(\xi)$ for some rational character $\chi : L \to \mathbb{C}^\times$. The quotient of two relative invariants with the same character would be an L–invariant rational function of $\mathfrak{u}_1$, hence constant, so a relative invariant f is determined up to scalar multiple by its character χ [SK, Proposition 3 in §4]. In particular all the $f_c : \xi \mapsto f(c\xi)$ are proportional, so f is a *homogeneous* polynomial. It will be convenient to denote

(4.1) $\mathcal{A}(L, \mathfrak{u}_1)$: the associative algebra of all relative invariants of $(L, \mathfrak{u}_1)$.

The *regular set* for $(L, \mathfrak{u}_1)$ is the open L–orbit $\mathcal{O}_0 := \mathrm{Ad}(L)\xi_0 \subset \mathfrak{u}_1$ and the *singular set* is its complement $\mathfrak{u}_1 \setminus \mathcal{O}_0$. Let $V_1, \dots, V_e$ be those components of the singular set that are of codimension 1 in $\mathfrak{u}_1$. For each i, V_i is the zero set of an irreducible polynomial f_i. The algebra of relative invariants for $(L, \mathfrak{u}_1)$ is the polynomial algebra $\mathbb{C}[f_1, \dots, f_e]$ [SK, Proposition 5 in §4]. In particular $(L, \mathfrak{u}_1)$ has a relative invariant if and only if the its singular set has a component of codimension 1. So far we haven't used irreducibility of L on $\mathfrak{u}_1$, but now we use it to see [SK, Proposition 12 in §4] that $e \leqq 1$, i.e. that either $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}$ (in other words $(L.\mathfrak{u}_1)$ has no relative invariant) or $\mathcal{A}(L, \mathfrak{u}_1)$ has form $\mathbb{C}[f]$.

4.2. Cases $SO(2p, r)$. We first consider the various cases where G_0 is the universal covering group of the indefinite orthogonal group $SO(2p, 2q)$ or $SO(2p, 2q+1)$. For convenience we write that as $SO(2p, r)$. Then L_0 consists of all $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where a is in the image of the standard embedding $\iota : U(p) \hookrightarrow SO(2p)$ and where $b \in SO(r)$. Here $\mathfrak{s}_0 = \left\{ \left(\begin{smallmatrix} 0 & x \\ {}^t x & 0 \end{smallmatrix}\right) \middle| \, x \in \mathbb{R}^{2p \times r} \right\} \cong \mathbb{R}^{2p \times r}$ and the (conjugation) action $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in L_0$ on $\mathfrak{s}_0$ is given by $x \mapsto axb^{-1}$. Now $\mathfrak{u}_1 \cong \mathbb{C}^{p \times r}$ with the action of $L \simeq GL(p; \mathbb{C}) \times SO(r; \mathbb{C})$ given by $\ell = \left(\begin{smallmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{smallmatrix}\right) : z \mapsto \ell_1 z \ell_2^{-1}$. Then $f(z) := \det(z \cdot {}^t z)$

transforms by $f(\ell(z)) = \det(\ell)^2 f(z)$. However it is a relative invariant only when it is not identically zero, i.e. when $p \leqq r$.

On the other hand, if $p > r$ then the $(SL(p;\mathbb{C}) \times SO(r;\mathbb{C}))$–orbit of $\begin{pmatrix} I_r \\ 0_{p-r} \end{pmatrix}$ is open in $\mathbb{C}^{p\times r}$, so there is no nonconstant $(SL(p;\mathbb{C}) \times SO(r;\mathbb{C}))$–invariant. It follows that there is no relative invariant for L.

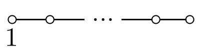
Summary: if $p \leqq r$ then $\mathcal{A}(L,\mathfrak{u}_1) = \mathbb{C}[f]$ where $f(z) := \det(z \cdot {}^t z)$, polynomial of degree $2p$. If $p > r$ then $\mathcal{A}(L,\mathfrak{u}_1) = \mathbb{C}$.

4.3. Cases $Sp(p,q)$. We next consider the cases where $G_0 = Sp(p,q)$. Then L_0 consists of all $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where a is in the image of the standard embedding $\iota : U(p) \hookrightarrow Sp(p)$ and where $b \in Sp(q)$. Here $\mathfrak{s}_0 = \left\{ \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right) \middle| x \in \mathbb{H}^{p\times q} \right\} \cong \mathbb{H}^{p\times q}$ and the (conjugation) action $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in L_0$ on $\mathfrak{s}_0$ is given by $x \mapsto axb^{-1}$. Now $\mathfrak{u}_1 \cong \mathbb{C}^{p\times 2q}$ with the action of $L \simeq GL(p;\mathbb{C}) \times Sp(q;\mathbb{C})$ given by $\ell = \left(\begin{smallmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{smallmatrix}\right) : z \mapsto \ell_1 z \ell_2^{-1}$. Let $J = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}$, so $Sp(q;\mathbb{C})$ is characterized by $b \cdot J \cdot {}^t b = J$. Let Pf denote the Pfaffian polynomial on the space of antisymmetric $2q \times 2q$ matrices, so $\mathrm{Pf}(m)^2 = \det(m)$. Then $f(z) := \mathrm{Pf}(z \cdot J \cdot {}^t z)$ transforms by $f(\ell z) = \det(\ell) f(z)$. And of course f is a relative invariant only when it is not identically zero. For that we must have the possibility that the $p \times p$ antisymmetric matrix $z \cdot J \cdot {}^t z$ is nonsingular, which is the case just when both that $p \leqq 2q$ and p is even.

As before, if $p > 2q$ then the $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))$–orbit of $\begin{pmatrix} I_{2q} \\ 0_{p-2q} \end{pmatrix}$ is open in $\mathbb{C}^{p\times 2q}$, so there is no nonconstant $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))$–invariant, and thus no relative invariant for L.

If $p \leqq 2q$ but p is odd we define $m \in \mathbb{C}^{p\times 2q}$ by $m_{i,i} = 1$ if $i \leqq p$ and i is odd, $m_{i,q+i} = 1$ if $i \leqq p$ and i is even, all other entries zero. The point is that the row space of m has dimension p and has nullity 1 relative to the bilinear form J that defines $Sp(q;\mathbb{C})$. Then $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))(m)$ consists of all elements of $\mathbb{C}^{p\times 2q}$ whose row space has dimension p and nullity 1 relative to J, and that is open in $\mathbb{C}^{p\times 2q}$. As above, it follows that there is no relative invariant for L.

Summary: if $p \leqq 2q$ and p is even then $\mathcal{A}(L,\mathfrak{u}_1) = \mathbb{C}[f]$ where $f(z) := \mathrm{Pf}(z \cdot J \cdot {}^t z)$, polynomial of degree p. If $p > 2q$ of if p is odd then $\mathcal{A}(L,\mathfrak{u}_1) = \mathbb{C}$.

We have completely described $\mathcal{A}(L\mathfrak{u}_1)$ when G is classical. Except for a few extreme cases, the representation $\tau_1|_{L'}$ of the derived group $L' = [L,L]$ failed to be self–dual because of a tensor factor  . In many of the exceptional group cases, $\tau_1|_{L'}$ is self–dual, so it has a bilinear invariant, and when that bilinear invariant is symmetric it generates $\mathcal{A}(L,\mathfrak{u}_1)$. Also, if that bilinear invariant is antisymmetric, then $\tau_1(L')$ is contained in the symplectic group J of the bilinear invariant, and if σ is the representation of J on $\mathfrak{u}_1$ then $S^2(S^2(\sigma))$ contains the trivial representation with multiplicity 1, and that gives a quartic invariant that generates $\mathcal{A}(L,\mathfrak{u}_1)$. Before formalizing these statements we look back at Section 3 to see the first five columns of

(4.4)

Case	G_0	$\deg \tau_1$	self–dual?	bilinear invariant	relative invariant
(3.10)	G_{2,A_1A_1}	4	yes	antisymmetric	degree 4
(3.11)	F_{4,A_1C_3}	14	yes	antisymmetric	degree 4
(3.12)	F_{4,B_4}	8	yes	symmetric	degree 2
(3.13)	$E_{6,A_1A_5,1}$	20	no	none	none
(3.14)	$E_{6,A_1A_5,2}$	20	yes	antisymmetric	degree 4
(3.15)	$E_{7,A_1D_6,1}$	32	yes	antisymmetric	degree 4
(3.16)	$E_{7,A_1D_6,2}$	16	no	none	none
(3.17)	E_{7,A_7}	35	no	none	degree 7
(3.18)	E_{8,D_8}	64	no	none	degree 8
(3.19)	E_{8,A_1E_7}	56	yes	antisymmetric	degree 4

.

The information of the last column of Table 4.4 is contained in [SK], but we can give a short direct proof of the cases where there is a relative invariant, as follows.

Lemma 4.5 *If $\tau_1|_{L'}$ is self–dual there are two possibilities. Either it has a nonzero symmetric bilinear invariant b and $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[b]$, or it has a nonzero antisymmetric bilinear invariant and $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 4. In the non self–dual case* (3.17) *we have $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 7, and in the non self–dual case* (3.18) *we have $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 8.*

Proof. If the bilinear invariant b is symmetric, then since it has degree 2 it must generate $\mathcal{A}(L, \mathfrak{u}_1)$. If b is antisymmetric, then in each of the five relevant cases of Table 4.4 we compute symmetric powers $S^2(\tau_1|_{L'})$, $S^3(\tau_1|_{L'})$ and $S^4(\tau_1|_{L'})$ to see that we first encounter a $\tau_1(L')$–invariant in degree 4. (This degree 4 semiinvariant can also be seen by a classification free argument [P, Proposition 1.4].)

Consider the two non self–dual cases of Table 4.4 for which we claim a $\tau_1|_{L'}$–invariant. In case (3.17) we compute the $S^r(\tau_1|_{L'})$ for $2 \leqq r \leqq 7$ to see that we first encounter a $\tau_1(L')$–invariant in degree 7, and and in case (3.18) we compute the $S^r(\tau_1|_{L'})$ for $2 \leqq r \leqq 8$ to see that we first encounter a $\tau_1(L')$–invariant in degree 8. $\square$

5 Negativity and K_0–types

In this section we discuss negativity of a homogeneous holomorphic vector bundle over G_0/L_0 and the K_0–types of the resulting discrete series representations.

Recall some notation from Section 2. The flag domain $D = G_0(z_0) \cong G_0/L_0$ is an open G_0–orbit in the complex flag manifold $Z = G/Q$, where $z_0 = 1Q$ is the base point and $L_0 = G_0 \cap Q$. The parabolic subgroup Q of G has Lie algebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}_-$ and its nilradical $\mathfrak{u}_-$ is opposite to $\mathfrak{u}_+$, which in turn represents the holomorphic tangent space to D at z_0. According the the multiplicity of the noncompact simple root, $\mathfrak{u}_+ = \mathfrak{u}_1 + \mathfrak{u}_2$. The maximal compact subvariety $Y = K_0(z_0) = K(z_0)$ has holomorphic tangent space at z_0 represented by $\mathfrak{u}_2$ and has holomorphic normal space represented by $V = \mathfrak{u}_1$. The group L acts irreducibly on both of them, and those representations were derived explicitly in Section 3. The variety Y is a complex flag manifold $K/(K \cap Q)$ in its own right, and is the fiber of the basic fibration (2.8) $D = G_0/L_0 \to G_0/K_0$.

Fix an irreducible representation τ_γ of L. Here γ is the highest weight, E_γ is the representation space, $\mathbb{E}_\gamma \to D$ is the associated homogeneous holomorphic vector bundle. and $\mathcal{O}(\mathbb{E}_\gamma) \to D$ is the sheaf of germs of holomorphic sections.

By $\mathcal{O}(\mathbb{E}_\gamma)|_Y \to D$ we mean the pull–back sheaf of $\mathcal{O}(\mathbb{E}_\gamma) \to D$ under $Y \hookrightarrow D$. It is a sheaf on D supported on Y. We filter it by order of vanishing in directions transverse to Y:

$$\mathcal{F}^n(\mathbb{E}_\gamma) = \bigl\{ f \in \mathcal{O}(\mathbb{E}_\gamma)|_Y \bigm| f \text{ vanishes to order } \geqq n \text{ in directions transverse to } Y \bigr\}. \tag{5.6}$$

We also need the notation

$$\begin{aligned} &\mathbb{N}_Y \to Y: \text{ holomorphic normal bundle to } Y \text{ in } D,\\ &\mathbb{N}_Y^* \to Y: \text{ holomorphic conormal bundle to } Y \text{ in } D \text{ and}\\ &\mathcal{S}^n(\mathbb{N}_Y^*) = \mathcal{O}(S^n(\mathbb{N}_Y^*)) \text{ where } S^n(\mathbb{N}_Y^*) \to Y \text{ is the } n^{th} \text{ symmetric power of } \mathbb{N}_Y^* \to Y. \end{aligned} \tag{5.7}$$

Then $\mathbb{N}_Y \to Y$ is the homogeneous holomorphic vector bundle over Y with fiber represented by $V = \mathfrak{u}_1$, its dual $\mathbb{N}_Y^* \to Y$ is the homogeneous holomorphic vector bundle with fiber $V^* = \mathfrak{u}_{-1}$, similarly for the third bundle with fibers $S^n(V^*)$, and we view $\mathcal{S}^n(\mathbb{N}_Y^*)$ as a sheaf on D supported on Y. Now we have short exact sequences

$$0 \to \mathcal{F}^{n+1}(\mathbb{E}_\gamma) \to \mathcal{F}^n(\mathbb{E}_\gamma) \to \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*)) \to 0 \tag{5.8}$$

of sheaves on D supported in Y. This leads to the long exact sequences

$$
(5.9)\qquad
\begin{aligned}
0 \to & H^0(D;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^0(D;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^0(D;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& H^1(D;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^1(D;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^1(D;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& H^{s-1}(D;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s-1}(D;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s-1}(D;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& H^{s}(D;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s}(D;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s}(D;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} 0.
\end{aligned}
$$

where a and b are coefficient morphisms from (5.8), δ is the coboundary, and $s = \dim_{\mathbb{C}} Y$. If a sheaf on a locally compact space (such as D) is supported on a closed subspace (such as Y) then the inclusion induces a natural isomorphism of cohomologies [G, Corollary to Lemma 4.9.2]. So we can rewrite (5.9) as

$$
(5.10)\qquad
\begin{aligned}
0 \to & H^0(Y;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^0(Y;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^0(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& H^1(Y;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^1(Y;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^1(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& H^{s-1}(Y;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s-1}(Y;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s-1}(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
& H^{s}(Y;\mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s}(Y;\mathcal{F}^{n}(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s}(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} 0.
\end{aligned}
$$

Note that (5.10) is an exact sequence of K–modules.

Remark 5.11 Let $n, j \geqq 0$. Then $H^j(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = H^j(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y)) \otimes S^n(\mathfrak{u}_{-1})$ as K_2–module. If $\mathbb{E}_\gamma \to Y$ is a line bundle then K_2 acts trivially on the first factor $H^j(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y))$.

Proof. The group K_2 acts trivially on Y, so its action on $S^n(\mathbb{N}_Y^*)$ factors out of the cohomology. Recall that $\mathbb{N}_Y^* \to Y$ is the K_0–homogeneous vector bundle based on the L_0–module $\mathfrak{u}_{-1}$. If $\mathbb{E}_\gamma \to Y$ is a line bundle then K_2 acts trivially on each $H^j(Y;\mathcal{O}(\mathbb{E}_\gamma|_Y))$ because it is semisimple. $\diamondsuit$

Recall that the positive compact roots are those for which the coefficient of ν, as a linear combination from $\Psi = \Psi_G$, is 0 or 2. The ones of coefficient 0 are roots of $(\mathfrak{l}, \mathfrak{t})$. The others, forming the set Δ_2 of the discussion after (2.10), are the complementary compact positive roots. They give the holomorphic tangent space of Y. Let $\rho_{\mathfrak{k}}$ denote half the sum of the positive compact roots (positive roots of $\mathfrak{k}$). Then the proof of (2.11a) gives us

$$
(5.12)\qquad \langle \gamma + \rho_{\mathfrak{k}}, \alpha\rangle < 0 \text{ for all } \alpha \in \Delta_2 \text{ if and only if } \langle \gamma + \rho_{\mathfrak{k}}, \mu\rangle < 0.
$$

If $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$ then $\alpha_1 + \alpha_2$ is not a root, because it would have coefficient 3 at ν. Thus $\langle \alpha_1, \alpha_2\rangle \geqq 0$. That gives us

Lemma 5.13 *If $\alpha_2 \in \Delta_2$ then $\langle \gamma + \rho_{\mathfrak{k}}, \alpha_2\rangle \leqq \langle \gamma + \rho_{\mathfrak{g}}, \alpha_2\rangle$. In particular if $\alpha_2 \in \Delta_2$ then $\langle \gamma + \rho_{\mathfrak{g}}, \alpha_2\rangle < 0$ implies $\langle \gamma + \rho_{\mathfrak{k}}, \alpha_2\rangle < 0$. Thus the G_0–negativity condition*
$$\langle \gamma + \rho_{\mathfrak{g}}, \alpha\rangle < 0 \textit{ for all } \alpha \in \Delta_1 \cup \Delta_2$$
implies the K_0–negativity condition
$$\langle \gamma + \rho_{\mathfrak{k}}, \alpha\rangle < 0 \textit{ for all } \alpha \in \Delta_2.$$

We are going to need the following fact about tensor products of irreducible finite dimensional representations. It appears in [H] as Exercise 12 to Section 24, based on [Ko1].

Lemma 5.14 *Let E_{γ_1} and E_{γ_2} be irreducible L_0–modules, where γ_i is the highest weight of E_{γ_i}. Then every irreducible summand of $E_{\gamma_1} \otimes E_{\gamma_2}$ has highest weight of the form $\gamma_1 + \varphi$ for some weight φ of E_{γ_2}.*

Now the K_0–negativity condition gives a vanishing result in (5.10), as follows, where we take (5.12) into account.

Theorem 5.15 *Suppose that $\langle \gamma + \rho_{\mathfrak{k}}, \mu \rangle < 0$. Then $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = 0$ whenever $j \neq s$ and $n \geqq 0$.*

Proof. Note that $\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*) \to Y$ is the K_0–homogeneous bundle based on the representation of L_0 on $E_\gamma \otimes S^n(\mathfrak{u}_{-1})$. In view of Lemma 5.14 that L_0–module is the sum of irreducibles with highest weights of the form $\gamma + \varphi$ where φ is a weight of $S^n(\mathfrak{u}_{-1})$. Thus, as a homogeneous holomorphic vector bundle, $\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*)$ has composition series with composition factors of the form $\mathbb{E}_{\gamma+\alpha_1+\cdots+\alpha_n}$ where the $\alpha_i \in \Delta_{-1}$.

Let $\alpha \in \Delta_2$. Then (5.12) shows that $\langle \gamma + \rho_{\mathfrak{k}}, \alpha \rangle < 0$. The coefficient of ν in α is 2, so $\alpha - \alpha_i$ cannot be a root. This forces $\langle \alpha_i, \alpha \rangle \leqq 0$. Now $\langle \gamma + \rho_{\mathfrak{k}}, \alpha \rangle < 0$ forces $\langle \gamma + \alpha_1 + \cdots + \alpha_n + \rho_{\mathfrak{k}}, \alpha \rangle < 0$. The Bott–Borel–Weil Theorem now tells us that $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = 0$ for $j \neq s$. □

Corollary 5.16 *Suppose that $\langle \gamma + \rho_{\mathfrak{k}}, \mu \rangle < 0$. Then $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y)) = 0$ whenever $j \neq s$.*

Following [S2] and [W4], with the result [S3] that $\gamma + \rho_{\mathfrak{g}}$ need only be nonsingular (instead of "sufficiently nonsingular"), one has the following vanishing theorem.

Theorem 5.17 *If $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ whenever $\alpha \in \Delta_1 \cup \Delta_2$, then $H^j(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $j \neq s$.*

Recall the decomposition of (2.10): $\gamma = \gamma_0 + t\nu^*$ where $\langle \gamma_0, \nu \rangle = 0$ and $t \in \mathbb{R}$. In view of (2.11a), (2.11b) and Theorem 2.12, we reformulate Theorem 5.17 as follows.

Theorem 5.18 *Let $\gamma = \gamma_0 + t\nu^*$ as in (2.10). If $\langle \gamma + \rho_{\mathfrak{g}}, \mu \rangle < 0$ and $\langle \gamma + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle < 0$, in other words if $t < -\frac{1}{2}\langle \gamma_0 + \rho_{\mathfrak{g}}, \mu \rangle$ and $t < -\langle \gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle$, then $H^j(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $j \neq s$.*

Definition 5.19 To facilitate use of these vanishing theorems we will say that $\mathbb{E}_\gamma \to D$ is *sufficiently negative* if $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ whenever α is a complementary positive root, i.e. whenever $\alpha \in \Delta_1 \cup \Delta_2$. This means that $\mathbb{E}_\gamma \otimes \mathbb{K}^{1/2} \to D$ is negative in the sense of differential or algebraic geometry, where $\mathbb{K} \to D$ is the canonical line bundle. ♢

In the presence of sufficient negativity Theorem 5.18 trivializes the long exact sequences (5.9) and (5.10) as follows.

Proposition 5.20 *Suppose that $\mathbb{E}_\gamma \to D$ is sufficiently negative. Then*

$$H^q(Y; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \cong H^q(Y; \mathcal{F}^n(\mathbb{E}_\gamma)) \qquad \text{for } 0 \leqq q < s, \qquad \text{and}$$
$$H^s(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \cong H^s(Y; \mathcal{F}^n(\mathbb{E}_\gamma))/H^s(Y; \mathcal{F}^{n+1}(\mathbb{E}_\lambda)).$$

Now we may apply the above case-by-case analysis and diagrams to understand what amounts to the structure and geometric quantization of the coadjoint elliptic orbits corresponding to the particular discrete series of representations we have in mind, the so–called *Borel – de Siebenthal discrete series*. Also, the results above on the filtration are crucial for the construction of the cohomology groups carrying these representations; they are the analytic counterparts of the Vogan–Zuckerman derived functor modules that are constructed purely algebraically. Thus we wish to construct the Borel – de Siebenthal discrete series by direct analysis on orbits, and using the above results analyze the K_0–types explicitly (without subscript K denotes the complexified group); after this we shall end the paper with some remarks and immediate consequences, and treat the (interesting) analytic continuation of this particular discrete series in a later paper.

As is clear from the above discussion there is some variation in the meaning of "discrete series". Initially the discrete series of G_0 meant the family of (equivalence classes of) irreducible unitary representation π of G_0 that are discrete summands of the left regular representation.

This is equivalent to the condition that the matrix coefficients $f_{u,v}(g) = \langle u, \pi(v)\rangle$ of π belong to $L^2(G)$. That is how they are treated in the work of Harish–Chandra, and there the discrete series representations are also treated as Harish–Chandra modules. Later one had the construction of discrete series representation as the action of G_0 on cohomology spaces $H^q(D;\mathbb{E})$ both as nuclear Fréchet spaces ([S2], [SW2]) and as Hilbert spaces [W4], and still later they appeared algebraically as Zuckerman derived functor modules. The underlying Harish–Chandra module is the same for all these constructions, and we will use the cohomology constructions.

We first recall some results about the discrete series representations in general. See [Kn1, Theorem 9.20] where Harish-Chandra's parameterization is recalled: Here there is given a *standard root order* (which is not the same as we are working with in the diagrams above) of the root system as follows:

$$\Delta^+_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \alpha\rangle > 0\} \tag{5.21}$$

where Δ is the root system, and λ is the *Harish-Chandra parameter* for the discrete series representation π_λ. The Harish–Chandra parameter $\lambda \in (i\mathfrak{t})'$ satisfies the integrality condition that $\lambda + \rho_\mathfrak{g}$ is analytically integral, in other words that $\exp(\lambda + \rho_\mathfrak{g})$ is a well defined character on the maximal torus of K_0. It also satisfies the nonsingularity condition that $\langle \lambda, \alpha\rangle \neq 0$ for all α in Δ. Two such representations are equivalent if and only if their parameters are conjugate under the compact Weyl group $W_\mathfrak{k}$. Thus one could normalize the Harish–Chandra parameter by the condition that $\langle \lambda, \alpha\rangle < 0$ for all compact positive roots.

The parameter λ of course determines the positive root system Δ^+_λ of the standard root order (5.21), and conversely to each Weyl chamber of $\mathfrak{g}$, modulo the action of $W_\mathfrak{k}$, we associate a family of discrete series representations. The family we are interested in is in some sense the smallest possible kind of discrete series representations of G_0.

Of particular interest is the lowest K_0–type contained in the (Harish-Chandra module for) π_λ given by its highest weight (in the standard root order (5.21))

$$\Lambda = \lambda + \rho_\mathfrak{g} - 2\rho_\mathfrak{k}$$

in terms of the usual half sums of positive roots. This K_0–type has multiplicity one, and other K_0–types have highest weights of the form

$$\Lambda' = \Lambda + \sum_{\alpha\in\Delta^+} n_\alpha \alpha$$

for integers $n_\alpha \geqq 0$. In the general theory of discrete series this statement about the K_0–types only amounts to an inclusion, whereas our results above analyzing the cohomology groups in terms of restriction and Taylor expansion in the normal direction (V) gives a concrete list of the K_0–types. We shall formulate this precisely below.

Let us first see how these parameters fit with the description in [GW] of the quaternionic discrete series π^q_λ. They write β for the maximal root, but we translate that to our notation of μ in describing their results. Thus the Harish–Chandra (and infinitesimal character) parameter of their π^q_λ is of the form $\lambda = -\frac{k}{2}\mu + \rho_\mathfrak{g}$ where the integer $k \geq 2d+1$ and $\dim_\mathbb{R} G_0/K_0 = 4d$. We consider the corresponding standard root order Δ^+_λ. Dividing as usual into compact and noncompact roots we have $\rho_\mathfrak{g} = \rho_\mathfrak{k} + \rho_{\mathfrak{g}/\mathfrak{k}}$, $\rho_\mathfrak{k} = \rho_\mathfrak{l} + \frac{\mu}{2}$ and $\rho_{\mathfrak{g}/\mathfrak{k}} = \frac{d}{2}\mu$, where $\rho_{\mathfrak{g}/\mathfrak{k}}$ is half the sum of the noncompact positive roots and $\rho_\mathfrak{l}$ is half sum of positive roots of $\mathfrak{l}$. Similarly for the standard root order we have $\rho'_\mathfrak{k} = \rho_\mathfrak{l} - \frac{\mu}{2}$ and $\rho'_{\mathfrak{g}/\mathfrak{k}} = -\frac{d}{2}\mu$, so the lowest K_0–type in the standard root order Δ^+_λ is $\Lambda = -\frac{k}{2}\mu + \rho_\mathfrak{g} + \rho'_\mathfrak{g} - 2\rho'_\mathfrak{k}$. That simplifies to $\Lambda = \frac{-k+2}{2}\mu$. This is exactly the highest weight for the $(k-1)$–dimensional representation of the simple $SU(2)$ factor in K found as the lowest K_0–type by Gross and Wallach. In the following we shall find the analogous lowest K_0–type for the Borel – de Siebenthal discrete series, and at the same time realize it (and in fact all K_0–types) as cohomology groups on the compact Hermitian symmetric space Y.

Now recall the noncompact simple root ν from Section 2. As before, ν^* denote the dual to ν in the system of fundamental simple weights (2.9). The parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ may also be defined by means of ν^*, and the centralizer of ν^* is $\mathfrak{l}$. Thus the coadjoint orbit $\mathrm{Ad}^*(G_0)(\nu^*)$ is our space G_0/L_0 and is fibered by Y. Multiples of this ν^* will define the line bundles we shall need, and the representations in the Borel – de Siebenthal discrete series are then the cohomology groups in degree $s = \dim_{\mathbb{C}} Y$ with coefficients in the bundle.

Recall the maximal compact subgroup $K_0 = K_1 \times K_2$ explicit in the classification of Section 3, where the "small" factor K_1 corresponds to the component of the simple root system $\Psi_{\mathfrak{k}} = (\Psi \setminus \{\nu\}) \cup \{-\mu\}$ that contains $\{-\mu\}$. In the quaternionic case $L_1 = Sp(1)$. Now $Y = K_0/L_0 = (K_1 \times K_2)/(L_1 \times K_2) = K_1/L_1$. Thus, as far as induced representations and cohomology, the action of the K_2 factor will be rather simple. This we will make explicit below. Also, it is important that the factor L_1 in L_0 contains the center of L_0, and that the action of that center on the holomorphic normal space $V = \mathfrak{u}_1$ is given explicitly in the case by case diagrams of Section 2.

Let $\mathbb{E}_{\gamma_k} \to D$ be the holomorphic vector bundle induced from the representation of L_0 with highest weight $\gamma_k = \gamma_0 - k\nu^*$, $k \in \mathbb{N}$. (As G is simply connected $\exp(\gamma_k)$ is the highest weight of a representation of L_0). Denote

(5.22) $\quad \pi_{\lambda_k}$: representation of G_0 on $H^s(D, \mathcal{O}(\mathbb{E}_{\gamma_k}))$ where $\lambda_k = \gamma_k + \rho_{\mathfrak{g}} = \gamma_0 - k\nu^* + \rho_{\mathfrak{g}}$.

We will say that the integer k is *sufficiently positive* if $\lambda_k = \gamma_k + \rho_{\mathfrak{g}} = \gamma_0 - k\nu^* + \rho_{\mathfrak{g}}$ is sufficiently negative in the sense of Definition 5.19 and Theorem 2.12. Using the filtration (5.6) and arguments analogous to those of [GW] we characterize the line bundle valued Borel – de Siebenthal discrete series as follows.

Theorem 5.23 *The Borel – de Siebenthal discrete series representations of G_0 are the π_{λ_k} of* (5.22) *for which $k \in \mathbb{N}$ is sufficiently positive. As a K–module, the underlying Harish–Chandra module is*

$$\sum_{m \geqq 0} H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_k} \otimes S^m(V^*))),$$

and it not only is K_0–admissible but is K_1–admissible. Further, the lowest K_0–type (corresponding to $m = 0$) *is given by*

$$W_{\lambda_k} = H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_k})),$$

and it has multiplicity 1 *in* π_{λ_k}.

Remark. The L_0–modules $S^m(V^*)$ are not always multiplicity free, though they are multiplicity free in many cases. For example for the group of type D_9 and $m = 6$, calculation with the computer program LiE produces multiplicities, while there are none for F_4. Thus even in the scalar case, where $\mathbb{E}_{\gamma_n} \to D$ is a line bundle, i.e. when $\gamma_0 = 0$, π_{λ_k} need not be K_0–multiplicity free. This is of course in contrast the the K_0–multiplicity free property of the line bundle holomorphic discrete series. $\diamondsuit$

Proof. We use the filtration (5.6) and the exact sequences (5.9) and (5.10), together with the fact that Y is a compact hermitian symmetric space for K_1. The action of K_2 is part of the holomorphically induced representation, and the action of L_1 on V and its dual V^* is given as above. Finally the admissibility can be read off from the K_0–types directly: each $H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_k} \otimes S^m(V^*)))$ is a sum of irreducible representations of K_1, disjoint for different m, and the $S^m(V^*)$ are finite dimensional representations of K_2 and also of L_0.

Consider the parabolic subgroup $Q \cap K = LU_{-2}$ of K. Whenever M is a finite dimensional $(Q \cap K)$–module, the space of K–finite vectors in the induced representation $\mathrm{Ind}_{Q \cap K}^K(M)$ is $\sum_{\delta \in \widehat{K}} V_\delta \otimes (V_\delta^* \otimes M)^{Q \cap K}$. In particular the multiplicity of a K_0–type δ is equal to the number of times the highest weight vector of M occurs as a highest weight vector for L in V_δ. In our case the highest weights of M will grow with m in $S^m(V^*)$, and they are distinguished by the action of the center of L_0. Thus each K_1–type only occurs finitely many times, so π_{λ_k} is K_1–admissible. That, of course, implies admissibility for K_0. □

Remark. We compare our parameter for the lowest K_0–type with the general description mentioned for the scalar quaternionic case. In that scalar quaternionic case the infinitesimal character of the representation π_{λ_k} is given by

$$\lambda_k = -k\nu^* + \rho_{\mathfrak{g}},\ \rho_{\mathfrak{k}} = \rho_{\mathfrak{l}} + c_2\nu^* \text{ and } \rho_{\mathfrak{g}/\mathfrak{k}} = c_1\nu^*$$

for positive constants c_1 and c_2 depending only on the root system. Then for the standard root order $\Delta^+_{\lambda_k}$ we get

$$\rho'_{\mathfrak{k}} = \rho_{\mathfrak{l}} - c_2\nu^* \text{ and } \rho'_n = -c_1\nu^*$$

so that the lowest K_0–type has highest weight

$$\Lambda = -k\nu^* + \rho_{\mathfrak{g}} + \rho'_{\mathfrak{g}} - 2\rho'_{\mathfrak{k}} = -k\nu^* + 2\rho_{\mathfrak{k}/\mathfrak{l}}$$

where $\rho_{\mathfrak{k}/\mathfrak{l}} = c_2\nu^*$ is exactly the shift coming from the square root of the canonical bundle $\mathbb{K} \to Y$. Thus this corresponds to the lowest K_0–type above, viz. $W = H^s(Y, \mathcal{O}(\mathbb{L}_{-k}))$. ♢

It is an interesting problem to study the structure, including unitarity, of π_{λ_k} for smaller values of k, and to relate this to the projective varieties defined by the relative invariants - this will be taken up in a sequel to this paper. In particular the ring of regular functions on L_0–orbits will be important, as in the paper by Gross and Wallach for the case of the quaternionic discrete series. For now we remark as an application of the admissibility above, that branching problems will be manageable in a way similar to the case of holomorphic discrete series. This will require that the embedding of the smaller group respects the relevant structure, i.e. that the orderings are compatible. For example if we want to branch to a symmetric subgroup H_0 of G_0, then the embedding will be compatible provided the symmetry fixes K_1. Namely, we simply use the admissibility of the action of K_1, so that admissibility for the branching law to H_0 will follow for the Borel – de Siebenthal discrete series. In this case a Borel – de Siebenthal discrete series will branch as a direct sum of Borel – de Siebenthal discrete series representations.

Remark. As is evident from the case of indefinite orthogonal groups as in [Kn3], the question of continuation of the discrete series modules is closely connected with the geometry of the relative invariants for the holomorphic normal (to the maximal compact subvariety) V. Already the case where G is of type E_8 and K of type D_8 is an interesting example; here V is of dimension 64 and admits a relative invariant of degree 8, and the maximal compact subvariety is the Grassmannian of 2–planes in 16–space. Let us here be a little more explicit about this example:

Let

$$\gamma_0 = n_2\xi_2 + ... + n_8\xi_8$$

and we find

$$\rho_{\mathfrak{k}} = 14\psi_1 + 28\psi_2 + 35\psi_3 + 55\psi_4 + 46\psi_5 + 36\psi_6 + 25\psi_7 + 13\psi_8$$

and

$$\rho_{\mathfrak{g}} = 46\psi_1 + 68\psi_2 + 91\psi_3 + 135\psi_4 + 110\psi_5 + 84\psi_6 + 57\psi_7 + 29\psi_8$$

so the "sufficient G_0–negativity" condition of Theorems 2.12 and 5.18 is, using

$$\mu = 2\psi_1 + 3\psi_2 + 4\psi_3 + 6\psi_4 + 5\psi_5 + 4\psi_6 + 3\psi_7 + 2\psi_8$$

that both $t < -\frac{1}{2}(3n_2 + 4n_3 + 6n_4 + 5n_5 + 4n_6 + 3n_7 + 2n_8) - \frac{29}{2}$ and $t < -\langle\gamma_0 + \rho_{\mathfrak{g}}, w^0_{\mathfrak{l}}(\nu)\rangle$. Now the range from G_0–sufficiently negative to K_0–sufficiently negative is indicated by the condition (as in Corollary 5.16) $t < -\frac{1}{2}(3n_2 + 4n_3 + 6n_4 + 5n_5 + 4n_6 + 3n_7 + 2n_8) + \frac{1}{2}$.

Hence we see that there is an interval, where the K_0–types still exist as cohomology groups, even though the large cohomology group carrying the G_0–representation ceases to exist. We shall study in more detail what happens here in a sequel to the present paper.

References

[BE] R. J. Baston & M. G. Eastwood, "The Penrose Transform: Its Interaction with Representation Theory", Clarendon Press, Oxford, 1989

[BoS] A. Borel & J. de Siebenthal, Les sous–groupes fermés de rang maximum des groupes de Lie clos, Comment. Math. Helv. **23** (1949), 200–221.

[EW] M. G. Eastwood & J. A. Wolf, The Range of the Double Fibration Transform, in preparation.

[FHW] G. Fels, A. T. Huckleberry & J. A. Wolf, "Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint", Birkhauser, 2005.

[G] R. Godement, "Topologie Algébrique et Théorie des Faisceaux", Hermann & Cie, 1958, 1964.

[GW] B. Gross & N. Wallach, On quaternionic discrete series representations, and their continuations, J. reine angew. Math. **481** (1996), 73–123.

[H] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory", Springer, 1972, 1980.

[Kn1] A. W. Knapp, "Lie Groups Beyond an Introduction, Second Edition", Birkhauser, 2002.

[Kn2] A. W. Knapp, Analytic continuation of nonholomorphic discrete series for classical groups. In "Noncommutative harmonic analysis", Birkhäuser, Progress in Mathematics, **220** (2004), 253–289.

[Kn3] A. W. Knapp, Nilpotent orbits and some small unitary representations of indefinite orthogonal groups, J. Funct. Anal. **209** (2004), 36–100.

[Ko1] B. Kostant, A formula for the multiplicity of a weight, Trans. Amer. Math. Soc. **93** (1959), 53–73.

[Ko2] B. Kostant, Lie algebra cohomology and the generalized Borel–Weil theorem, Ann. of Math. **74** (1961), 329–387.

[Ko3] B. Kostant, The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations, Internat. Math. Res. Notices **1998**, 225–252.

[P] D. Panushev, Some amazing properties of spherical nilpotent orbits, Math. Z. **245** (2003), 557–580.

[SK] M. Sato & T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math J. **65** (1977), 1–155.

[S1] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, thesis, University of California at Berkeley, 1967.

[S2] W. Schmid, On a conjecture of Langlands, Ann. of Math. **93** (1971), 1–42.

[S3] W. Schmid, Some properties of square integrable representations of semisimple Lie groups, Ann. of Math. **102** (1975), 535–564.

[SW1] W. Schmid & J. A. Wolf, A vanishing theorem for open orbits on complex flag manifolds, Proc. Amer. Math. Soc. **92** (1984), 461–464

[SW2] W. Schmid & J. A. Wolf, Geometric quantization and derived functor modules for semisimple Lie groups, J. Funct. Analysis **90** (1990), 48–112.

[WaW] N. R. Wallach & J. A. Wolf, Completeness of Poincaré series for automorphic forms associated to the integrable discrete series. In "Representation Theory of Reductive Groups" Birkhäuser, Progress in Mathematics, **40** (1983), 265–281.

[WeW] R. O. Wells, Jr., & J. A. Wolf, Poincaré series and automorphic cohomology on flag domains. Annals of Math. **105** (1977), 397–448.

[W0] J. A. Wolf, Complex homogeneous contact manifolds and quaternionic symmetric spaces, J. Math. and Mechanics **14** (1965), 1033–1048.

[W1] J. A. Wolf, "Classification and Fourier Inversion for Parabolic Subgroups with Square Integrable Nilradical", Memoirs AMS 225, 1979.

[W2] J. A. Wolf, "Spaces of Constant Curvature, Fifth Edition", Publish or Perish, 1984.

[W3] J. A. Wolf, The action of a real semisimple Lie group on a complex flag manifold, I: Orbit structure and holomorphic arc components, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

[W4] J. A. Wolf, The action of a real semisimple Lie group on a complex flag manifold, II: Unitary representations on partially holomorphic cohomology spaces, Memoirs Amer. Math. Soc. **138**, 1974.