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Infinite Dimensional Multiplicity Free Spaces II: Limits of Commutative Nilmanifolds

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Abstract. We study direct limits $(G, K) = \varinjlim (G_n, K_n)$ of Gelfand pairs of the form $G_n = N_n \rtimes K_n$ with N_n nilpotent, in other words pairs (G_n, K_n) for which G_n/K_n is a commutative nilmanifold. First, we extend the criterion of [**W4**] for a direct limit representation to be multiplicity free. Then we study direct limits $G/K = \varinjlim G_n/K_n$ of commutative nilmanifolds and look to see when the regular representation of $G = \varinjlim G_n$ on an appropriate Hilbert space $\varinjlim L^2(G_n/K_n)$ is multiplicity free. One knows that the N_n are commutative or 2–step nilpotent. In many cases where the derived algebras $[\mathfrak{n}_n, \mathfrak{n}_n]$ are of bounded dimension we construct G_n–equivariant isometric maps $\zeta_n : L^2(G_n/K_n) \to L^2(G_{n+1}/K_{n+1})$ and prove that the left regular representation of G on the Hilbert space $L^2(G/K) := \varinjlim\{L^2(G_n/K_n), \zeta_n\}$ is a multiplicity free direct integral of irreducible unitary representations. The direct integral and its irreducible constituents are described explicitly. One constituent of our argument is an extension of the classical Peter–Weyl Theorem to parabolic direct limits of compact groups.

1. Introduction

Gelfand pairs (G, K), and the corresponding "commutative" homogeneous spaces G/K, form a natural extension of the class of riemannian symmetric spaces. Let G be a locally compact topological group, K a compact subgroup, and $M = G/K$. Then the following conditions are equivalent; see [**W3**, Theorem 9.8.1].

1. (G, K) is a Gelfand pair, i.e. $L^1(K\backslash G/K)$ is commutative under convolution.
2. If $g, g' \in G$ then $\mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$ (convolution of measures on $K\backslash G/K$).
3. $C_c(K\backslash G/K)$ is commutative under convolution.
4. The measure algebra $\mathcal{M}(K\backslash G/K)$ is commutative.
5. The representation of G on $L^2(M)$ is multiplicity free.

If G is a connected Lie group one can also add

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6. The algebra of G–invariant differential operators on M is commutative.

Conditions 1, 2, 3 and 4 depend on compactness of K so that integration on M and $K\backslash G/K$ corresponds to integration on G. Condition 5 makes sense as long as K is unimodular in G, and condition 6 remains meaningful (and useful) whenever G is a connected Lie group.

In this note we look at some cases where G and K are not locally compact, in fact are infinite dimensional, and show in those cases that the multiplicity–free condition 5 is satisfied. The main results are Theorems 5.10 and 9.10. We first discuss a multiplicity free criterion which extends that of [**W4**]. We used that criterion in [**W4**] for a number of direct systems $\{(G_n, K_n)\}$ where the G_n are compact Lie groups and the K_n are closed subgroups. Here we use its extension for direct systems $\{(G_n, K_n)\}$ in which a (closed connected) nilpotent subgroup N_n of G_n acts transitively on G_n/K_n. Then (see [**W3**, Chapter 13]) N_n is the nilradical of G_n and G_n is the semidirect product group $N_n \rtimes K_n$. Further, the group N_n is commutative or 2–step nilpotent so its structure and representation theory can be clearly understood in terms of the familiar Heisenberg groups.

Section 3 pins down the structure of direct limit of Heisenberg group, and Section 4 provides an infinite dimensional analog of the Peter–Weyl Theorem that turns out to be a necessary tool for our multiplicity free arguments.

In Section 5 we look closely at the cases where N_n is the $(2n+1)$–dimensional Heisenberg group $H_n = \operatorname{Im}\mathbb{C} + \mathbb{C}^n$ and K_n (viewed as a subgroup of $U(n)$) acts irreducibly on $\mathbb{C}^n$. The classification of these cases is well known; see Table 5.1. It leads to 16 direct systems, listed in Table 5.2a,b. In Theorem 5.10 we see that, for each of these systems, the limit space $L^2(G/K)$ is well defined and the action of G on $L^2(G/K)$ is multiplicity free. The methods here are explicit and they guide the arguments in the more general cases.

Our basic arguments go in two steps. First we examine the limit $L^2(N) := \varinjlim L^2(N_n)$ as an $(N \times N)$–module, and then we look at the right action of K on $L^2(N)$ in order to analyze $L^2(G/K) := \varinjlim L^2(G_n/K_n)$ as a left G–module. The first step is carried out in Section 6. Section 7 works out some preliminaries for understanding the right action of K. Then in Sections 8 and 9 specify the precise requirements for the multiplicity free condition. See Theorems 8.6 and 9.1. The latter reduces some considerations to the case where N_n is a Heisenberg group.

Tables 9.6 and 9.15 list the key cases of direct systems of nilpotent commutative pairs not based directly on Heisenberg groups. The corresponding multiplicity free results are Theorems 9.10 and 9.20.

Finally, Appendix A is a small discussion of formal degree for induced representations, and Appendix B indicates how some of our results can be made more explicit using branching rules for representations of the K_n.

Our arguments require the direct system $\{(G_n, K_n)\}$ to have the property that the $\dim[\mathfrak{n}_n, \mathfrak{n}_n]$ have an upper bound. It seems probable that this condition is not necessary, but that will require a new idea.

2. Direct Limit Groups and Representations

We consider direct limit groups $G = \varinjlim G_n$ and direct limit representations $\pi = \varinjlim \pi_n$ of them. This means that π_n is a representation of G_n on a vector space V_n, that the V_n form a direct system, and that π is the representation of G on $V = \varinjlim V_n$ given by $\pi(g)v = \pi_n(g_n)v_n$ whenever n is sufficiently large that $V_n \hookrightarrow V$ and $G_n \hookrightarrow G$ send v_n to v and g_n to g. The formal definition amounts to saying that π is well defined.

Here we will only consider unitary representations. Thus the V_n all will be Hilbert spaces and the inclusions $V_n \hookrightarrow V_{n+1}$ will preserve norms (and inner products).

It is clear that a direct limit of irreducible representations is irreducible, but there are irreducible representations of direct limit groups that cannot be formulated as direct limits of irreducible finite dimensional representations. This is a combinatoric matter and is discussed extensively in [**DPW**]. Dealing with this matter is the crux of the problem of proving multiplicity–free properties. The following definition is closely related to the relevant combinatorics but applies to a somewhat simpler situation.

Definition 2.1. We say that a unitary representation π of $G = \varinjlim G_n$ is **limit–aligned** if it is a direct limit $\varinjlim \pi_n$ of unitary representations in such a way that (i) each group G_n is of type I, (ii) π_n is a continuous direct sum $\int \zeta_n \, d\nu_n(\zeta_n)$ of mutually disjoint primary representations, and (iii) in the corresponding inclusions $V_{\pi_n} = \int V_{\zeta_n} \, d\nu_n(\zeta_n) \hookrightarrow \int V_{\zeta_{n+1}} \, d\nu_{n+1}(\zeta_{n+1}) = V_{\pi_{n+1}}$ of representation spaces map ν_n–almost-every primary integrand V_{ζ_n} of V_{π_n} into a primary integrand $V_{\zeta_{n+1}}$ of $V_{\pi_{n+1}}$. $\diamondsuit$

Theorem 2.2. *Let $\pi = \varinjlim \pi_n$ be a limit–aligned unitary representation of $G = \varinjlim G_n$. Suppose that the π_n are multiplicity free. Then π is multiplicity free. In other words the commuting algebra of π is commutative.*

Proof. Let $V = \varinjlim V_{\pi_n}$ be the representation spaces. Consider the primary decompositions $V_{\pi_n} = \int V_{\zeta_n} \, d\nu_n(\zeta_n)$. Here ν_n is a non–negative measure on the analytic Borel space $\widehat{G_n}$, the ζ_n are mutually inequivalent irreducible unitary representations of G_n, and $\pi_n = \int \zeta_n \, d\nu_n(\zeta_n)$.

Since π is limit–aligned we have maps $b_n : (\widehat{G_n}, \nu_n) \to (\widehat{G_{n+1}}, \nu_{n+1})$ of measure spaces such that, for ν_n–almost all $\zeta_n \in \widehat{G_n}$, $V_{\pi_n} \hookrightarrow V_{\pi_{n+1}}$ maps V_{ζ_n} into $V_{b_n(\zeta_n)}$. Thus we have direct systems

$$V_{\zeta_n} \to V_{b_n(\zeta_n)} \to V_{b_{n+1}(b_n(\zeta_n))} \to V_{b_{n+2}(b_{n+1}(b_n(\zeta_n)))} \to V_{b_{n+3}(b_{n+2}(b_{n+1}(b_n(\zeta_n))))} \to \cdots$$

of irreducible unitary representation spaces for $G = \varinjlim G_n$. Let V_ζ denote the corresponding direct limit Hilbert space and ζ the (necessarily irreducible) direct limit unitary representation of G on V_ζ.

Write $\widehat{G_n}'$ for the support of ν_n in the hull–kernel topology. From the considerations just above we see that $b_n(\widehat{G_n}') \subset \widehat{G_{n+1}}'$. That defines a space $\widehat{G}' = \varinjlim \widehat{G_n}'$, a measure class $\nu = \varinjlim \nu_n$ on $\widehat{G}'$, and a decomposition $V = \int V_\zeta \, d\nu(\zeta)$. Since the ζ are irreducible, the closed $\pi(G)$–invariant subspaces of V are just the $V_S = \int_S V_\zeta \, d\nu(\zeta)$ where S is a measurable subset of $\widehat{G}'$. Thus any two projections in the commuting algebra of π commute with each other. Since the commuting algebra is a W^* algebra, thus generated by projections, it is commutative. We have proved that π is multiplicity free. $\square$

3. Direct Limits of Heisenberg Groups

In this section we work out the structure and properties of the (2–sided) regular representation of the infinite Heisenberg groups. That is the foundation for study of the multiplicity free property for direct limits of commutative nilmanifolds.

Recall that the (ordinary) Heisenberg group is the group $H_n = \mathrm{Im}\,\mathbb{C} + \mathbb{C}^n$ with composition $(z,w)(z',w') = (z+z'+\mathrm{Im}\,(w\cdot w'), w+w')$. Here $w\cdot w'$ refers to the standard positive definite hermitian inner product on $\mathbb{C}^n$ and $\mathrm{Im}\,v = \frac{1}{2}(v-\overline{v})$ is the imaginary *component* of a complex number v. Thus $\mathrm{Im}\,\mathbb{C}$ is both the center and the derived group and $H_n/\mathrm{Im}\,\mathbb{C}$ is a vector group.

For t nonzero and real, $\pi_{n,t}$ denotes the irreducible unitary representation of H_n with central character $e^t : (z,0) \mapsto e^{tz}$ (we use the fact that $z \in \mathrm{Im}\,\mathbb{C}$ is pure imaginary). Then $\pi_{n,t}$ is square integrable modulo the center of H_n in the sense that its coefficients $f_{u,v}(h) = \langle u, \pi_{n,t}(h)v\rangle$ satisfy $|f| \in L^2(H_n/\mathrm{Im}\,\mathbb{C})$. For the appropriate normalization of Haar measure $\pi_{n,t}$ has formal degree $|t|^n$ in the sense of the orthogonality relation $\langle f_{u,v}, f_{u',v'}\rangle_{L^2(H_n/\mathrm{Im}\,\mathbb{C})} = |t|^{-n}\langle u,u'\rangle\,\overline{\langle v,v'\rangle}$.

The representation space of $\pi_{n,t}$ is the Fock space

$$\mathcal{H}_{n,t} = \left\{ f : \mathbb{C}^n \to \mathbb{C} \text{ holomorphic} \;\middle|\; \int |f(w)|^2 \exp(-|t||w|^2)d\lambda(w) < \infty \right\}$$

where λ is Lebesgue measure. The representation is

$$[\pi_t(z,v)f](w) = e^{tz \pm t\mathrm{Im}\,(w-v/2)\cdot v} f(w-v)$$

where $\pm$ is the sign of $t/|t|$.

For each multi–index $\mathbf{m} = (m_1,\dots,m_n)$, $m_i \geqq 0$, we have the monomial $w^{\mathbf{m}} = w_1^{m_1}\dots w_n^{m_n}$. One computes $\int w^{\mathbf{m}}\overline{w^{\mathbf{m}'}}\exp(-|t||w|^2)d\lambda(w)$ to see that it is equal to 0 for $\mathbf{m} \neq \mathbf{m}'$, and if $\mathbf{m} = \mathbf{m}'$ it is equal to $c\,\mathbf{m}!$ for a positive constant c independent of $\mathbf{m}$. Here $\mathbf{m}!$ means $\prod(m_i!)$. Thus we can (and do) normalize the inner product on $\mathcal{H}_t$ so that the $w[\mathbf{m}] := w^{\mathbf{m}}/\sqrt{\mathbf{m}!}$ form a complete orthonormal set in $\mathcal{H}_{n,t}$. The corresponding space of matrix coefficients is $\mathcal{E}_{n,t} = \mathcal{H}_{n,t}\widehat{\otimes}\mathcal{H}_{n,t}^*$. It is spanned by the functions $f_{\mathbf{l},\mathbf{m};t} : g \mapsto \langle w[\mathbf{l}], \pi_{n,t}(g)z[\mathbf{w}]\rangle$. These coefficients belong to the Hilbert space

$$L^2(H_n/\mathrm{Im}\,\mathbb{C}; e^t) = \{f : H_n \to \mathbb{C} \mid |f| \in L^2(H_n/\mathrm{Im}\,\mathbb{C}) \text{ and } f(z,w) = e^{-tz}f(0,w)\}$$

with inner product $\langle f, f'\rangle = \int_{\mathbb{C}^n} f(z,w)\overline{f'(z,w)}d\lambda(w)$.

Since $|t|^n$ is the formal degree of $\pi_{n,t}$ the orthogonality relations say that the inner product in $\mathcal{E}_{n,t}$ is given by $\langle f_{\mathbf{l},\mathbf{m};t}, f_{\mathbf{l}',\mathbf{m}';t}\rangle = |t|^{-n}$ if $\mathbf{l} = \mathbf{l}'$ and $\mathbf{m} = \mathbf{m}'$, 0 otherwise. Now the $|t|^{n/2}f_{\mathbf{l},\mathbf{m};t}$ form a complete orthonormal set in $\mathcal{E}_{n,t}$, and $\mathcal{E}_{n,t}$ consists of the functions $\Phi_{n,t,\varphi}$ given by

$$\Phi_{n,t,\varphi}(h) = \sum_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|t|^{n/2} f_{\mathbf{l},\mathbf{m};t}(h) \tag{3.1}$$

where the numbers $\varphi_{\mathbf{l},\mathbf{m}}(t)$ satisfy $\sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 < \infty$.

The Hilbert space $L^2(H_n)$ is the direct integral $\int_{-\infty}^{\infty} \mathcal{E}_{n,t}|t|^n\,dt$ based on the complete orthonormal sets $\{|t|^{n/2} f_{\mathbf{l},\mathbf{m};t}\}$ in the $\mathcal{E}_{n,t}$. Thus it consists of all functions $\Psi_{n,\varphi}$ given by

$$(3.2)\quad \Psi_{n,\varphi}(h) = \int_{-\infty}^{\infty} \Phi_{n,t,\varphi}(h)|t|^n\,dt = \int_{-\infty}^{\infty} \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|t|^{n/2} f_{\mathbf{l},\mathbf{m};t}(h)\right)|t|^n\,dt$$

such that the functions $\varphi_{\mathbf{l},\mathbf{m}} : \mathbb{R} \to \mathbb{C}$ are measurable, $\sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 < \infty$ for almost all t, and $\sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 \in L^1(\mathbb{R}, |t|^n dt)$. Note that

$$(3.3)\quad \begin{aligned} ||\Psi_{n,\varphi}||^2_{L^2(H_n)} &= \int_{-\infty}^{\infty} ||\Phi_{n,t,\varphi}||^2_{\mathcal{E}_{n,t}}\,|t|^n\,dt \\ &= \int_{-\infty}^{\infty} \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2\right)|t|^n\,dt = \sum\nolimits_{\mathbf{l},\mathbf{m}} ||\varphi_{\mathbf{l},\mathbf{m}}||^2_{L^2(\mathbb{R},|t|^{n/2}dt)} \end{aligned}$$

The left/right representation of $H_n \times H_n$ on $\mathcal{E}_{n,t}$ is the exterior tensor product $\pi_{n,t} \boxtimes \pi^*_{n,t}$. It is irreducible. The corresponding representation of $H_n \times H_n$ on $L^2(H_n)$ is the left/right regular representation $\Pi_n := \int_{-\infty}^{\infty} (\pi_{n,t} \boxtimes \pi^*_{n,t})|t|^n dt$. (The factor $|t|^n$ is not relevant to the equivalence class of the representation, but it is crucial to expansion of functions.)

LEMMA 3.4. *The left/right regular representation* Π_n *of* $H_n \times H_n$ *on* $L^2(H_n)$ *is multiplicity free.*

Proof. Any invariant subspace must be invariant under the center $\operatorname{Im}\mathbb{C} \times \operatorname{Im}\mathbb{C}$ of $H_n \times H_n$, so it is of the form $\int_S \mathcal{E}_{n,t}\,|t|^n\,dt$ where S is a measurable subset of $\mathbb{R}$. Now any two projections in the commuting algebra of Π_n must commute with each other. The commuting algebra is a von Neumann algebra, so the projections generate a dense subalgebra. Thus the commuting algebra is commutative. $\square$

Suppose $m \geqq n$. We view an n–tuple $\mathbf{w}$ as an m–tuple by appending $m - n$ zeroes; then $w^{\mathbf{m}}$ and $w[\mathbf{m}]$ have the same meaning as functions on $\mathbb{C}^n$ and on $\mathbb{C}^m$. Thus the coefficient function $f_{\mathbf{l},\mathbf{m};t} : H_n \to \mathbb{C}$ is the restriction of $f_{\mathbf{l},\mathbf{m};t} : H_m \to \mathbb{C}$. From (3.2) and (3.3) we see that the map

$$(3.5)\quad \begin{aligned} &\zeta'_{m,n} : \mathcal{E}_{n,t} \to \mathcal{E}_{m,t} \text{ defined by} \\ &\qquad \zeta'_{m,n}(|t|^{n/2} f_{\mathbf{l},\mathbf{m};t}) = |t|^{m/2} f_{\mathbf{l},\mathbf{m};t} \text{ and } \zeta'_{m,n}(\Psi_{n,\varphi}) = \Psi_{m,|t|^{(n-m)/2}\varphi} \end{aligned}$$

is an isometric $(H_n \times H_n)$–equivariant injection of $\mathcal{E}_{n,t}$ into $\mathcal{E}_{m,t}$. Specifically, it maps a complete orthonormal set in $\mathcal{E}_{n,t}$ to an orthonormal set in $\mathcal{E}_{m,t}$.

REMARK 3.6. The adjoint of the isometric injection $\zeta'_{m,n} : \Psi_{n,\varphi} \mapsto \Psi_{m,|t|^{(n-m)/2}\varphi}$ of $L^2(H_n)$ into $L^2(H_m)$ is orthogonal projection of $L^2(H_m)$ to the image of the injection. It is the scalar multiple (by $|t|^{(m-n)/2}$) of restriction of functions on each direct integrand $\mathcal{E}_{m,t}$ of $L^2(H_m)$, but the scalar varies with m, n and t. $\diamondsuit$

Now let's consider convergence. Let M_n denote the set of multi–indices $\mathbf{m}$ of length n. In order that $\Phi_{n,t} \in \mathcal{E}_{n,t}$ we needed that $\varphi(t) \in L^2(M_n)$, using counting measure. That gives the function $||\varphi||^2(t)$. Now for $\Psi_{n,\varphi}$ to be in $L^2(H_n)$ as n grows, we need $||\varphi||^2 \in L^1(\mathbb{R}, |t|^n dt)$ as n grows. These conditions are satisfied whenever only finitely many of the $\varphi_{\mathbf{l},\mathbf{m}}$ are not identically zero, and the nonzero ones are Schwartz class functions of t. Thus we have a dense subset of $L^2(H_n)$ that maps in a norm–preserving way into $L^2(H_m)$, whenever $m \geqq n$, and extends by continuity to give a well defined unitary injection $L^2(H_n) \to L^2(H_m)$.

All the ingredients in the construction of this injection are $(H_n \times H_n)$–equivariant, so the unitary injection $L^2(H_n) \to L^2(H_m)$ is equivariant for the left/right regular representation Π_n of $H_n \times H_n$.

THEOREM 3.7. *There is a strict direct system* $\{L^2(H_n), \zeta'_{m,n}\}$ *of* L^2 *spaces of the Heisenberg groups, whose maps* $\zeta'_{m,n} : L^2(H_n) \to L^2(H_m)$ *are* $(H_n \times H_n)$*–equivariant unitary injections. Let* Π_n *denote the left/right regular representation of* $H_n \times H_n$ *on* $L^2(H_n)$ *and let* H *denote the infinite dimensional Heisenberg group* $\varinjlim H_n$. *Then we have a well defined Hilbert space* $L^2(H) := \varinjlim\{L^2(H_n), \zeta'_{m,n}\}$ *and a natural unitary representation* $\Pi = \varinjlim \Pi_n$ *of* $H \times H$ *on* $L^2(H)$. *Further, that representation* Π *is multiplicity–free.*

Proof. All the assertions except the multiplicity–free assertion have just been proved. Now it remains only to prove that $\Pi = \varinjlim \Pi_n$ is limit–aligned, for then Lemma 3.4 and Theorem 2.2 complete the proof. That alignment is immediate because the unitary injections $L^2(H_n) \to L^2(H_m)$ are equivariant for the action of the center of $H_n \times H_n$, and that action specifies the direct integrands $\mathcal{E}_{n,t}$ and $\mathcal{E}_{m,t}$ within $L^2(H_n)$ and $L^2(H_m)$. $\square$

4. The Peter–Weyl Theorem for Direct Limits of Compact Groups

We will apply Theorem 3.7 to the study of direct limits $\varinjlim\{(H_n \rtimes K_n, K_n)\}$. There K_n is a compact connected group of automorphisms of the Heisenberg group H_n. In order to do that we need to extend Theorem 3.7 from the H_n to the semidirect product groups $H_n \rtimes K_n$, and for that we must first prove the analog of Theorem 3.7 for the direct systems $\{K_n\}$. This analog, the Peter–Weyl Theorem for $\{K_n\}$, is the subject of this section. As we will see in Section 5 we need only study the restricted class of direct systems $\{K_n\}$ given as follows.

DEFINITION 4.1. Let $\{K_n\}$ be a strict direct system of compact connected Lie groups, $\{(K_n)_{\mathbb{C}}\}$ the direct system of their complexifications. Suppose that, for each n,

$$\text{the semisimple part } [(\mathfrak{k}_n)_{\mathbb{C}}, (\mathfrak{k}_n)_{\mathbb{C}}] \text{ of the reductive algebra } (\mathfrak{k}_n)_{\mathbb{C}} \text{ is the semisimple component of a parabolic subalgebra of } (\mathfrak{k}_{n+1})_{\mathbb{C}}. \tag{4.2}$$

Then we say that the direct systems $\{K_n\}$ and $\{(K_n)_{\mathbb{C}}\}$ are **parabolic** and that $\varinjlim K_n$ and $\varinjlim (K_n)_{\mathbb{C}}$ are **parabolic direct limits**. This is a small variation on the definitions of parabolic and weakly parabolic direct limits in [**W2**]. $\diamondsuit$

Now let $\{K_n\}$ be a strict direct system of compact connected Lie groups that is parabolic. We recursively construct Cartan subalgebras $\mathfrak{t}_n \subset \mathfrak{k}_n$ with $\mathfrak{t}_1 \subset \mathfrak{t}_2 \subset \cdots \subset \mathfrak{t}_n \subset \mathfrak{t}_{n+1} \subset \dots$ and (using the parabolic property) simple root systems $\Psi_n = \Psi((\mathfrak{k}_n)_{\mathbb{C}}, (\mathfrak{t}_n)_{\mathbb{C}})$ such that each simple root for $(\mathfrak{k}_n)_{\mathbb{C}}$ is the restriction of exactly one simple root for $(\mathfrak{k}_{n+1})_{\mathbb{C}}$. Then we may assume that $\Psi_n = \{\psi_{n,1}, \dots, \psi_{n,p(n)}\}$ in such a way that each $\psi_{n,j}$ is the $(\mathfrak{t}_n)_{\mathbb{C}}$–restriction of $\psi_{n+1,j}$ and of no other element of Ψ_{n+1}. The corresponding sets $\Xi_n = \{\xi_{n,1}, \dots, \xi_{n,p(n)}\}$ of of fundamental highest weights satisfy: $\xi_{n+1,j}$ is the unique element of Ξ_{n+1} whose $(\mathfrak{t}_n)_{\mathbb{C}}$–restriction is $\xi_{n,j}$, for $1 \leqq j \leqq p(n)$. Now the inverse limit $\Xi = \varprojlim \Xi_n$ consists of all "strings" $\xi_j = (\xi_{n,j}, \xi_{n+1,j}, \xi_{n+2,j}, \dots)$ where $\xi_{n,j}$ is first defined, i.e., where $p(n-1) < j \leqq p(n)$. Note that Ξ is countable.

Let λ be a non–negative integral linear combination of the ξ_j, say $\lambda = \sum_1^\infty \ell_j \xi_j$. Write $\kappa_{n,\lambda}$ for the irreducible representation of K_n with highest weight $\lambda_n := \sum_1^{p(n)} \ell_j \xi_{n,j}$ and $\mathcal{F}_{n,\lambda}$ for its representation space. Choose highest weight unit vectors $v_{n,\lambda} \in \mathcal{F}_{n,\lambda}$. Note that the Lie algebra inclusion $\mathfrak{k}_n \hookrightarrow \mathfrak{k}_{n+1}$ defines an inclusion $\mathcal{U}(\mathfrak{k}_n) \hookrightarrow \mathcal{U}(\mathfrak{k}_{n+1})$ of enveloping algebras. Now the map $\mathcal{A}(v_{n,\lambda}) \mapsto \mathcal{A}(v_{n+1,\lambda})$, $\mathcal{A} \in \mathcal{U}(\mathfrak{k}_n)$, defines a K_n–equivariant injection $\mathcal{F}_{n,\lambda} \hookrightarrow \mathcal{F}_{n+1,\lambda}$, and that injection is unitary because it sends the unit vector $v_{n,\lambda}$ to the unit vector $v_{n+1,\lambda}$. Thus we can simply regard $\mathcal{F}_{n,\lambda}$ as the subspace of $\mathcal{F}_{n+1,\lambda}$ generated by the highest weight unit vector $v_{n,\lambda} = v_{n+1,\lambda}$.

The matrix coefficients of $\kappa_{n,\lambda}$ are the $f_{x,y}(k) = \langle x, \kappa_{n,\lambda}(k)y\rangle$, and the Schur Orthogonality Relations say that their $L^2(K_n)$ inner product is $\langle f_{x,y}, f_{x',y'}\rangle = \deg(\kappa_{n,\lambda})^{-1}\langle x, x'\rangle\overline{\langle y, y'\rangle}$. Write $\mathcal{L}_{n,\lambda}$ for that space of matrix coefficients. For $m \geqq n$ let

$$\zeta''_{m,n} : f_{x,y} \text{ (function on } K_n) \mapsto (\deg(\kappa_{m,\lambda})/\deg(\kappa_{n,\lambda}))^{1/2} f_{x,y} \text{ (function on } K_m).$$

Then

$$\zeta''_{m,n} : \mathcal{L}_{n,\lambda} \to \mathcal{L}_{m,\lambda} \text{ is a } (K_n \times K_n)\text{–equivariant isometric map.}$$

The Peter–Weyl Theorem $L^2(K_n) = \sum_\lambda \mathcal{L}_{n,\lambda}$ assembles the $\mathcal{L}_{n,\lambda} \hookrightarrow \mathcal{L}_{n+1,\lambda}$ into a $(K_n \times K_n)$–equivariant isometric injection $\zeta''_{n,m} : L^2(K_n) \to L^2(K_m)$. We now come to the following result, corresponding to Theorem 3.7.

Theorem 4.3. (Peter–Weyl Theorem for parabolic direct limit groups.) *Let $\{K_n\}$ be a strict direct system of compact connected Lie groups. Suppose that $\{K_n\}$ is parabolic. Then there is a strict direct system $\{L^2(K_n), \zeta''_{m,n}\}$ of L^2 spaces, whose maps $\zeta''_{m,n} : L^2(K_n) \to L^2(K_m)$ are $(K_n \times K_n)$–equivariant unitary injections. Let Γ_n denote the left/right regular representation of $K_n \times K_n$ on $L^2(K_n)$ and let $K = \varinjlim K_n$. Then we have a well defined Hilbert space $L^2(K) := \varinjlim\{L^2(K_n), \zeta''_{m,n}\}$ and a natural unitary representation $\Gamma = \varinjlim \Gamma_n$ of $K \times K$ on $L^2(K)$.*

The space $L^2(K)$ is the Hilbert space orthogonal direct sum of its subspaces $\mathcal{L}_\lambda := \varinjlim \mathcal{L}_{n,\lambda}$ and the action of $K \times K$ on $\mathcal{L}_\lambda$ is irreducible with highest weight λ. In particular the left/right regular representation Γ is multiplicity–free.

Proof. As in the Heisenberg group setting, it remains only to prove that $\Gamma = \varinjlim \Gamma_n$ is limit–aligned, for then Lemma 3.4 and Theorem 2.2 complete the proof. Evidently $L^2(K)$ contains the mutually orthogonal spaces $\mathcal{L}_\lambda := \varinjlim \mathcal{L}_{n,\lambda}$. If $f \in L^2(K)$ is orthogonal to all of them, we interpret f as a function on K, and then $f|_{K_n} = 0$ for all n, so $f = 0$. Thus $L^2(K)$ is the (Hilbert space closure of the) orthogonal direct sum of the $\mathcal{L}_\lambda := \varinjlim \mathcal{L}_{n,\lambda}$. As λ determines the direct summand $\mathcal{L}_\lambda$, now Γ is limit–aligned. $\square$

Remark 4.4. As we noted in Remark 3.6, the adjoint to the injective map $\zeta''_{m,n} : L^2(K_n) \to L^2(K_m)$ is orthogonal projection to the image of that injection, and on each $(K_m \times K_m)$–irreducible summand of $L^2(K_m)$ it is a scalar multiple of restriction of functions. $\diamondsuit$

5. Commutative Spaces for Heisenberg Groups

In this section we put together the results of Sections 3 and 4 to study direct systems $\{(G_n, K_n)\}$ of Gelfand pairs where G_n is the semidirect product $H_n \rtimes K_n$

of a Heisenberg group H_n with a compact subgroup $K_n \subset \mathrm{Aut}(H_n)$. Then K_n is a closed subgroup of the maximal compact subgroup $U(n) \in \mathrm{Aut}(H_n)$. The concrete results in this section will require that K_n be connected and that its action on $\mathbb{C}^n$ be irreducible.

The classification goes as follows for the cases where K_n is connected and is irreducible on $\mathbb{C}^n$. Carcano's theorem ([**C**]; or see [**BJR**, Theorem 4.6] or [**W3**, Theorem 13.2.2]) says that (G_n, K_n) is a Gelfand pair if and only if the representation of $(K_n)_{\mathbb{C}}$, on polynomials on $\mathbb{C}^n$, is multiplicity free. Those groups were classified by Kač [**K**, Theorem 3] in another context. Benson, Jenkins and Ratcliff put it together for a classification of these "irreducible Heisenberg" Gelfand pairs (G_n, K_n). See [**BJR**, Theorem 4.6]. In Section 9 below we'll look at some cases where K_n need not be irreducible on $\mathbb{C}^n$.

Kač' list (as formulated in [**W3**, (13.2.5)]) is

(5.1)

Irreducible connected groups $K_n \subset U(n)$ multiplicity free on polynomials on $\mathbb{C}^n$				
	Group K_n	Group $(K_n)_{\mathbb{C}}$	Acting on	Conditions on n
1	$SU(n)$	$SL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$GL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \times Sp(m)$	$\mathbb{C}^* \times Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
5	$U(1) \times SO(n)$	$\mathbb{C}^* \times SO(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
6	$U(m)$	$GL(m;\mathbb{C})$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$
7	$SU(m)$	$SL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	m odd, $n = \frac{1}{2}m(m-1)$
8	$U(m)$	$GL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	$n = \frac{1}{2}m(m-1)$
9	$SU(\ell) \times SU(m)$	$SL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m,\ \ell \neq m$
10	$U(\ell) \times SU(m)$	$GL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m$
11	$U(2) \times Sp(m)$	$GL(2;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$	$n = 4m$
12	$SU(3) \times Sp(m)$	$SL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
13	$U(3) \times Sp(m)$	$GL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
14	$U(4) \times Sp(4)$	$GL(4;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^8$	$n = 32$
15	$SU(m) \times Sp(4)$	$SL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
16	$U(m) \times Sp(4)$	$GL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
17	$U(1) \times Spin(7)$	$\mathbb{C}^* \times Spin(7;\mathbb{C})$	$\mathbb{C}^8$	$n = 8$
18	$U(1) \times Spin(9)$	$\mathbb{C}^* \times Spin(9;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
19	$Spin(10)$	$Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
20	$U(1) \times Spin(10)$	$\mathbb{C}^* \times Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
21	$U(1) \times G_2$	$\mathbb{C}^* \times G_{2,\mathbb{C}}$	$\mathbb{C}^7$	$n = 7$
22	$U(1) \times E_6$	$\mathbb{C}^* \times E_{6,\mathbb{C}}$	$\mathbb{C}^{27}$	$n = 27$

Now we have the direct systems

(5.2a)

Direct systems $\{(H_n \rtimes K_n, K_n)\}$ of Gelfand pairs, K_n connected and irreducible on $\mathbb{C}^n$			
	Group K_n	Acting on	Conditions on n
1	$SU(n)$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \times Sp(m)$	$\mathbb{C}^n$	$n = 2m$

.... table continued on next page

.... table continued from previous page

(5.2b)

5a	$U(1) \times SO(2m)$	$\mathbb{C}^{2m}$	$n = 2m \geqq 2$
5b	$U(1) \times SO(2m+1)$	$\mathbb{C}^{2m+1}$	$n = 2m+1 \geqq 3$
6	$U(m)$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$
7	$SU(m)$	$\Lambda^2(\mathbb{C}^m)$	m odd, $n = \frac{1}{2}m(m-1)$
8	$U(m)$	$\Lambda^2(\mathbb{C}^m)$	$n = \frac{1}{2}m(m-1)$
9	$SU(\ell) \times SU(m)$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m,\ \ell \neq m$
10	$S(U(\ell) \times U(m))$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m$
11	$U(2) \times Sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$	$n = 4m$
12	$SU(3) \times Sp(m)$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
13	$U(3) \times Sp(m)$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
15	$SU(m) \times Sp(4)$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
16	$U(m) \times Sp(4)$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$

In each case the direct system $\{K_n\}$ is both strict and parabolic. (We separated entry 5 of Table 5.1 into entries 5a and 5b of Table 5.2a,b in order to have the parabolic property.)

We now suppose that $\{K_n\}$ is one of the strict parabolic direct system, for example one of the 16 systems given by the rows of Table 5.2a,b. We retain the notation of Section 4 for the representations, highest weights, unitary inclusions, etc., associated to $\{K_n\}$.

As $U(n)$ acts on $H_n = \mathrm{Im}\,\mathbb{C} + \mathbb{C}^n$ by $k : (z,v) \mapsto (z, kv)$ it preserves the equivalence class of each of the square integrable representations $\pi_{n,t}$ of H_n. The Mackey obstruction vanishes and $\pi_{n,t}$ extends to a unitary representation $\widetilde{\pi_{n,t}}$ of $H_n \rtimes U(n)$ on $\mathcal{H}_{n,t}$. See [**W1**, Section 4] for a geometric proof. We will also write $\widetilde{\pi_{n,t}}$ for its restriction, the extension of $\pi_{n,t}$ to a unitary representation of $G_n = H_n \rtimes K_n$.

Denote $\pi_{n,t,\lambda} = \widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,\lambda}}$. Then $\mathcal{H}_{n,t,\lambda} := \mathcal{H}_{n,t} \otimes \mathcal{F}_{n,\lambda}$ denotes its representation space. Fix an orthonormal basis $\{u_i\}$ of $\mathcal{F}_{n,\lambda}$. Then $\{w[\mathbf{m}] \otimes u_i\}$ is a complete orthonormal set in $\mathcal{H}_{n,t,\lambda}$. Denote the matrix coefficients by

$$f_{\mathbf{l},\mathbf{m},i,j;t}(h,k) = \langle (w[\mathbf{l}] \otimes u_i), ((\widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,\lambda}})(h,k))(w[\mathbf{m}] \otimes u_j)\rangle.$$

The formal degree $\deg \pi_{n,t,\lambda} = |t|^n \deg(\kappa_{n,\lambda})$, so the $|t|^{n/2} \deg(\kappa_{n,\lambda})^{1/2} f_{\mathbf{l},\mathbf{m},i,j;t}$ form a complete orthonormal set in the space $\mathcal{E}_{n,t,\lambda} = \mathcal{H}_{n,t,\lambda} \widehat{\otimes} \mathcal{H}^*_{n,t,\lambda}$ of matrix coefficient functions. Given a coefficient set $\varphi = (\varphi_{\mathbf{l},\mathbf{m},i,j}(t,\lambda))$ we have the functions $\Phi_{n,t,\lambda,\varphi}$ on G_n given by

$$\Phi_{n,t,\lambda,\varphi}(hk) = \sum_{\mathbf{l},\mathbf{m},i,j} \varphi_{\mathbf{l},\mathbf{m},i,j}(t,\lambda) |t|^{n/2} \deg(\kappa_{n,\lambda})^{1/2} f_{\mathbf{l},\mathbf{m},i,j;t}(h,k) \tag{5.3}$$

for $h \in H_n$ and $k \in K_n$. Here $||\Phi_{n,t,\lambda,\varphi}||^2_{\mathcal{E}_{n,t,\lambda}} = \sum_{\mathbf{l},\mathbf{m},i,j} |\varphi_{\mathbf{l},\mathbf{m},i,j}(t,\lambda)|^2$. We sum the $\Phi_{n,t,\lambda,\varphi}$ to form L^2 functions $\Psi_{n,\varphi}$ on $H_n \rtimes K_n$, given by

$$\Psi_{n,\varphi}(hk) = \sum_{\kappa_{n,\lambda} \in \widehat{K_n}} \deg \kappa_{r,\lambda} \left(\int_{-\infty}^{\infty} \Phi_{n,t,\lambda,\varphi}(h,k) |t|^n dt \right). \tag{5.4}$$

As before, for $m \geqq n$ we have a $(G_n \times G_n)$–equivariant isometric injection of $L^2(G_n)$ into $L^2(G_m)$ given by

$$\zeta_{m,n}(\Psi_{n,\varphi}) = \Psi_{m,|t|^{(n-m)/2}(\deg \kappa_{n,\lambda}/\deg \kappa_{m,\lambda})^{1/2}\varphi} \quad . \tag{5.5}$$

This gives us

Theorem 5.6. *For $n > 0$ let K_n be a compact connected subgroup of* $\mathrm{Aut}(H_n)$ *such that $\{K_n\}$ is a strict parabolic direct system. Define $G_n = H_n \rtimes K_n$. Then there is a strict direct system $\{L^2(G_n), \zeta_{r,n}\}$ of L^2 spaces, in which the maps $\zeta_{m,n} : L^2(G_n) \to L^2(G_m)$ are $(G_n \times G_n)$–equivariant unitary injections. Let Π_n denote the left/right regular representation of $G_n \times G_n$ on $L^2(G_n)$. Note that $G := \varinjlim(G_n) = H \rtimes K$ where $H = \varinjlim H_n$ and $K = \varinjlim K_n$. Thus we have a well defined Hilbert space $L^2(G) := \varinjlim\{L^2(G_n), \zeta_{m,n}\}$ and a natural unitary representation $\Pi = \varinjlim \Pi_n$ of $G \times G$ on $L^2(G)$. Further, Π is the multiplicity–free direct integral of the irreducible representations $\pi_{t,\lambda} := \varinjlim \pi_{n,t,\lambda}$.*

Now suppose that $\{K_n\}$ is in fact one of the 16 systems of Table 5.2a,b. We will use its specific properties in order to pass from the left/right representation of $G_n \times G_n$ on $L^2(G_n)$ to the left regular representation of G_n on $L^2(G_n/K_n)$.

Define $G_n = H_n \rtimes K_n$. Since (G_n, K_n) is a Gelfand pair with K_n irreducible on $\mathbb{C}^n$, Carcano's Theorem [**C**] says that the action of K_n on the polynomial ring $\mathbb{C}[C^n]$ is multiplicity free, and it picks out the right K_n–invariants in $L^2(G_n)$, as follows.

Lemma 5.7. *Recall the notation of* Section 4 *for Ξ, λ and $\kappa_{n,\lambda} \in \widehat{K_n}$. Define $\widetilde{\kappa_{n,\lambda}} \in \widehat{G_n}$ by $\widetilde{\kappa_{n,\lambda}}(hk) = \kappa_{n,\lambda}(k)$ for $h \in H_n$ and $k \in K_n$. Then $\pi_{n,t,\lambda} := \widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,\lambda}}$ has a nonzero K_n–fixed vector if and only if $\kappa^*_{n,\lambda}$ occurs as a subrepresentation of $\widetilde{\pi_{n,t}}|_{K_n}$, and in that case the space of K_n–fixed vectors has dimension* 1.

Proof. This is essentially the argument in [**W1**, Section 14.5A]. Decompose $\widetilde{\pi_{n,t}}|_{K_n} = \sum_{\gamma \in \widehat{K_n}} m_\gamma\, \gamma$. Carcano's Theorem ([**C**], or see [**W3**, Theorem 13.2.2]) says that each m_γ is either 0 or 1. The K_n–fixed vectors of $\widetilde{\kappa} \otimes \widetilde{\pi_{n,t}}$ all occur in $\kappa \otimes (m_{\kappa^*}\kappa^*)$, and they form a space of dimension m_{κ^*}. The assertion follows. $\square$

Since K_n is compact, we can view $L^2(G_n/K_n)$ as the space of right–K_n–invariant functions in $L^2(G_n)$. With Lemma 5.7 in mind we set

$$\widehat{K_n}^{\dagger} = \{\kappa_{n,\lambda} \in \widehat{K_n} \mid \kappa^*_{n,\lambda} \text{ occurs in the space of polynomials on } \mathbb{C}^n\}.$$

Given $\kappa_{n,\lambda} \in \widehat{K_n}^{\dagger}$ the right K_n–invariant in $\mathbb{C}[\mathbb{C}^n] \otimes \mathcal{F}^*_{n,\lambda}$ is the sum over a basis of the $\kappa_{n,\lambda}$–subspace of $\mathbb{C}[\mathbb{C}^n]$ times the dual basis of $\mathcal{F}^*_{n,\lambda}$. Normalize it to a unit vector $u_{n,t,\lambda}$. Then the (left regular) representation of G_n on $L^2(G_n/K_n)$ is $\sum_{\kappa_{n,\lambda} \in \widehat{K_n}^\dagger} \int_{-\infty}^{\infty} \widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,\lambda}}$. In this decomposition the representation space is $\sum_{\kappa_{n,\lambda} \in \widehat{K_n}^\dagger} \int_{-\infty}^{\infty} (\mathcal{H}_{n,t,\lambda} \otimes u_{n,t,\lambda}\mathbb{C})\, dt$.

Proposition 5.8. *If $m \geqq n$ and $\kappa_{n,\lambda} \in \widehat{K_n}^{\dagger}$ then $\kappa_{m,\lambda} \in \widehat{K_m}^{\dagger}$, and $\kappa_{n,\lambda}$ and $\kappa_{m,\lambda}$ have the same highest weight λ space.*

Proof. The group K_n acts on $\mathbb{C}^n$ by some representation γ_n, so the representation of K_n on polynomials of degree d is the symmetric power $S^d(\gamma_n^*)$. Thus we can compute the set $X_{n,d}$ of highest weights of K_n on the space $P_{n,d}$ of polynomials of degree d on $\mathbb{C}^n$. Running through the 16 cases of Table 5.2a,b we see that $X_{n,d} \subset X_{m,d}$. For example (Line 3 of Table 5.2a,b) the representation of $Sp(m)$ on polynomials of degree d in C^{2m} is the irreducible representation with highest weight $\xi_{2m,1}$, and (Lines 5a and 5b of Table 5.2a,b) the representation of $U(1) \times SO(n)$ on polynomials of degree d in C^n is the $-d^{th}$ power of the usual representation of $U(1)$ by scalars on $\mathbb{C}^n$ with the multiplicity–free sum of the irreducibles of highest weights $\xi_{n,1}$, $3\xi_{n,1}$, $5\xi_{n,1}$, ... , $d\xi_{n,1}$ of $SO(n)$ if d is odd, or with the the multiplicity–free sum of the irreducible representations of highest weights $0\xi_{n,1}$, $2\xi_{n,1}$, $4\xi_{n,1}$, ... , $d\xi_{n,1}$ of $SO(n)$ in case d is even.

Now let $\lambda \in X_{n,d}$. Let $v_{n,\lambda}$ denote a (nonzero) highest weight λ vector for $\mathfrak{k}_n$ in $P_{n,d}$, and similarly let $v_{m,\lambda}$ denote a (nonzero) highest weight λ vector for $\mathfrak{k}_m$. Divide up the variables of $\mathbb{C}^m$ to $\{w_1, \ldots, w_n\}$ for $\mathbb{C}^n$ and $\{z_{n+1}, \ldots, z_m\}$ for its complement in $\mathbb{C}^m$. Express $v_{m,\lambda} = \sum_{A,B} b_{A,B} w^A z^B$ where each term has total degree $|A| + |B| = d$. Note that K_n treats the z_i as constants. Evaluating the z_i at arbitrary constant values $C = (c_{n+1}, \ldots, c_m)$ we have a highest weight λ vector for $\mathfrak{k}_n$. By Carcano's Theorem it is a multiple of $v_{n,\lambda}$. In other words $v_{m,\lambda}|_{\{z=C\}} = m_{_C} v_{n,\lambda}$. The terms $b_{A,B} w^A z^B$ with z–degree $|B| > 0$ yield evaluations of w–degree $|A| < d$, and cannot contribute to any $m_{_C} v_{n,\lambda}$. Now $b_{A,B} w^A z^B = 0$ whenever $|B| > 0$. This shows that $v_{m,\lambda}$ is a homogeneous polynomial of degree d in the w_j, as is $v_{n,\lambda}$. We conclude that $v_{m,\lambda}$ is a nonzero multiple of $v_{n,\lambda}$. □

Corollary 5.9. *Let $m \geqq n$. Then every K_n–invariant vector in $\mathcal{H}_{n,t,\lambda}$ is the image of a K_m–invariant vector in $\mathcal{H}_{m,t,\lambda}$ under the adjoint of the unitary map $\mathcal{H}_{n,t,\lambda} \hookrightarrow \mathcal{H}_{m,t,\lambda}$.*

Proof. Retain the notation $X_{n,d}$ for those λ such that $\tau_{n,\lambda}$ occurs on the space $P_{n,d}$ of polynomials of degree d on $\mathbb{C}^n$. If $\lambda \notin X_{n,d}$ there are no nonzero K_n–invariant vectors in $\mathcal{H}_{n,t,\lambda}$, so the assertion is vacuous. Now assume $\lambda \in X_{n,d}$ and choose an orthonormal basis $\{x_1, \ldots, x_{q(n)}\}$ of the representation space for $\tau_{n,\lambda}$ in $P_{n,d}$. According to Proposition 5.8 that representation space is contained in the representation space for $\tau_{m,\lambda}$ in $P_{m,d}$, so the latter has an orthonormal basis $\{x_1, \ldots, x_{q(n)}, x_{q(n)+1}, \ldots, x_{q(m)}\}$. Let $\{x_1^*, \ldots, x_{q(n)}^*, x_{q(n)+1}^*, \ldots, x_{q(m)}^*\}$ and $\{x_1^*, \ldots, x_{q(n)}^*\}$ be the corresponding dual bases of $\mathcal{L}_{m,\lambda}$ and $\mathcal{L}_{n,\lambda}$. the K_n–invariant vectors in $\mathcal{H}_{n,t,\lambda}$ are the multiples of $\sum_1^{q(n)} x_i \otimes x_i^*$, and the K_m–invariant vectors in $\mathcal{H}_{m,t,\lambda}$ are the multiples of $\sum_1^{q(m)} x_i \otimes x_i^*$. The adjoint of unitary inclusion is orthogonal projection, which sends $\sum_1^{q(m)} x_i \otimes x_i^*$ to $\sum_1^{q(n)} x_i \otimes x_i^*$. □

Combining Theorem 5.6, Lemma 5.7 and Corollary 5.9 we have

Theorem 5.10. *Let $\{(H_n \rtimes K_n, K_n)\}$ be one of the* 16 *direct systems of* Table 5.2a,b. *We denote $G_n = H_n \rtimes K_n$, $G = \varinjlim G_n$ and $K = \varinjlim K_n$. Then the unitary direct system $\{L^2(G_n), \zeta_{m,n}\}$ of* Theorem 5.6 *restricts to a unitary direct system $\{L^2(G_n/K_n), \zeta_{m,n}\}$, the Hilbert space $L^2(G/K) := \varinjlim\{L^2(G_n/K_n), \zeta_{m,n}\}$ is the subspace of $L^2(G) := \varinjlim L^2(G_n)$ of right–K–invariant functions, and the natural unitary representation of G on $L^2(G/K)$ is a multiplicity free direct integral of lim–irreducible representations.*

Remark 5.11. As in Remarks 3.6 and 4.4, whenever $m \geqq n$ the adjoint of the direct system map $\zeta_{m,n} : L^2(G_n/K_n) \to L^2(G_m/K_m)$ is orthogonal projection to the image subspace, and on each $(G_m \times G_m)$–irreducible direct integrand it is a scalar multiple of restriction of functions. The scalar is given by the formal degree, so it depends on the integrand. $\diamondsuit$

6. Extension to Certain Classes of Nilpotent Groups

Theorem 3.7 depends on four basic facts. First, the $\pi_{n,t}$ are determined by their central character. Second, we have good models $\mathcal{H}_{n,t}$ for the representation spaces, such that n does not appear explicitly in the formulae for the actions of the group elements. Third, the injections $H_n \hookrightarrow H_m$ restrict to isomorphisms $Z_n \cong Z_m$ of the centers. And fourth, we have complete information on the Plancherel measure for the H_n. In this section we consider a somewhat larger class of nilpotent direct systems that satisfy these conditions.

We will need the theory of square integrable representations of nilpotent Lie groups ([**MW**], or see [**W3**, Section 14.2] for a short exposition). Let N be a connected, simply connected nilpotent Lie group and $\mathfrak{n}$ its Lie algebra. Decompose $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ and $N = Z\exp(\mathfrak{v})$ where $\mathfrak{z}$ is the center of $\mathfrak{n}$. Then $Z = \exp(\mathfrak{z})$ is the center of N. We say that an irreducible unitary representation π of N is *square integrable* if its coefficient functions $f_{u,v}(g) = \langle u, \pi(g)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. In that case π is determined by its central character, $\pi = \pi_t$ where $t \in \mathfrak{z}^*$ and the central character is $\exp(\zeta) \mapsto e^{it(\zeta)}$. In terms of geometric quantization, π_t corresponds to the coadjoint orbit in $\mathfrak{n}^*$ consisting of all linear functionals on $\mathfrak{n}$ whose restriction to $\mathfrak{z}$ is t. Further, the antisymmetric bilinear form $b_t(\xi,\eta) = t([\xi,\eta])$ on $\mathfrak{v}$ is nondegenerate, and (up to a positive constant that depends only on normalizations of Haar measures) the formal degree of π_t is $|\mathrm{Pf}(b_t)|$, where $\mathrm{Pf}(b_t)$ is the Pfaffian[1] of $b_t : \mathfrak{v} \times \mathfrak{v} \to \mathbb{R}$. In fact, if π_s is the representation of N that corresponds to $\mathrm{Ad}^*(N)s \subset \mathfrak{n}^*$, then π_s is square integrable if and only if $\mathrm{Pf}(b_{s|_{\mathfrak{z}^*}}) \neq 0$. In any case, $\mathrm{Pf}(b_t)$ is a polynomial function of t, and (again up to a constant that depends on normalizations) $|\mathrm{Pf}(b_t)|$ is the Plancherel density. It follows that if one irreducible unitary representation of N is square integrable then Plancherel–almost–all are. In the case of the Heisenberg group H_n, where we identified $\mathfrak{z}^*$ with $\mathbb{R}$, the Pfaffian corresponding to $\pi_{n,t}$ is t^n.

The point of this, from the viewpoint of commutative spaces, is that many Gelfand pairs are of the form $(N \rtimes K, K)$ where N is a connected simply connected Lie group, K is a compact subgroup of $\mathrm{Aut}(N)$, and N has square integrable representations. See [**W3**, Theorem 14.4.3]. In quite a few cases the groups N of [**W3**, Theorem 14.4.3] fall naturally into direct systems for which we can apply the techniques of Section 3. This is simplified by the 2–step Nilpotent Theorem [**W3**, Theorem 13.1.1] of Benson-Jenkins-Ratcliff and Vinberg, which says that N must be abelian or 2–step nilpotent. In a certain sense representations treat those groups as Heisenberg groups:

Lemma 6.1. ([**W3**, Lemma 14.4.1]) *Let N be a connected simply connected 2–step nilpotent Lie group with 1–dimensional center. Then N is isomorphic to the Heisenberg group H_n where $n = \frac{1}{2}(\dim_{\mathbb{R}} \mathfrak{n} - 1)$, and in particular N has square integrable representations.*

[1]Strictly speaking, $\mathrm{Pf}(b_t)$ depends on a choice of basis of $\mathfrak{v}$, for a basis change of determinant a_t multiplies $\det b_t|_{\mathfrak{v}\times\mathfrak{v}}$ by $\det a_t^2$ and multiplies $\mathrm{Pf}(b_t)$ by $\det a_t$.

PROPOSITION 6.2. ([**W3**, Proposition 14.4.2]) *Let N be a connected simply connected 2–step nilpotent Lie group. Let $f \in \mathfrak{n}^*$ such that $f|_{\mathfrak{z}} \neq 0$. Denote $\mathfrak{w}_f = \{z \in \mathfrak{z} \mid f(z) = 0\}$ and $W_f := \exp(\mathfrak{w}_f)$. Then*

1. *W_f is a closed subgroup of Z, hence a closed normal subgroup of N.*
2. *The functional f is the pullback of a linear functional $f' \in (\mathfrak{n}/\mathfrak{w}_f)^*$ and is nonzero on the central subalgebra $\mathfrak{z}/\mathfrak{w}_f$ of $\mathfrak{n}/\mathfrak{w}_f$.*
3. *The representation $[\pi_f]$ is the pullback to N of the class $[\pi_{f'}] \in \widehat{N/W_f}$.*
4. *If the representation $[\pi_f]$ is square integrable then $[\pi_{f'}]$ is square integrable, and in that case N/W_f has center Z/W_f and is isomorphic to a Heisenberg group H_n where $n = \frac{1}{2}\dim(\mathfrak{n}/\mathfrak{z})$.*

We now consider a strict direct system $\{N_n\}$ of 2–step nilpotent connected, simply connected Lie groups that have square integrable representations, where the inclusions $\mathfrak{n}_n \to \mathfrak{n}_m$ map the center $\mathfrak{z}_n \hookrightarrow \mathfrak{z}_m$ and the complement $\mathfrak{v}_n \hookrightarrow \mathfrak{v}_m$ in decompositions $\mathfrak{n}_n = \mathfrak{z}_n + \mathfrak{v}_n$. Then the direct limit algebra $\mathfrak{n} := \varinjlim \mathfrak{n}_n$ has center $\mathfrak{z} := \varinjlim \mathfrak{z}_n$ and $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ where $\mathfrak{v} = \varinjlim \mathfrak{v}_n$. On the group level, $Z = \varinjlim Z_n = \exp(\mathfrak{z})$ is the center of $N := \varinjlim N_n$ and we have $N = Z\exp(\mathfrak{v})$.

We further assume that the dimensions $\dim Z_n$ of the centers are bounded. Since they are non–decreasing we may assume that they are eventually constant. Passing to a cofinal sequence,

$$\mathfrak{n}_n \hookrightarrow \mathfrak{n}_m \text{ maps } \mathfrak{z}_n \cong \mathfrak{z}_m. \tag{6.3}$$

Under that identification we write $\mathfrak{z}$ for all the $\mathfrak{z}_n$, $\mathfrak{z}^*$ for all the $\mathfrak{z}_n^*$, and Z for all the Z_n.

Let $t \in \mathfrak{z}^*$. Write $b_{n,t}$ for the bilinear form $(\xi, \eta) \mapsto t([\xi, \eta])$ on $\mathfrak{v}_n$. Then t corresponds to a square integrable representation $\pi_{n,t}$ of N_n just when the Pfaffian $\mathrm{Pf}(b_{n,t}) \neq 0$. For purposes of comparing the Pfaffians as n varies, we note that $\mathrm{Pf}(b_{n,t})$ is specified by t and a basis of $\mathfrak{v}_n$, so we simply assume that these bases are nested in the sense that the basis of $\mathfrak{v}_{n+1}$ consists the basis of $\mathfrak{v}_n$ together with some elements that are $b_{n+1,t}$–orthogonal to $\mathfrak{v}_n$. Thus, if $\mathrm{Pf}(b_{n+1,t}) \neq 0$ then $\mathrm{Pf}(b_{n,t}) \neq 0$. The converse fails in general, but the following lemma deals with the possibility that $\mathrm{Pf}(b_{n,t}) \neq 0 = \mathrm{Pf}(b_{m,t})$. It depends on the fact [**MW**] that each $\mathrm{Pf}(b_{n,t})$ is a polynomial function on $\mathfrak{z}^*$.

LEMMA 6.4. *Let $\mathfrak{a}_n \in \mathfrak{z}^*$ denote the zero set of* $\mathrm{Pf}(b_{n,t})$ *and set $\mathfrak{a} = \bigcup \mathfrak{a}_n$. Then $\mathfrak{a}$ is a set of Lebesgue measure zero in $\mathfrak{z}^*$.*

Proof. Since N_n has square integrable representations, the Pfaffian $\mathrm{Pf}(b_{n,t})$ is a nontrivial polynomial function of $t \in \mathfrak{z}^*$, so $\mathfrak{a}_n$ is a finite union of lower–dimensional subvarieties of $\mathfrak{z}_n^*$. Now the set $\mathfrak{a}$ is a countable union of sets $\mathfrak{a}_n$ of Lebesgue measure zero. □

For convenience we define

$$T = \{t \in \mathfrak{z}^* \mid \text{ each } \mathrm{Pf}(b_{n,t}) \neq 0\} = \mathfrak{z}^* \setminus \mathfrak{a}. \tag{6.5}$$

By construction, for every $t \in T$ and every index n we have a square integrable representation $\pi_{n,t} \in \widehat{N_n}$.

Fix $t \in T$. Then we have the hyperplane $\mathfrak{w}_t$ in $\mathfrak{z}$, and $W_t := \exp(\mathfrak{w}_t)$ is a closed subgroup of Z. Lemma 6.1 and Proposition 6.2 tell us that each quotient N_n/W_t is isomorphic to a Heisenberg group $H_{d(n)}$ and that $\pi_{n,t}$ factors through to the square integrable representation of N_n/W_t with central character e^{it}. Now the various (as t varies in T) $\pi_{n,t}$ act on the same Fock space $\mathcal{H}_{d(n),t}$, $d(n) = \frac{1}{2}\dim \mathfrak{v}_n$, by formulae independent of $d(n)$.

We normalize the inner products on the $\mathcal{H}_{d(n),t}$ as before, so the $w[\mathbf{m}]$ form a complete orthonormal set, and realize the space $\mathcal{E}_{n,t} = \mathcal{H}_{d(n),t}\widehat{\otimes}\mathcal{H}^*_{d(n),t}$ of matrix coefficients as the closed span of the functions $f_{\mathbf{l},\mathbf{m},t} : g \mapsto \langle w[\mathbf{l}], \pi_{n,t}(g)w[\mathbf{m}]\rangle$, as in Section 3. The orthogonality relations say that the inner product on $\mathcal{E}_{n,t}$ is given by $\langle f_{\mathbf{l},\mathbf{m},t}, f_{\mathbf{l}',\mathbf{m}',t}\rangle = |\mathrm{Pf}(b_{n,t})|^{-1}$ if $\mathbf{l} = \mathbf{l}'$ and $\mathbf{m} = \mathbf{m}'$, and is 0 otherwise. Now the $|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m},t}$ form a complete orthonormal set in $\mathcal{E}_{n,t}$, and as before $\mathcal{E}_{n,t}$ consists of the functions $\Phi_{n,t,\varphi}$ on $H_{d(n)}$ given by

$$\Phi_{n,t,\varphi}(h) = \sum_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m};t}(h) \text{ with } \sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 < \infty. \tag{6.6}$$

Now $L^2(N_n)$ is the direct integral $\int_{\mathfrak{z}_n^*} \mathcal{E}_{n,t}|\mathrm{Pf}(b_{n,t})|dt = \int_T \mathcal{E}_{n,t}|\mathrm{Pf}(b_{n,t})|dt$. It consists of all functions $\Psi_{n,\varphi}$ defined by

$$\begin{aligned} \Psi_{n,\varphi}(h) &= \int_{\mathfrak{z}_n^*} \Phi_{n,t,\varphi}(h)|\mathrm{Pf}(b_{n,t})|dt \\ &= \int_T \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m},t}\right) |\mathrm{Pf}(b_{n,t})|dt \end{aligned} \tag{6.7}$$

such that the functions $\varphi_{\mathbf{l},\mathbf{m}} : \mathfrak{z}_n^* \to \mathbb{C}$ are measurable with $\sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 < \infty$ for almost all $t \in T$ and $\sum_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2 \in L^1(\mathfrak{z}_n^*, |\mathrm{Pf}(b_{n,t})|dt)$. The norms are

$$\begin{aligned} ||\Psi_{n,\varphi}||^2_{L^2(N_n)} &= \int_T ||\Phi_{n,t,\varphi}||^2_{\mathcal{E}_{n,t}} |\mathrm{Pf}(b_{n,t})|dt \\ &= \int_T \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2\right) |\mathrm{Pf}(b_{n,t})|dt \\ &= \sum\nolimits_{\mathbf{l},\mathbf{m}} ||\varphi_{\mathbf{l},\mathbf{m}}||^2_{L^2(\mathfrak{z}^*, |\mathrm{Pf}(b_{n,t})|dt)} \quad . \end{aligned} \tag{6.8}$$

As in the Heisenberg group case, the left/right representation of $N_n \times N_n$ on $\mathcal{E}_{n,t}$ is the exterior tensor product $\pi_{n,t} \boxtimes \pi^*_{n,t}$; it is irreducible and the left/right regular representation of $N_n \times N_n$ on $L^2(N_n)$ is $\Pi_n = \int_{\mathfrak{z}^*} (\pi_{n,t} \boxtimes \pi^*_{n,t})|\mathrm{Pf}(b_{n,t})|dt$. The argument of Lemma 3.4 goes through without change, proving

LEMMA 6.9. *The left/right regular representation Π_n of $N_n \times N_n$ on $L^2(N_n)$ is a multiplicity free direct integral of the irreducible unitary representations $\pi_{n,t}$.*

REMARK 6.10. From the considerations just described, one sees that Lemma 6.9 holds for every 2–step nilpotent Lie group that has square integrable representations.

We now continue the argument as in the Heisenberg group case. Suppose that the index $m \geqq n$. Then $|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m},t} \mapsto |\mathrm{Pf}(b_{m,t})|^{1/2} f_{\mathbf{l},\mathbf{m},t}$ defines an isometric injection $\Phi_{n,\varphi}(t) \mapsto \Phi_{m,\varphi}(t)$ of $\mathcal{E}_{n,t}$ into $\mathcal{E}_{m,t}$. The norm computation just above gives

$$
\begin{aligned}
&||\Psi_{m,|\mathrm{Pf}(b_{n,t})/\mathrm{Pf}(b_{m,t})|^{1/2}\varphi}||^2_{L^2(N_m)} \\
(6.11)\qquad &= \int_T \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} |(\mathrm{Pf}(b_{n,t})/\mathrm{Pf}(b_{m,t})|\ |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2\right) |\mathrm{Pf}(b_{m,t})|dt \\
&= \int_T \left(\sum\nolimits_{\mathbf{l},\mathbf{m}} |\varphi_{\mathbf{l},\mathbf{m}}(t)|^2\right) |\mathrm{Pf}(b_{n,t})|dt = ||\Psi_{n,\varphi}||^2_{L^2(N_n)}.
\end{aligned}
$$

Thus we have an $(N_n \times N_n)$–equivariant isometric injection

$$(6.12)\qquad \zeta'_{m,n} : L^2(N_n) \to L^2(N_m) \text{ by } \zeta'_{m,n}(\Psi_{n,\varphi}) = \Psi_{m,|\mathrm{Pf}(b_{n,t})/\mathrm{Pf}(b_{m,t})|^{1/2}\varphi}.$$

On the level of coefficients it is given by $\zeta'_{m,n}(\Phi_{n,t,\varphi}) = \Phi_{m,t,|\mathrm{Pf}(b_{n,t})/\mathrm{Pf}(b_{m,t})|^{1/2}\varphi}$. In other words $\zeta'_{m,n}$ sends the function $\sum_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m};t}$ on N_n to the function on N_m given by

$$
\begin{aligned}
&\sum_{\mathbf{l},\mathbf{m}} (|\mathrm{Pf}(b_{n,t})/\mathrm{Pf}(b_{m,t})|^{1/2}\varphi_{\mathbf{l},\mathbf{m}}(t))(|\mathrm{Pf}(b_{m,t})|^{1/2} f_{\mathbf{l},\mathbf{m};t}) \\
&\qquad\qquad\qquad = \sum_{\mathbf{l},\mathbf{m}} \varphi_{\mathbf{l},\mathbf{m}}(t)|\mathrm{Pf}(b_{n,t})|^{1/2} f_{\mathbf{l},\mathbf{m};t}.
\end{aligned}
$$

As in the Heisenberg case we now have

THEOREM 6.13. *There is a strict direct system* $\{L^2(N_n), \zeta'_{m,n}\}$ *of* L^2 *spaces. The direct system maps* $\zeta'_{m,n} : L^2(N_n) \to L^2(N_m)$ *are* $(N_n \times N_n)$*–equivariant unitary injections. Let* Π_n *denote the left/right regular representation of* $N_n \times N_n$ *on* $L^2(N_n)$ *and let* $N = \varinjlim N_n$*. Then we have a well defined Hilbert space* $L^2(N) := \varinjlim\{L^2(N_n), \zeta'_{m,n}\}$ *and a natural unitary representation* $\Pi = \varinjlim \Pi_n$ *of* $N \times N$ *on* $L^2(N)$*. Further, that representation* Π *is multiplicity–free.*

REMARK 6.14. As in Remark 3.6, the adjoint of $\zeta'_{m,n} : L^2(N_n) \to L^2(N_m)$ is orthogonal projection, and on each $(N_m \times N_m)$–irreducible direct integrand of $L^2(N_m)$ it is a scalar multiple of restriction of functions. $\diamondsuit$

7. Structural Preliminaries

In this section and the next we work out some structural results for a strict direct system $\{K_n, \varphi_{m,n}\}$ of compact connected Lie groups and a consistent family $\{\gamma_n\}$ of representations of the K_n on a fixed finite dimensional vector space $\mathfrak{z}$. In Section 8 we will use that information to extend Theorems 5.6 and 5.10 to a larger family of strict direct systems of nilmanifold Gelfand pairs.

As just indicated, $\{K_n, \varphi_{m,n}\}$ is a strict direct system of compact connected Lie groups. Denote $K = \varinjlim\{K_n, \varphi_{m,n}\}$. Its Lie algebra is $\mathfrak{k} = \varinjlim\{\mathfrak{k}_n, d\varphi_{m,n}\}$. The $\gamma_n : K_n \to U(\mathfrak{z})$ are unitary representations of the K_n on the finite dimensional vector space $\mathfrak{z}$. They are consistent in the sense that $\gamma_m \cdot \varphi_{m,n} = \gamma_n$, so they define the direct limit representation $\gamma = \varinjlim \gamma_n$ of K on $\mathfrak{z}$.

Let $U_n = \gamma_n(K_n)$. The U_n form an increasing sequence of compact connected subgroups of dimension $\leqq (\dim \mathfrak{z})^2$ in the unitary group $U(\mathfrak{z})$, so from some index on they are all the same compact connected subgroup U of $U(\mathfrak{z})$. Truncating the index set we may assume that every $\gamma_n(K_n) = U$, in particular that, in the limit, $U = \gamma(K)$. Let $K_n^\dagger$ denote the identity component of the kernel of γ_n. Then $\varphi_{m,n}(K_n^\dagger) \subset K_m^\dagger$, so we have $K^\dagger = \varinjlim\{K_n^\dagger, \varphi_{m,n}|_{K_n^\dagger}\}$, and $K^\dagger$ is the identity component of the kernel of γ. In particular its Lie algebra $\mathfrak{k}^\dagger = \varinjlim\{\mathfrak{k}_n^\dagger, d\varphi_{m,n}|_{\mathfrak{k}_n^\dagger}\}$ is the kernel of $d\gamma : \mathfrak{k} \to \mathfrak{u}$.

Since K_n is compact and connected, and $K_n^\dagger$ is a closed connected normal subgroup, K_n has another closed connected normal subgroup L_n such that K_n is locally isomorphic to the direct product $K_n^\dagger \times L_n$. On the Lie algebra level, $\mathfrak{k}_n = \mathfrak{k}_n^\dagger \oplus \mathfrak{l}_n$, direct sum of ideals. The semisimple part, $\mathfrak{l}_n' = [\mathfrak{l}_n, \mathfrak{l}_n]$, is the direct sum of all the simple ideals of $\mathfrak{k}_n$ that are not contained in $\mathfrak{k}_n^\dagger$. Thus $\mathfrak{l}_n'$ is independent of the choice of L_n, and $d\varphi_{m,n}(\mathfrak{l}_n') = \mathfrak{l}_m'$.

Proposition 7.1. *One can choose the groups L_n so that $\varphi_{m,n}(L_n) = L_m$ for $m \geqq n \gg 0$.*

Proof. We argue by induction on $r = \dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])$. If $r = 0$ then the $\mathfrak{l}_n = \mathfrak{l}_n'$. so $d\varphi_{m,n}(\mathfrak{l}_n') = \mathfrak{l}_m'$ says $d\varphi_{m,n}(\mathfrak{l}_n) = \mathfrak{l}_m$, and it follows that $\varphi_{m,n}(L_n) = L_m$.

The group $U = \gamma(K)$ is compact and connected, and the identity component of its center is a torus T of dimension r. Suppose $r > 0$ let let S be a subtorus of dimension $r-1$ in T. Now define codimension 1 subgroups $'K_n = \gamma_n^{-1}(S) \subset K_n$ and $'K = \gamma^{-1}(S) \subset K$. Since $\gamma_m \cdot \varphi_{m,n} = \gamma_n$ we have $\varphi_{m,n}('K_n) \subset {}'K_m$, and $'K = \varinjlim\{'K_n, \varphi_{m,n}|_{'K_n}\}$. By induction on r we have closed connected normal subgroups $'L_n \subset {}'K_n$ such that $'K_n$ isomorphic to $K_n^\dagger \times {}'L_n$ and $\varphi_{m,n}('L_n) = {}'L_m$. We now have well defined limits $'L = \varinjlim {}'L_n$ and $'\mathfrak{l} = \varinjlim {}'\mathfrak{l}_n$. Here $'K$ is locally isomorphic to $K^\dagger \times {}'L$ and $'\mathfrak{k} = \mathfrak{k}^\dagger \oplus {}'\mathfrak{l}$.

Let $\mathfrak{w}^\dagger$ denote the center of $\mathfrak{k}^\dagger$ and $'\mathfrak{w}$ the center of $'\mathfrak{l}$. Let $\mathfrak{w}$ denote the centralizer of $'\mathfrak{k}$ in $\mathfrak{k}$. Then $\mathfrak{w}$ is an abelian ideal in $\mathfrak{k}$ that contains $\mathfrak{w}^\dagger \oplus {}'\mathfrak{w}$ as a subalgebra of codimension 1. We choose a 1–dimensional subalgebra $''\mathfrak{w} \subset \mathfrak{w}$ not contained in $\mathfrak{w}^\dagger \oplus {}'\mathfrak{w}$ and such that the 1–parameter subgroup $''W = \exp(''\mathfrak{w})$ is closed in L. Define $\mathfrak{l} = {}''\mathfrak{w} \oplus {}'\mathfrak{l}$. The corresponding analytic subgroup L is a closed subgroup of K. For n sufficiently large, say $n \geqq n_0$, the direct limit maps $\varphi_n : K_n \hookrightarrow K$ satisfy $''W \subset \varphi_n(K_n)$. As indicated above, the induction hypothesis gives us $'L \subset \varphi_n(K_n)$. Thus $L \subset \varphi_n(K_n)$ for $n \geqq n_0$. As the $\varphi_n : K_n \hookrightarrow K$ are injective we how have well defined closed connected subgroups $L_n = \varphi_n^{-1}(L)$ for $n \geqq n_0$, and $\gamma_n : L_n \to U$ is surjective with finite kernel. Thus the $\varphi_{m,n}(L_n) = L_m$, and K_n is locally isomorphic to $K_n^\dagger \times L_n$, for $m \geqq n \geqq n_0$. That completes the proof of Proposition 7.1. $\square$

For convenience of formulation we again truncate the index set, this time so that $L_n = \varphi_n^{-1}(L)$ for all indices n.

Corollary 7.2. *Let $t \in \mathfrak{z}$, and let $K_{n,t}$ be its stabilizer in K_n. If one of the direct systems $\{K_n, \varphi_{m,n}\}$, $\{K_{n,t}, \varphi_{m,n}|_{K_{n,t}}\}$, or $\{K_n^\dagger, \varphi_{m,n}|_{K_n^\dagger}\}$ is parabolic, then the other two also are parabolic.*

Proof. Let L_t denote the stabilizer of t in L. Up to local isomorphism,

$$\{K_n, \varphi_{m,n}\} = \{K_n^\dagger \times L, \varphi_{m,n}|_{K_n^\dagger} \times 1\} \text{ and } \{K_{n,t}, \varphi_{m,n}|_{K_{n,t}}\} = \{K_n^\dagger \times L_t, \varphi_{m,n}|_{K_n^\dagger} \times 1\}.$$

Each of the direct systems of Corollary 7.2 is parabolic if and only if $\{K_n^\dagger, \varphi_{m,n}|_{K_n^\dagger}\}$ is parabolic. $\square$

Now Theorem 4.3 gives us

COROLLARY 7.3. *Let $t \in \mathfrak{z}$, and let $K_{n,t}$ be its stabilizer in K_n. Suppose that the direct system $\{K_n, \varphi_{m,n}\}$ is parabolic. Then there are natural isometric injections $\mathcal{F}_{n,t,\lambda} \hookrightarrow \mathcal{F}_{m,t,\lambda}$ for $m \geqq n$, from the highest weight λ representation space of $K_{n,t}$ to that of $K_{m,t}$, and corresponding isometric injections*

$$\zeta''_{m,n} : f \mapsto \left((\deg \kappa_{m,t,\lambda})/(\deg \kappa_{n,t,\lambda})\right)^{1/2} f$$

on spaces of coefficient functions.

Another immediate consequence of Proposition 7.1 is

COROLLARY 7.4. *Let $t \in \mathfrak{z}$, and let $K_{n,t}$ be its stabilizer in K_n. Then $L := \varinjlim L_n$ is compact, $K = K^\dagger L$, and K is locally isomorphic to $K^\dagger \times L$. In particular K acts on $\mathfrak{z}$ as a compact linear group and $\mathfrak{z}$ has a $\gamma(K)$–invariant positive definite inner product.*

8. A Class of Commutative Nilmanifolds, I: Group Structure

In this section and the next, we make use of the results of Sections 6 and 7 in order to extend Theorems 5.6 and 5.10 to strict direct systems $\{(G_n, K_n)\}$ of Gelfand pairs that satisfy

$$\begin{aligned}
&\text{(i) } G_n = N_n \rtimes K_n, \text{ semidirect product, where } N_n \text{ is a connected,}\\
&\quad\text{simply connected, nilpotent Lie group with square integrable}\\
&\quad\text{representations and } K_n \text{ is connected,}\\
&\text{(ii) the } K_n \text{ form a parabolic strict direct system,}\\
&\text{(iii) the inclusions } \mathfrak{n}_n \hookrightarrow \mathfrak{n}_{n+1} \text{ maps centers } \mathfrak{z}_n \cong \mathfrak{z}_{n+1} \text{ and maps}\\
&\quad\text{complements } \mathfrak{v}_n \hookrightarrow \mathfrak{v}_{n+1}, \text{ and}\\
&\text{(iv) for each } n \text{ the complement } \mathfrak{v}_n \text{ is } \mathrm{Ad}(K_n)\text{–invariant.}
\end{aligned} \tag{8.1}$$

We identify each $\mathfrak{z}_n$ with $\mathfrak{z} := \varinjlim \mathfrak{z}_n$. Let $K_n^\dagger$ denote the identity component of the kernel of the action of K_n on $\mathfrak{z}$. The image $\mathrm{Ad}(K_n)|_{\mathfrak{z}}$ of that action is a compact connected group of linear transformations of $\mathfrak{z}$. Its dimension is bounded because $\dim \mathfrak{z} < \infty$. We may assume that each $\mathrm{Ad}(K_n)|_{\mathfrak{z}} = U$ for some compact connected group U of linear transformations of $\mathfrak{z}$. Proposition 7.1 gives us complementary closed connected normal subgroups $L_n \subset K_n$ that inject isomorphically under $K_n \hookrightarrow K_{n+1}$, so each L_n is equal to $L := \varinjlim L_n$. Thus we have decompositions $K_n = K_n^\dagger \cdot L$ and $K = K^\dagger \cdot L$ where $K_n^\dagger$ is the kernel of the adjoint action of K_n on $\mathfrak{z}$, $K = \varinjlim K_n$, and $K^\dagger = \varinjlim K_n^\dagger$. For each n, Ad_{K_n} maps $L = L_n$ onto U with finite kernel.

If $t \in \mathfrak{z}^*$ write $\mathcal{O}_t$ for the orbit $\mathrm{Ad}^*(L)(t)$. We denote the respective stabilizers of t in G_n, K_n and L_n by $G_{n,t}$, $K_{n,t}$ and $L_{n,t}$. Then $\mathrm{Ad}^*(G)(t) = \mathrm{Ad}^*(G_n)(t) = \mathrm{Ad}^*(K_n)(t) = \mathcal{O}_t \cong L/L_t$ for each n. Since $\mathrm{Ad}^*(G_n)$ acts on $\mathcal{O}_t$ as the compact group L there is an invariant measure ν_t derived from Haar measure on L; we normalize ν_t to total mass 1. Given $t \in \mathfrak{z}^*$ its stabilizers $G_t = \{g \in G \mid \mathrm{Ad}^*(g)t = t\}$, $K_t = K \cap G_t$ and $L_t = L \cap G_t$. Their pullbacks in G_n are $G_{n,t}$, $K_{n,t}$ and $L_{n,t}$. Note that $K_t = K^\dagger \cdot L_t$ and $K_{n,t} = K_n^\dagger \cdot L_{n,t}$.

Let $T = \{t \in \mathfrak{z}^* \mid \text{ each } \mathrm{Pf}(b_{n,t}) \neq 0\}$, as in Section 6, and fix $t \in T$. As in Section 5 the square integrable representation $\pi_{n,t}$ extends to a unitary representation $\widetilde{\pi_{n,t}}$ of $G_{n,t} := N_n \rtimes K_{n,t}$ on the same representation space $\mathcal{H}_{n,t}$. If $\kappa_{n,t,\lambda} \in \widehat{K_{n,t}}$

has representation space $\mathcal{F}_{n,t,\lambda}$ we write $\widetilde{\kappa_{n,t,\lambda}}$ for its extension to a representation of $G_{n,t}$ on $\mathcal{F}_{n,t,\lambda}$ that annihilates N_n. Then we have the irreducible unitary representation $\pi^{\diamondsuit}_{n,t,\lambda} := \widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,t,\lambda}}$ of $G_{n,t}$ on $\mathcal{H}^{\diamondsuit}_{n,t,\lambda} := \mathcal{H}_{n,t} \otimes \mathcal{F}_{n,t,\lambda}$. That gives us the unitary representation $\pi_{n,t,\lambda} = \mathrm{Ind}_{G_{n,t}}^{G_n}(\pi^{\diamondsuit}_{n,t,\lambda})$ of G_n. Its representation space

$$\mathcal{H}_{n,t,\lambda} := \int_{\mathcal{O}_t} (\mathcal{H}_{n,\mathrm{Ad}^*(k)t} \otimes \mathcal{F}_{n,t,\lambda})\, d\nu_t(k(t))$$

consists of all measurable functions $\varphi : G_n \to \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ such that

(i) $\varphi(gm) = \pi^{\diamondsuit}_{n,t,\lambda}(m)^{-1}\varphi(\ell)$ for $g \in G_n$ and $m \in G_{n,t}$ and

(ii) $\displaystyle\int_{\mathcal{O}_t} ||\varphi(g)||^2 d\nu_t(\mathrm{Ad}^*(g)(t)) < \infty.$

In other words $\mathcal{H}_{n,t,\lambda}$ is the space $L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{n,t,\lambda})$ of L^2 sections of the homogeneous Hilbert space bundle $\mathbb{H}^{\diamondsuit}_{n,t,\lambda} \to \mathcal{O}_t$ with fiber $\mathcal{H}^{\diamondsuit}_{n,t,\lambda}$. The action $\pi_{n,t,\lambda}$ of G_n on $\mathcal{H}_{n,t,\lambda}$ is $[(\pi_{n,t,\lambda}(g))(\varphi)](g') = \varphi(g^{-1}g')$. The inner product on $\mathcal{H}_{n,t,\lambda}$ is $\langle \varphi, \psi \rangle_{\mathcal{H}_{n,t,\lambda}} = \int_{\mathcal{O}_t} \langle \varphi(g), \psi(g) \rangle_{\mathcal{H}^{\diamondsuit}_{n,t,\lambda}} d\nu_t(\mathrm{Ad}^*(g)(t))$.

According to the Mackey little group theory, (i) $\pi_{n,t,\lambda}$ is irreducible, (ii) $\pi_{n,t,\lambda}$ is equivalent to $\pi_{n,t',\lambda'}$ if and only if $t' \in \mathcal{O}_t$, say $t' = \mathrm{Ad}^*(\ell)t$ where $\ell \in L$, and $\mathrm{Ad}^*(\ell)$ carries λ to λ', and (iii) Plancherel–almost–all irreducible unitary representations of G_n are of the form $\pi_{n,t,\lambda}$ where $t \in T$ and $\kappa_{n,t,\lambda} \in \widehat{K_{n,t}}$.

Corollary 7.2 tells us that the system $\{K_{n,t}\}$ is parabolic. As noted in Corollary 7.3 that gives us isometric injections $\mathcal{F}_{n,t,\lambda} \hookrightarrow \mathcal{F}_{m,t,\lambda}$ of representation spaces and corresponding isometric injections of the spaces of coefficient functions. Those injections combine with the corresponding maps of Section 6 to give us isometric injections $\zeta''_{m,n} : f \mapsto \bigl((|\mathrm{Pf}(b_{m,t})| \deg \kappa_{m,t,\lambda})/(|\mathrm{Pf}(b_{n,t})| \deg \kappa_{n,t,\lambda})\bigr)^{1/2} f$ from the space of coefficient functions of $\pi^{\diamondsuit}_{n,t,\lambda}$ to that of $\pi^{\diamondsuit}_{m,t,\lambda}$. Those come out of unitary injections $\mathcal{H}^{\diamondsuit}_{n,t,\lambda} \hookrightarrow \mathcal{H}^{\diamondsuit}_{m,t,\lambda}$ of the representation spaces. The representation space injections define unitary (on each fiber) injections $\mathbb{H}^{\diamondsuit}_{n,t,\lambda} \hookrightarrow \mathbb{H}^{\diamondsuit}_{m,t,\lambda}$ of the corresponding homogeneous Hilbert space bundles over the orbit $\mathcal{O}_t$. Spaces of L^2 sections correspond by

$$L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{n,t,\lambda}) = \{\varphi \in L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{m,t,\lambda}) \mid \varphi(\ell(t)) \in \ell(\mathcal{H}^{\diamondsuit}_{n,t,\lambda}) \text{ for all } \ell \in L\}.$$

The inner products on $\mathcal{H}_{n,t,\lambda} = L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{n,t,\lambda})$ and on $\mathcal{H}_{m,t,\lambda} = L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{m,t,\lambda})$ are G_n–invariant, and G_n is irreducible on $L^2(\mathcal{O}_t; \mathbb{H}^{\diamondsuit}_{n,t,\lambda})$, so there is a real scalar $c_{m,n} > 0$ such that $\varphi \mapsto c_{m,n}\varphi$ gives a G_n–equivariant isometric injection of $\mathcal{H}_{n,t,\lambda}$ into $\mathcal{H}_{m,t,\lambda}$. Summarizing to this point,

PROPOSITION 8.2. *As just described we have a strict direct system $\{\mathcal{H}_{n,t,\lambda}\}$ based on G_n–equivariant isometric injections $\mathcal{H}_{n,t,\lambda} \hookrightarrow \mathcal{H}_{m,t,\lambda}$.*

Now consider the spaces $\mathcal{E}_{n,t,\lambda} := \mathcal{H}_{n,t,\lambda} \boxtimes \mathcal{H}^*_{n,t,\lambda}$, Hilbert space completion of the space of coefficient functions $f_{\varphi,\psi} : g \mapsto \langle \varphi, \pi_{n,t,\lambda}(g)\psi \rangle_{\mathcal{H}_{n,t,\lambda}}$ for $\varphi, \psi \in \mathcal{H}_{n,t,\lambda}$. The $(G_n \times G_n)$–invariant inner product on $\mathcal{E}_{n,t,\lambda}$ is

$$\langle \varphi \boxtimes \psi, \varphi' \boxtimes \psi' \rangle_{\mathcal{E}_{n,t,\lambda}} = \tfrac{1}{d_{n,t,\lambda}} \langle \varphi, \varphi' \rangle_{\mathcal{H}_{n,t,\lambda}} \overline{\langle \psi, \psi' \rangle_{\mathcal{H}_{n,t,\lambda}}} \tag{8.3}$$

for some number $d_{n,t,\lambda} > 0$, which we interpret as the formal degree $\deg \pi_{n,t,\lambda}$. See Appendix A, specifically Theorem A.1 below, for a discussion of this. In any case, the right/left action of $G_n \times G_n$ on $\mathcal{E}_{n,t,\lambda}$ is an irreducible unitary representation, and the isometric embeddings $\mathcal{H}_{n,t,\lambda} \hookrightarrow \mathcal{H}_{m,t,\lambda}$ define our isometric embeddings $\zeta_{m,n} : f \mapsto (\deg \pi_{n+1,t,\lambda}/ \deg \pi_{n,t,\lambda})^{1/2} f$ of $\mathcal{E}_{n,t,\lambda}$ into $\mathcal{E}_{m,t,\lambda}$. That gives us

PROPOSITION 8.4. *As just described we have a strict direct system* $\{\mathcal{E}_{n,t,\lambda}, \zeta_{m,n}\}$, *whose spaces are the Hilbert spaces of coefficients of the representations* $\pi_{n,t,\lambda}$, *and whose maps are* $(G_n \times G_n)$*–equivariant isometric embeddings.*

The group K acts on $\mathfrak{z}^*$ through its compact subgroup L, so $\mathfrak{z}^*$ has an $\mathrm{Ad}^*(K)$–invariant inner product. Let S be the unit sphere. Since L and S are compact there are only finitely many orbit types $L_{s_1}, \dots, L_{s_\ell}$ of L on S ([**Yan**], or see [**P**, Theorem 1.7.25]). In other words every isotropy subgroup of L on S is conjugate to exactly one of the L_{s_i}. Thus every isotropy subgroup of K on S is conjugate to exactly one of the $K_{s_i} = K^\dagger L_{s_i}$. If $t \in \mathfrak{z}^*$ and $r \neq 0$ then the isotropy groups $K_t = K_{rt}$. Now every isotropy subgroup of K on $\mathfrak{z}^* \setminus \{0\}$ is conjugate to exactly one of the K_{s_i}.

Decompose $S \cap T = S_1 \cup \cdots \cup S_m$ where S_i is the union of the orbits $\mathrm{Ad}^*(K)(s)$ in $S \cap T$ with isotropy $K_s = K_{s_i}$. Locally S_i contains a smooth section to the action of K. Thus there is a measurable section $\sigma_i : K\backslash S_i \to S_i$ to the action of K, such that for each n the isotropy $K_{n,\sigma_i(x)}$ is K_{n,s_i}. Let $\Sigma_i = \sigma_i(K\backslash S_i)$. According to the Mackey little group theory, Plancherel almost all irreducible representations of G_n are of the form $\pi_{n,t,\lambda}$ where $t = rs_i$ with $1 \leqq i \leqq m$ and $r > 0$ and with $\kappa_{n,t,\lambda} \in \widehat{K_{n,t}}$. That gives

$$L^2(G_n) = \sum_{i=1}^{m} \sum_{\widehat{K_{n,s_i}}} \int_{r=0}^{\infty} \int_{s \in \Sigma_i} \mathcal{E}_{n,rs,\lambda}\, ds\, dr \tag{8.5}$$

As n increases we have the isometric equivariant injections $\zeta_{m.n} : \mathcal{E}_{n,t,\lambda} \to \mathcal{E}_{m,t,\lambda}$ of Proposition 8.4. When we form the discrete and continuous sums of (8.5), the $\zeta_{m.n}$ act on the summands, where they fit together to define isometric equivariant injections (which we also denote $\zeta_{m.n}$) from $L^2(G_n)$ to $L^2(G_m)$, $m \geqq n$. That yields the first assertion of

THEOREM 8.6. *Let* $\{(G_n, K_n)\}$ *be a strict direct system of commutative pairs that satisfy* (8.1). *Then the* G_n*–equivariant maps* $\mathcal{E}_{n,t,\lambda} \to \mathcal{E}_{m,t,\lambda}$ *of* Proposition 8.4 *define* $(G_n \times G_n)$*–equivariant isometric injections* $\zeta_{m.n} : L^2(G_n) \to L^2(G_m)$. *That gives a direct system* $\{L^2(G_n), \zeta_{m,n}\}$ *of Hilbert spaces and equivariant isometric injections. Let* Π_n *denote the left/right regular representation of* $G_n \times G_n$ *on* $L^2(G_n)$ *and let* $G = \varinjlim G_n$. *Then we have a well defined Hilbert space* $L^2(G) := \varinjlim \{L^2(G_n), \zeta_{m,n}\}$ *and a natural unitary representation* $\Pi = \varinjlim \Pi_n$ *of* $G \times G$ *on* $L^2(G)$. *Further, that representation* Π *is multiplicity–free.*

9. A Class of Commutative Nilmanifolds, II: Manifold Structure

We now pass from $L^2(G)$ to $L^2(G/K)$ for strict direct systems of commutative spaces that satisfy (8.1). Retain the notation of Section 8. The first step is

THEOREM 9.1. *Let* $t \in T$. *Then* $(N_n \rtimes K_{n,t}, K_{n,t})$ *is a Gelfand pair. In particular* $K_{n,t}$ *is multiplicity free on* $\mathbb{C}[\mathfrak{v}_n]$.

Note the similarity between the statement of Theorem 9.1 and Yakimova's commutativity criterion ([**Y3**, Theorem 1], or see [**W3**, Theorem 15.1.1]).

Proof. We may assume that $t = s_i$, representing one of the orbit types of L on $S \cap T$. Suppose that $(N_n \rtimes K_{n,t}, K_{n,t})$ is not a Gelfand pair. Then the commuting algebra $\mathcal{A}$ for the representation λ_t of $N_n \rtimes K_{n,t}$ on $L^2((N_n \rtimes K_{n,t})/K_{n,t})$ is not commutative. Let $A_1, A_2 \in \mathcal{A}$ with $A_1A_2 \neq A_2A_1$. Here $L^2((N_n \rtimes K_n)/K_n)$ is the sum (over j) of the representation spaces of the $\widetilde{\lambda_j} := \mathrm{Ind}\,_{(N_n \rtimes K_{n,s_j})}^{(N_n \rtimes K_n)}(\lambda_{s_j})$. In other words it is the sum of the spaces $L^2_j((N_n \rtimes K_n)/K_n)$ where $L^2_j((N_n \rtimes K_n)/K_n)$ consists of the $L^2(K_n/K_{n,t})$ functions $\varphi : N_n \rtimes K_n \to L^2((N_n \rtimes K_{n,s_j})/K_{n,s_j})$ such that $\varphi(gh) = \lambda_{s_j}(h)^{-1}\varphi(g)$ for $g \in N_n \rtimes K_n$ and $h \in N_n \rtimes K_{n,s_j}$. Now define $\widetilde{A_1}$ and $\widetilde{A_2}$ by $(\widetilde{A_u}\varphi)(g) = A_u(\varphi(g))$. Then $\widetilde{A_u}(\varphi)(gh) = A_u(\varphi(gh)) = A_u(\lambda_{s_j}(h)^{-1}\varphi(g)) = \lambda_{s_j}(h)^{-1}(A_u(\varphi(g))) = \lambda_{s_j}(h)^{-1}(\widetilde{A_u}\varphi)(g)$, so $\widetilde{A_u}$ is a well defined linear transformation of $L^2_j((N_n \rtimes K_n)/K_n)$. Further, $[\widetilde{\lambda_{s_j}}(g) \cdot \widetilde{A_u}(\varphi)](g_1) = (\widetilde{A_u}(\varphi))(g^{-1}g_1) = A_u(\varphi(g^{-1}g_1)) = A_u([\widetilde{\lambda_{s_j}}(g)\varphi](g_1)) = [\widetilde{A_u}(\widetilde{\lambda_{s_j}}(g)\varphi)](g_1)$, so $\widetilde{A_u}$ is an intertwining operator for $\widetilde{\lambda_{s_j}}$. As the A_u do not commute, neither do the $\widetilde{A_u}$. Since $(N_n \rtimes K_n, K_n)$ is a Gelfand pair this is a contradiction. We conclude that $(N_n \rtimes K_{n,t}, K_{n,t})$ is a Gelfand pair. In particular, now, $K_{n,t}$ is multiplicity free on $\mathbb{C}[\mathfrak{v}_n]$ by Carcano's Theorem. $\square$

The Hilbert bundle model for the induced representation $\pi_{n,t,\lambda} \in \widehat{G_{n,t}}$ is given by $\pi_{n,t,\lambda} = \mathrm{Ind}\,_{G_{n,t}}^{G_n}(\pi^{\diamondsuit}_{n,t,\lambda})$. The representation space $\mathcal{H}_{n,t,\lambda}$ of $\pi_{n,t,\lambda}$ consists of all $L^2(K_n/K_{n,t})$ sections of the homogeneous bundle $p : \mathbb{H}^{\diamondsuit}_{n,t,\lambda} \to G_n/G_{n,t} = K_n/K_{n,t}$ whose typical fiber is the representation space $\mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ of $\pi^{\diamondsuit}_{n,t,\lambda}$. Given $k \in K_n$ we write $k \cdot \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ for the fiber $p^{-1}(kK_{n,t})$. Let $u \in \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ be a $\pi^{\diamondsuit}_{n,t,\lambda}(K_{n,t})$–fixed unit vector. Then u belongs to the fiber $1 \cdot \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$, and $k \cdot u \in k \cdot \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ depends only on the coset $kK_{n,t}$. Define a section

$$\sigma_u : K_n/K_{n,t} \to \mathbb{H}^{\diamondsuit}_{n,t,\lambda} \text{ by } \sigma_u(kK_{n,t}) = k \cdot u. \tag{9.2}$$

Then σ_u is a $\pi_{n,t,\lambda}(K_n)$–invariant unit vector in the Hilbert space $\mathcal{H}_{n,t,\lambda}$. (We will also write φ_u for the corresponding function $G_n \to \mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ such that $\varphi_u(gg_t) = \pi^{\diamondsuit}_{n,t,\lambda}(g_t)^{-1}(\varphi_u(g))$ for $g \in G_n$ and $g_t \in G_{n,t}$.) Conversely if σ is a $\pi_{n,t,\lambda}(K_n)$–invariant unit vector in $\mathcal{H}_{n,t,\lambda}$, then $\sigma(1K_{n,t}) = cu$ where $|c| = 1$ by $K_{n,t}$–invariance, and then $\sigma = c\sigma_u$ by K–invariance. In summary,

Lemma 9.3. *Let $t \in T$ and let u be the unique* (up to scalar multiple) $\pi^{\diamondsuit}_{n,t,\lambda}(K_{n,t})$–*fixed unit vector in $\mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ Then the section σ_u, given by* (9.2), *is the unique* (up to scalar multiple) $\pi_{n,t,\lambda}(K)$–*fixed unit vector in $\mathcal{H}_{n,t,\lambda}$.*

By Theorem 9.1 we can apply Proposition 8.4 to the function spaces $\mathcal{E}^{\diamond}_{n,t,\lambda} = \mathcal{H}^{\diamond}_{n,t,\lambda} \boxtimes (\mathcal{H}^{\diamond}_{n,t,\lambda})^*$ on the groups $G_{n,t} = N_n \rtimes K_{n,t}$. Now combining Proposition 8.4 and Lemma 9.3 we have

Proposition 9.4. *If orthogonal projection $\mathcal{E}^{\diamondsuit}_{n+1,t,\lambda} \to \mathcal{E}^{\diamondsuit}_{n,t,\lambda}$ sends a nonzero right $K_{n+1,t}$–invariant function to a nonzero right $K_{n,t}$–invariant function, then orthogonal projection $\mathcal{E}_{n+1,t,\lambda} \to \mathcal{E}_{n,t,\lambda}$ sends a nonzero right K_{n+1}–invariant function to a nonzero right K_n–invariant function.*

Vinberg ([**V1**], [**V2**]; or see [**W3**, Table 13.4.1]) classified the maximal irreducible nilpotent Gelfand pairs. A Gelfand pair (G_n, K_n) is called *maximal* if it is not obtained from another Gelfand pair (G'_n, K'_n) by the construction $(G_n, K_n) = (G'_n/C, K'_n/(K'_n \cap C))$ for any nontrivial closed connected central subgroup C of G'_n. And (G_n, K_n) is called *irreducible* if $\mathrm{Ad}(K_n)$ is irreducible on $\mathfrak{v}_n = \mathfrak{n}_n/\mathfrak{z}$. Here is Vinberg's classification of maximal irreducible nilpotent Gelfand pairs; see [**W3**] for the notation.

(9.5)

Maximal Irreducible Nilpotent Gelfand Pairs $(N_n \rtimes K_n, K_n)$ ([**V1**], [**V2**])					
	Group K_n	$\mathfrak{v}_n$	$\mathfrak{z}$	$U(1)$ is needed if	max requires
1	$SO(n)$	$\mathbb{R}^n$	$\mathrm{Skew}\,\mathbb{R}^{n\times n} = \mathfrak{so}(n)$		
2	$Spin(7)$	$\mathbb{R}^8 = \mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
3	G_2	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
4	$U(1)\cdot SO(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		$n \neq 4$
5	$(U(1)\cdot)SU(n)$	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C}$	n odd	
6	$SU(n), n$ odd	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
7	$SU(n), n$ odd	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		
8	$U(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{n\times n} = \mathfrak{u}(n)$		
9	$(U(1)\cdot)Sp(n)$	$\mathbb{H}^n$	$\mathrm{Re}\,\mathbb{H}_0^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}$		
10	$U(n)$	$S^2\mathbb{C}^n$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n), n \geqq 3$	$\Lambda^2\mathbb{C}^n$	$\mathbb{R}$	n even	
12	$U(1)\cdot Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
13	$U(1)\cdot Spin(9)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
14	$(U(1)\cdot)Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
15	$U(1)\cdot G_2$	$\mathbb{C}^7$	$\mathbb{R}$		
16	$U(1)\cdot E_6$	$\mathbb{C}^{27}$	$\mathbb{R}$		
17	$Sp(1)\times Sp(n)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2)\times Sp(n)$	$\mathbb{H}^{2\times n}$	$\mathrm{Im}\,\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m)\times SU(n)$ $m, n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20	$(U(1)\cdot)SU(2)\times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$	$n = 2$	
21	$(U(1)\cdot)Sp(2)\times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2)\times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3)\times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

As noted in [**W3**], in Table 9.5 one often can replace K_n by a smaller group in such a way that (G_n, K_n) continues to be a Gelfand pair. For example, in Table 9.5, Item 2, where N_n is the octonionic Heisenberg group $H_{\mathbb{O},1}$, the pairs $(N_n \rtimes Spin(7), Spin(7))$, $(N_n \rtimes Spin(6), Spin(6))$ and $(N_n \rtimes Spin(5), Spin(5))$ all are Gelfand pairs; see [**L**, Proposition 5.6]. This is the tip of the iceberg for the classification of commutative nilmanifolds. A systematic analysis is given in [**Y2**]; or see [**W3**, Chapter 15].

The strict direct systems in Table 9.5, with $\dim \mathfrak{z}_n$ bounded, are as follows. Here we split entry line 4 of Table 9.5 so that $\{K_n\}$ is parabolic, and we split entry 20 into two essentially different cases.

(9.6)

Direct Systems of Maximal Irreducible Nilpotent Gelfand Pairs $(N_n \rtimes K_n, K_n)$					
	Group K_n	$\mathfrak{v}_n$	$\mathfrak{z}_n$	$U(1)$ is needed if	max requires
4a	$U(1) \cdot SO(2n)$	$\mathbb{C}^{2n}$	$\mathrm{Im}\,\mathbb{C}$		$n \neq 2$
4b	$U(1) \cdot SO(2n+1)$	$\mathbb{C}^{2n+1}$	$\mathrm{Im}\,\mathbb{C}$		
7	$SU(n), n$ odd	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		
10	$U(n)$	$S^2\mathbb{C}^n$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n), n \geqq 3$	$\Lambda^2\mathbb{C}^n$	$\mathbb{R}$	n even	
17	$Sp(1) \times Sp(n)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2) \times Sp(n)$	$\mathbb{H}^{2\times n}$	$\mathrm{Im}\,\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m) \times SU(n)$ $m, n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20a	$SU(2) \times SU(n), n \geqq 3$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
20b	$U(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
21	$(U(1)\cdot)Sp(2) \times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3) \times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

In each case of Table 9.6, [**W3**, Theorem 14.4.3] says that N_n has square integrable representations. In the cases $\dim \mathfrak{z} > 1$ of Table 9.6 we have $K_n = K' \cdot K''_n$ where the big factor K''_n acts trivially on $\mathfrak{z}$ and the small factor K' acts on $\mathfrak{z}$ by its adjoint representation. Summarizing these observations,

PROPOSITION 9.7. *Each of the thirteen direct systems* $\{(G_n, K_n)\}$ *of* Table 9.6 *has the properties* (i) $\{K_n\}$ *is parabolic* (ii) *the* $\{K_{n,s_i}\}$ *are parabolic and* (iii) N_n *has square integrable representations.*

The following result is immediate from the multiplicity free part of Theorem 9.1 and the argument of Lemma 5.7.

COROLLARY 9.8. *Let* $\{(G_n, K_n)\}$ *be one of the thirteen direct systems of* Table 9.6 *and let* $t \in T$. *Let* $\kappa_{n,\lambda} \in \widehat{K_{n,t}}$. *Define* $\widetilde{\kappa_{n,t,\lambda}} \in \widehat{N_n \rtimes K_{n,t}}$ *by* $\widetilde{\kappa_{n,t,\lambda}}(h,k) = \kappa_{n,t,\lambda}(k)$. *Then* $\pi^{\diamondsuit}_{n,t,\lambda} := \widetilde{\pi_{n,t}} \otimes \widetilde{\kappa_{n,t,\lambda}}$ *has a nonzero* $K_{n,t}$*–fixed vector if and only if* $\kappa^*_{n,t,\lambda}$ *occurs as a subrepresentation of* $\widetilde{\pi_{n,t}}|_{K_{n,t}}$, *and in that case the space of* $K_{n,t}$*–fixed vectors has dimension* 1.

Corollary 9.8 lets us apply the argument of Corollary 5.9 to the spaces $\mathcal{H}^{\diamondsuit}_{n,t,\lambda} := \mathcal{H}_{n,t} \otimes \mathcal{F}_{n,t,\lambda}$ of the $\pi^{\diamondsuit}_{n,t,\lambda}$. If $m \geqq n$ it shows that every $K_{n,t}$–invariant vector in $\mathcal{H}^{\diamondsuit}_{n,t,\lambda}$ is the image of a $K_{m,t}$–invariant vector under the adjoint of the unitary map $\mathcal{H}^{\diamondsuit}_{n,t,\lambda} \to \mathcal{H}^{\diamondsuit}_{m,t,\lambda}$. Combining this with Proposition 8.4 we see that orthogonal projection $\mathcal{E}^{\diamondsuit}_{m,t,\lambda} \to \mathcal{E}^{\diamondsuit}_{n,t,\lambda}$ sends nonzero right $K_{m,t}$–invariant functions to nonzero right $K_{n,t}$–invariant functions. (Here recall the space $\mathcal{E}^{\diamondsuit}_{n,t,\lambda} := \mathcal{H}^{\diamondsuit}_{n,t,\lambda} \otimes (\mathcal{H}^{\diamondsuit}_{n,t,\lambda})^*$ of functions on $G_{n,t}$.) Now Proposition 9.4 gives us

PROPOSITION 9.9. *Let* $\{(G_n, K_n)\}$ *be one of the thirteen direct systems of* Table 9.6. *Let* $t \in T$ *and* $m \geqq n$. *Then orthogonal projection* $\mathcal{E}_{m,t,\lambda} \to \mathcal{E}_{n,t,\lambda}$ *sends nonzero right* K_m*–invariant invariant functions to nonzero right* K_n*–invariant functions.*

Combining Theorem 8.6 with Corollary 9.8 and Proposition 9.9 we arrive at

THEOREM 9.10. *Let* $\{(G_n, K_n)\}$ *be one of the thirteen direct systems of* Table 9.6. *Denote* $G = \varinjlim G_n$ *and* $K = \varinjlim K_n$. *Then the unitary direct system* $\{L^2(G_n)\}$

of Theorem 8.6 *restricts to a unitary direct system* $\{L^2(G_n/K_n)\}$*, the Hilbert space* $L^2(G/K) := \varinjlim L^2(G_n/K_n)$ *is the subspace of* $L^2(G) := \varinjlim L^2(G_n)$ *consisting of right–*K*–invariant functions, and the natural unitary representation of* G *on* $L^2(G/K)$ *is a multiplicity free direct integral of lim–irreducible representations.*

Now we go past the cases that require irreducibility of K_n on $\mathfrak{v}_n$.

In the Table 9.14a below, $\mathfrak{h}_{n;\mathbb{F}}$ denotes the generalized Heisenberg algebra $\operatorname{Im}\mathbb{F}+\mathbb{F}^n$ of real dimension $1 + n\dim_{\mathbb{R}}\mathbb{F}$ where $\mathbb{F}$ denotes the complex number field $\mathbb{C}$, the quaternion algebra $\mathbb{H}$, or the octonion algebra $\mathbb{O}$. It is the Lie algebra of the generalized Heisenberg group $H_{n;\mathbb{F}}$ given by

$$(9.11)\qquad \begin{aligned} H_{n,\mathbb{F}}:\ &\text{real vector space } \operatorname{Im}\mathbb{F}+\mathbb{F}^n \text{ with group composition}\\ &(z,w)(z',w') = (z+z'+\operatorname{Im}h(w,w'), w+w') \end{aligned}$$

where h is the standard positive definite hermitian form on $\mathbb{F}^n$. The generalized Heisenberg groups $H_{n;\mathbb{F}}$ all have square integrable representations [**W3**, Theorem 14.3.1].

In Table 9.14a we have direct sum decompositions

$$(9.12)\quad \mathfrak{n} = \mathfrak{n}'\oplus\mathfrak{z}'' \text{ where } \mathfrak{n}' \text{ has center } \mathfrak{z}' = [\mathfrak{n},\mathfrak{n}],\ \mathfrak{n} \text{ has center } \mathfrak{z}'\oplus\mathfrak{z}'' = [\mathfrak{n},\mathfrak{n}]\oplus\mathfrak{z}''.$$

and K–stable vector space decompositions

$$(9.13)\qquad \mathfrak{n} = \mathfrak{z}+\mathfrak{v} \quad\text{and}\quad \mathfrak{n}' = \mathfrak{z}'+\mathfrak{v} = [\mathfrak{n},\mathfrak{n}]+\mathfrak{v}.$$

Finally, in Table 9.14a, $\mathfrak{su}(n)$ does not mean the Lie algebra, but simply denotes its underlying vector space, the space of $n\times n$ skew–hermitian complex matrices, as a module for $\operatorname{Ad}(U(n))$ or $\operatorname{Ad}(SU(n))$.

Here is a small reformulation of Yakimova's classification of indecomposable, principal, maximal and $Sp(1)$–saturated commutative pairs $(N\rtimes K, K)$, where the action of K on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is reducible. Compare [**W3**, Table 13.4.4]. See [**Y2**] or [**W3**] for the technical definitions; for our purposes it suffices to note that these are the basic building blocks for the complete classification described in [**Y2**] and [**W3**]. We omit the case $[\mathfrak{n},\mathfrak{n}] = 0$, where $N = \mathbb{R}^n$ and K is any closed subgroup of the orthogonal group $O(n)$.

(9.14a)

Maximal Indecomposable Principal Saturated Nilpotent Gelfand Pairs $(N\rtimes K, K)$, N Nonabelian Nilpotent, Where the Action of K on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is Reducible

	Group K	K–module $\mathfrak{v}$	module $\mathfrak{z}'=[\mathfrak{n},\mathfrak{n}]$	module $\mathfrak{z}''$	Algebra $\mathfrak{n}'$
1	$U(n)$	$\mathbb{C}^n$	$\mathbb{R}$	$\mathfrak{su}(n)$	$\mathfrak{h}_{n;\mathbb{C}}$
2	$U(4)$	$\mathbb{C}^4$	$\operatorname{Im}\mathbb{C}\oplus\Lambda^2\mathbb{C}^4$	$\mathbb{R}^6$	$\operatorname{Im}\mathbb{C}+\Lambda^2\mathbb{C}^4+\mathbb{C}^4$
3	$U(1)\times U(n)$	$\mathbb{C}^n\oplus\Lambda^2\mathbb{C}^n$	$\mathbb{R}\oplus\mathbb{R}$	0	$\mathfrak{h}_{n;\mathbb{C}}\oplus\mathfrak{h}_{n(n-1)/2;\mathbb{C}}$
4	$SU(4)$	$\mathbb{C}^4=\mathbb{H}^2$	$\operatorname{Im}\mathbb{C}\oplus\operatorname{Re}\mathbb{H}^{2\times 2}$	$\mathbb{R}^6$	$\operatorname{Im}\mathbb{C}+\operatorname{Re}\mathbb{H}^{2\times 2}+\mathbb{C}^4$
5	$U(2)\times U(4)$	$\mathbb{C}^{2\times 4}$	$\operatorname{Im}\mathbb{C}^{2\times 2}$	$\mathbb{R}^6$	$\operatorname{Im}\mathbb{C}^{2\times 2}+\mathbb{C}^{2\times 4}$
6	$S(U(4)\times U(m))$	$\mathbb{C}^{4\times m}$	$\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{4m;\mathbb{C}}$
7	$U(m)\times U(n)$	$\mathbb{C}^{m\times n}\oplus\mathbb{C}^m$	$\mathbb{R}\oplus\mathbb{R}$	0	$\mathfrak{h}_{mn;\mathbb{C}}\oplus\mathfrak{h}_{m;\mathbb{C}}$
8	$U(1)\times Sp(n)\times U(1)$	$\mathbb{C}^{2n}\oplus\mathbb{C}^{2n}$	$\mathbb{R}\oplus\mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}}\oplus\mathfrak{h}_{2n;\mathbb{C}}$
9	$Sp(1)\times Sp(n)\times U(1)$	$\mathbb{H}^n\oplus\mathbb{H}^n$	$\operatorname{Im}\mathbb{H}\oplus\mathbb{R}$	0	$\mathfrak{h}_{n;\mathbb{H}}\oplus\mathfrak{h}_{2n;\mathbb{C}}$
10	$Sp(1)\times Sp(n)\times Sp(1)$	$\mathbb{H}^n\oplus\mathbb{H}^n$	$\operatorname{Im}\mathbb{H}\oplus\operatorname{Im}\mathbb{H}$	0	$\mathfrak{h}_{n;\mathbb{H}}\oplus\mathfrak{h}_{n;\mathbb{H}}$
11	$Sp(n)\times\{Sp(1),U(1),\{1\}\}$ $\times Sp(m)$	$\mathbb{H}^n$	$\operatorname{Im}\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
12	$Sp(n)\times\{Sp(1),U(1),\{1\}\}$	$\mathbb{H}^n$	$\operatorname{Re}\mathbb{H}_0^{n\times n}$	$\operatorname{Im}\mathbb{H}$	$\mathfrak{h}_{n;\mathbb{H}}$
13	$Spin(7)\times\{SO(2),\{1\}\}$	$\mathbb{R}^8=\mathbb{O}$	$\mathbb{R}^7=\operatorname{Im}\mathbb{O}$	$\mathbb{R}^{7\times 2}$	$\mathfrak{h}_{1;\mathbb{O}}$
14	$U(1)\times Spin(7)$	$\mathbb{C}^7$	$\mathbb{R}$	$\mathbb{R}^8$	$\mathfrak{h}_{7;\mathbb{C}}$
15	$U(1)\times Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}$	$\mathbb{R}^7$	$\mathfrak{h}_{8;\mathbb{C}}$
16	$U(1)\times U(1)\times Spin(8)$	$\mathbb{C}^8_+\oplus\mathbb{C}^8_-$	$\mathbb{R}\oplus\mathbb{R}$	0	$\mathfrak{h}_{8;\mathbb{C}}\oplus\mathfrak{h}_{8;\mathbb{C}}$
17	$U(1)\times Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$	$\mathbb{R}^{10}$	$\mathfrak{h}_{16;\mathbb{C}}$

.... table continued on next page

.... table continued from previous page

(9.14b)

18	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
19	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2;\mathbb{C}}$
20	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2)\times$ $\{SU(m), U(m), U(1)Sp(\frac{m}{2})\}$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
21	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\}$ $\times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{8;\mathbb{C}}$
22	$U(4) \times U(2)$	$\mathbb{C}^{4\times 2}$	$\mathbb{R}$	$\mathbb{R}^6 \oplus \mathfrak{su}(2)$	$\mathfrak{h}_{8;\mathbb{C}}$
23	$U(4) \times U(2) \times U(4)$	$\mathbb{C}^{4\times 2} \oplus \mathbb{C}^{2\times 4}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6 \oplus \mathbb{R}^6$	$\mathfrak{h}_{8;\mathbb{C}} \oplus \mathfrak{h}_{8;\mathbb{C}}$
24	$U(1) \times U(1) \times SU(4)$	$\mathbb{C}^4 \oplus \mathbb{C}^4$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{4;\mathbb{C}} \oplus \mathfrak{h}_{4;\mathbb{C}}$
25	$(U(1)\cdot)SU(4)(\cdot SO(2))$	$\mathbb{C}^4$	$\mathbb{R}^{6\times 2}$	$\mathbb{R}$	$\mathfrak{h}_{4;\mathbb{C}}$

In each case of Table 9.14, the group $N = N' \times Z''$ has square integrable representations [**W3**, Theorem 14.3.1]. In fact, if $t \in \mathfrak{z}^*$ we decompose $t = t' + t''$ where $t'(\mathfrak{z}'') = 0 = t''(\mathfrak{z}')$, and then $\mathrm{Pf}(b_t) = \mathrm{Pf}(b_{t'})$, independent of t''.

The strict direct systems in Table 9.14 with $\dim \mathfrak{z}'_n$ bounded, are as follows. Here the index ℓ can be n or (m, n), the group $G_\ell = N_\ell \rtimes K_\ell$, and the subgroup $G'_\ell = N'_\ell \rtimes K_\ell$.

(9.15)

Strict Direct Systems $\{(G_\ell, K_\ell)\}$ and $\{(G'_\ell, K'_\ell)\}$ of Gelfand Pairs From Table 9.14a with $\dim \mathfrak{z}'_\ell$ Bounded

	Group K_ℓ	K_ℓ–module $\mathfrak{v}_\ell$	$\mathfrak{z}'_\ell = [\mathfrak{n}_\ell, \mathfrak{n}_\ell]$	module $\mathfrak{z}''_\ell$	Algebra $\mathfrak{n}'_\ell$
1	$U(n)$	$\mathbb{C}^n$	$\mathbb{R}$	$\mathfrak{su}(n)$	$\mathfrak{h}_{n;\mathbb{C}}$
3	$U(1) \times U(n)$	$\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{n;\mathbb{C}} \oplus \mathfrak{h}_{n(n-1)/2;\mathbb{C}}$
6	$S(U(4) \times U(m))$	$\mathbb{C}^{4\times m}$	$\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{4m;\mathbb{C}}$
7	$U(m) \times U(n)$	$\mathbb{C}^{m\times n} \oplus \mathbb{C}^m$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{mn;\mathbb{C}} \oplus \mathfrak{h}_{m;\mathbb{C}}$
8	$U(1) \times Sp(n) \times U(1)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2n;\mathbb{C}}$
9	$Sp(1) \times Sp(n) \times U(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} \oplus \mathbb{R}$	0	$\mathfrak{h}_{n;\mathbb{H}} \oplus \mathfrak{h}_{2n;\mathbb{C}}$
10	$Sp(1) \times Sp(n) \times Sp(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$	0	$\mathfrak{h}_{n;\mathbb{H}} \oplus \mathfrak{h}_{n;\mathbb{H}}$
11a	$Sp(n) \times Sp(1) \times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
11b	$Sp(n) \times U(1) \times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
11c	$Sp(n) \times \{1\} \times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
18a	$SU(n) \times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
18b	$U(n) \times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
18c	$U(1)Sp(\frac{n}{2}) \times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
19a	$SU(n) \times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2;\mathbb{C}}$
19b	$U(n) \times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2;\mathbb{C}}$
19c	$U(1)Sp(\frac{n}{2}) \times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2;\mathbb{C}}$
20aa	$SU(n) \times SU(2) \times SU(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20ab	$SU(n) \times SU(2) \times U(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20ac	$SU(n) \times SU(2) \times U(1)Sp(\frac{m}{2})$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20ba	$U(n) \times SU(2) \times SU(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20bb	$U(n) \times SU(2) \times U(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20bc	$U(n) \times SU(2) \times U(1)Sp(\frac{m}{2})$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20ca	$U(1)Sp(\frac{n}{2}) \times SU(2) \times SU(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20cb	$U(1)Sp(\frac{n}{2}) \times SU(2) \times U(m)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
20cc	$U(1)Sp(\frac{n}{2}) \times SU(2)\times$ $U(1)Sp(\frac{m}{2})\}$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	0	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{2m;\mathbb{C}}$
21a	$SU(n) \times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{8;\mathbb{C}}$
21b	$U(n) \times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{8;\mathbb{C}}$
21c	$U(1)Sp(\frac{n}{2}) \times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}} \oplus \mathfrak{h}_{8;\mathbb{C}}$

By inspection of each row of the table we arrive at

PROPOSITION 9.16. *Each of the* 28 *strict direct systems* $\{(G'_\ell, K_\ell)\}$ *of* Table 9.15 *satisfies* (8.1).

If $\ell \leqq k$ in the index set then $\mathfrak{z}'_\ell \hookrightarrow \mathfrak{z}'_k$ is surjective. We identify each of the $\mathfrak{z}'_\ell$ with $\mathfrak{z}' := \varinjlim \mathfrak{z}'_\ell$. As in Lemma 6.4 we have $\mathfrak{a}'_\ell$, the zero set of the polynomial

$\mathrm{Pf}(b_{\ell,t'})$ on $\mathfrak{z}'_\ell$, and $\mathfrak{a}' := \bigcup \mathfrak{a}'_\ell$ is a set of measure zero in $(\mathfrak{z}')^*$. And as in (6.5) we denote $T' = \{t' \in (\mathfrak{z}')^* \mid \text{ each } \mathrm{Pf}(b_{\ell,t'}) \neq 0\} = (\mathfrak{z}')^* \setminus \mathfrak{a}'$.

If $t' \in T'$ then $K_{\ell,t'}$, $G'_{\ell,t'} = N_\ell \rtimes K'_{\ell,t'}$ and $G_{\ell,t'} = N_\ell \rtimes K_{\ell,t'}$ are its respective stabilizers in K_ℓ, G'_ℓ and G_ℓ. Theorem 9.1 tells us that $(G'_{\ell,t'}, K_{\ell,t'})$ is a Gelfand pair, so in particular the action of $K_{\ell,t'}$ on $\mathbb{C}[\mathfrak{v}_\ell]$ is multiplicity free. As $G_{\ell,t'}$ and $G'_{\ell,t'}$ correspond to the same multiplicity free action of $K_{\ell,t'}$ on $\mathbb{C}[\mathfrak{v}_\ell]$, Carcano's Theorem says that $(G_{\ell,t'}, K_{\ell,t'})$ is a Gelfand pair. Now Lemma 9.3 and Proposition 9.4 apply. As above, this leads to

THEOREM 9.17. *Let $\{(G'_\ell, K_\ell)\}$ be one of the twenty eight direct systems of* Table 9.15. *Denote $G' = \varinjlim G'_\ell$ and $K = \varinjlim K_\ell$. Then the unitary direct system $\{L^2(G'_\ell), \zeta'_{\ell,\widetilde{\ell}}\}$ given by that of* Theorem 8.6, *restricts to a unitary direct system $\{L^2(G'_\ell/K_\ell), \zeta'_{\ell,\widetilde{\ell}}\}$, the Hilbert space $L^2(G'/K) := \varinjlim\{L^2(G'_\ell/K_\ell), \zeta'_{\ell,\widetilde{\ell}}\}$ is the subspace of $L^2(G') := \varinjlim\{L^2(G'_\ell), \zeta'_{\ell,\widetilde{\ell}}\}$ consisting of right–K–invariant functions, and the natural unitary representation of G' on $L^2(G'/K)$ is a multiplicity free direct integral of lim–irreducible representations.*

Now define $\mathfrak{z}'' = \varinjlim \mathfrak{z}''_\ell$ and let $Z'' := \varinjlim Z''_\ell$ denote the corresponding vector group. In Table 9.15 the $\mathfrak{z}''_\ell$ are constant except for the entries of row 1, where $\mathfrak{z}'' = \mathfrak{su}(\infty)$, and rows 11a,b,c, where $\mathfrak{z}''$ can be to any of $\mathbb{H}^{n\times\infty}$, $\mathbb{H}^{\infty\times m}$ and $\mathbb{H}^{\infty\times\infty}$. In any case, $(\mathfrak{z}'')^* = \varprojlim(\mathfrak{z}''_\ell)^*$.

In the cases where the $\mathfrak{z}''_\ell$ are zero, i.e. $\mathfrak{z}'' = 0$, we have $G_\ell = G'_\ell$, so the natural unitary representation of $G = \varinjlim G_\ell$ on $L^2(G/K)$ is multiplicity free by Theorem 9.17. The cases where the $\mathfrak{z}''_\ell$ are nonzero but constant are

(9.18)

	Group K_ℓ	K_ℓ–module $\mathfrak{v}_\ell$	module $\mathfrak{z}'_\ell = [\mathfrak{n}_\ell, \mathfrak{n}_\ell]$	module $\mathfrak{z}''_\ell$	Algebra $\mathfrak{n}'_\ell$
6	$S(U(4)\times U(m))$	$\mathbb{C}^{4\times m}$	$\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{4m;\mathbb{C}}$
18a	$SU(n)\times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
18b	$U(n)\times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
18c	$U(1)Sp(\frac{n}{2})\times SU(2)$	$\mathbb{C}^{n\times 2}$	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{h}_{2n;\mathbb{C}}$
21a	$SU(n)\times SU(2)\times U(4)$	$\mathbb{C}^{n\times 2}\oplus\mathbb{C}^{2\times 4}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}}\oplus\mathfrak{h}_{8;\mathbb{C}}$
21b	$U(n)\times SU(2)\times U(4)$	$\mathbb{C}^{n\times 2}\oplus\mathbb{C}^{2\times 4}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}}\oplus\mathfrak{h}_{8;\mathbb{C}}$
21c	$U(1)Sp(\frac{n}{2})\times SU(2)\times U(4)$	$\mathbb{C}^{n\times 2}\oplus\mathbb{C}^{2\times 4}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}^6$	$\mathfrak{h}_{2n;\mathbb{C}}\oplus\mathfrak{h}_{8;\mathbb{C}}$

In those cases $\{(G_\ell, K_\ell)\}$ satisfies (8.1), so the considerations leading to Theorem 9.17 apply directly to the $\{(G_\ell, K_\ell)\}$. We turn now to the other cases. They are given by

(9.19)

	Group K_ℓ	K_ℓ–module $\mathfrak{v}_\ell$	module $\mathfrak{z}'_\ell = [\mathfrak{n}_\ell, \mathfrak{n}_\ell]$	module $\mathfrak{z}''_\ell$	Algebra $\mathfrak{n}'_\ell$
1	$U(n)$	$\mathbb{C}^n$	$\mathbb{R}$	$\mathfrak{su}(n)$	$\mathfrak{h}_{n;\mathbb{C}}$
11a	$Sp(n)\times Sp(1)\times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
11b	$Sp(n)\times U(1)\times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$
11c	$Sp(n)\times\{1\}\times Sp(m)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H}$	$\mathbb{H}^{n\times m}$	$\mathfrak{h}_{n;\mathbb{H}}$

For entry 1 in Table 9.19 we have $T' = (\mathfrak{z}')^* \setminus \{0\} = \mathbb{R}\setminus\{0\}$. The generic orbits of $K_\ell = U(n)$ on $\mathfrak{z}''_\ell = \mathfrak{su}(n)$ are those for which all eigenvalues of the matrix in $\mathfrak{su}(n)$ are distinct, so the generic $t = (t', t'') \in T$ are those for which the stabilizer $K_{\ell,t}$ is a maximal torus in K_ℓ. Here note that the irreducible subspaces for $K_{\ell,t}$ on $\mathbb{C}[\mathfrak{v}_\ell] = \mathbb{C}[\mathbb{C}^n]$ each consists of the multiples of a monomial, so the action of $K_{\ell,t}$ on $\mathbb{C}[\mathfrak{v}_\ell]$ is multiplicity free. Now, exactly as in the considerations leading up to

Theorem 9.10, the natural unitary representation of G on $L^2(G/K)$ is a multiplicity free direct integral of lim–irreducible representations.

The argument is more or less the same for entries 11a,b,c in Table 9.19. Here $T' = (\mathfrak{z}')^*\backslash\{0\} = \operatorname{Im}\mathbb{H}\backslash\{0\}$ where $\operatorname{Im}\mathbb{H}$ is identified with its real dual using the inner product $\langle z_1', z_2'\rangle = \operatorname{Re}(z_1'\overline{z_2'})$. The action of $K_\ell = Sp(n)\times\{Sp(1), U(1), \{1\}\}\times Sp(m)$ on $\mathfrak{v}_\ell = \mathbb{H}^n$ is $(k_1, k_2, k_3) : v \mapsto k_1 v\overline{k_2}$, on $\mathfrak{z}_\ell' = \operatorname{Im}\mathbb{H}$ is $(k_1, k_2, k_3) : z' \mapsto k_2 z'\overline{k_2}$, and on $\mathfrak{z}_\ell'' = \mathbb{H}^{n\times m}$ is $(k_1, k_2, k_3) : z'' \mapsto k_1 z'' k_3^*$. Here the composition $\mathbb{H}^n \times \mathbb{H}^n \to \operatorname{Im}\mathbb{H}$ is $(v, w) \mapsto \operatorname{Im} v^* w$ and k_2 is a quaternionic scalar. The stabilizer of $t' \in T'$ in K_ℓ is $Sp(n) \times \{U(1), U(1) \text{ or } \{\pm 1\}, \{1\}\} \times Sp(m)$; generically that stabilizer is $Sp(n) \times \{U(1), \{\pm 1\}, \{1\}\} \times Sp(m)$. Again, as in the considerations leading up to Proposition 9.9 and Theorem 9.10, the natural unitary representation of G on $L^2(G/K)$ is a multiplicity free direct integral of lim–irreducible representations. In summary,

THEOREM 9.20. *Let $\{(G_\ell, K_\ell)\}$ be one of the twenty eight direct systems of* Table 9.15. *Then the unitary direct system $\{L^2(G_\ell), \zeta_{\ell,\widetilde{\ell}}\}$ analogous to that of* Theorem 8.6 *restricts to a unitary direct system $\{L^2(G_\ell/K_\ell), \zeta_{\ell,\widetilde{\ell}}\}$. The Hilbert space $L^2(G/K) := \varinjlim\{L^2(G_\ell/K_\ell), \zeta_{\ell,\widetilde{\ell}}\}$ is the subspace of $L^2(G) := \varinjlim\{L^2(G_\ell), \zeta_{\ell,\widetilde{\ell}}\}$ consisting of right–K–invariant functions, and the natural unitary representation of G on $L^2(G/K)$ is a multiplicity free direct integral of lim–irreducible representations.*

Appendix A: Formal Degrees of Induced Representations

In this section we work out the formal degree of an irreducible induced representation $\operatorname{Ind}_M^L(\gamma)$, where γ is a square integrable representation of M, L/M is compact, and L/M has a positive L–invariant measure. The result, which is suggested by (8.3), is not surprising, but does not seem to be in the literature.

Let L be a separable locally compact group, M a closed subgroup of L, and J a closed central subgroup of M that is normal in L. Suppose that γ is an irreducible square integrable (modulo J) unitary representation of M. Then γ has well defined formal degree $\deg\gamma$ in the usual sense: if u, v, u' and v' belong to the representation space $\mathcal{H}_\gamma$, and if we write $f_{u,v}$ for the coefficient $f_{u,v}(m) = \langle u, \gamma(m)v\rangle_{\mathcal{H}_\gamma}$, then $\langle f_{u,v}, f_{u',v'}\rangle_{L^2(M/J)} = \frac{1}{\deg\gamma}\langle u, u'\rangle\overline{\langle v, v'\rangle}$.

The modular functions Δ_M and Δ_L coincide on M. This is just another way of saying that we have a unique (up to scale) L–invariant Radon measure $d(\ell M)$ on L/M, and thus that we have an L–invariant integral $f \mapsto \int_{L/M} f(\ell)d(\ell M)$ for functions $f \in C_c(L/M)$.

Denote $\widetilde{\gamma} = \operatorname{Ind}_M^L(\gamma)$ and let $\widetilde{\mathcal{H}_\gamma}$ be its representation space. The elements of $\widetilde{\mathcal{H}_\gamma}$ are the measurable functions $\varphi : L \to \mathcal{H}_\gamma$ such that (i) $\varphi(\ell m) = \gamma(m)^{-1}\varphi(\ell)$ (for $\ell \in L$ and $m \in M$) and (ii) $\int_{L/M} ||\varphi(\ell)||^2 d(\ell M) < \infty$. The action $\widetilde{\gamma}$ of L on $\widetilde{\mathcal{H}_\gamma}$ is $[\widetilde{\gamma}(\ell)\varphi](\ell') = \varphi(\ell^{-1}\ell')$. The inner product on $\widetilde{\mathcal{H}_\gamma}$ is $\langle\varphi, \psi\rangle_{\widetilde{\mathcal{H}_\gamma}} = \int_{L/M}\langle\varphi(\ell), \psi(\ell)\rangle_{\mathcal{H}_\gamma} d(\ell M)$.

The group M acts on the fiber $\mathcal{H}_\gamma$ of $\widetilde{\mathcal{H}_\gamma} \to L/M$ at $1M$ by γ, and more generally the stabilizer $\ell M\ell^{-1}$ of the fiber $\ell\mathcal{H}_\gamma$ at ℓM acts on that fiber by $\gamma_\ell(\ell m\ell^{-1})(\ell v) =$

$\ell \cdot \gamma(m)v$. Since J is normal in L it sits in $\ell M \ell^{-1}$ and $\gamma_\ell|_J$ consists of scalar transformations of $\ell\mathcal{H}_\gamma$. Now the representations γ_ℓ all are square integrable mod J and have the same formal degree $\deg\gamma$.

If $u, v \in \mathcal{H}_\gamma$ we have the coefficient function $f_{u,v}(m) = \langle u, \gamma(m)v\rangle_{\mathcal{H}_\gamma}$ on M. If $\varphi, \psi \in \widetilde{\mathcal{H}_\gamma}$ we have the coefficient $\widetilde{f}_{\varphi,\psi}(\ell) = \langle \varphi, \widetilde{\gamma}(\ell)\psi\rangle_{\widetilde{\mathcal{H}_\gamma}}$ on L. Now let $\varphi, \psi \in \widetilde{\mathcal{H}_\gamma}$ such that, as functions from L to $\mathcal{H}_\gamma$, both φ and ψ are continuous. Their support is compact modulo M because L/M is compact. Now $\widetilde{f}_{\varphi,\psi} : L \to \mathbb{C}$ has support that is compact modulo M. For every $\ell \in L$, $\widetilde{f}_{\varphi,\psi}|_{\ell M}$ is a matrix coefficient of γ_ℓ and $\left|\widetilde{f}_{\varphi,\psi}|_{\ell M}\right|^2$ has integral (integrate over $\ell M \ell^{-1}/J$) equal to $\frac{1}{\deg\gamma}||\varphi(\ell)||^2_{\mathcal{H}_\gamma}||\psi(\ell)||^2_{\mathcal{H}_\gamma}$. The functions $\ell \mapsto ||\varphi(\ell)||^2_{\mathcal{H}_\gamma}$ and $\ell \mapsto ||\psi(\ell)||^2_{\mathcal{H}_\gamma}$ are continuous on L/M, so their $L^2(L/M)$ inner product converges. Now

$$\begin{aligned}\int_{L/J} |\widetilde{f}_{\varphi,\psi}(\ell)|^2 d(\ell M) &= \int_{L/M}\left(\int_{\ell M} |\widetilde{f}_{\varphi,\psi}(\ell m)|^2 d(mJ)\right) d(\ell M)\\ &= \frac{1}{\deg\gamma}\int_{L/M} ||\varphi(\ell)||^2_{\mathcal{H}_\gamma}||\psi(\ell)||^2_{\mathcal{H}_\gamma} d(\ell M) < \infty.\end{aligned}$$

Thus $\widetilde{\gamma}$ has a nonzero square integrable (mod J) coefficient. Since it is irreducible, all its coefficients are square integrable (mod J). In particular the formal degree $\deg\widetilde{\gamma}$ is defined. In summary,

Theorem A.1 *Let L be a separable locally compact group, $M \subset L$ a closed subgroup, and J a central subgroup of M that is normal in L. Suppose that L/M is compact and has a nonzero L–invariant Radon measure. Let γ be a square integrable* (modulo J) *irreducible unitary representation of M and* $\deg\gamma$ *its formal degree such that $\widetilde{\gamma} := \mathrm{Ind}_M^L(\gamma)$ is irreducible. Then $\widetilde{\gamma}$ is square integrable* (modulo J) *and it has a well defined formal degree* $\deg\widetilde{\gamma}$.

Of course, if L/M is finite, then Theorem A.1 becomes trivial, and there if we use counting measure on L/M then $\deg\widetilde{\gamma} = |L/M|\deg\gamma$. However, in effect we use the result in (8.3) where L/M is compact but infinite.

Appendix B: A Computational Argument For Theorem 9.1

In this appendix we give a computational proof of Theorem 9.1 for the spaces of Table 9.6. This provides somewhat more information and could be useful in studying the spherical functions. In order to align the K_n–invariants in $L^2(G_n)$ and pass to $L^2(G_n/K_n)$ we will need

Theorem B.1 *Let $\{(G_n, K_n)\}$ be one of the thirteen direct systems of* Table 9.6 *and let $t \in T$. Then the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.*

Proof. If $\dim\mathfrak{z} = 1$ the assertion follows from Carcano's Theorem. That leaves table entries 17, 18, 20a, 20b and 22. In those cases we explicitly decompose $\mathbb{C}[\mathfrak{v}_n]$ under the $K_{n,t}$.

In the case of entry 17, the big factor $K_n'' = Sp(n)$ is irreducible on $\mathfrak{v}_n = \mathbb{H}^n = \mathbb{C}^{2n}$, and $K_{n,t} = U(1) \times Sp(n)$. The representation of $K_{n,t}$ on $\mathbb{C}[gv_n]$ is the same as that of the Gelfand pair listed on row 4 of Table 5.1 and also row 4 of Table 5.2a,b. It follows that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

In the case of entry 20a we have $K_{n,t} = K'_{n,t} \times SU(n)$ where $K'_{n,t}$ is the circle group consisting of all $k_a = \left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right)$ with $|a| = 1$. Then $\mathfrak{v}_n = \mathbb{C}^n_+ \oplus \mathbb{C}^n_-$ where k_a acts on $\mathbb{C}^n_\pm$ as multiplication by $a^{\pm 1}$. Define (i) $\mathbb{P}_{n,1,m_1}$ is the space of polynomials of degree m_1 on $\mathbb{C}^n_+$, (ii) $\mathbb{P}_{n,1,m_2}$ is the space of polynomials of degree m_2 on $\mathbb{C}^n_-$, and (iii) $\mathbb{P}_{n,m} = \mathbb{P}_{n,1,m_1} \otimes \mathbb{P}_{n,1,m_2}$ where $m = (m_1, m_2)$. Then $K_{n,t}$ acts on $\mathbb{P}_{n,1,m_1}$ by $\overset{m_1}{\times}\quad \overset{m_1}{\circ}\!-\!\!\!-\!\circ\!-\!\!\!-\cdots-\!\!\!-\!\circ$ and on $\mathbb{P}_{n,2,m_2}$ by $\overset{-m_2}{\times}\quad \overset{m_2}{\circ}\!-\!\!\!-\!\circ\!-\!\!\!-\cdots-\!\!\!-\!\circ$. Thus the representation of $K_{n,t}$ on $\mathbb{P}_{n,m}$ is $\sum_{p+2q=m_1+m_2}\Big(\overset{m_1-m_2}{\times}\quad \overset{p}{\circ}\!-\!\!\!-\!\overset{q}{\circ}\!-\!\!\!-\!\circ\!-\!\!\!-\cdots-\!\!\!-\!\circ\Big)$. Multiplicities in the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ would lead to equations $m'_1 - m'_2 = m_1 - m_2$, $p' = p$ and $q' = q$ with $p' + 2q' = m'_1 + m'_2$ and $p + 2q = m_1 + m_2$, forcing $m'_1 = m_1$ and $m'_2 = m_2$. That is a contradiction because representation of $K_{n,t}$ on $\mathbb{P}_{n,m}$ is multiplicity free. We conclude that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

In the case of entry 20b, the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free as a consequence of the multiplicity free result for entry 20a.

In the case of entry 22 we view $\mathfrak{v}_n$ as $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$. Here $K_{n,t} = K'_{n,t} \times Sp(n)$ where $K'_{n,t}$ is the 2–torus group consisting of all $k_{a,b} = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $|a| = |b| = 1$. Then $\mathfrak{v}_n = \mathbb{C}^n_+ \oplus \mathbb{C}^n_-$ where $k_{a,b}$ acts on $\mathbb{C}^n_+$ as multiplication by a and on $\mathbb{C}^n_-$ as multiplication by b. The representation of $Sp(n)$ on either of $\mathbb{C}^n_\pm$ is $\overset{1}{\circ}\!-\!\!\!-\!\circ\!-\!\!\!-\cdots-\!\!\!-\!\circ\!\Leftarrow\!\circ$, so the representation on polynomials of degree d is the symmetric power

$$S^d\big(\overset{1}{\circ}-\circ-\cdots-\circ\Leftarrow\circ\big) = \overset{d}{\circ}-\circ-\cdots-\circ\Leftarrow\circ .$$

Now the branching rule (with $r \leqq s$)

$$\overset{r}{\circ}-\circ-\cdots-\circ\Leftarrow\circ \;\otimes\; \overset{s}{\circ}-\circ-\cdots-\circ\Leftarrow\circ$$
$$= \sum_{v=0}^{r-s} \sum_{u=0}^{v} \big(\overset{r+s-2v}{\circ}-\overset{u}{\circ}-\circ-\cdots-\circ\Leftarrow\circ\big)$$

shows that the representation of $K_{n,t}$ on $\mathbb{P}_{n,m}$ is

$$\sum_{v=0}^{|m_1-m_2|} \sum_{u=0}^{v} \big(\overset{m_1}{\times}\quad \overset{m_2}{\times}\quad \overset{m_1+m_2-2v}{\circ}-\overset{u}{\circ}-\circ-\cdots-\circ\Leftarrow\circ\big).$$

If two such irreducible summands are equivalent, for example for (m_1, m_2, u, v) and (m'_1, m'_2, u', v'), then evidently $m_1 = m'_1$, $m_2 = m'_2$, $m_1 + m_2 - 2v = m'_1 + m'_2 - 2v'$ and $u = u'$. We conclude that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

In the case of entry 18 the $K_{n,t}$ correspond to the centralizers of tori (of dimensions 0, 1 or 2) in $Sp(2)$. If we view $Sp(2)$ from its diagram $\overset{\alpha}{\circ}\!\Leftarrow\!\overset{\beta}{\circ}$ the centralizers $K_{n,t} = K'_{n,t} \times Sp(n)$ are given up to conjugacy by $K'_{n,t} = Sp(2)$, $K'_{n,t} \cong U(2)$ with simple root α, $K'_{n,t} \cong U(2)$ with simple root β, and $K'_{n,t} = T^2$ maximal torus of $Sp(2)$. Specifically, if we realize $\mathfrak{sp}(2)$ as the space of 2×2 quaternionic matrices ξ with $\xi + \xi^* = 0$, it has basis

$$\left(\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} j & 0 \\ 0 & 0 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & i \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & j \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right),\ \left(\begin{smallmatrix} 0 & j \\ j & 0 \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} 0 & k \\ k & 0 \end{smallmatrix}\right).$$

Write the complexifying $\mathbb{C}$ as $\mathbb{R} + b\mathbb{R}$ where $b^2 = -1$, to avoid the notation of the quaternions used to express $\mathfrak{sp}(2)$, and let $\varepsilon_1, \varepsilon_2$ be the usual linear functional (that pick out the diagonal entries of a matrix). Using the Cartan subalgebra spanned by $\left(\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & i \end{smallmatrix}\right)$,

$$\begin{aligned}
&\left(\begin{smallmatrix} j \pm bk & 0 \\ 0 & 0 \end{smallmatrix}\right) \text{ is a root vector for } \pm\varepsilon_1;\\
&\left(\begin{smallmatrix} 0 & 0 \\ 0 & j \pm bk \end{smallmatrix}\right) \text{ is a root vector for } \pm\varepsilon_2;\\
&\left(\begin{smallmatrix} 0 & 1 \pm bi \\ -1 \pm bi & 0 \end{smallmatrix}\right) \text{ is a root vector for } \pm(\varepsilon_1 - \varepsilon_2) \text{ and}\\
&\left(\begin{smallmatrix} 0 & j \pm bk \\ j \pm bk & 0 \end{smallmatrix}\right) \text{ is a root vector for } \pm(\varepsilon_1 + \varepsilon_2).
\end{aligned}$$

Thus the centralizer of $z_a := \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)$ is the direct sum of the Cartan subalgebra with the span of

(if $a_1 = 0 \neq a_2$): $\left(\begin{smallmatrix} j & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so the centralizer of z_a is $\left(\begin{smallmatrix} \mathfrak{sp}(1) & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & i\mathbb{R} \end{smallmatrix}\right)$;
(if $a_1 \neq 0 = a_2$): $\left(\begin{smallmatrix} 0 & 0 \\ 0 & j \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$, so the centralizer of z_a is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathfrak{sp}(1) \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} i\mathbb{R} & 0 \\ 0 & 0 \end{smallmatrix}\right)$;
(if $0 \neq a_1 \neq a_2 \neq 0$): the centralizer of z_a is the Cartan $\left(\begin{smallmatrix} i\mathbb{R} & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & i\mathbb{R} \end{smallmatrix}\right)$;
(if $a_1 = a_2 \neq 0$): $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, so the centralizer of z_a is $\mathfrak{u}(2)$;
(if $a_1 = -a_2 \neq 0$): $\left(\begin{smallmatrix} 0 & j \\ j & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & k \\ k & 0 \end{smallmatrix}\right)$, so the centralizer of z_a is $\cong \mathfrak{u}(2)$.

Here the first two cases are $Sp(2)$–conjugate, and the last two cases are $Sp(2)$–conjugate, because the Weyl group consists of all signed permutations of the ϵ_ℓ. Thus we really only have three cases up to $Sp(2)$–conjugacy. The nilpotent group N_n is the (very) generalized Heisenberg group $H_{2,n,0;\mathbb{H}}$ of [**W3**, Section 14.3], so [**W3**, Theorem 14.3.1] shows that z_a corresponds to a square integrable representation of N_n precisely when $a_1 \neq 0 \neq a_2$. Now we need only consider the cases (i) $a_1 = a_2 \neq 0$ and (ii) $0 \neq a_1 \neq a_2 \neq 0$.

Suppose $a_1 = a_2 \neq 0$. Then the corresponding $K_{n,t} = SU(2) \times Sp(n)$. Since $\dim_{\mathbb{R}} \mathfrak{v}_n = \dim_{\mathbb{R}} \mathbb{H}^{2\times n} = 8n$ we are now viewing $\mathfrak{v}$ as $\mathbb{C}^{2\times 2n}$, and the representation of $K_{n,t}$ (as a subgroup of $U(n)$) on $\mathfrak{v}_n$ is the irreducible $(k', k'') : v \mapsto k'v(k'')^{-1}$. As $(N_n \rtimes K_{n,t}, K_{n,t})$ is a Gelfand pair, Carcano's Theorem says that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

Suppose $0 \neq a_1 \neq a_2 \neq 0$. Then $K_{n,t} = U(1) \times U(1) \times Sp(n)$ and we view $\mathfrak{v}_n$ as $\mathbb{C}^{2\times 2n} = \mathbb{C}^{2n}_1 \oplus \mathbb{C}^{2n}_2$ where the action τ of $K_{n,t}$ is $\tau(k'_1, k'_2, k'')(v_1 \oplus v_2) = (k'_1 v_1 (k'')^{-1} \oplus k'_2 v_2 (k'')^{-1})$. The representation τ has diagram

$$\Big(\overset{1}{\times} \quad \times \quad \overset{1}{\circ}\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circ\!\Leftarrow\!\circ \Big) \oplus \Big(\times \quad \overset{1}{\times} \quad \overset{1}{\circ}\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\circ\!\Leftarrow\!\circ \Big).$$

Now decomposition of the symmetric powers $S^m(\tau)$ goes exactly as in the case of entry 22, and we conclude that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

We have shown, by direct computation for each of the series of Table 9.6, that the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free. □

One can carry out similar calculations for the series of Table 9.15 showing by direct computation that in each case the representation of $K_{n,t}$ on $\mathbb{C}[\mathfrak{v}_n]$ is multiplicity free.

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