Infinite-Dimensional Multiplicity-Free Spaces I: Limits of Compact Commutative Spaces

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Summary. We study direct limits $(G, K) = \varinjlim (G_n, K_n)$ of compact Gelfand pairs. First, we develop a criterion for a direct limit representation to be a multiplicity-free discrete direct sum of irreducible representations. Then we look at direct limits $G/K = \varinjlim G_n/K_n$ of compact riemannian symmetric spaces, where we combine our criterion with the Cartan–Helgason theorem to show in general that the regular representation of $G = \varinjlim G_n$ on a certain function space $\varinjlim L^2(G_n/K_n)$ is multiplicity-free. That method is not applicable for direct limits of nonsymmetric Gelfand pairs, so we introduce two other methods. The first, based on "parabolic direct limits" and "defining representations", extends the method used in the symmetric space case. The second uses some (new) branching rules from finite-dimensional representation theory. In both cases we define function spaces $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$ to which our multiplicity-free criterion applies.

Key words: Lie group, Gelfand pair, commutative space, direct limit representation, multiplicity-free representation.

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1 Introduction

Gelfand pairs (G, K), and the corresponding "commutative" homogeneous spaces G/K, form a natural extension of the class of riemannian symmetric spaces. We recall some of their basic properties. Let G be a locally compact topological group, K a compact subgroup, and $M = G/K$. Then the following conditions are equivalent; see [W2007, Theorem 9.8.1].

1. (G, K) is a Gelfand pair, i.e., $L^1(K\backslash G/K)$ is commutative under convolution.

2. If $g, g' \in G$ then $\mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$ (convolution of Dirac measures on $K\backslash G/K$).

3. $C_c(K\backslash G/K)$ is commutative under convolution.

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4. The measure algebra $\mathcal{M}(K\backslash G/K)$ is commutative.
5. The representation of G on $L^2(M)$ is multiplicity-free.

If G is a connected Lie group one can also add

6. The algebra of G-invariant differential operators on M is commutative.

When we drop the requirement that K be compact, conditions 1, 2, 3, and 4 lose their meaning because integration on M or $K\backslash G/K$ no longer corresponds to integration on G. Condition 5 still makes sense as long as K is unimodular in G. Condition 6 remains meaningful (and useful) whenever G is a connected Lie group; there one speaks of "generalized Gelfand pairs".

In this paper we look at some cases where G and K are not locally compact, in fact are infinite dimensional, and show in those cases that the multiplicity-free condition 5 is satisfied. We first discuss a multiplicity-free criterion that can be viewed as a variation on some of the combinatoric considerations of [DPW2002]; it emerged from some discussions with Ivan Penkov in another context. We then apply the criterion in the setting of symmetric spaces, proving that direct limits of compact symmetric spaces are multiplicity-free. This applies in particular to infinite-dimensional real, complex, and quaternionic Grassmann manifolds, and it uses some basic symmetric space structure theory. In particular, our argument for direct limits of compact riemannian symmetric spaces makes essential use of the Cartan–Helgason theorem, and thus does not extend to direct limits of nonsymmetric Gelfand pairs.

In order to extend the multiplicity-free result to at least some direct limits of nonsymmetric Gelfand pairs, we define the notion of "defining representation" for a direct system $\{(G_n, K_n)\}$, where the G_n are compact Lie groups and the K_n are closed subgroups. We show how a defining representation for $\{(G_n, K_n)\}$ leads to a direct system $\{\mathcal{A}(G_n/K_n)\}$ of $\mathbb{C}$-valued polynomial function algebras, a continuous function completion $\{\mathcal{C}(G_n/K_n)\}$, and a Lebesgue space completion $\{L^2(G_n/K_n)\}$. The direct limit spaces $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$ are the function spaces on $G/K = \varinjlim G_n/K_n$ which we study as G-modules.

Next, we prove the multiplicity-free property, for the action of G on $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$, when $\{(G_n, K_n)\}$ is one of several families of Gelfand pairs related to spheres and Grassmann manifolds. We prove the multiplicity-free property for three other types of direct limits of Gelfand pairs.

Finally we summarize the results, extending them slightly by including the possibility of enlarging the K_n within their G_n-normalizers without losing the property that $\{K_n\}$ is a direct system.

Our proofs of the multiplicity-free condition, for some direct limits of nonsymmetric Gelfand pairs, use a number of branching rules, new and old, for finite-dimensional representations. This lends a certain *ad hoc* flavor which I hope can be avoided in the future.

Direct limits $(G, K) = \varinjlim(G_n, K_n)$ of riemannian symmetric spaces were studied by Ol'shanskii from a very different viewpoint [Ol1990]. He viewed the

G_n inside dual reductive pairs and examined their action on Hilbert spaces of Hermite polynomials. Ol'shanskii made extensive use of factor representation theory and Gaussian measure, obtaining analytic results on limit-spherical functions. See Faraut [Fa2006] for a discussion of spherical functions in the setting of direct limit pairs. In contrast to the work of Ol'shanskii and Faraut, we use the rather simple algebraic method of renormalizing formal degrees of representations to obtain isometric embeddings $L^2(G_n/K_n) \hookrightarrow L^2(G_{n+1}/K_{n+1})$. That leads directly to our multiplicity-free results.

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2 Direct limit groups and representations

We consider direct limit groups $G = \varinjlim G_n$ and, their direct limit representations $\pi = \varinjlim \pi_n$. This means that π_n is a representation of G_n on a vector space V_n, that the V_n form a direct system, and that π is the representation of G on $V = \varinjlim V_n$ given by $\pi(g)v = \pi_n(g_n)v_n$ whenever n is sufficiently large that $V_n \hookrightarrow V$ and $G_n \hookrightarrow G$ send v_n to v and g_n to g. The formal definition amounts to saying that π is well defined.

It is clear that a direct limit of irreducible representations is irreducible, but there are irreducible representations of direct limit groups that cannot be formulated as direct limits of irreducible finite-dimensional representations. This is a combinatoric matter and is discussed extensively in [DPW2002]. The following definition is closely related to those combinatorics but applies to a somewhat simpler situation.

Definition 1 We say that a representation π of G is *limit aligned* if it has form $\varinjlim \pi_n$ in such a way that (i) each π_n is a direct sum of primary representations, and (ii) the corresponding representation spaces $V = \varinjlim V_n$ have the property that every primary subspace of V_n is contained in a primary subspace of V_{n+1}.

Theorem 2 *A limit-aligned representation $\pi = \varinjlim \pi_n$ of $G = \varinjlim G_n$ is a direct sum of primary representations. If the π_n are multiplicity free, then π is a multiplicity-free direct sum of irreducible representations.*

Proof. Let $V = \varinjlim V_n$ be the representation spaces. Decompose $V_n = \sum_{\alpha \in I_n} V_{n,\alpha}$, where the $V_{n,\alpha}$ are the subspaces for the primary summands of π_n. Write $\pi_{n,\alpha}$ for the representation of G_n on $V_{n,\alpha}$, so $\pi_n = \sum_{\alpha \in I_n} \pi_{n,\alpha}$.

Since π is limit aligned, i.e., since each $V_{n,\alpha} \subset V_{n+1,\beta}$ for some $\beta \in I_{n+1}$, we may assume $I_n \subset I_{n+1}$ in such a way that each $V_{n,\alpha} \subset V_{n+1,\alpha}$ for every $\alpha \in I_n$. Now $V = \sum_{\alpha \in I} V_\alpha$, discrete sum, where $I = \bigcup I_n$ and $V_\alpha = \bigcup V_{n,\alpha}$. The sum is direct, for if $u_1 + u_2 + \cdots + u_r = 0$ where $u_i \in V_{\alpha_i}$ for distinct indices $\alpha_1, \ldots, \alpha_r$, then we take n sufficiently large so that each $u_i \in V_{n,\alpha_i}$ and conclude that $u_1 = u_2 = \cdots = u_r = 0$. Thus π is the discrete direct sum of the representations $\pi_\alpha = \varinjlim \pi_{n,\alpha}$ of G on V_α.

Let $C_\alpha = \{X : V_\alpha \to V_\alpha \text{ linear } \mid X\pi_\alpha(g) = \pi_\alpha(g)X \text{ for all } g \in G\}$, the commuting algebra of π_α. If π_α fails to be primary, then C_α contains nontrivial commuting ideals C'_α and C''_α. Then for n large, the stabilizer $N_{C_\alpha}(V_{n,\alpha})$ of $V_{n,\alpha}$ in C_α contains nontrivial commuting ideals $N_{C'_\alpha}(V_{n,\alpha})$ and $N_{C''_\alpha}(V_{n,\alpha})$. That is impossible because $\pi_{n,\alpha}$ is primary. We have proved that π is the discrete direct sum of primary representations π_α.

If the π_n are multiplicity free, then the $\pi_{n,\alpha}$ are irreducible and it is immediate that the $\pi_\alpha = \varinjlim \pi_{n,\alpha}$ are irreducible. This completes the proof of Theorem 2. □

A direct limit of irreducible representations is irreducible, but it is not immediate that every irreducible direct limit representation can be rewritten as a direct limit of irreducible representations. With this and Theorem 2 in mind, we extend Definition 1 as follows.

Definition 3 A representation π of $G = \varinjlim G_n$ is *lim irreducible* if it has form $\pi = \varinjlim \pi_n$ where each π_n is an irreducible representation of G_n. Similarly, π is *lim primary* if it has form $\pi = \varinjlim \pi_n$ where each π_n is a primary representation of G_n.

Theorem 4 *Consider a representation $\pi = \varinjlim \pi_n$ of $G = \varinjlim G_n$ with representation space $V = \varinjlim V_n$. Suppose that each π_n is a multiplicity-free direct sum of irreducible highest weight representations. Suppose for $n \gg 0$ that the direct system map $V_{n-1} \hookrightarrow V_n$ sends G_{n-1}-highest weight vectors to G_n-highest weight vectors. Then π is a multiplicity-free direct sum of lim-irreducible representations of G.*

Proof. By hypothesis each π_n is a direct sum of primary representations which, in fact, are irreducible highest weight representations. We recursively choose highest weight vectors so that $\pi_{n-1} = \sum \pi_{\lambda,n-1}$, where $\pi_{\lambda,n-1}$ has highest weight vector $v_{\lambda,n-1} \in V_{n-1}$ that maps to a highest weight vector $v_{\lambda,n} \in V_n$ of an irreducible constituent $\pi_{\lambda,n}$ of π_n. This exhibits π as a limit-aligned direct sum because it embeds the summand $V_{\lambda,n-1}$ of V_{n-1} into the irreducible summand of V_n that contains $v_{\lambda,n}$. Now Theorem 2 shows that π is a multiplicity-free direct sum of lim-irreducible representations of G. □

3 Limit theorem for symmetric spaces

We now apply Theorems 2 and 4 to direct limits of compact riemannian symmetric spaces. Fix a direct system of compact connected Lie groups G_n and subgroups K_n such that each (G_n, K_n) is an irreducible riemannian symmetric pair. Suppose that the corresponding compact symmetric spaces

$M_n = G_n/K_n$ are connected and simply connected. Up to renumbering and passage to a common cofinal subsequence, the only possibilities are as given in the following table.

compact irreducible riemannian symmetric $M_n = G_n/K_n$				
	G_n	K_n	RankM_n	DimM_n
1	$SU(n) \times SU(n)$	diagonal $SU(n)$	$n-1$	n^2-1
2	$Spin(2n+1) \times Spin(2n+1)$	diagonal $Spin(2n+1)$	n	$2n^2+n$
3	$Spin(2n) \times Spin(2n)$	diagonal $Spin(2n)$	n	$2n^2-n$
4	$Sp(n) \times Sp(n)$	diagonal $Sp(n)$	n	$2n^2+n$
5	$SU(p+q),\ p=p_n, q=q_n$	$S(U(p) \times U(q))$	$\min(p,q)$	$2pq$
6	$SU(n)$	$SO(n)$	$n-1$	$\frac{(n-1)(n+2)}{2}$
7	$SU(2n)$	$Sp(n)$	$n-1$	$2n^2-n-1$
8	$SO(p+q),\ p=p_n, q=q_n$	$SO(p) \times SO(q)$	$\min(p,q)$	pq
9	$SO(2n)$	$U(n)$	$[\frac{n}{2}]$	$n(n-1)$
10	$Sp(p+q),\ p=p_n, q=q_n$	$Sp(p) \times Sp(q)$	$\min(p,q)$	$4pq$
11	$Sp(n)$	$U(n)$	n	$n(n+1)$

(5)

Fix one of the direct systems $\{(G_n, K_n)\}$ of Table 5. Then we have involutive automorphisms θ_n of G_n such that the Lie algebras decompose into ± 1 eigenspaces of the θ_n,

$$\mathfrak{g}_n = \mathfrak{k}_n + \mathfrak{s}_n \text{ in such a way that } \mathfrak{k}_n = \mathfrak{g}_n \cap \mathfrak{k}_{n+1} \text{ and } \mathfrak{s}_n = \mathfrak{g}_n \cap \mathfrak{s}_{n+1}.$$

Then we recursively construct a system of maximal abelian subspaces

$$\mathfrak{a}_n: \text{ maximal abelian subspace of } \mathfrak{s}_n \text{ such that } \mathfrak{a}_n = \mathfrak{g}_n \cap \mathfrak{a}_{n+1}.$$

The restricted root systems

$$\Sigma_n = \Sigma_n(\mathfrak{g}_n, \mathfrak{a}_n): \text{ the system of } \mathfrak{a}_n\text{-roots on } \mathfrak{g}_n$$

form an inverse system of linear functionals: $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is the system $\varprojlim \Sigma_n$ of linear functionals on $\mathfrak{a} = \varinjlim \mathfrak{a}_n$. In this inverse system, the multiplicities of the restricted roots will increase without bound, but we can make consistent choices of positive subsystems

$$\Sigma_n^+ = \Sigma_n^+(\mathfrak{g}_n, \mathfrak{a}_n): \text{ system of positive } \mathfrak{a}_n\text{-roots on } \mathfrak{g}_n$$

so that $\Sigma_n^+ \subset \Sigma_m^+|_{\mathfrak{a}_n}$ for $m \geqq n \geqq n_0$. Consider the reduced root system

$$\Sigma_{0,n} = \{\alpha \in \Sigma_n \mid 2\alpha \notin \Sigma_n\}$$

and its positive subsystem $\Sigma_{0,n}^+ := \Sigma_{0,n} \cap \Sigma_n^+$. Examining the tables of Araki ([Ar1962], or referring to [He1978, pp. 532–534] or [W1980, pp. 90–93]), we see the following.

Lemma 6 *Suppose that G_n is simple. Then there are only two possibilities.*
(a) $\Sigma_{0,n} = \Sigma_n$*; in other words, if* $\alpha \in \Sigma_n$ *then* $2\alpha \notin \Sigma_n$.

(b) $\Sigma_{0,n} \neq \Sigma_n$*; there is exactly one simple root* $\psi_{1,n}$ *for* Σ_n^+ *such that* $2\psi_{1,n} \in \Sigma_n$*, and* $\psi_{1,n}$ *is at the end of the Dynkin diagram of* Σ_n^+ *opposite to the end where roots are added to obtain the diagram of* Σ_{n+1}^+.

Then the corresponding simple root systems for $\Sigma_{0,n}^+$, which we denote

$$\Psi_n = \Psi_n(\mathfrak{g}_n, \mathfrak{a}_n) = \{\psi_{1,n}, \ldots, \psi_{r_n,n}\}: \text{ simple reduced } \mathfrak{a}_n\text{-roots on } \mathfrak{g}_n$$

satisfy $\Psi_n \subset \Psi_m|_{\mathfrak{a}_n}$ for $m \geqq n \geqq n_0$ as well. Here $r_n = \dim \mathfrak{a}_n$, rank of M_n.

In case (a) of Lemma 6, Ψ_n is a simple root system for Σ_n^+, but in case (b) the corresponding simple root system for Σ_n^+ is $\{\frac{1}{2}\psi_{1,n}, \psi_{2,n}, \ldots, \psi_{r_n,n}\}$. In both cases $\Psi_n \subset \Psi_m|_{\mathfrak{a}_n}$ for $m \geqq n \geqq n_0$. More precisely, if $\psi_{j,n} \in \Psi_n$ and $m \geqq n$, then there is just one element $\psi \in \Psi_m$ with $\psi|_{\mathfrak{a}_n} = \psi_{j,n}$. In other words, we may (and do) recursively enumerate the simple root systems Ψ_n so that

$$\text{if } m \geqq n \text{ and } \psi_{j,n} \in \Psi_n, \text{ then } \psi_{j,m} \in \Psi_m \text{ satisfies } \psi_{j,m}|_{\mathfrak{a}_n} = \psi_{j,n},$$

retaining the convention that in case (b) of Lemma 6 the $\frac{1}{2}\psi_{1,n}$ are roots. Later we will use the fact that

$$\text{in case (b) of Lemma 6, if } m \geqq n \text{ and } \tfrac{1}{2}\psi_{1,n} \in \Sigma_n^+, \text{ then } \tfrac{1}{2}\psi_{1,m} \in \Sigma_m^+. \tag{7}$$

Recursively define θ_n-stable Cartan subalgebras of $\mathfrak{h}_n = \mathfrak{t}_n + \mathfrak{a}_n$ of $\mathfrak{g}_n$ with $\mathfrak{h}_n = \mathfrak{g}_n \cap \mathfrak{h}_{n+1}$. Here $\mathfrak{t}_n$ is a Cartan subalgebra of the centralizer $\mathfrak{m}_n$ of $\mathfrak{a}_n$ in $\mathfrak{k}_n$. Now recursively construct positive root systems $\Sigma^+(\mathfrak{m}_n, \mathfrak{t}_n)$ such that if $\alpha \in \Sigma^+(\mathfrak{m}_{n+1}, \mathfrak{t}_{n+1})$, then either $\alpha|_{\mathfrak{t}_n} = 0$ or $\alpha|_{\mathfrak{t}_n} \in \Sigma^+(\mathfrak{m}_n, \mathfrak{t}_n)$. Then we have positive root systems

$$\Sigma^+(\mathfrak{g}_n, \mathfrak{h}_n) = \{\alpha \in i\mathfrak{h}_n^* \mid \alpha|_{\mathfrak{a}_n} = 0 \text{ or } \alpha|_{\mathfrak{a}_n} \in \Sigma_n^+(\mathfrak{g}_n, \mathfrak{a}_n)\},$$

the corresponding simple root systems, and the resulting systems of fundamental highest weights.

The Cartan–Helgason theorem says that the irreducible representation π_λ of $\mathfrak{g}_n$ of highest weight λ gives a summand of the representation of G_n on $L^2(M_n)$ if and only if (i) $\lambda|_{\mathfrak{t}_n} = 0$, so we may view λ as an element of $i\mathfrak{a}_n^*$, and (ii) if $\alpha \in \Sigma_n^+(\mathfrak{g}_n, \mathfrak{a}_n)$ then $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is an integer $\geqq 0$. Condition (i) persists under restriction $\lambda \mapsto \lambda|_{\mathfrak{h}_{n-1}}$ because $\mathfrak{t}_{n-1} \subset \mathfrak{t}_n$. Given (i), condition (ii) says that $\frac{1}{2}\lambda$ belongs to the weight lattice of $\mathfrak{g}_n$, so its restriction to $\mathfrak{h}_{n-1}$ exponentiates to a well-defined function on the corresponding maximal torus of G_{n-1} and thus belongs to the weight lattice of $\mathfrak{g}_{n-1}$. Given condition (i) now (7) says that condition (ii) persists under restriction $\lambda \mapsto \lambda|_{\mathfrak{h}_{n-1}}$. With this in mind, we define linear functionals $\xi_{n,j} \in i\mathfrak{a}_n^*$ by

$$\frac{\langle\xi_{n,i},\psi_{n,j}\rangle}{\langle\psi_{n,j},\psi_{n,j}\rangle} = \delta_{i,j} \text{ for } 1 \leqq j \leqq r_n, \text{ except that } \frac{\langle\xi_{n,1},\psi_{n,1}\rangle}{\langle\psi_{n,1},\psi_{n,1}\rangle} = 2 \text{ if } 2\psi_{n,1} \in \Sigma_n.$$

The weights $\xi_{n,j}$ are the *class 1 fundamental highest weights* for $(\mathfrak{g}_n, \mathfrak{k}_n)$. We denote

$$\Xi_n = \Xi_n(\mathfrak{g}_n, \mathfrak{t}_n, \mathfrak{a}_n) = \{\xi_{n,1}, \ldots, \xi_{n,r_n}\}\,.$$

Define

$$\Lambda_n = \Lambda(\mathfrak{g}_n, \mathfrak{k}_n, \mathfrak{a}_n) = \left\{\sum n_k \xi_k \mid \xi_k \in \Xi_n \text{ and } n_k \in \mathbb{Z}, n_k \geqq 0\right\}.$$

This is the set of highest weights for representations of G_n on $L^2(M_n)$, and we have just verified that $\Lambda_n|_{\mathfrak{a}_{n-1}} \subset \Lambda_{n-1}$.

Lemma 8 *For n sufficiently large, and passing to a cofinal subsequence, if $\xi \in \Xi_{n-1}$ there is a unique $\xi' \in \Xi_n$ such that $\xi'|_{a_{n-1}} = \xi$.*

Proof. In the group manifold cases, lines 1, 2, 3, and 4 of Table 5, express $G_n = L_n \times L_n$, and note that the complexification $(L_{n-1})_{\mathbb{C}}$ is the semisimple component of a parabolic subgroup of $(L_n)_{\mathbb{C}}$. The restricted root and weight systems of (G_n, K_n) are the same as the unrestricted root and weight systems of L_n, and the assertion follows.

In the Grassmann manifold cases, lines 5, 8, and 10 of Table 5, we first consider the case where $\{p_n\}$ is bounded. Then we may assume $p_n = p$ constant and q_n increasing for $n \gg 0$. Thus $\mathfrak{a}_{n-1} = \mathfrak{a}_n$, $\Psi_{n-1} = \Psi_n$ (though the multiplicities of the restricted roots will increase), and $\Xi_{n-1} = \Xi_n$. The assertion now is immediate.

In the Grassmann manifold cases we may now assume that both p_n and q_n are unbounded. If $p_n = q_n$ on a cofinal sequence of indices n we may assume $p_n = q_n$ for all n, so Ψ_n is always of type C_{r_n}. Then we interpolate pairs and renumber so that $p_n = q_n = p_{n-1} + 1 = q_{n-1} + 1$ for all n and notice that the Dynkin diagram inclusions $C_{r-1} \subset C_r$ are uniquely determined by the integer r. If $p_n = q_n$ for only finitely many n and $p_n < q_n$ on a cofinal sequence of indices n we may assume that $r_n = p_n < q_n$ for all n, so Ψ_n is always of type B_{r_n}. Then we interpolate $(p_{n-1}, qn - 1), (p_{n-1}, q_n), (p_{n-1}+1, q_n), \ldots, (p_n, q_n)$ and renumber so that we always have $r_n = r_{n-1}$ or $r_n = r_{n-1} + 1$ and notice that the Dynkin diagram inclusions $B_{r-1} \subset B_r$ are uniquely determined by the integer r. If $p_n = q_n$ for only finitely many n and also $p_n = q_n$ for only finitely many n, then $p_n > q_n$ on a cofinal sequence of indices n, and we may assume $p_n > q_n = r_n$ for all n. We interpolate as before, exchanging the rôles of p_ℓ and q_ℓ, and we note again that the Dynkin diagram inclusions $B_{r-1} \subset B_r$ are uniquely determined by the integer r. Thus in all cases the fundamental highest weights restrict as asserted.

In the lower rank cases, lines 6 and 7 of Table 5, Ψ_n is of type A_{n-1}, so again restriction to $\mathfrak{a}_{n-1}$ has the required property. In the hermitian symmetric case, line 11 of Table 5, $\mathfrak{a}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$ and $\mathfrak{g}_{n-1}$ complexifies to the semisimple part of a parabolic subalgebra of $(\mathfrak{g}_n)_{\mathbb{C}}$, so the assertion follows as in the group manifold cases. In the remaining case, line 9 of Table 5, Ψ_n is of type $C_{n/2}$ for n even, type $B_{(n-1)/2}$ for n odd. Passing to a cofinal subsequence we may assume n always even or always odd, and we may interpolate as necessary by pairs so that n increases in steps of 2. Then, again, there is no choice about the restriction, and the assertion follows. $\square$

In view of Lemma 8, after passage to a cofinal subsequence and renumbering, we may assume the sets Ξ_n ordered so that

$$\begin{aligned}\Xi_n = \Xi(\mathfrak{g}_n, \mathfrak{k}_n, \mathfrak{a}_n) = &\{\xi_{1,n}, \dots, \xi_{r_n,n}\} \text{ with} \\ &\xi_{\ell,n-1} = \xi_{\ell,n}|_{\mathfrak{a}_{n-1}} \text{ for } 1 \leqq \ell \leqq r_{n-1}.\end{aligned} \tag{9}$$

Now define

$$\begin{aligned}&\mathcal{I}_n : \text{ all } r_n\text{-tuples } I = (i_1, \dots, i_{r_n}) \text{ of non-negative integers,} \\ &\mathcal{I} = \varinjlim \mathcal{I}_n \text{ where } \mathcal{I}_n \hookrightarrow \mathcal{I}_m \text{ by } (i_1, \dots, i_{r_n}) \mapsto (i_1, \dots, i_{r_n}, 0, \dots, 0), \\ &\pi_{I,n} : \text{ rep of } G_n \text{ with highest weight } \xi_I = i_1\xi_1 + \dots + i_p\xi_{r_n}, \\ &\pi_I = \varinjlim \pi_{I,n} \text{ for } I \in \mathcal{I}.\end{aligned} \tag{10}$$

According to the Cartan–Helgason theorem, the $\pi_{I,n}$ exhaust the representations of G_n on $L^2(M_n)$. Denote

$$V_{I,n} : \text{ representation space for the abstract representation } \pi_{I,n}. \tag{11}$$

Then $V_{I,n}$ occurs with multiplicity 1 in the representation of G_n on $L^2(M_n)$. In effect, the representation of G_n on $L^2(M_n)$ is multiplicity-free, and $L^2(M_n) \cong \bigoplus_{I\in\mathcal{I}} V_{I,n}$ as a G_n-module. However, in the following we must distinguish between $\bigoplus_{I\in\mathcal{I}} V_{I,n}$ as a G_n-module and $L^2(M_n)$ as a space of functions.

Let $\mathcal{U}(\mathfrak{g}_n)$ denote the (complex) universal enveloping algebra of $\mathfrak{g}_n$. Let v_{n+1} be a highest weight unit vector in $V_{I,n+1}$ for the action of G_{n+1}. Then we have the G_n-submodule $\mathcal{U}(\mathfrak{g}_n)(v_{n+1}) \subset V_{I,n+1} \subset L^2(M_{n+1})$.

If $u, v \in V_{I,n}$ we write $f_{u,v;I,n}$ for $g \mapsto \langle u, \pi_{I,n}(g)v\rangle$, the matrix coefficient function on G_n. These matrix coefficient functions span a space $E_{I,n}$ that is invariant under left and right translations by elements of G_n. As a $(G_n \times G_n)$-module $E_{I,n} \cong V_{I,n} \boxtimes V_{I,n}^*$. If u_n^* is the (unique up to scalar multiplication) K_n-fixed unit vector in $V_{I,n}^*$, then the right K_n-fixed functions in $E_{I,n}$ form the left G_n-module $E_{I,n}^{K_n} \cong V_{I,n} \otimes u_n^*\mathbb{C} \cong V_{I,n}$.

In the following, it is crucial to distinguish between the abstract representation space $V_{I,n}$ and the space $E_{I,n}^{K_n}$ of functions on G_n/K_n.

We normalize the Haar measure on G_n (and the resulting measure in M_n) to total mass 1. If $u, v, u', v' \in V_{I,n}$, then we have the Schur orthogonality relation $\langle f_{u,v;I,n}, f_{u',v';I,n}\rangle|_{L^2(G_n)} = (\deg \pi_{I,n})^{-1}\langle u, u'\rangle\overline{\langle v, v'\rangle}$.

Theorem 12 *The space $E_{I,n}^{K_n}$ of functions on G_n/K_n is G_n-module equivalent to $\mathcal{U}(\mathfrak{g}_n)(v_{n+1} \otimes u_{n+1}^*) \subset E_{I,n+1}^{K_{n+1}}$. We map $E_{I,n}^{K_n}$ into $E_{I,n+1}^{K_{n+1}}$ as follows. Let $\{w_j\}$ be a basis of $V_{I,n}$ and define*

$$\psi'_{n+1,n}\left(\sum c_j\, f_{w_j,u_n^*;I,n}\right) = (\deg \pi_{I,n+1}/\deg \pi_{I,n})^{1/2}\sum c_j\, f_{w_j,u_{n+1}^*;I,n+1} \in E_{I,n+1}^{K_{n+1}}. \tag{13}$$

Then $\psi'_{n+1,n} : E_{I,n}^{K_n} \to E_{I,n+1}^{K_{n+1}}$ is G_n-equivariant and is isometric for L^2 norms on G_n/K_n and G_{n+1}/K_{n+1}. In particular, as I varies with n fixed, $\psi'_{n+1,n} : L^2(G_n/K_n) \to L^2(G_{n+1}/K_{n+1})$ is a G_n-equivariant isometry.

Proof. We have $a(v_{n+1}) = \xi_I(a)v_{n+1}$ for all $a \in \mathfrak{a}$. The inclusion $G_n \hookrightarrow G_{n+1}$ is G_n-equivariant, so restriction of functions is G_n-equivariant and thus is A-equivariant, and $(v_{n+1} \otimes u_{n+1}^*)|_{M_n}$ is a ξ_I-weight vector in $L^2(M_n)$. If α is a positive restricted root for G_{n+1} and $e_\alpha \in \mathfrak{g}_{n+1}$ is an α root vector, then $e_\alpha(v_{n+1}) = 0$. If α is already a root for G_n and if $e_\alpha \in \mathfrak{g}_n$, then we have $e_\alpha((v_{n+1} \otimes u_{n+1}^*)|_{M_n}) = 0$. Thus either the restriction $(v_{n+1} \otimes u_{n+1}^*)|_{M_n} = 0$ or $(v_{n+1} \otimes u_{n+1}^*)|_{M_n}$ is a highest weight vector in $E_{I,n}^{K_n}$.

Suppose that $(v_{n+1} \otimes u_{n+1}^*)|_{M_n} = 0$ as a function on $M_n = G_n/K_n$. Denote $V'_n = \mathcal{U}(\mathfrak{g}_n)(v_{n+1})$. It is a cyclic highest weight module for G_n with highest weight ξ_I, and $(V'_n \otimes u_{n+1}^*\mathbb{C})|_{M_n} = 0$, and it contains a unique (up to scalar multiple) K_n-invariant unit vector u'_n. The coefficient function $\varphi(g) := \langle u'_n, \pi_{I,n+1}(g)u'_n\rangle_{V'_n} = \int_{G_n}(u'_n \otimes u_n^*)(x)\overline{(u'_n \otimes u_n^*)(x^{-1}g)}dx$ is identically zero because the $u'_n(x)$ factor in the integrand vanishes for $x \in G_n$. But $\varphi|_{G_n}$ is the positive definite (G_n, K_n)-spherical function on G_n for the representation $\pi_{I,n}$, and in particular $\varphi(1) = 1$. That is a contradiction. We conclude that $(v_{n+1} \otimes u_{n+1}^*)|_{M_n} \neq 0$, so $(v_{n+1} \otimes u_{n+1}^*)|_{M_n}$ is a highest weight vector in $E_{I,n}^{K_n}$. In particular, $E_{I,n}^{K_n} \cong (V'_n \otimes u_{n+1}^*\mathbb{C})|_{M_n} \subset E_{I,n+1}^{K_{n+1}}|_{M_n}$. That is the equivariant map assertion. The unitary map assertion follows by Schur orthogonality. □

Theorem 12 gives isometric embeddings $\psi'_{m,n} : L^2(M_n) \to L^2(M_m)$ for $n \leqq m$. By construction, $\psi'_{m,n}$ is G_n-equivariant. Define

$$L^2(G/K) = \varinjlim\{L^2(G_n/K_n), \psi'_{m,n}\} : \text{ direct limit in the category of Hilbert spaces and unitary injections.} \tag{14}$$

We emphasize the renormalizations of Theorem 12. Without those renormalizations we lose the Hilbert space structure of $L^2(G/K)$.

Theorem 15 *The left regular representation of G on $L^2(G/K)$ is a multiplicity-free discrete direct sum of lim-irreducible representations. Specifically, that left regular representation is $\sum_{I\in\mathcal{I}} \pi_I$, where $\pi_I = \varinjlim \pi_{I,n}$ is the irreducible representation of G with highest weight $\xi_I := \sum i_r\xi_r$. This applies to all the direct systems of* Table 5. *In particular, we have the thirteen infinite-dimensional multiplicity-free spaces*

1. $SU(\infty) \times SU(\infty)/\mathit{diag}\, SU(\infty)$, *group manifold* $SU(\infty)$,
2. $Spin(\infty) \times Spin(\infty)/\mathit{diag}\, Spin(\infty)$, *group manifold* $Spin(\infty)$,
3. $Sp(\infty) \times Sp(\infty)/\mathit{diag}\, Sp(\infty)$, *group manifold* $Sp(\infty)$,
4. $SU(p+\infty)/S(U(p) \times U(\infty))$, $\mathbb{C}^p$ *subspaces of* $\mathbb{C}^\infty$,
5. $SU(2\infty)/[S(U(\infty) \times U(\infty))]$, $\mathbb{C}^\infty$ *subspaces of infinite codim in* $\mathbb{C}^\infty$,
6. $SU(\infty)/SO(\infty)$,
7. $SU(2\infty)/Sp(\infty)$,
8. $SO(p+\infty)/[SO(p) \times SO(\infty)]$, *oriented* $\mathbb{R}^p$ *subspaces of* $\mathbb{R}^\infty$,
9. $SO(2\infty)/[SO(\infty) \times SO(\infty)]$, $\mathbb{R}^\infty$ *subspaces of infinite codim in* $\mathbb{R}^\infty$,
10. $SO(2\infty)/U(\infty)$,
11. $Sp(p+\infty)/[Sp(p) \times Sp(\infty)]$, $\mathbb{H}^p$ *subspaces of* $\mathbb{H}^\infty$,
12. $Sp(2\infty)/[Sp(\infty) \times Sp(\infty)]$, $\mathbb{H}^\infty$ *subspaces of infinite codim in* $\mathbb{H}^\infty$,
13. $Sp(\infty)/U(\infty)$.

Proof. λ is limit aligned by Theorem 12. Denote $V_I = \bigcup V_{I,n} = \varinjlim V_{I,n}$. Then G acts irreducibly on it by $\pi_I = \varinjlim \pi_{I,n}$, and the various π_I are mutually inequivalent because they have different highest weights $\xi_I := \sum i_r \xi_r$, and are lim irreducible by construction. Now let $V = \sum_{I \in \mathcal{I}} V_I$. Then $V = \varinjlim L^2(G_n/K_n) = L^2(G/K)$. □

4 Gelfand pairs and defining representations

In this section we set the stage for the extension of Theorem 15 to a number of direct systems $\{(G_n, K_n)\}$ of compact nonsymmetric Gelfand pairs. A glance at [Ya2004] or [W2007] reveals many such pairs, but here we will only consider those for which the compact groups G_n are simple. The following table shows the Krämer classification of Gelfand pairs corresponding to compact simple Lie groups (see [Kr1979] or [Ya2004] or [W2007, Table 12.7.1]).

$M_n = G_n/H_n$ weakly symmetric				G_n/K_n symmetric
	G_n	H_n	Conditions	K_n with $H_n \subset K_n \subset G_n$
1	$SU(m+n)$	$SU(m) \times SU(n)]$	$n > m \geqq 1$	$S[U(m) \times U(n)]$
2	$SO(2n)$	$SU(n)$	n odd, $n \geqq 3$	$U(n)$
3	E_6	$Spin(10)$		$Spin(10) \cdot Spin(2)$
4	$SU(2n+1)$	$Sp(n)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
5	$SU(2n+1)$	$Sp(n) \times U(1)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
6	$Spin(7)$	G_2		(there is none)
7	G_2	$SU(3)$		(there is none)
8	$SO(10)$	$Spin(7) \times SO(2)$		$SO(8) \times SO(2)$
9	$SO(9)$	$Spin(7)$		$SO(8)$
10	$Spin(8)$	G_2		$Spin(7)$
11	$SO(2n+1)$	$U(n)$	$n \geqq 2$	$SO(2n)$
12	$Sp(n)$	$Sp(n-1) \times U(1)$	$n \geqq 1$	$Sp(n-1) \times Sp(1)$

(16)

This gives us the nonsymmetric direct systems $\{(G_n, K_n)\}$, where

$$\begin{aligned}
&\text{(a) } G_n = SU(p_n + q_n) \text{ and } K_n = SU(p_n) \times SU(q_n),\ p_n < q_n \\
&\text{(b) } G_n = SO(2n) \text{ and } K_n = SU(n),\ n \text{ odd},\ n \geqq 3 \\
&\text{(c) } G_n = SU(2n+1) \text{ and } K_n = Sp(n),\ n \geqq 1 \\
&\text{(d) } G_n = SU(2n+1) \text{ and } K_n = U(1) \times Sp(n),\ n \geqq 1 \\
&\text{(e) } G_n = SO(2n+1) \text{ and } K_n = U(n),\ n \geqq 2 \\
&\text{(f) } G_n = Sp(n) \text{ and } K_n = U(1) \times Sp(n-1),\ n \geqq 2.
\end{aligned} \tag{17}$$

Definition 18 Let $\{(G_n, K_n)\}$ be a direct system of Lie groups and closed subgroups. Suppose that $\pi = \varinjlim \pi_n$ is a lim-irreducible representation of $G = \varinjlim G_n$, with representation space $V = \varinjlim V_n$, such that (i) $\pi_n(K_n)$ is the $\pi_n(G_n)$-stabilizer of a vector $v_n \in V_n$ and (ii) each $v_{n+1} = v_n + w_{n+1}$ where $\pi_n(G_n)$ leaves w_{n+1} fixed. (Thus the v_n give a coherent system of embeddings of the G_n/K_n.) Suppose further that for $n \gg 0$ the π_n have the same highest weight vector. Then we say that $\pi = \varinjlim \pi_n$ is a *defining representation* for $\{(G_n, K_n)\}$.

Now let's consider some important examples of defining representations. We will use these examples later.

Example 19 $G_n = SU(p_n + q_n)$ and $K_n = SU(p_n) \times SU(q_n)$, $p_n < q_n$, in (17). Let $\{e_1, \ldots, e_{p_n+q_n}\}$ denote the standard orthonormal basis of $\mathbb{C}^{p_n+q_n}$. Then K_n is the G_n-stabilizer of $e_1 \wedge \cdots \wedge e_{p_n}$ in the representation $\pi_n = \Lambda^{p_n}(\tau)$, where τ is the standard (vector) representation of $SU(p_n + q_n)$ on $\mathbb{C}^{p_n+q_n}$. In the usual notation, $e_1 \wedge \cdots \wedge e_{p_n}$ also is the highest weight vector, and the highest weight is $\varepsilon_1 + \cdots + \varepsilon_{p_n}$. If the p_n are bounded, so that we may assume each $p_n = p < \infty$, then $\pi = \varinjlim \pi_n$ is well defined and is a defining representation for $\{(G_n, K_n)\}$. ♢

Example 20 $G_n = SU(2n+1)$ and $K_n = U(1) \times Sp(n)$, $n \geqq 1$, in (17). Again, $\{e_1, \ldots, e_{2n+1}\}$ is the standard orthonormal basis of $\mathbb{C}^{2n+1}$. Now K_n is the G_n-stabilizer of $\sum_{\ell=1}^{n} e_{2\ell} \wedge e_{2\ell+1}$ in the representation $\pi_n = \Lambda^2(\tau)$, where τ is the standard (vector) representation of $SU(2n+1)$ on $\mathbb{C}^{2n+1}$. Here $e_1 \wedge e_2$ is the highest weight vector and the highest weight is $\varepsilon_1 + \varepsilon_2$. Thus $\pi = \varinjlim \pi_n$ is well defined and is a defining representation for $\{(G_n, K_n)\}$. ♢

Example 21 $G_n = SO(2n+1)$ and $K_n = U(n)$, $n \geqq 2$, in (17). Let $\{e_1, \ldots, e_{2n+1}\}$ denote the standard orthonormal basis of $\mathbb{R}^{2n+1}$. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then K_n is the G_n-stabilizer of $\mathrm{diag}\{0, J, \ldots, J\} \in \mathfrak{g}_n$ in the adjoint representation of G_n; in other words (in this case), it is the G_n-stabilizer of $\sum_{\ell=1}^{n} e_{2\ell} \wedge e_{2\ell+1}$ in the representation $\pi_n = \Lambda^2(\tau)$, where τ is the standard (vector) representation of $SO(2n+1)$ on $\mathbb{R}^{2n+1}$. As in the previous example, $e_1 \wedge e_2$ is the highest weight vector and the highest weight is $\varepsilon_1 + \varepsilon_2$. Thus $\pi = \varinjlim \pi_n$ is well defined and is a defining representation for $\{(G_n, K_n)\}$. ♢

Example 22 $G_n = Sp(n)$ and $K_n = U(1) \times Sp(n-1)$, $n \geqq 2$, in (17). In quaternion matrices, K_n is the G_n-commutator of $\mathrm{diag}\{i, 0, 0, \ldots, 0\}$. In $2n \times 2n$ complex matrices, it is the G_n-commutator of $\mathrm{diag}\{J, 0, 0, \ldots, 0\}$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. There, G_n consists of all elements $g \in U(2n)$ such that $g\widetilde{J}g^t = \widetilde{J}$, where $\widetilde{J} = \mathrm{diag}\{J, J, \ldots, J\}$. Thus $\mathfrak{g}_n$ is given by $x\widetilde{J} + \widetilde{J}x^t = 0$, and in particular $\mathrm{diag}\{J, 0, 0, \ldots, 0\} \in \mathfrak{g}_n$. Now K_n is the G_n-stabilizer of $\mathrm{diag}\{J, 0, 0, \ldots, 0\}$ in the adjoint representation π_n of G_n. That adjoint representation is the symmetric square of the standard (vector) representation of G_n on $\mathbb{C}^{2n}$, so it has highest weight $2\varepsilon_1$ and highest weight vector e_1^2. Thus $\pi = \varinjlim \pi_n$ is well defined and is a defining representation for $\{(G_n, K_n)\}$. $\diamondsuit$

5 Function algebras

Fix a defining representation $\pi = \varinjlim \pi_n$ for $\{(G_n, K_n)\}$. We are going to define algebras

$$\begin{aligned} &\mathcal{A}(G_n) \text{ and } \mathcal{A}(G) = \bigcup \mathcal{A}(G_n); \\ &\qquad \mathcal{A}(G_n/K_n) \text{ and } \mathcal{A}(G/K) = \bigcup \mathcal{A}(G_n/K_n) \end{aligned}$$

of complex-valued polynomial functions and look at their relations to square integrable functions. Let $d_n = \dim_{\mathbb{R}} V_n$. Then we can consider G_n to be a group of real $d_n \times d_n$ matrices. Since the G_n are reductive linear algebraic groups, this lets us define

$$\begin{aligned} &\mathcal{A}(G_n) : \text{ the algebra of all } \mathbb{C}\text{-valued functions} \\ &\qquad f|_{G_n} \text{ where } f : \mathbb{R}^{d_n \times q_n} \to \mathbb{C} \text{ is a polynomial,} \\ &r_n : \mathcal{A}(G_n) \to \mathcal{A}(G_{n-1}) : \text{ restriction of functions,} \\ &S_n : \text{ kernel of the algebra homomorphism } r_n, \\ &T_n : G_{n-1}\text{-invariant complement to } S_n \text{ in } \mathcal{A}(G_n). \end{aligned} \tag{23}$$

The following is immediate.

Lemma 24 *The restriction $r_n|_{T_n} : T_n \to \mathcal{A}(G_{n-1})$ is a G_{n-1}-equivariant vector space isomorphism. In other words we have a G_{n-1}-equivariant injection $(r_n|_{T_n})^{-1} : \mathcal{A}(G_{n-1}) \hookrightarrow \mathcal{A}(G_n)$ of vector spaces with image complementary to the kernel of the restriction $r_n : \mathcal{A}(G_n) \to \mathcal{A}(G_{n-1})$ of functions.*

Lemma 24 gives us

$$\mathcal{A}(G) = \varinjlim \mathcal{A}(G_n) = \bigcup \mathcal{A}(G_n).$$

Taking the right-invariant functions we arrive at

$$\begin{aligned}\mathcal{A}(G_n/K_n) &:= \{h \in \mathcal{A}(G_n) \mid h(xk) = h(x) \text{ for } x \in G_n,\ k \in K_n\},\\ \mathcal{A}(G/K) &= \bigcup \mathcal{A}(G_n/K_n) \\ &= \{h \in \mathcal{A}(G) \mid h(xk) = h(x) \text{ for } x \in G,\ k \in K\}.\end{aligned} \tag{25}$$

These are our basic function algebras.

The algebra $\mathcal{A}(G_n)$ contains the constants, separates points on G_n, and is stable under complex conjugation. The Stone–Weierstrass theorem is the main component of the following lemma.

Lemma 26 *The algebra $\mathcal{A}(G_n)$ is dense in $\mathcal{C}(G_n)$, the algebra of continuous functions $G_n \to \mathbb{C}$ with the topology of uniform convergence. Let S_n' and T_n' denote the uniform closures of S_n and T_n in $\mathcal{C}(G_n)$. Then r_n extends by continuity to the restriction map $r_n' : \mathcal{C}(G_n) \to \mathcal{C}(G_{n-1})$, that extension r_n' restricts to a G_{n-1}-equivalence $T_n' \cong \mathcal{C}(G_{n-1})$, $\mathcal{C}(G_n)$ is the vector space direct sum of closed G_{n-1}-invariant subspaces S_n' and T_n', and this identifies $\mathcal{C}(G_{n-1})$ as a G_{n-1}-submodule of $\mathcal{C}(G_n)$.*

Proof. The density is exactly the Stone–Weierstrass theorem in this setting. Since S_n and T_n involve different sets of variables, so do S_n' and T_n'. Now r_n extends to r_n' as asserted and S_n' is the kernel of r_n'. Similarly, $S_n' \cap T_n' = 0$, and the induced algebra homomorphism $r_n' : \mathcal{C}(G_n) \to \mathcal{C}(G_{n-1})$ restricts to a G_{n-1}-equivariant map $r_n' : T_n' \cong \mathcal{C}(G_{n-1})$. Finally, $S_n' + T_n'$ is closed in $\mathcal{C}(G_n)$ and contains $\mathcal{A}(G_n)$. Thus $\mathcal{C}(G_n) = S_n' \oplus T_n'$ and we can identify $\mathcal{C}(G_{n-1})$ with the closed G_{n-1}-invariant subspace T_n' of $\mathcal{C}(G_n)$. □

We use the identifications $\mathcal{C}(G_{n-1}) \subset \mathcal{C}(G_n)$ of Lemma 26 to form the union $\bigcup \mathcal{C}(G_n)$. Note that $\bigcup \mathcal{C}(G_n)$ is the algebra of continuous functions on G that depend on only finitely many variables. Now use the sup norm, and thus the topology of uniform convergence, and define a Banach algebra

$$\begin{gathered}\mathcal{C}(G) : \text{functions } f : G \to \mathbb{C} \text{ in the uniform limit closure of } \bigcup \mathcal{C}(G_n)\\ \text{with sup norm and topology of uniform convergence.}\end{gathered}$$

Passing to the right K_n-invariant functions we have Banach function algebras

$$\begin{aligned}\mathcal{C}(G_n/K_n) &:= \{h \in \mathcal{C}(G_n) \mid h(xk) = h(x) \text{ for } x \in G_n,\ k \in K_n \ \text{ and}\\ \mathcal{C}(G/K) &= \bigcup \mathcal{C}(G_n/K_n) \\ &= \{h \in \mathcal{C}(G) \mid h(xk) = h(x) \text{ for } x \in G,\ k \in K\}.\end{aligned} \tag{27}$$

Here $\mathcal{A}(G_n/K_n)$ is the subalgebra consisting of all G_n-finite functions in $\mathcal{C}(G_n/K_n)$, and consequently $\mathcal{A}(G/K)$ is the subalgebra consisting of all G-finite functions in $\mathcal{C}(G/K)$.

We pass to L^2 limits more or less in the same way as in (25) and (27), except that we must rescale to preserve L^2 norms as in Theorem 12. For this we need some machinery from [W2009]. Let $\{G_n\}$ be a strict direct system of compact connected Lie groups, and $\{(G_n)_{\mathbb{C}}\}$ the direct system of their complexifications. Suppose that, for each n,

$$\begin{aligned}&\text{the semisimple part } [(\mathfrak{g}_n)_{\mathbb{C}}, (\mathfrak{g}_n)_{\mathbb{C}}] \text{ of the reductive algebra } (\mathfrak{g}_n)_{\mathbb{C}}\\ &\text{is the semisimple component of a parabolic subalgebra of } (\mathfrak{g}_{n+1})_{\mathbb{C}}.\end{aligned} \tag{28}$$

Then we say that the direct systems $\{G_n\}$ and $\{(G_n)_{\mathbb{C}}\}$ are *parabolic* and that $\varinjlim G_n$ and $\varinjlim (G_n)_{\mathbb{C}}$ are *parabolic direct limits.* This is a special case of the definition of parabolic direct limit in [W2005].

Now let $\{G_n\}$ be a strict direct system of compact connected Lie groups that is parabolic. We recursively construct Cartan subalgebras $\mathfrak{t}_n \subset \mathfrak{g}_n$ with $\mathfrak{t}_1 \subset \mathfrak{t}_2 \subset \cdots \subset \mathfrak{t}_n \subset \mathfrak{t}_{n+1} \subset \dots$ and simple root systems $\Psi_n = \Psi((\mathfrak{g}_n)_{\mathbb{C}}, (\mathfrak{t}_n)_{\mathbb{C}})$ such that each simple root for $(\mathfrak{g}_n)_{\mathbb{C}}$ is the restriction of exactly one simple root for $(\mathfrak{g}_{n+1})_{\mathbb{C}}$. Then we may assume that $\Psi_n = \{\psi_{n,1}, \dots, \psi_{n,p(n)}\}$ in such a way that each $\psi_{n,j}$ is the $(\mathfrak{t}_n)_{\mathbb{C}}$-restriction of $\psi_{n+1,j}$ and of no other element of Ψ_{n+1}. The corresponding sets $\Xi_n = \{\xi_{n,1}, \dots, \xi_{n,p(n)}\}$ of fundamental highest weights can be ordered so that they satisfy: $\xi_{n+1,j}$ is the unique element of Ξ_{n+1} whose $(\mathfrak{t}_n)_{\mathbb{C}}$-restriction is $\xi_{n,j}$, for $1 \leqq j \leqq p(n)$. Exactly as in Theorem 12 this gives us isometric G_n-equivariant injections $\psi_{m,n} : L^2(G_n) \to L^2(G_m)$ for $n \leqq m$. The associated direct limit maps $\psi_n : L^2(G_n) \to \varinjlim\{L^2(G_n), \psi_{m,n}\}$ define the direct limit in the category of Hilbert spaces and unitary maps as the Hilbert space completion

$$L^2(G) = \varinjlim_{unitary} \{L^2(G_n), \psi_{m,n}\} = \left(\bigcup \psi_n(L^2(G_n))\right)^{completion}.$$

Lemma 29 *Let* $\{(G_n, K_n)\}$ *be one of the systems of* Examples 19, 20, 21*, or* 22*. Then* $\{G_n\}$ *is parabolic and the* G_n*-equivariant maps*

$$\psi_{m,n} : L^2(G_n) \hookrightarrow L^2(G_m)$$

send right-K_n*-invariants to right-*K_{n+1}*-invariants, resulting in* G_n*-equivariant unitary injections* $\psi'_{m,n} : L^2(G_n/K_n) \to L^2(G_m/K_m)$*.*

Proof. We use the defining relations given in Examples 19, 20, 21, and 22. In each case we look at the subspaces of L^2 given by polynomials of degree $\leqq d$; those are finite-dimensional invariant subspaces of the $\mathcal{A}(G_n/K_n)$. We observed above that $\mathcal{A}(G_n) \hookrightarrow \mathcal{A}(G_{n+1})$ maps right-K_n-invariants to right-K_{n+1}-invariants. On each irreducible summand, the $L^2(G_n) \hookrightarrow L^2(G_{n+1})$ differ only by scale from the corresponding summands of $\mathcal{A}(G_n)$ and $\mathcal{A}(G_{n+1})$, so they also map right-K_n-invariants to right-K_{n+1}-invariants. □

Now we have some L^2 analogues of (25) and (27).

$$\begin{aligned}L^2(G_n/K_n) &:= \{h \in L^2(G_n) \mid h(xk) = h(x) \text{ for } x \in G_n,\ k \in K_n\},\\ L^2(G/K) &= \left(\bigcup \psi'_n(L^2(G_n/K_n))\right)^{completion}\\ &= \{h \in L^2(G) \mid h(xk) = h(x) \text{ for } x \in G,\ k \in K\}.\end{aligned} \tag{30}$$

We have $\mathcal{A}(G/K) \subset \mathcal{C}(G/K) \subset L^2(G/K)$ for these spaces, and $\mathcal{A}(G/K)$ is the set of polynomial elements in $L^2(G/K)$.

Theorem 31 *Let $\{(G_n, K_n)\}$ be one of the direct systems of nonsymmetric Gelfand pairs given by* Examples 19, 20, 21, *and* 22. *Then the left regular representations of G on $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$ are multiplicity-free discrete direct sums of lim-irreducible representations. In the notation of* (9), (10), *and* (11), *those left regular representations are $\sum_{I\in\mathcal{I}} \pi_I$, where $\pi_I = \varinjlim \pi_{I,n}$ is the irreducible representation of G with highest weight $\xi_I := \sum i_r \xi_r$. Thus we have the infinite-dimensional multiplicity-free spaces*

(1) $SU(p+\infty)/(SU(p)\times SU(\infty))$ *for* $1 \leqq p \leqq \infty$,
(2) $SU(1+2\infty)/(U(1)\times Sp(\infty))$,
(3) $SO(1+2\infty)/U(\infty)$, *and*
(4) $Sp(1+\infty)/(U(1)\times Sp(\infty))$

Proof. Examples 19, 20, 21, and 22 have defining representations and well-defined function spaces $\mathcal{A}(G/K)$ and $\mathcal{C}(G/K)$. The same holds for $L^2(G/K)$ by Lemma 29. In these examples $\{G_n\}$ is parabolic, so the left regular representations are limit aligned by Theorem 12. Now the proof of Theorem 15 holds for these four examples, resulting in the multiplicity-free property for their left regular representations. □

6 Pairs related to spheres and Grassmann manifolds

In dealing with nonsymmetric Gelfand pairs we have to be very specific about the embeddings $G_{n-1} \hookrightarrow G_n$, so we review a few of those embeddings.

Orthogonal groups. Let $G_n = SO(n_0 + 2n)$, the special orthogonal group for the bilinear form $h(u, v) = \sum_1^{n_0+2n} u_i v_i$. The embeddings are given by $G_n \hookrightarrow G_{n+1}$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $G = \varinjlim G_n$ is the classical direct limit group $SO(\infty)$. It doesn't matter what n_0 is here, but sometimes we have to distinguish between the cases of even or odd n_0, and in any case we want $\{G_n\}$ to be parabolic, so we jump by two 1's instead of just one. Specifically, this direct system consists either of groups of type B (when the $n_0 + 2n$ are odd) or of type D (when the $n_0 + 2n$ are even). In this section $K_n = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) | x \in SO(n_0 + 2n - 1)\} \subset G_n$. Then G_n/K_n is the sphere S^{n_0+2n-1}, $G = \varinjlim G_n = SO(\infty)$, and we express $K = \varinjlim K_n$ as $SO(1) \times SO(\infty - 1)$ to indicate the embedding $K \hookrightarrow G$.

A defining representation for $\{(G_n, K_n)\}$ is given by the family of standard (vector) representations π_n of $SO(n_0 + 2n)$ on $\mathbb{R}^{n_0+2n}$. Here $\{SO(n_0 + 2n)\}$ is a parabolic direct system. In the standard orthonormal basis the π_n all have the same highest weight vector e_1 and highest weight ε_1. Following the considerations of Section 5, this defining representation $\pi = \varinjlim \pi_n$

defines the function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$. The π_n share a highest weight vector so we have natural equivariant inclusions $\mathcal{A}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{C}(G_n/K_n)$, and $L^2(G_{n-1}/K_{n-1}) \hookrightarrow L^2(G_n/K_n)$, and thus the limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. Thus we have the regular representation of $G = SO(\infty)$ on those limit spaces.

Unitary groups. Fix $p > 0$ and define $G_n = SU(p+n)$, the special unitary group for the complex hermitian form $h(u,v) = \sum_1^{p+n} u_i \bar{v}_i$. The embedding $G_n \hookrightarrow G_{n+1}$ is given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $G = \varinjlim G_n$ is the classical parabolic direct limit group $SU(\infty)$. In this section $K_n = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) | \, x \in SU(p),\ y \in SU(n)\}$. Then G_n/K_n is a circle bundle over the Grassmann manifold of p-dimensional linear subspaces of $\mathbb{C}^{p+n}$, $G = \varinjlim G_n = SU(\infty)$, and we sometimes express $K = \varinjlim K_n$ as $SU(p) \times SU(\infty - p)$ to indicate the embedding $K \hookrightarrow G$. If $p = 1$ then G_n/K_n is the sphere S^{2n+1}, the complex Grassmann manifold is a complex projective space, and the circle bundle projection is the Hopf fibration.

Here the defining representation is essentially that of Example 19. Let π_{ξ_1} denote the usual vector representation of G_n on $\mathbb{C}^{p+n}$. Write π_{ξ_p} for its p^{th} alternating power, the representation of G_n on $\Lambda^p(\mathbb{C}^{p+n})$; it is the first representation of G_n with a vector fixed under K_n. That vector is $e_1 \wedge \cdots \wedge e_p$ relative to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, and K_n is its G_n-stabilizer. Thus the $\pi_n = \pi_{\xi_p}$ give a defining representation for $\{(G_n, K_n)\}$. Note that the π_n all have the same highest weight vector $e_1 \wedge \cdots \wedge e_p$ and highest weight $\varepsilon_1 + \cdots + \varepsilon_p$. Following the considerations of Section 5, this defining representation $\pi = \varinjlim \pi_n$ defines the function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$. The π_n share a highest weight vector, so we have natural equivariant inclusions $\mathcal{A}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{C}(G_n/K_n)$, and $L^2(G_{n-1}/K_{n-1}) \hookrightarrow L^2(G_n/K_n)$, and thus the limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. That gives us the regular representation of $G = SU(\infty)$ on those limit spaces.

Symplectic groups. Here $Sp(n)$ is the unitary group of the quaternion-hermitian form $h(u,v) = \sum_1^n u_i \bar{v}_i$ on the quaternionic vector space $\mathbb{H}^n$. We then have $G_n = Sp(n) \times Sp(1)$, where the $Sp(1)$ acts by quaternion scalars on $\mathbb{H}^n$. We will also look at its subgroup $Sp(n) \times U(1)$, where $U(1)$ is any (they are all conjugate) circle subgroup of $Sp(1)$, say $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$. In both cases the embeddings $G_n \hookrightarrow G_{n+1}$ are specified by the maps $Sp(n) \hookrightarrow Sp(n+1)$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. (We are using quaternionic matrices.) Then $G = \varinjlim G_n$ is the classical direct limit group $Sp(\infty) \times Sp(1)$ and $G' = \varinjlim G'_n$ is $Sp(\infty) \times U(1)$. (We need the $Sp(1)$ or the $U(1)$ factor because otherwise, as we will see below, the multiplicity-free property will fail.)

Symplectic 1. First consider the parabolic direct system given by $G_n = Sp(n) \times Sp(1)$. Given n we have two $Sp(1)$ groups to deal with at the same time, so we avoid confusion by denoting the $Sp(1)$ factor of G_n as $Sp(1)_{ext,n}$ (ext for external) and the identity component of the centralizer of $Sp(n-1)$

in $Sp(n)$ by $Sp(1)_{int,n}$ (*int* for internal). In our matrix descriptions of G_n, the group $Sp(1)_{diag,n}$ is the diagonal subgroup in $Sp(1)_{int,n} \times Sp(1)_{ext,n}$. Then $G_n = Sp(n) \times Sp(1)_{ext,n}$ and $G = \varinjlim G_n = Sp(\infty) \times Sp(1)$. Now let $K_n = \{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) | \, x \in Sp(n-1)\} \times Sp(1)_{diag,n}$ and $K = \varinjlim K_n$. Then G_n/K_n is the sphere S^{4n-1}, in other words, the Hopf fibration 3-sphere bundle over quaternion projective space $P^{n-1}(\mathbb{H})$. In order to indicate the embedding $K \hookrightarrow G$ we express K as $\{1\} \times Sp(\infty - 1) \times Sp(1)$.

A defining representation for $\{(G_n, K_n)\}$ is given by the family of standard (vector) representations π_n of $Sp(n)$ on $\mathbb{C}^{2n}$ tensored with the standard 2-dimensional representation of $Sp(1)$ on $\mathbb{C}^2$. That representation has an invariant real form $\mathbb{R}^{4n}$. Consider the standard orthonormal basis $\{e_i \otimes f_j\}$ of $\mathbb{C}^{2n} \otimes \mathbb{C}^2$. The representations π_n of G_n there have the same highest weight vector $e_1 \otimes f_1$ and highest weight $(\varepsilon_1)_{Sp(n)} + (\varepsilon_1)_{Sp(1)}$. They give a defining representation for $\{(G_n, K_n)\}$. Following the considerations of Section 5, this defining representation $\pi = \varinjlim \pi_n$ defines the function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$. The π_n have the same highest weight vector so we have natural equivariant inclusions $\mathcal{A}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{C}(G_n/K_n)$, and $L^2(G_{n-1}/K_{n-1}) \hookrightarrow L^2(G_n/K_n)$, and thus the limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. So we have the regular representation of $G = Sp(\infty) \times Sp(1)$ on those limit spaces.

Symplectic 2. Next consider the parabolic direct system given by $G_n = Sp(n) \times U(1)$, where the $Sp(1)$ factor of $Sp(n) \times Sp(1)$ is replaced by the circle subgroup $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$. Given n we have two $U(1)$ groups, the $U(1)_{ext,n}$ that is the $U(1)$ factor of G_n and the corresponding circle subgroup $U(1)_{int,n}$ of $Sp(1)_{int,n}$. Then of course we have the diagonal $U(1)_{diag,n}$. As above we define K_n to be the product group $\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) | \, x \in Sp(n-1)\} \times U(1)_{diag,n}$ and we set $K = \varinjlim K_n$. Then G_n/K_n again is the sphere S^{4n-1}. We express K as $\{1\} \times Sp(\infty - 1) \times U(1)$.

A defining representation for $\{(G_n, K_n)\}$ is given by the family of standard (vector) representations π_n of $Sp(n)$ on $\mathbb{C}^{2n}$ tensored with the standard 1-dimensional representation of $U(1)$ on $\mathbb{C}$. The representations π_n of G_n there have the same highest weight vector $e_1 \otimes f_1$. The corresponding highest weight is $(\varepsilon_1)_{Sp(n)} + (\varepsilon_1)_{U(1)}$, and the π_n give a defining representation for $\{(G_n, K_n)\}$. Following the considerations of Section 5, this defining representation $\pi = \varinjlim \pi_n$ defines the function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$. The π_n have the same highest weight vector, so we have natural equivariant inclusions $\mathcal{A}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_{n-1}/K_{n-1}) \hookrightarrow \mathcal{C}(G_n/K_n)$, and $L^2(G_{n-1}/K_{n-1}) \hookrightarrow L^2(G_n/K_n)$, and thus the limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. So we have the regular representation of $G = Sp(\infty) \times U(1)$ on those limit spaces.

Symplectic 3. A variation on the case just considered is where $K_n = \{ \left(\begin{smallmatrix} z & 0 \\ 0 & x \end{smallmatrix}\right) | \, z \in U(1), x \in Sp(n-1)\} \times U(1)$, and $K = \varinjlim K_n$. Then the $U(1)$ factor of G_n is contained in K_n so it acts trivially on G_n/K_n. Thus G_n/K_n is a 2-sphere bundle over $P^{n-1}(\mathbb{H})$ exactly as in the "Symplectic 2" case.

We express K as $U(1) \times Sp(\infty - 1) \times U(1)$. The groups K_n are larger than the case "Symplectic 2" just considered, so the present function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$ are subspaces of those of "Symplectic 2", and the same holds for their limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. Now we have the regular representation of $G = Sp(\infty) \times Sp(1)$ on those limit spaces.

Symplectic 4. A variation on the "Symplectic 1" case is where $G_n = Sp(n) \times Sp(1)$ and $K_n = \{ \left(\begin{smallmatrix} z & 0 \\ 0 & x \end{smallmatrix} \right) | \, z \in U(1), x \in Sp(n-1)\} \times Sp(1)$ and $K = \varinjlim K_n$. Then the $Sp(1)$ factor of G_n is contained in K_n, so it acts trivially on G_n/K_n. Thus $G_n/K_n = Sp(n)/[U(1) \times Sp(n-1)]$ is a 2-sphere bundle over $P^{n-1}(\mathbb{H})$, exactly as in the "Symplectic 3" case above. We express K as $U(1) \times Sp(\infty - 1) \times Sp(1)$ and we note that the function spaces $\mathcal{A}(G_n/K_n)$, $\mathcal{C}(G_n/K_n)$, and $L^2(G_n/K_n)$ are exactly the same as those of "Symplectic 3", so the same holds for their limits $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$. Thus we have the regular representation of $G = Sp(\infty) \times Sp(1)$ on those limit spaces.

The classifications of Krämer [Kr1979] and Yakimova [Ya2004] (see [W2007]) show that the six direct systems just described, one orthogonal, one unitary, and four symplectic, all consist of Gelfand pairs.

7 Limits related to spheres and Grassmann manifolds

In this section we prove the multiplicity-free property for the direct limits of Gelfand pairs described in Section 6.

Theorem 32 *Let $(G, K) = \varinjlim \{(G_n, K_n)\}$, where $\{(G_n, K_n)\}$ is one of the six systems described in* Section 6*. Let $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$ be as described there. Then the regular representations of G on $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$ are multiplicity-free discrete direct sums of lim-irreducible representations.*

Proof. We run through the proof of Theorem 32 for the three types of limit groups G. In each case we do this by examining the representation of G_n on $\mathcal{A}(G_n/K_n)$, verifying the limit-aligned condition, and applying Theorem 4 to the regular representation of G on $\mathcal{A}(G/K)$. We already know the result for the orthogonal group case, where the (G_n, K_n) are symmetric pairs, but we need the representation-theoretic information from that case in order to deal with the other cases.

Orthogonal group case. Here we shift the index so that $G_n = SO(n)$ and $K_n = SO(n-1)$. Then G_n/K_n is the unit sphere in $\mathbb{R}^n$. The G_n-finite functions on G_n/K_n are just the restrictions of polynomial functions on $\mathbb{R}^n$. Let $\psi_{1;n}$ denote the usual representation of G_n on $\mathbb{R}^n$ and let ξ denote its highest weight. Choose orthonormal linear coordinates $\{x_1, \ldots, x_n\}$ of that $\mathbb{R}^n$ such that the monomial x_1 is a highest weight vector. Then the representation of G_n on the space of polynomials of pure degree ℓ is of the form $\psi_{\ell;n} \oplus \gamma_{\ell;n}$, where $\psi_{\ell;n}$ is the irreducible representation of highest weight $\ell\xi$ and highest weight vector x_1^ℓ. Then $\gamma_{\ell;n}$ is the sum of the $\psi_{\ell-2j;n}$ for $1 \leqq j \leqq [\ell/2]$, and the representation space of that $\psi_{\ell-2j;n}$ consists of the polynomial functions on $\mathbb{R}^n$ divisible by $||x||^{2j}$ but not by $||x||^{2j+2}$. Write $E_{\ell;n}$ for the space of functions on G_n/K_n obtained by restricting those polynomials of degree ℓ contained in the representation space for $\psi_{\ell;n}$. Then $\mathcal{A}(G_n/K_n) = \sum_{\ell \geqq 0} E_{\ell;n}$.

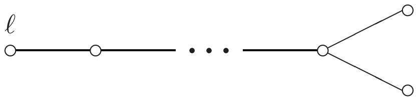
We now verify that the inclusions $\mathcal{A}(G_n/K_n) \hookrightarrow \mathcal{A}(G_{n+1}/K_{n+1})$ send $E_{\ell;n}$ into $E_{\ell;n+1}$, so that the representation of G on $\mathcal{A}(G/K)$ is limit aligned and Theorem 4 shows that $\varinjlim \mathcal{A}(G_n/K_n)$ is the multiplicity-free direct sum of lim-irreducible G-modules $E_\ell = \varinjlim E_{\ell;n}$. For that, note that the restriction $\mathcal{A}(G_{n+1}/K_{n+1}) \to \mathcal{A}(G_n/K_n)$ is obtained by setting x_{n+1} equal to zero. Thus the inclusions $E_{\ell,n} \hookrightarrow \mathcal{A}(G_{n+1}/K_{n+1})$ are given by identifying the function $x_1^\ell : \mathbb{R}^n \to \mathbb{R}$ with the function $x_1^\ell : \mathbb{R}^{n+1} \to \mathbb{R}$ and applying G_n-equivariance. Now $\mathcal{A}(G/K) = \varinjlim \mathcal{A}(G_n/K_n)$ is the direct sum of the $E_\ell = \varinjlim E_{\ell;n}$, and the representations of G on the E_ℓ are the mutually inequivalent lim-irreducible $\varinjlim \psi_{\ell;n}$. That gives an elementary proof for the case $G = SO(\infty)$ and $K = SO(\infty - 1)$.

Unitary group cases. Here we shift the index so that $G_n = SU(n)$ and $K_n = SU(p) \times SU(n-p)$, $n > p$. So $G = SU(\infty)$ and $K = SU(p) \times SU(\infty - p)$. Without loss of generality assume $n > 2p$ so that the (G_n, K_n) are Gelfand pairs. Recall the defining representation $\pi = \varinjlim \pi_n$ where $\pi_n = \pi_{\xi_p}$, the p^{th} exterior power of the vector representation of G_n on $\mathbb{C}^n$. So K_n is the G_n-stabilizer of $e_{I_0} := e_1 \wedge \cdots \wedge e_p$, resulting in the map $G_n/K_n \hookrightarrow \Lambda^p(\mathbb{C}^n)$ by $gK_n \mapsto g(e_{I_0})$.

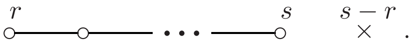
We have $\mathbb{C}$-linear functions z_I on $\Lambda^p(\mathbb{C}^n)$ dual to the basis of $\Lambda^p(\mathbb{C}^n)$ consisting of the e_I with $I = (i_1, \dots, i_p)$, where $1 \leqq i_1 < \cdots < i_p \leqq n$. (Here $I_0 = (1, 2, \dots, p)$.) Their real and imaginary parts generate the algebra $\mathcal{A}(G_n/K_n)$. Relative to the diagonal Cartan subalgebra of $\mathfrak{g}_n$ the e_I are weight vectors, and e_{I_0} is the highest weight vector, for π_{ξ_p}. Now the action of G_n on the polynomials of degree ℓ in the z_I and the $\overline{z_I}$ is $\sum_{r+s=\ell} \pi_{r\xi_p + s\xi_{n-p}}$, where $\pi_{r\xi_p+s\xi_{n-p}}$ has highest weight $r\xi_p + s\xi_{n-p}$ and highest weight vector $\overline{z_{I_0}^r} z_{I_0}^s$. Those representations are mutually inequivalent, using $n > 2p$, and $\mathcal{A}(G_n/K_n) = \sum_{\ell \geqq 0} \sum_{r+s=\ell} E_{r,s;n}$, where G_n acts on $E_{r,s;n}$ by $\pi_{r\xi_p+s\xi_{n-p}}$. The $\mathcal{A}(G_n/K_n) \hookrightarrow \mathcal{A}(G_{n+1}/K_{n+1})$ are given on the level of $E_{r,s;n} \hookrightarrow E_{r,s;n+1}$ by identifying $\overline{z_{I_0}^r} z_{I_0}^s : \Lambda^p(\mathbb{C}^n) \to \mathbb{C}$ with $\overline{z_{I_0}^r} z_{I_0}^s : \Lambda^p(\mathbb{C}^{n+1}) \to \mathbb{C}$. In view of Theorem 4, it follows that the representation of G on $\mathcal{A}(G/K)$ is a limit-aligned discrete direct sum of mutually inequivalent lim-irreducible representations.

We will need the case $p = 1$ when we look at the symplectic group cases. There $G_n = SU(n)$ and $K_n = \{1\} \times SU(n-1)$, and the G_n-finite functions on G_n/K_n are just the restrictions of finite linear combinations of the functions $z^r \bar{z}^s$. We saw how to decompose $\mathcal{A}(S^{2n-1})$ into irreducible modules for $SO(2n)$: it is the sum of the spaces $E_{\ell;2n}$ described above with highest weight

$\ell\xi$ and highest weight vector x_1^ℓ, where, of course, $x_j = \frac{1}{2}(z_j + \bar{z}_j)$. In terms of the Dynkin diagram that representation is

ψ_ℓ : 

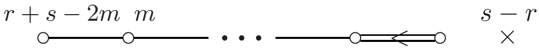
and $\psi_{\ell;2n}|_{U(n)} = \sum_{r+s=\ell} \psi_{r,s;n}$, where $\psi_{r,s;n}$ has diagram

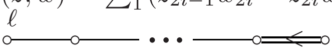
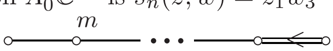


Both $\psi_{r,s;n}$ and $\psi_{r,s;n}|_{SU(n)}$ have highest weight vector $z_1^r \bar{z}_1^s$. Let $E_{r,s;n}$ denote the representation space for $\psi_{r,s;n}$. Now $\mathcal{A}(G_n/K_n) = \sum_{\ell \geqq 0} \sum_{r+s=\ell} E_{r,s;n}$.

Symplectic group cases. First suppose $G_n = Sp(n) \times U(1)$. There are two cases: (i) $K_n = \{1\} \times Sp(n-1) \times U(1)_{diag,n}$ and (ii) $K_n = U(1) \times Sp(n-1) \times U(1)$. The assertions for case (i) will imply them for case (ii), so we may assume that $K_n = \{1\} \times Sp(n-1) \times U(1)_{diag,n}$. Then $(G, K) = \varinjlim\{(G_n, K_n)\}$ has defining representation $\pi = \varinjlim \pi_n$ where π_n is the representation  of G_n on $\mathbb{C}^{2n}$.

Note that π_n factors through the vector representation of $U(2n)$ on $\mathbb{C}^{2n}$. We saw how $U(2n)$ acts irreducibly on the space $E_{r,s;2n}$ by the representation $\psi_{r,s;2n}$, which has diagram

We now need two facts. First, $G_n \hookrightarrow U(2n)$ sends the $U(1)$ factor of G_n to the center of $U(n)$. Second, $\psi_{r,s;2n}|_{Sp(n)} = \sum_{0 \leqq m \leqq \min(r,s)} {}'\varphi_{r,s,m;n}$, where ${}'\varphi_{r,s,m;n}$ has diagram  . That gives us $\psi_{r,s;2n}|_{Sp(n)U(1)} = \sum_{0 \leqq m \leqq \min(r,s)} \varphi_{r,s,m;n}$, where $\varphi_{r,s,m;n}$ is the representation of $Sp(n)U(1)$ with diagram  , and ${}'\varphi_{r,s,m;n}$ has the same representation space (call it $E_{r,s,m;n}$) as $\varphi_{r,s,m;n}$. The $E_{r,s,m;n}$ are irreducible and inequivalent under $Sp(n)U(1)$; in other words, the irreducible representations $\varphi_{r,s,m;n}$ all are mutually inequivalent. Note, however, that ${}'\varphi_{r,s,m;n} \simeq {}'\varphi_{r+t,s-t,m;n}$ for all t such that $r+t, s-t \geqq 0$; this reflects the fact that $(Sp(n), Sp(n-1))$ is not a Gelfand pair.

To trace the inclusions let $\{z_1, \ldots, z_{2n}\}$ be the coordinates of $\mathbb{C}^{2n}$, all weight vectors, where z_1 is the highest weight vector, $z_2 = e_{-\alpha_1} z_1$ is the next highest, and so on, and the antisymmetric bilinear invariant of $Sp(n)$ on $\mathbb{C}^{2n}$ is $v_n(z, w) = \sum_1^n (z_{2i-1} w_{2i} - z_{2i} w_{2i-1})$. Then z_1^ℓ is the highest weight vector of  and $\Lambda^2\mathbb{C}^{2n}$ is the sum $\Lambda_0^2\mathbb{C}^{2n} \oplus v_n\mathbb{C}$ of its irreducible component and its trivial component under the action of $Sp(n)$. Here v_n has matrix diag $\left\{ \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \ldots \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \right\}$ and we work with the maximal toral subalgebra that consists of all matrices diag $\{a_1, -a_1; \ldots; a_n, -a_n\}$; thus the highest weight vector on $\Lambda_0^2\mathbb{C}^{2n}$ is $s_n(z, w) = z_1 w_3 - z_3 w_1$. Now s_n^m is the highest weight vector of  . The corresponding highest weight vector of $\varphi_{r,s,m;n}$ is $z_1^{r-m}\bar{z}_1^{s-m}s_n^m$. Now the restriction $\mathcal{A}(G_{n+1}/K_{n+1}) \to \mathcal{A}(G_n/K_n)$ maps the highest weight vector $z_1^{r-m}\bar{z}_1^{s-m}s_{n+1}^m$ of $\varphi_{r,s,m;n+1}$ to the highest weight vector $z_1^{r-m}\bar{z}_1^{s-m}s_n^m$. This proves that the representation of G on $\mathcal{A}(G/K)$ is limit aligned. Theorem 4 shows that $\varinjlim \mathcal{A}(G_n/K_n)$ is the multiplicity-free direct sum of lim-irreducible G-modules $E_{r,s,m} := \varinjlim E_{r,s,m;n}$.

Finally, we suppose $G_n = Sp(n) \times Sp(1)$. Again there are two cases: (i) that $K_n = \{1\} \times Sp(n-1) \times Sp(1)_{diag,n}$ and (ii) $K_n = U(1) \times Sp(n-1) \times Sp(1)$. The function algebras and group actions in case (ii) are exactly the same as those of the setting $(G_n, K_n) = (Sp(n) \times U(1), U(1) \times Sp(n-1) \times U(1))$ above, where the assertions are proved. Thus we need only consider case (i), $K_n = \{1\} \times Sp(n-1) \times Sp(1)_{diag,n}$. Then $(G, K) = \varinjlim\{(G_n, K_n)\}$ has defining representation $\pi = \varinjlim \pi_n$ described in "Symplectic 1" above. Those π_n satisfy the condition of Theorem 4 because $Sp(n) \times Sp(1)$ simply puts together representation spaces $E_{r-m,s-m,m;n}$ of $Sp(n) \times U(1)$ on $\mathcal{A}(Sp(n)U(1)/Sp(n-1)U(1))$. This assembly maintains total degree $\ell = (r-m) + (s-m) + 2m$, views the $U(1)$ factor of $Sp(n) \times U(1)$ as a maximal torus of the $Sp(1)$ factor of $Sp(n) \times Sp(1)$, and sums the spaces for the $\overset{s-r}{\times}$ to form the space for the irreducible representation (call it β_ℓ) of $Sp(1)$ of degree $\ell + 1$. It has diagram $\overset{\ell}{\circ}$. Now the irreducible spaces for $Sp(n) \times Sp(1)$ are the $F_{\ell,m;n} := \sum_{r+s=\ell} E_{r-m,s-m,m;n}$ and the corresponding representations are the $\varphi_{\ell,m,n} := \sum_{r+s=\ell}{}' \varphi_{r-m,s-m,m;n}$. This proves that the representation of G on $\mathcal{A}(G/K)$ is limit aligned. Theorem 4 shows that $\varinjlim \mathcal{A}(G_n/K_n)$ is the multiplicity-free direct sum of lim-irreducible G-modules $F_{\ell,m} := \varinjlim F_{\ell,m;n}$.

We have proved Theorem 32. □

Remark 33 Alternatively, the systems (d), (e), and (f) from the list (17), and also (a) when the $\{p_n\}$ are bounded, can be treated by the method of Sections 6 and 7. That gives an alternative proof of the multiplicity-free property for the pairs

(1) $SU(p + \infty)/(SU(p) \times SU(\infty))$ for $1 \leqq p \leqq \infty$,
(2) $SU(1 + 2\infty)/(U(1) \times Sp(\infty))$,
(3) $SO(1 + 2\infty)/U(\infty)$, and
(4) $Sp(1 + \infty)/(U(1) \times Sp(\infty))$

of Theorem 31. ♢

8 Conclusions

We have proved that the regular representations of G on $\mathcal{A}(G/K)$, $\mathcal{C}(G/K)$, and $L^2(G/K)$, are multiplicity-free discrete direct sums of lim-irreducible representations in the following cases. In addition, in these cases it is always permissible to enlarge the groups K_n, say to $F \cdot K_n$ where F is a closed subgroup of the normalizer $N_{G_n}(K_n)$, because $\mathcal{A}(G_n/[F \cdot K_n])$ is a G_n-submodule of $\mathcal{A}(G_n/K_n)$.

Limits of riemannian symmetric spaces. We have the multiplicity-free property for the thirteen cases described in Theorem 15, as well as some obvious variations. The latter include

$$SO(\infty) \times SO(\infty)/\operatorname{diag} SO(\infty) = \varinjlim SO(n) \times SO(n)/\operatorname{diag} SO(n) \quad \text{and}$$
$$SO(p+\infty)/[S(O(p) \times O(\infty))] = \varinjlim SO(n)/[S(O(p) \times O(n-p))] \,.$$

Limits of a few systems of Gelfand pairs. We have the multiplicity-free property for the four cases described in Theorem 31,

(1) $SU(p+\infty)/(SU(p) \times SU(\infty))$ for $1 \leqq p \leqq \infty$,
(2) $SU(1+2\infty)/(U(1) \times Sp(\infty))$,
(3) $SO(1+2\infty)/U(\infty)$, and
(4) $Sp(1+\infty)/(U(1) \times Sp(\infty))$.

We also have the multiplicity-free property for spaces that interpolate between $(SU(p+\infty), SU(p) \times SU(\infty))$ and the limit Grassmannian $(U(p+\infty), \varinjlim U(p) \times U(n)$.

Fix a closed subgroup F of $U(1)$. Then we have the multiplicity-free property for the pairs $(G, K) = \varinjlim\{(G_n, K_n)\}$, where $G_n = SU(p+n)$ and

$$K_n = \left\{ \left(\begin{smallmatrix} k_n' & 0 \\ 0 & k_n'' \end{smallmatrix} \right) \middle| \, k_n' \in U(p), k_n'' \in SU(n), \det k_n' \in F \right\}.$$

Limits of Gelfand pairs related to spheres and Grassmann manifolds. We have the multiplicity-free property for the six cases described in Theorem 32, four of which are nonsymmetric, as well as some obvious variations. Fix a closed subgroup F of $U(1)$; it can be any finite cyclic group or the entire circle group $U(1)$. As a result we have the multiplicity-free property for the nonsymmetric pairs

$$\begin{aligned}
&SU(\infty)/[SU(p) \times SU(\infty - p)] \\
&\qquad = \varinjlim SU(n)/[SU(p) \times SU(n-p)] \,, \\
&[Sp(\infty) \times U(1)]/[F \times Sp(\infty - 1) \times U(1)_{diag}] \\
&\qquad = \varinjlim [Sp(n) \times U(1)]/[F \times Sp(n-1) \times U(1)_{diag}] \,, \\
&[Sp(\infty) \times Sp(1)]/[\{1\} \times Sp(\infty - 1) \times Sp(1)_{diag}] \\
&\qquad = \varinjlim [Sp(n) \times Sp(1)]/[\{1\} \times Sp(n-1) \times Sp(1)_{diag}] \,, \text{ and} \\
&[Sp(\infty) \times Sp(1)]/[\{\pm 1\} \times Sp(\infty - 1) \times Sp(1)_{diag}] \\
&\qquad = \varinjlim [Sp(n) \times Sp(1)]/[\{\pm 1\} \times Sp(n-1) \times Sp(1)_{diag}] \,.
\end{aligned}$$

What we don't have. There is a huge number of direct systems $\{(G_n, K_n)\}$ of Gelfand pairs where the G_n are compact connected Lie groups. We have only verified the multiplicity-free condition for a few of them. We have not, for example, checked it for the interesting cases

$$G_n = SU(2n+1) \text{ and } K_n = F \times Sp(n), \quad F \subset U(1) \text{ finite cyclic,}$$

and

$$G_n = SO(2n) \text{ and } K_n = F \times SU(n),\ n \text{ odd},\ n \geqq 3.$$

Also, we have not checked it for the *very* interesting case

$$G_n = Sp(a_n) \times Sp(b_n) \text{ and } K_n = Sp(a_n - 1) \times Sp(1) \times Sp(b_n - 1),$$

which is a prototype for nonsymmetric irreducible direct systems $\{(G_n, K_n)\}$ with the G_n semisimple but not simple. In that case $K_n \hookrightarrow G_n$ is given by $(k_1, a, k_2) \mapsto \left(\left(\begin{smallmatrix} k_1 & 0 \\ 0 & a \end{smallmatrix} \right), \left(\begin{smallmatrix} a & 0 \\ 0 & k_2 \end{smallmatrix} \right) \right)$, so G_n/K_n fibers over $P^{a_n-1}(\mathbb{H}) \times P^{b_n-1}(\mathbb{H})$ with fiber $(Sp(1) \times Sp(1))/(diagonal) = S^3$.

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