

Parabolic Subgroups of Real Direct Limit Lie Groups

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ABSTRACT. Let $G_{\mathbb{R}}$ be a classical real direct limit Lie group, and $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. The parabolic subalgebras of the complexification $\mathfrak{g}_{\mathbb{C}}$ were described by the first two authors. In the present paper we extend these results to $\mathfrak{g}_{\mathbb{R}}$. This also gives a description of the parabolic subgroups of $G_{\mathbb{R}}$. Furthermore, we give a geometric criterion for a parabolic subgroup $P_{\mathbb{C}}$ of $G_{\mathbb{C}}$ to intersect $G_{\mathbb{R}}$ in a parabolic subgroup. This criterion involves the $G_{\mathbb{R}}$ -orbit structure of the flag ind-manifold $G_{\mathbb{C}}/P_{\mathbb{C}}$.

1. Introduction and Basic Definitions

We start with the three classical simple locally finite countable-dimensional Lie algebras $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{n, \mathbb{C}}$, and their real forms $\mathfrak{g}_{\mathbb{R}}$. The Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are the classical direct limits, $\mathfrak{sl}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C}) = \varinjlim \mathfrak{so}(2n; \mathbb{C}) = \varinjlim \mathfrak{so}(2n + 1; \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sp}(n; \mathbb{C})$, where the direct systems are given by the inclusions of the form $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. See [1] or [2]. We often consider the locally reductive algebra $\mathfrak{gl}(\infty; \mathbb{C}) = \varinjlim \mathfrak{gl}(n; \mathbb{C})$ along with $\mathfrak{sl}(\infty; \mathbb{C})$.

The real forms of these classical simple locally finite countable-dimensional complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$ have been classified by A. Baranov in [1]. A slight reformulation of [1, Theorem 1.4] says that the following is a complete list of the real forms of $\mathfrak{g}_{\mathbb{C}}$.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(\infty; \mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

- $\mathfrak{sl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sl}(n; \mathbb{R})$, the real special linear Lie algebra,
- $\mathfrak{sl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{sl}(n; \mathbb{H})$, the quaternionic special linear Lie algebra, given by $\mathfrak{sl}(n; \mathbb{H}) := \mathfrak{gl}(n; \mathbb{H}) \cap \mathfrak{sl}(2n; \mathbb{C})$,
- $\mathfrak{su}(p, \infty) = \varinjlim \mathfrak{su}(p, n)$, the complex special unitary Lie algebra of real rank p ,
- $\mathfrak{su}(\infty, \infty) = \varinjlim \mathfrak{su}(p, q)$, complex special unitary algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty; \mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

- $\mathfrak{so}(p, \infty) = \varinjlim \mathfrak{so}(p, n)$, the real orthogonal Lie algebra of finite real rank p ,
- $\mathfrak{so}(\infty, \infty) = \varinjlim \mathfrak{so}(p, q)$, the real orthogonal Lie algebra of infinite real rank,
- $\mathfrak{so}^*(2\infty) = \varinjlim \mathfrak{so}^*(2n)$, with

$$\mathfrak{so}^*(2n) = \{ \xi \in \mathfrak{sl}(n; \mathbb{H}) \mid \kappa_n(\xi x, y) + \kappa_n(x, \xi y) = 0 \ \forall x, y \in \mathbb{H}^n \}$$

where $\kappa_n(x, y) := \sum_{\ell} x^{\ell} i \bar{y}^{\ell} = {}^t x i \bar{y}$.

Equivalently, $\mathfrak{so}^*(2n) = \mathfrak{so}(2n; \mathbb{C}) \cap \mathfrak{u}(n, n)$ with

$$\mathfrak{so}(2n; \mathbb{C}) \text{ defined by } \langle u, v \rangle = \sum_1^n (u_{2j-1} v_{2j} + u_{2j} v_{2j-1}) \text{ and}$$

$$\mathfrak{u}(n, n) \text{ defined by } \langle u, v \rangle = \sum_1^n (u_{2j-1} \bar{v}_{2j-1} - u_{2j} \bar{v}_{2j}).$$

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If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(\infty; \mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

- $\mathfrak{sp}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sp}(n; \mathbb{R})$, the real symplectic Lie algebra,
- $\mathfrak{sp}(p, \infty) = \varinjlim \mathfrak{sp}(p, n)$, the quaternionic unitary Lie algebra of real rank p ,
- $\mathfrak{sp}(\infty, \infty) = \varinjlim \mathfrak{sp}(p, q)$, quaternionic unitary Lie algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\infty; \mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

- $\mathfrak{gl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{gl}(n; \mathbb{R})$, the real general linear Lie algebra,
- $\mathfrak{gl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{gl}(n; \mathbb{H})$, the quaternionic general linear Lie algebra,
- $\mathfrak{u}(p, \infty) = \varinjlim \mathfrak{u}(p, n)$, the complex unitary Lie algebra of finite real rank p ,
- $\mathfrak{u}(\infty, \infty) = \varinjlim \mathfrak{u}(p, q)$, the complex unitary Lie algebra of infinite real rank.

The *defining representations* of $\mathfrak{g}_{\mathbb{C}}$ are characterized as direct limits of minimal-dimensional nontrivial representations of simple subalgebras. It is well known that that $\mathfrak{sl}(\infty; \mathbb{C})$ and $\mathfrak{gl}(\infty; \mathbb{C})$ have two inequivalent defining representations V and W , whereas each of $\mathfrak{so}(\infty; \mathbb{C})$ and $\mathfrak{sp}(\infty; \mathbb{C})$ has only one (up to equivalence) V . In particular the restrictions to $\mathfrak{so}(\infty; \mathbb{C})$ or $\mathfrak{sp}(\infty; \mathbb{C})$ of the two defining representations of $\mathfrak{sl}(\infty; \mathbb{C})$ are equivalent. The real forms $\mathfrak{g}_{\mathbb{R}}$ listed above also have *defining representations*, as detailed below, which are particular restrictions of the defining representations of $\mathfrak{g}_{\mathbb{C}}$. We denote an element of $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $*$.

Suppose that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{sl}(\infty; \mathbb{R})$ or $\mathfrak{gl}(\infty; \mathbb{R})$. The defining representation spaces of $\mathfrak{g}_{\mathbb{R}}$ are the finitary (i.e. with finitely many nonzero entries) column vectors $V_{\mathbb{R}} = \mathbb{R}^{\infty}$ and the finitary row vectors $W_{\mathbb{R}} = \mathbb{R}^{\infty}$. The algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$ or $W_{\mathbb{R}}$ is \mathbb{R} . The restriction of the pairing of V and W is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$ -invariant \mathbb{R} -bilinear pairing of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$.

The defining representation space $V_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(*, \infty)$ consists of the finitary real column vectors. The algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$ (the commuting algebra) is \mathbb{R} . The restriction of the symmetric form on V to $V_{\mathbb{R}}$ is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$ -invariant symmetric \mathbb{R} -bilinear form.

The defining representation space $V_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(\infty; \mathbb{R})$ consists of the finitary real column vectors. The algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$ is \mathbb{R} . The restriction of the antisymmetric form on V to $V_{\mathbb{R}}$ is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$ -invariant antisymmetric \mathbb{R} -bilinear form.

In both of these cases the defining representation of $\mathfrak{g}_{\mathbb{R}}$ is a real form of the defining representation of $\mathfrak{g}_{\mathbb{C}}$, i.e. $V = V_{\mathbb{R}} \otimes \mathbb{C}$.

Suppose that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{su}(*, \infty)$ or $\mathfrak{u}(*, \infty)$. Then $\mathfrak{g}_{\mathbb{R}}$ has two defining representations, one on the space $V_{\mathbb{R}} = \mathbb{C}^{*, \infty}$ of finitary complex column vectors and the other on the space $W_{\mathbb{R}}$ of finitary complex row vectors. Thus the two defining representations of $\mathfrak{g}_{\mathbb{C}}$ remain irreducible as a representations of $\mathfrak{g}_{\mathbb{R}}$, the respective algebras of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ are \mathbb{C} , and $V = V_{\mathbb{R}}$ and $W = W_{\mathbb{R}}$. The pairing of V and W defines a $\mathfrak{g}_{\mathbb{R}}$ -invariant hermitian form of signature $(*, \infty)$ on $V_{\mathbb{R}}$.

Suppose that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{sl}(\infty; \mathbb{H})$ or $\mathfrak{gl}(\infty; \mathbb{H})$. The two defining representation spaces of $\mathfrak{g}_{\mathbb{R}}$ consist of the finitary column vectors $V_{\mathbb{R}} = \mathbb{H}^{\infty}$ and finitary row vectors $W_{\mathbb{R}} = \mathbb{H}^{\infty}$. The algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$ or $W_{\mathbb{R}}$ is \mathbb{H} . The defining representations of $\mathfrak{g}_{\mathbb{C}}$ on V and W restrict to irreducible representations of $\mathfrak{g}_{\mathbb{R}}$, and $V_{\mathbb{R}} = \mathbb{H}^{\infty} = \mathbb{C}^{\infty} + \mathbb{C}^{\infty}j = \mathbb{C}^{2\infty} = V$. The pairing of V and W is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$ -invariant \mathbb{R} -bilinear pairing of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$.

The defining representation space $V_{\mathbb{R}} = \mathbb{H}^{*, \infty}$ of $\mathfrak{sp}(*, \infty)$ consists of the finitary quaternionic vectors. The algebra of $\mathfrak{sp}(*, \infty)$ -endomorphisms of $V_{\mathbb{R}}$ is \mathbb{H} . The

form on $V_{\mathbb{R}}$ is a nondegenerate $\mathfrak{sp}(*, \infty)$ -invariant quaternionic-hermitian form of signature $(*, \infty)$. In this case $V_{\mathbb{R}} = \mathbb{H}^{*, \infty} = \mathbb{C}^{2*, 2\infty} = V$.

The defining representation space $V_{\mathbb{R}} = \mathbb{H}^{\infty}$ of $\mathfrak{so}^*(2\infty)$ consists of the finitary quaternionic vectors. The algebra of $\mathfrak{so}^*(2\infty)$ -endomorphisms of $V_{\mathbb{R}}$ is \mathbb{H} . The form on $V_{\mathbb{R}}$ is the nondegenerate $\mathfrak{so}^*(2\infty)$ -invariant quaternionic-skew-hermitian form κ which is the limit of the forms κ_n . In this case again $V_{\mathbb{R}} = \mathbb{H}^{\infty} = \mathbb{C}^{2\infty} = V$.

The Lie ind-group (direct limit group) corresponding to $\mathfrak{gl}(\infty; \mathbb{C})$ is the general linear group $GL(\infty; \mathbb{C})$, which consists of all invertible linear transformations of V of the form $g = g' + \text{Id}$ where $g' \in \mathfrak{gl}(\infty; \mathbb{C})$. The subgroup of $GL(\infty; \mathbb{C})$ corresponding to $\mathfrak{sl}(\infty; \mathbb{C})$ is the special linear group $SL(\infty; \mathbb{C})$, consisting of elements of determinant 1. The connected ind-subgroups of $GL(\infty; \mathbb{C})$ whose Lie algebras are $\mathfrak{so}(\infty; \mathbb{C})$ and $\mathfrak{sp}(\infty; \mathbb{C})$ are denoted by $SO_0(\infty; \mathbb{C})$ and $Sp(\infty; \mathbb{C})$.

In Section 2 we recall the structure of parabolic subalgebras of complex finitary Lie algebras from [4]. A *parabolic subalgebra* of a complex Lie algebra is by definition a subalgebra that contains a maximal locally solvable (that is, *Borel*) subalgebra. Parabolic subalgebras of complex finitary Lie algebras are classified in [4]. We recall the structural result that every parabolic subalgebra is a subalgebra (technically: defined by infinite trace conditions) of the stabilizer of a *taut couple* of generalized flags in the defining representations, and we strengthen this result by studying the non-uniqueness of the flags in the case of the orthogonal Lie algebra. As in the finite-dimensional case, we define a *parabolic subalgebra* of a real locally reductive Lie algebra $\mathfrak{g}_{\mathbb{R}}$ as a subalgebra $\mathfrak{p}_{\mathbb{R}}$ whose complexification $\mathfrak{p}_{\mathbb{C}}$ is parabolic in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. It is a well-known fact that already in the finite-dimensional case a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ does not necessarily contain a subalgebra whose complexification is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

In Section 3 we prove our main result. It extends the classification in [4] to the real case. The key difference from the complex case is that one must take into account the additional structure of a defining representation space of $\mathfrak{g}_{\mathbb{R}}$ as a module over its algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms.

In Section 4 we give a geometric criterion for a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ to be the complexification of a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. The criterion is based on an observation of one of us from the 1960's, concerning the structure of closed real group orbits on finite-dimensional complex flag manifolds. We recall that result, appropriately reformulated, and indicate its extension to flag ind-manifolds.

2. Complex Parabolic Subalgebras

2A. Generalized Flags. Let V and W be countable-dimensional right vector spaces over a real division algebra $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , together with a nondegenerate bilinear pairing $\langle \cdot, \cdot \rangle : V \times W \rightarrow \mathbb{D}$. Then V and W are endowed with the Mackey topology, and the closure of a subspace $F \subset V$ is $F^{\perp\perp}$, where \perp refers to the pairing $\langle \cdot, \cdot \rangle$. A set of \mathbb{D} -subspaces of V (or W) is called a *chain* in V (or W) if it is totally ordered by inclusion. A \mathbb{D} -*generalized flag* is a chain in V (or W) such that each subspace has an immediate predecessor or an immediate successor in the inclusion ordering, and every nonzero vector of V (or W) is caught between an immediate predecessor-successor pair [5].

DEFINITION 2.1. [4] A \mathbb{D} -generalized flag \mathcal{F} in V (or W) is said to be semi-closed if for every immediate predecessor-successor pair $F' \subset F''$ the closure of F' is either F' or F'' . ◇

If \mathcal{C} is a chain in V (or W), then we denote by \mathcal{C}^\perp the chain in W (or V) consisting of the perpendicular complements of the subspaces of \mathcal{C} .

We fix an identification of V and W with the defining representations of $\mathfrak{gl}(\infty; \mathbb{D})$ as follows. To identify V and W with the defining representations of $\mathfrak{gl}(\infty; \mathbb{D})$, it suffices to find bases in V and W dual with respect to the pairing $\langle \cdot, \cdot \rangle$. If $\mathbb{D} \neq \mathbb{H}$, the existence of dual bases in V and W with respect to any nondegenerate \mathbb{D} -bilinear pairing is a result of Mackey [9, p. 171]. Now suppose that $\mathbb{D} = \mathbb{H}$. Then there exist \mathbb{C} -subspaces $V_{\mathbb{C}} \subset V$ and $W_{\mathbb{C}} \subset W$ such that $V = V_{\mathbb{C}} \oplus V_{\mathbb{C}}j$ and $W = W_{\mathbb{C}} \oplus W_{\mathbb{C}}j$. The restriction of $\langle \cdot, \cdot \rangle$ to $V_{\mathbb{C}} \times W_{\mathbb{C}}$ is a nondegenerate \mathbb{C} -bilinear pairing. The result of Mackey therefore implies the existence of dual bases in $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$, which are also dual bases of V and W over \mathbb{H} . In all cases we identify the right multiplication of vectors in V by elements of \mathbb{D} with the action of the algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$.

DEFINITION 2.2. [4] *Let \mathcal{F} and \mathcal{G} be \mathbb{D} -semiclosed generalized flags in V and W , respectively. We say \mathcal{F} and \mathcal{G} form a taut couple if \mathcal{F}^\perp is stable under the $\mathfrak{gl}(\infty; \mathbb{D})$ -stabilizer of \mathcal{G} and \mathcal{G}^\perp is stable under the $\mathfrak{gl}(\infty; \mathbb{D})$ -stabilizer of \mathcal{F} . If we have a fixed isomorphism $f : V \rightarrow W$ then we say that \mathcal{F} is self-taut if \mathcal{F} and $f(\mathcal{F})$ form a taut couple. \diamond*

If one has a fixed isomorphism between V and W , then there is an induced bilinear form on V . A semiclosed generalized flag \mathcal{F} in V is self-taut if and only if \mathcal{F}^\perp is stable under the $\mathfrak{gl}(\infty; \mathbb{D})$ -stabilizer of \mathcal{F} , where \mathcal{F}^\perp is taken with respect to the form on V .

REMARK 2.3. Fix a nondegenerate bilinear form on V . If V is finite dimensional, a self-taut generalized flag in V consists of a finite number of isotropic subspaces together with their perpendicular complements. In this case, the stabilizer of a self-taut generalized flag equals the stabilizer of its isotropic subspaces. If V is infinite dimensional, the non-closed non-isotropic subspaces in a self-taut generalized flag in V influence its stabilizer, but it is still true that every subspace is either isotropic or coisotropic. Indeed, let \mathcal{F} be a self-taut generalized flag, and let $F \in \mathcal{F}$. By [4, Proposition 3.2], F^\perp is a union of elements of \mathcal{F} if it is a nontrivial proper subspace of V . Hence $\mathcal{F} \cup \{F^\perp\}$ is a chain that contains both F and F^\perp . Thus either $F \subset F^\perp$ or $F^\perp \subset F$, so F is either isotropic or coisotropic. \diamond

We will need the following lemma when we pass to consideration of real parabolic subalgebras.

LEMMA 2.4. *Suppose that $\mathbb{D} = \mathbb{H}$. Fix \mathbb{H} -generalized flags \mathcal{F} in V and \mathcal{G} in W . Then \mathcal{F} and \mathcal{G} form a taut couple if and only if they are form taut couple as \mathbb{C} -generalized flags.*

PROOF. It is immediate from the definition that \mathcal{F} and \mathcal{G} are semiclosed \mathbb{C} -generalized flags if and only if they are semiclosed \mathbb{H} -generalized flags. The proof of [4, Proposition 3.2] holds in the quaternionic case as well. Thus if \mathcal{F} and \mathcal{G} form a taut couple as either \mathbb{C} -generalized flags or \mathbb{H} -generalized flags, then as long as F^\perp is a nontrivial proper subspace of W , it is a union of elements of \mathcal{G} for any $F \in \mathcal{F}$. Thus F^\perp is stable under both the $\mathfrak{gl}(\infty; \mathbb{C})$ -stabilizer and the $\mathfrak{gl}(\infty; \mathbb{H})$ -stabilizer of \mathcal{G} for any $F \in \mathcal{F}$. Similarly, if $G \in \mathcal{G}$ then G^\perp is stable under both the $\mathfrak{gl}(\infty; \mathbb{C})$ -stabilizer and the $\mathfrak{gl}(\infty; \mathbb{H})$ -stabilizer of \mathcal{F} . \square

2B. Trace Conditions. Let \mathfrak{g} be a locally finite Lie algebra over a field of characteristic zero. A subalgebra of \mathfrak{g} is *locally solvable* (resp. *locally nilpotent*) if every finite subset of \mathfrak{g} is contained in a solvable (resp. nilpotent) subalgebra. The sum of all locally solvable ideals is again a locally solvable ideal, the *locally solvable radical* of \mathfrak{g} . If \mathfrak{r} is the locally solvable radical of \mathfrak{g} then $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$ is a locally nilpotent ideal in \mathfrak{g} . Indeed, note that $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] = \bigcup_n (\mathfrak{r} \cap [\mathfrak{g}_n, \mathfrak{g}_n]) \cap \mathfrak{g}_n$ for any exhaustion $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ by finite-dimensional subalgebras \mathfrak{g}_n , and furthermore $(\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]) \cap \mathfrak{g}_n$ is nilpotent for all n by standard finite-dimensional Lie theory.

Let \mathfrak{g} be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{D})$, that is, a subalgebra containing the Jordan components of its elements), and let \mathfrak{r} be its locally solvable radical. The *linear nilradical* \mathfrak{m} of \mathfrak{g} is defined to be the set of all nilpotent elements in \mathfrak{r} .

LEMMA 2.5. *Let \mathfrak{g} be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{D})$. Then its linear nilradical \mathfrak{m} is a locally nilpotent ideal. If $\mathbb{D} = \mathbb{R}$, then the complexification $\mathfrak{m}_{\mathbb{C}}$ is the linear nilradical of $\mathfrak{g}_{\mathbb{C}}$.*

PROOF. If $\xi, \eta \in \mathfrak{m}$ they are both contained in the solvable radical of a finite-dimensional subalgebra of \mathfrak{g} , so $\xi + \eta$ and $[\xi, \eta]$ are nilpotent. Thus, by Engel’s Theorem, \mathfrak{m} is a locally nilpotent subalgebra of \mathfrak{g} . Although it is only stated for complex Lie algebras, [4, Proposition 2.1] shows that $\mathfrak{m} \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$, so $[\mathfrak{m}, \mathfrak{g}] \subset [\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$, and thus \mathfrak{m} is an ideal in \mathfrak{g} . This proves the first statement. For the second let \mathfrak{r} be the locally solvable radical of \mathfrak{g} and note that $\mathfrak{r}_{\mathbb{C}}$ is the locally solvable radical of $\mathfrak{g}_{\mathbb{C}}$, so the assertion follows from finite-dimensional theory. \square

DEFINITION 2.6. *Let \mathfrak{g} be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{F})$ where \mathbb{F} is \mathbb{R} or \mathbb{C} , and let \mathfrak{m} be its linear nilradical. A subalgebra \mathfrak{p} of \mathfrak{g} is defined by trace conditions on \mathfrak{g} if $\mathfrak{m} \subset \mathfrak{p}$ and*

$$[\mathfrak{g}, \mathfrak{g}]/\mathfrak{m} \subset \mathfrak{p}/\mathfrak{m} \subset \mathfrak{g}/\mathfrak{m},$$

in other words if there is a family Tr of Lie algebra homomorphisms $f : \mathfrak{g} \rightarrow \mathbb{F}$ with joint kernel equal to \mathfrak{p} . Further, \mathfrak{p} is defined by infinite trace conditions if every $f \in \text{Tr}$ annihilates every finite-dimensional simple ideal in $[\mathfrak{g}, \mathfrak{g}]/\mathfrak{m}$. \diamond

We write $\text{Tr}^{\mathfrak{p}}$ for the maximal family Tr of Definition 2.6. On the group level we have corresponding *determinant conditions* and *infinite determinant conditions*. Note that infinite trace conditions and infinite determinant conditions do not occur when \mathfrak{g} and G are finite dimensional.

2C. Complex Parabolic Subalgebras. Recall that a *parabolic subalgebra* of a complex Lie algebra is by definition a subalgebra that contains a Borel subalgebra, i.e. a maximal locally solvable subalgebra.

THEOREM 2.7. [4] *Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{gl}(\infty, \mathbb{C})$ or $\mathfrak{sl}(\infty, \mathbb{C})$, and let V and W be its defining representation spaces. A subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. subgroup of $G_{\mathbb{C}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer (resp. $G_{\mathbb{C}}$ -stabilizer) of a (necessarily unique) taut couple of \mathbb{C} -generalized flags \mathcal{F} in V and \mathcal{G} in W .*

Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{so}(\infty, \mathbb{C})$ or $\mathfrak{sp}(\infty, \mathbb{C})$. and let V be its defining representation space. A subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. subgroup of $G_{\mathbb{C}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer (resp. $G_{\mathbb{C}}$ -stabilizer) of a self-taut \mathbb{C} -generalized flag \mathcal{F} in V . In the $\mathfrak{sp}(\infty, \mathbb{C})$ case the flag \mathcal{F} is necessarily unique.

In contrast to the finite dimensional case, the normalizer of a parabolic subalgebra can be larger than the parabolic algebra. For example, Theorem 2.7 implies that $\mathfrak{sl}(\infty, \mathbb{C})$ is parabolic in $\mathfrak{gl}(\infty; \mathbb{C})$, since it is the elements of the stabilizer of the trivial generalized flags $\{0, V\}$ and $\{0, W\}$ whose usual trace is 0. To understand the origins of this example, one should consider the explicit construction in [6] of a locally nilpotent Borel subalgebra of $\mathfrak{gl}(\infty; \mathbb{C})$. The normalizer of a parabolic subalgebra equals the stabilizer of the corresponding generalized flags [4], which is in general larger than the parabolic subalgebra because of the infinite determinant conditions. The self-normalizing parabolics are thus those for which $\text{Tr}^p = 0$. This is in contrast to the finite-dimensional setting, where there are no infinite trace conditions, and all parabolic subalgebras are self-normalizing.

In [4] the uniqueness issue was discussed for $\mathfrak{gl}(\infty, \mathbb{C})$, $\mathfrak{sl}(\infty, \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C})$, but not for $\mathfrak{so}(\infty, \mathbb{C})$. In the orthogonal setting one can have three different self-taut generalized flags with the same stabilizer (see [3] and [7], where the non-uniqueness is discussed in special cases.)

THEOREM 2.8. *Let \mathfrak{p} be a parabolic subalgebra given by infinite trace conditions on the $\mathfrak{so}(\infty; \mathbb{C})$ -stabilizer of a self-taut generalized flag \mathcal{F} in V . Then there are two possibilities:*

- (1) \mathcal{F} is uniquely determined by \mathfrak{p} ;
- (2) there are exactly three self-taut generalized flags with the same stabilizer as \mathcal{F} .

The latter case occurs precisely when there exists an isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^{\perp}/L = 2$. The three flags with the same stabilizer are then

- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\}$
- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_1$
- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_2$

where M_1 and M_2 are the two maximal isotropic subspaces containing L .

PROOF. The main part of the proof is to show that \mathfrak{p} determines all the subspaces in \mathcal{F} , except a maximal isotropic subspace under the assumption that \mathcal{F} has a closed isotropic subspaces L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$.

Let A denote the set of immediate predecessor-successor pairs of \mathcal{F} such that both subspaces in the pair are isotropic. Let F'_{α} denote the predecessor and F''_{α} the successor of each pair $\alpha \in A$. Let M denote the union of all the isotropic subspaces in \mathcal{F} , i.e. $M = \bigcup_{\alpha \in A} F''_{\alpha}$. If $M \neq M^{\perp}$, then M has an immediate successor W in \mathcal{F} . Note that W is not isotropic, by the definition of M . Furthermore, one has $W^{\perp} = M$ since \mathcal{F} is a self-taut generalized flag. If $M = M^{\perp}$, let us take $W = 0$.

Let C denote the set of all $\gamma \in A$ such that F'_{γ} is closed. For each $\gamma \in C$, it is seen in [4] that the coisotropic subspace $(F'_{\gamma})^{\perp}$ has an immediate successor in \mathcal{F} . For each $\gamma \in C$, let G''_{γ} denote the immediate successor of $(F'_{\gamma})^{\perp}$ in \mathcal{F} . It is also shown in [4] that $(G''_{\gamma})^{\perp} = F'_{\gamma}$.

Since \mathcal{F} is a self-taut generalized flag, \mathcal{F} is uniquely determined by the set of subspaces

$$\{F''_{\alpha} \mid \alpha \in A\} \cup \{G''_{\gamma} \mid \gamma \in C \text{ such that } G''_{\gamma} \text{ is not closed}\} \cup \{W\}.$$

We use separate arguments for these three kinds of subspaces to show that they are determined by \mathfrak{p} , except for a maximal isotropic subspace and W under the assumption that \mathcal{F} has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$. We

must also show that we can determine from \mathfrak{p} whether or not \mathcal{F} has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$.

Let $\tilde{\mathfrak{p}}$ denote the normalizer in $\mathfrak{so}(\infty; \mathbb{C})$ of \mathfrak{p} . We use the classical identification $\mathfrak{so}(\infty; \mathbb{C}) \cong \Lambda^2(V)$ where $u \wedge v$ corresponds to the linear transformation $x \mapsto \langle x, v \rangle u - \langle x, u \rangle v$. With this identification, following [4] one has

$$\tilde{\mathfrak{p}} = \sum_{\alpha \in A \setminus C} F''_{\alpha} \wedge (F'_{\alpha})^{\perp} + \sum_{\gamma \in C} F''_{\gamma} \wedge G''_{\gamma} + \Lambda^2(W).$$

Let $\alpha \in A$, and let $x \in F''_{\alpha} \setminus F'_{\alpha}$. Then one may compute

$$\begin{aligned} \tilde{\mathfrak{p}} \cdot x &= \left(\sum_{\alpha \in A \setminus C} F''_{\alpha} \wedge (F'_{\alpha})^{\perp} + \sum_{\gamma \in C} F''_{\gamma} \wedge G''_{\gamma} + \Lambda^2(W) \right) \cdot x \\ &= \left(\sum_{\alpha \in A \setminus C} F''_{\alpha} \otimes (F'_{\alpha})^{\perp} + \sum_{\gamma \in C} F''_{\gamma} \otimes G''_{\gamma} \right) \cdot x \\ &= \left(\bigcup_{x \notin (F'_{\alpha})^{\perp\perp}} F''_{\alpha} \right) \cup \left(\bigcup_{x \notin (G''_{\gamma})^{\perp}} F''_{\gamma} \right). \end{aligned}$$

As a result

$$\tilde{\mathfrak{p}} \cdot x = \begin{cases} F'_{\alpha} & \text{if } \alpha \in A \setminus C \\ F''_{\alpha} & \text{if } \alpha \in C. \end{cases}$$

So far we have shown the following. If $x \in \tilde{\mathfrak{p}} \cdot x$, then $F''_{\alpha} = \tilde{\mathfrak{p}} \cdot x$. If $x \notin \tilde{\mathfrak{p}} \cdot x$, then $F''_{\alpha} = (\tilde{\mathfrak{p}} \cdot x)^{\perp\perp}$. Furthermore, if $x \notin M$, then $\tilde{\mathfrak{p}} \cdot x$ is not isotropic, unless there exists a closed isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^{\perp}/L = 2$, and x is an element of M_1 or M_2 . We now consider the union of the subspaces $\tilde{\mathfrak{p}} \cdot x$, where the union is taken over $x \in V$ for which $\tilde{\mathfrak{p}} \cdot x$ is isotropic. If there does not exist L as described, then these subspaces will be the nested isotropic subspaces computed above, and indeed their union is M . If L exists, then these subspaces will exhaust L , and furthermore M_1 and M_2 will both appear in the union. Hence the union of the isotropic subspaces of the form $\tilde{\mathfrak{p}} \cdot x$ for $x \in V$ when L exists is L^{\perp} . As a result, if the union of all the isotropic subspaces of the form $\tilde{\mathfrak{p}} \cdot x$ for $x \in V$ is itself isotropic, then we conclude that no such L exists and we have constructed the subspace M . If that union is not isotropic, then we conclude that there exists a closed isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^{\perp}/L = 2$, and the union is L^{\perp} . In the latter case, L is recoverable from \mathfrak{p} , as it equals $L^{\perp\perp}$. We have now shown that we can determine whether \mathcal{F} has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$, that F''_{α} is determined by \mathfrak{p} for all $\alpha \in A$ in the latter case, and that F''_{α} is determined by \mathfrak{p} for all $\alpha \in A$ such that $F''_{\alpha} \subset L$ in the former case.

We now turn our attention to a non-closed subspace G''_{γ} for $\gamma \in C$. Since G''_{γ} is not closed, the codimension of F''_{γ} in G''_{γ} is infinite. Thus if there exists $L \in \mathcal{F}$ as above, then $F''_{\gamma} \subset L$. So we have already shown that F''_{γ} , and indeed F'_{γ} as well, are recoverable from \mathfrak{p} whether or not there exists $L \in \mathcal{F}$. Let $x \in (F'_{\gamma})^{\perp} \setminus (F''_{\gamma})^{\perp}$. Then there exists $v \in F''_{\gamma}$ such that $\langle v, x \rangle \neq 0$, and one has

$$(v \wedge G''_{\gamma}) \cdot x = \{(v \wedge y) \cdot x \mid y \in G''_{\gamma}\} = \{\langle x, y \rangle v - \langle x, v \rangle y \mid y \in G''_{\gamma}\}.$$

Since $v \wedge G''_{\gamma} \subseteq \tilde{\mathfrak{p}}$ and $v \in F''_{\gamma}$, we see that $G''_{\gamma} = (v \wedge G''_{\gamma}) \cdot x + F''_{\gamma} \subset \tilde{\mathfrak{p}} \cdot x + F''_{\gamma} \subset G''_{\gamma}$. Hence $G''_{\gamma} = \tilde{\mathfrak{p}} \cdot x + F''_{\gamma}$, and we conclude that G''_{γ} is recoverable from \mathfrak{p} .

Finally, we must show that \mathfrak{p} determines W under the assumption that no subspace $L \in \mathcal{F}$ as above exists. We have already shown that M is recoverable

from \mathfrak{p} under this assumption. If $M = M^\perp$, then $W = 0$. We claim that $W = \tilde{\mathfrak{p}} \cdot x + M$ for any $x \in M^\perp \setminus M$ when $M \neq M^\perp$. Indeed, let X be any vector space complement of M in W . Since $x \notin M$ and $W^\perp = M$, one has $\langle x, X \rangle \neq 0$. Furthermore, the restriction of the symmetric bilinear form on V to X is symmetric and nondegenerate. Then $\Lambda^2(X) \cdot x = X$ because $\dim_{\mathbb{C}} X \geq 3$. Since $\Lambda^2(X) \subset \tilde{\mathfrak{p}}$, we conclude that $\tilde{\mathfrak{p}} \cdot x + M = W$. Thus W can be recovered from \mathfrak{p} .

If \mathcal{F} is a self-taut generalized flag without any isotropic subspace $L \in \mathcal{F}$ such that $\dim_{\mathbb{C}} L^\perp/L = 2$, then we have now shown that \mathcal{F} is uniquely determined by \mathfrak{p} . Finally, suppose that there does exist an isotropic subspace $L \in \mathcal{F}$ such that $\dim_{\mathbb{C}} L^\perp/L = 2$. Then we have shown that every subspace of \mathcal{F} which does not lie strictly between L and L^\perp is determined by \mathfrak{p} . There are exactly two maximal isotropic subspaces M_1 and M_2 containing L , and both M_1 and M_2 are stable under the $\mathfrak{so}(\infty; \mathbb{C})$ -stabilizer of L . Hence the three self-taut generalized flags listed in the statement are precisely the self-taut generalized flags whose stabilizers equal the stabilizer of \mathcal{F} . \square

3. Real Parabolic Subalgebras

Recall that a *parabolic subalgebra* of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is a subalgebra whose complexification is a parabolic subalgebra of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{g}_{\mathbb{C}}$ be one of $\mathfrak{gl}(\infty, \mathbb{C})$, $\mathfrak{sl}(\infty, \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C})$, and let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{R}}$ be the corresponding connected real subgroup of $G_{\mathbb{C}}$. When $\mathfrak{g}_{\mathbb{R}}$ has two inequivalent defining representations, we denote them by $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$, and when $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation, we denote it by $V_{\mathbb{R}}$. Let \mathbb{D} denote the algebra of $\mathfrak{g}_{\mathbb{R}}$ -endomorphisms of $V_{\mathbb{R}}$.

THEOREM 3.1. *Suppose that $\mathfrak{g}_{\mathbb{R}}$ has two inequivalent defining representations. A subalgebra of $\mathfrak{g}_{\mathbb{R}}$ (resp. subgroup of $G_{\mathbb{R}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{R}}$ -stabilizer (resp. $G_{\mathbb{R}}$ -stabilizer) of a taut couple of \mathbb{D} -generalized flags \mathcal{F} in $V_{\mathbb{R}}$ and \mathcal{G} in $W_{\mathbb{R}}$.*

Suppose that $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation. A subalgebra of $\mathfrak{g}_{\mathbb{R}}$ (resp. subgroup) of $G_{\mathbb{R}}$ is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{R}}$ -stabilizer (resp. $G_{\mathbb{R}}$ -stabilizer) of a self-taut \mathbb{D} -generalized flag \mathcal{F} in $V_{\mathbb{R}}$.

PROOF. We will prove the statements for the Lie algebras in question. The statements on the level of Lie ind-groups follow immediately, since infinite determinant conditions on a Lie ind-group are equivalent to infinite trace conditions on its Lie algebra.

Suppose that $\mathfrak{p}_{\mathbb{R}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. By definition, the complexification $\mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is defined by infinite trace conditions $\text{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer of a taut couple of generalized flags in V and W or on a self-taut generalized flag in V . As $\text{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ is stable under complex conjugation it is the complexification of the real subspace $(\text{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}} := \{t \in \text{Tr}^{\mathfrak{p}_{\mathbb{C}}} \mid \tau(t) = t\}$ where τ comes from complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}_{\mathbb{R}}$. We will use this to show case by case that $\mathfrak{p}_{\mathbb{R}}$ is defined by trace conditions on the $\mathfrak{g}_{\mathbb{R}}$ -stabilizer of the appropriate generalized flag(s).

The first cases we treat are those where the defining representation space $V_{\mathbb{R}}$ is the fixed point set of a complex conjugation $\tau : V \rightarrow V$. The real forms fitting this description are $\mathfrak{sl}(\infty; \mathbb{R})$, $\mathfrak{so}(\infty, \infty)$, $\mathfrak{so}(p, \infty)$, $\mathfrak{sp}(\infty; \mathbb{R})$, and $\mathfrak{gl}(\infty; \mathbb{R})$. Consider the

$\mathfrak{sl}(\infty; \mathbb{R})$ case, and note that the proof also holds in the $\mathfrak{gl}(\infty; \mathbb{R})$ case. Let \mathcal{F} and \mathcal{G} be the taut couple of generalized flags in V and W given in Theorem 2.7, and note that $W_{\mathbb{R}}$ is the fixed points of complex conjugation $\tau : W \rightarrow W$. Evidently $\tau(\mathfrak{p}_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}}$, so $\tau(\mathcal{F}) = \mathcal{F}$ and $\tau(\mathcal{G}) = \mathcal{G}$ by the uniqueness claim of Theorem 2.7. Since the generalized flags \mathcal{F} and \mathcal{G} are τ -stable, every subspace in them is τ -stable. (Explicitly, for any $F \in \mathcal{F}$, we have $\tau(F) \in \mathcal{F}$, so either $\tau(F) \subset F$ or $F \subset \tau(F)$. Since $\tau^2 = \text{Id}$, we have $F = \tau(F)$ for any $F \in \mathcal{F}$.) Hence every subspace in \mathcal{F} and \mathcal{G} has a real form, obtained as the intersection with $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$, respectively. The generalized flags $\mathcal{F}_{\mathbb{R}} := \{F \cap V_{\mathbb{R}} \mid F \in \mathcal{F}\}$ and $\mathcal{G}_{\mathbb{R}} := \{G \cap W_{\mathbb{R}} \mid G \in \mathcal{G}\}$ form a taut couple as \mathbb{R} -generalized flags in $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. Now $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\text{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the $\mathfrak{sl}(\infty; \mathbb{R})$ -stabilizer of the taut couple $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ of generalized flags in $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$.

If $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{so}(*, \infty)$ or $\mathfrak{sp}(\infty; \mathbb{R})$, Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is defined by infinite trace conditions on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer a self-taut generalized flag \mathcal{F} in V . The arguments of the $\mathfrak{sl}(\infty; \mathbb{R})$ case show that \mathcal{F} is τ -stable, provided that $\tau(\mathfrak{p}_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}}$ forces $\tau(\mathcal{F}) = \mathcal{F}$. That is ensured by the uniqueness claim in Theorem 2.7 for the symplectic case, and by Theorem 2.8 in the orthogonal cases where uniqueness holds. Uniqueness fails precisely when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(\infty, \infty)$ and there exists an isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}}(L^{\perp}/L) = 2$. We may assume that \mathcal{F} is the first of the three generalized flags listed in the statement of Theorem 2.8. Then $\tau(\mathcal{F})$ is one of the three generalized flags listed in the statement of Theorem 2.8, and since \mathcal{F} is contained in any of those three, the subspaces of \mathcal{F} are all τ -stable. Finally, the generalized flag $\mathcal{F}_{\mathbb{R}} := \{F \cap V_{\mathbb{R}} \mid F \in \mathcal{F}\}$ in $V_{\mathbb{R}}$ is self-taut, and $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\text{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on its $\mathfrak{g}_{\mathbb{R}}$ -stabilizer.

Second, suppose that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(*, \infty)$. Note that the arguments for $\mathfrak{su}(*, \infty)$ apply without change to $\mathfrak{u}(*, \infty)$. By Theorem 2.7, $\mathfrak{p}_{\mathbb{C}}$ is given by infinite trace conditions $\text{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ on the $\mathfrak{gl}(\infty; \mathbb{C})$ -stabilizer of a taut couple \mathcal{F} and \mathcal{G} of generalized flags in V and W . There exists an isomorphism of $\mathfrak{g}_{\mathbb{R}}$ -modules $f : V \rightarrow W$. Both \mathcal{G} and $f(\mathcal{F})$ are stabilized by $\mathfrak{p}_{\mathbb{R}}$, hence also by $\mathfrak{p}_{\mathbb{C}}$, so the uniqueness claim of Theorem 2.7 tells us that $\mathcal{G} = f(\mathcal{F})$. Thus \mathcal{F} is self-taut. We conclude that $\mathfrak{p}_{\mathbb{R}}$ is given by the infinite trace conditions $(\text{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the stabilizer of the self-taut generalized flag \mathcal{F} .

The third case we consider is that of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(\infty; \mathbb{H})$. Note that the $\mathfrak{gl}(\infty; \mathbb{H})$ case is proved in the same manner. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2\infty; \mathbb{C})$, where we have the identifications $V = \mathbb{C}^{2\infty} = \mathbb{C}^{\infty} + \mathbb{C}^{\infty}j = \mathbb{H}^{\infty} = V_{\mathbb{R}}$ and $W = W_{\mathbb{R}}$. The quaternionic scalar multiplication $v \mapsto vj$ is a complex conjugate-linear transformation J of $\mathbb{C}^{2\infty}$ of square $-\text{Id}$, and the complex conjugation τ of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}_{\mathbb{R}}$ is given by $\xi \mapsto J\xi J^{-1} = J^{-1}\xi J$. Let \mathcal{F} and \mathcal{G} be the unique taut couple given by Theorem 2.7. Since $\mathfrak{p}_{\mathbb{C}} = \tau(\mathfrak{p}_{\mathbb{C}})$, we have $\mathcal{F} = J(\mathcal{F})$ and $\mathcal{G} = J(\mathcal{G})$. Since $J^2 = -\text{Id}$, every subspace of \mathcal{F} and \mathcal{G} is preserved by J . In other words \mathcal{F} and \mathcal{G} consist of \mathbb{H} -subspaces of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. The fact that \mathcal{F} and \mathcal{G} form a taut couple of \mathbb{C} -generalized flags in V and W implies via Lemma 2.4 that they form a taut couple of \mathbb{H} -generalized flags in $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. Hence $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\text{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the stabilizer of the taut couple \mathcal{F}, \mathcal{G} .

The fourth case we consider is that of $\mathfrak{sp}(*, \infty)$. Then $V_{\mathbb{R}}$ has an invariant quaternion-hermitian form of signature $(*, \infty)$ and a complex conjugate-linear transformation J of square $-\text{Id}$ as described above. Let \mathcal{F} be the unique self-taut generalized flag in V given by Theorem 2.7. By the uniqueness of \mathcal{F} , we have

$\mathcal{F} = J(\mathcal{F})$, so as before \mathcal{F} consists of \mathbb{H} -subspaces of $V_{\mathbb{R}}$. Lemma 2.4 implies that \mathcal{F} is self-taut when considered as an \mathbb{H} -generalized flag in $V_{\mathbb{R}}$. Hence $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the stabilizer of \mathcal{F} .

The fifth and final case and is that of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2\infty)$. Any subspace of V which is stable under the \mathbb{C} -conjugate linear map J which corresponds to $x \mapsto xj$ is an \mathbb{H} -subspace of $V_{\mathbb{R}}$. Let \mathcal{F} be a self-taut generalized flag in V as given by Theorem 2.7. Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty; \mathbb{C})$, Theorem 2.8 says that either \mathcal{F} is unique or there are exactly three possibilities for \mathcal{F} . When \mathcal{F} is unique, we must have $\mathcal{F} = J(\mathcal{F})$, so \mathcal{F} is an \mathbb{H} -generalized flag. When \mathcal{F} is not unique, we may assume that \mathcal{F} is the first of the three generalized flags listed in the statement of Theorem 2.8, the one with an immediate predecessor-successor pair $L \subset L^{\perp}$ where L is closed and $\dim_{\mathbb{C}}(L^{\perp}/L) = 2$. Then $J(\mathcal{F})$ has the same property so $J(\mathcal{F}) = \mathcal{F}$. In all cases Lemma 2.4 implies that \mathcal{F} is self-taut when considered as an \mathbb{H} -generalized flag. Hence $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the $\mathfrak{so}^*(2\infty)$ -stabilizer of the self-taut \mathbb{H} -generalized flag \mathcal{F} .

Conversely, suppose that $\mathfrak{p}_{\mathbb{R}}$ is defined by infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}}$ on the $\mathfrak{g}_{\mathbb{R}}$ -stabilizer of a taut couple $\mathcal{F}_{\mathbb{R}}, \mathcal{G}_{\mathbb{R}}$ or a self-taut generalized flag $\mathcal{F}_{\mathbb{R}}$, as appropriate. Either $V = V_{\mathbb{R}} \otimes \mathbb{C}$ or $V = V_{\mathbb{R}}$.

Suppose first that $V = V_{\mathbb{R}} \otimes \mathbb{C}$. Let $\mathcal{F} := \{F \otimes \mathbb{C} \mid F \in \mathcal{F}_{\mathbb{R}}\}$. If $\mathfrak{g}_{\mathbb{C}}$ has only one defining representation V , then \mathcal{F} is a self-taut generalized flag in V , and $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer of \mathcal{F} . Now suppose that $\mathfrak{g}_{\mathbb{C}}$ has two inequivalent defining representations. If $\mathfrak{g}_{\mathbb{R}}$ also has two inequivalent defining representations, let $\mathcal{G} := \{G \otimes \mathbb{C} \mid G \in \mathcal{G}_{\mathbb{R}}\}$. If $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation, then let \mathcal{G} be the image of \mathcal{F} under the $\mathfrak{g}_{\mathbb{R}}$ -module isomorphism $V \rightarrow W$. Then \mathcal{F}, \mathcal{G} are a taut couple, and $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer of \mathcal{F}, \mathcal{G} .

Suppose that $V = V_{\mathbb{R}}$. Then $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ have the same number of defining representations. If $\mathfrak{g}_{\mathbb{R}}$ has two defining representations, then Lemma 2.4 implies that $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ are a taut couple when considered as \mathbb{C} -generalized flags. Then $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer of $\mathcal{F}_{\mathbb{R}}, \mathcal{G}_{\mathbb{R}}$. If $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation, then Lemma 2.4 implies that $\mathcal{F}_{\mathbb{R}}$ is a self-taut generalized flag when considered as a \mathbb{C} -generalized flag. Thus $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$ -stabilizer of $\mathcal{F}_{\mathbb{R}}$.

In each case, Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, so by definition $\mathfrak{p}_{\mathbb{R}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. \square

THEOREM 3.2. *Let $\mathfrak{p}_{\mathbb{R}}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. If $\mathfrak{g}_{\mathbb{R}} \not\cong \mathfrak{so}(\infty, \infty)$, then there is a unique taut couple or self-taut generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ by Theorem 3.1. The real analogue of Theorem 2.8 holds for $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(\infty, \infty)$.*

PROOF. If there is a unique taut couple or self-taut generalized flag associated to $\mathfrak{p}_{\mathbb{C}}$, then the uniqueness of the taut couple or self-taut generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ is immediate from the proof of Theorem 3.1. If $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(\infty, \infty)$, then each of the \mathbb{C} -generalized flags of Theorem 2.8 has a real form, hence the real analogue of Theorem 2.8 holds in this case. Now suppose that $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}^*(2\infty)$ and the self-taut generalized flag \mathcal{F} associated to $\mathfrak{p}_{\mathbb{C}}$ has a closed isotropic subspace L with $\dim_{\mathbb{C}}(L^{\perp}/L) = 2$. The proof of Theorem 3.1 shows that L and L^{\perp} are \mathbb{H} -subspaces, and the quaternionic codimension of L in L^{\perp} is 1. Hence the \mathbb{H} -generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ has no subspaces strictly between L and L^{\perp} , which forces it to be unique. \square

REMARK 3.3. Theorem 3.1 simplifies sharply in the $\mathfrak{su}(p, \infty)$, $\mathfrak{so}(p, \infty)$, $\mathfrak{sp}(p, \infty)$, and $\mathfrak{u}(p, \infty)$ cases when $p \in \mathbb{Z}_{\geq 0}$. Because p is the maximal dimension of an isotropic subspace of $V_{\mathbb{R}}$ (and thus the maximal codimension of a closed coisotropic subspace), a self-taut generalized flag must be finite. No infinite trace conditions arise. The stabilizer of such a self-taut generalized flag coincides with the joint stabilizer of its isotropic subspaces and at most one non-closed coisotropic subspace. (The perpendicular complement of the single non-closed coisotropic subspace, when it occurs, is the largest isotropic subspace.) \diamond

REMARK 3.4. The special case where the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (or $\mathfrak{g}_{\mathbb{R}}$) is a direct limit of parabolics of the $\mathfrak{g}_{n, \mathbb{C}}$ (or the $\mathfrak{g}_{n, \mathbb{R}}$) has been studied in a number of contexts such as [8] and [10], and in particular in connection with direct limits of principal series representations [12]. Any direct limit of parabolic subalgebras is a parabolic subalgebra in the general sense of this paper. \diamond

4. A Geometric Interpretation

Our geometric interpretation is modeled on a criterion from the finite-dimensional case. Let $G_{\mathbb{C}}$ be a finite-dimensional classical Lie ind-group, and $G_{\mathbb{R}}$ a real form of $G_{\mathbb{C}}$. Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup, and let $Z := G_{\mathbb{C}}/P$ be the corresponding flag manifold. Then $G_{\mathbb{R}}$ acts on Z as a subgroup of $G_{\mathbb{C}}$. One knows [11, Theorem 3.6] that there is a unique closed $G_{\mathbb{R}}$ -orbit F on Z , and that $\dim_{\mathbb{R}} F \geq \dim_{\mathbb{C}} Z$, with equality precisely when F is a real form of Z . Thus real and complex dimensions satisfy $\dim_{\mathbb{R}} F = \dim_{\mathbb{C}} Z$ if and only if F is a totally real submanifold of Z . This is the motivation for our geometric interpretation, for F is a totally real submanifold of Z if and only if $G_{\mathbb{R}}$ has a parabolic subgroup whose complexification is $G_{\mathbb{C}}$ -conjugate to P . Then that real parabolic subgroup is the $G_{\mathbb{R}}$ -stabilizer of a point of the closed orbit F . Here note that if any $G_{\mathbb{R}}$ -orbit in Z is totally real then it has real dimension $\leq \dim_{\mathbb{C}} Z$, so it must be the closed orbit.

Let now $G_{\mathbb{C}}$ be one of the Lie ind-groups $GL(\infty; \mathbb{C})$, $SL(\infty; \mathbb{C})$, $SO_0(\infty; \mathbb{C})$ and $Sp(\infty; \mathbb{C})$. Fix an exhaustion of $G_{\mathbb{C}}$ by classical connected finite-dimensional subgroups $G_{n, \mathbb{C}}$, and let $G_{n, \mathbb{R}}$ be nested real forms of $G_{n, \mathbb{C}}$. Then $G_{\mathbb{R}} := \varinjlim G_{n, \mathbb{R}}$ is a real form of $G_{\mathbb{C}}$. Let $P_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$. As described in Section 2C, $P_{\mathbb{C}}$ is defined by infinite determinant conditions on the stabilizer $\widetilde{P}_{\mathbb{C}}$ of a taut couple or a self-taut generalized flag. Here $\widetilde{P}_{\mathbb{C}}$ is the normalizer of $P_{\mathbb{C}}$ in $G_{\mathbb{C}}$. We use the usual notation for the Lie algebras of all these Lie ind-groups.

LEMMA 4.1. *Consider the homogeneous space $Z = G_{\mathbb{C}}/\widetilde{P}_{\mathbb{C}}$. Write z_0 for the identity coset $1 \cdot \widetilde{P}_{\mathbb{C}}$ in Z and define $Z_n = G_{n, \mathbb{C}}(z_0)$. Then each Z_n is a (finite-dimensional) complex homogeneous space and Z is the complex ind-manifold $\varinjlim Z_n$ (direct limit in the category of complex manifolds and holomorphic maps.)*

PROOF. $\widetilde{P}_{\mathbb{C}}$ is a complex subgroup of $G_{\mathbb{C}}$, and $\widetilde{P}_{\mathbb{C}} = \varinjlim (G_{n, \mathbb{C}} \cap \widetilde{P}_{\mathbb{C}})$. Each finite-dimensional orbit Z_n is a complex manifold because $G_{n, \mathbb{C}} \cap \widetilde{P}_{\mathbb{C}}$ is a complex subgroup of $G_{n, \mathbb{C}}$, and the inclusions $Z_n \hookrightarrow Z_{n+1}$ are holomorphic embeddings. As in [10] now $Z = \varinjlim Z_n$ is a strict direct limit in the category of complex manifolds and holomorphic maps. In other words a function f on an open subset $U \subset Z$ is holomorphic if and only if each of the $f|_{U \cap Z_n} : U \cap Z_n \rightarrow \mathbb{C}$ is holomorphic. Note that separately holomorphic functions on open subsets $U \subset Z$ are jointly

holomorphic because each $f|_{U \cap Z_n}$ is jointly holomorphic (and thus continuous) by Hartogs' Theorem. \square

LEMMA 4.2. *Let $Y = G_{\mathbb{R}}(z_0)$ and $Y_n = G_{n,\mathbb{R}}(z_0)$. Then Y is a totally real submanifold of Z if and only if each Y_n is a totally real submanifold of Z_n .*

PROOF. Let J denote the complex structure operator for Z , linear transformation of square $-\text{Id}$ on the complexified tangent space $T := T_{z_0,\mathbb{C}}(Z)$ of Z at z_0 . Then J preserves each of the $T_n := T_{z_0,\mathbb{C}}(Z_n)$. Now Y is totally real if and only if the real tangent space $T_{\mathbb{R}} := T_{z_0}(Z)$ satisfies $J(T_{\mathbb{R}}) \cap T_{\mathbb{R}} = 0$, and Y_n is totally real if and only if the real tangent space $T_{n,\mathbb{R}} := T_{z_0}(Z_n)$ satisfies $J(T_{n,\mathbb{R}}) \cap T_{n,\mathbb{R}} = 0$. Since $T_{\mathbb{R}} = \varinjlim T_{n,\mathbb{R}}$ the assertion follows. \square

LEMMA 4.3. *$G_{n,\mathbb{R}} \cap \widetilde{P}_{\mathbb{C}}$ is a real form of $G_{n,\mathbb{C}} \cap \widetilde{P}_{\mathbb{C}}$ if and only if Y_n is totally real in Z_n .*

PROOF. Denote $H_{n,\mathbb{C}} = G_{n,\mathbb{C}} \cap \widetilde{P}_{\mathbb{C}}$ and $H_{n,\mathbb{R}} = G_{n,\mathbb{R}} \cap \widetilde{P}_{\mathbb{C}}$. Suppose first that Y_n is totally real in Z_n . Then $\dim_{\mathbb{R}} G_{n,\mathbb{R}} - \dim_{\mathbb{R}} H_{n,\mathbb{R}} = \dim_{\mathbb{R}} Y_n \leq \dim_{\mathbb{C}} Z_n = \dim_{\mathbb{C}} G_{n,\mathbb{C}} - \dim_{\mathbb{C}} H_{n,\mathbb{C}}$, so $\dim_{\mathbb{R}} H_{n,\mathbb{R}} \geq \dim_{\mathbb{C}} H_{n,\mathbb{C}}$, forcing $\dim_{\mathbb{R}} H_{n,\mathbb{R}} = \dim_{\mathbb{C}} H_{n,\mathbb{C}}$. Now $H_{n,\mathbb{R}}$ is a real form of $H_{n,\mathbb{C}}$.

Conversely suppose that $H_{n,\mathbb{R}}$ is a real form of $H_{n,\mathbb{C}}$. Then the real tangent space to Y_n at z_0 is represented by any vector space complement $\mathfrak{m}_{n,\mathbb{R}}$ to $\mathfrak{h}_{n,\mathbb{R}}$ in $\mathfrak{g}_{n,\mathbb{R}}$, while the real tangent space to Z_n at z_0 is represented by the vector space complement $\mathfrak{m}_{n,\mathbb{R}} \otimes \mathbb{C}$ to $\mathfrak{h}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$, so Y_n is totally real in Z_n . \square

Putting all this together, we have our geometric characterization of parabolic subgroups of the classical real Lie ind-groups.

THEOREM 4.4. *Fix a parabolic subgroup $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ and consider the flag ind-manifold $Z = G_{\mathbb{C}}/\widetilde{P}_{\mathbb{C}}$. Then $P_{\mathbb{C}} \cap G_{\mathbb{R}}$ is a parabolic subgroup of $G_{\mathbb{R}}$ if and only if the following two conditions hold:*

- (i) *the orbit $G_{\mathbb{R}}(z_0)$ of the base point $z_0 = \widetilde{P}_{\mathbb{C}}$ is a totally real submanifold of Z ;*
- (ii) *the set of all infinite trace conditions on $\widetilde{\mathfrak{p}}_{\mathbb{C}}$ satisfied by $\mathfrak{p}_{\mathbb{C}}$ is stable under the complex conjugation τ of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}_{\mathbb{R}}$.*

PROOF. Lemmas 4.2 and 4.3 show that the orbit $G_{\mathbb{R}}(z_0)$ is a totally real submanifold of Z if and only if $G_{\mathbb{R}} \cap \widetilde{P}_{\mathbb{C}}$ is parabolic in $G_{\mathbb{R}}$.

If $G_{\mathbb{R}} \cap P_{\mathbb{C}}$ is parabolic in $G_{\mathbb{R}}$ then $G_{\mathbb{R}} \cap \widetilde{P}_{\mathbb{C}}$ is parabolic because it contains $G_{\mathbb{R}} \cap P_{\mathbb{C}}$, and the corresponding real set of infinite trace conditions complexifies to the set of infinite trace conditions by which $\mathfrak{p}_{\mathbb{C}}$ is defined from $\widetilde{\mathfrak{p}}_{\mathbb{C}}$. Thus (i) and (ii) follow.

Conversely assume (i) and (ii). From (i), $G_{\mathbb{R}} \cap \widetilde{P}_{\mathbb{C}}$ is a parabolic subgroup of $G_{\mathbb{R}}$, and from (ii), $\{x \in \mathfrak{g}_{\mathbb{R}} \cap \widetilde{\mathfrak{p}}_{\mathbb{C}} \mid x \text{ satisfies } \text{Tr}^{\mathfrak{p}_{\mathbb{C}}}\} \otimes \mathbb{C} = \{x \in \widetilde{\mathfrak{p}}_{\mathbb{C}} \mid x \text{ satisfies } \text{Tr}^{\mathfrak{p}_{\mathbb{C}}}\}$, where $\text{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ denotes the set of infinite trace conditions described in Definition 2.6. \square

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