# Complex Forms of Quaternionic Symmetric Spaces*

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*Dedicated to Professor Lieven Vanhecke*

**Summary.** We give a complete classification of the complex forms of quaternionic symmetric spaces.

## 1 Introduction

Some years ago, H. A. Jaffee found the real forms of Hermitian symmetric spaces ([J1], [J2]; or see [HÓ]). That classification turns out to be related to the classification of causal symmetric spaces. This was first observed by I. Satake ([S, Remark 2 on page 30] and [S, Remark on page 87]). Somewhat later, it was independently observed by J. Hilgert, G. Ólafsson and B. Ørsted; see [HÓ], especially Chapter 3 and the Notes at the end of that Chapter. I learned about that from Bent Ørsted. He and Gestur Ólafsson had informally discussed complex forms of quaternionic symmetric spaces and found examples for the classical groups, for $G_2$ , and perhaps for $F_4$. Ørsted told me about the classical ones, and we rediscovered examples for $G_2$ and $F_4$. I thank Bent Ørsted for agreeing to my incorporating those examples into this note. Later I used the computer program LiE [L] to find examples for $E_6$, $E_7$ and $E_8$.

In this note, I write down a complete classification for complex forms $L/V$ of quaternionic symmetric spaces $G/K$. The definitions and some preliminary results are in Sections 2 and 3, the main results are stated in Section 4, and the proofs are in Sections 5, 6, 7 and 8. The case where $G$ is a classical group and $\mathrm{rank}(L) = \mathrm{rank}(G)$ is handled, essentially by matrix considerations, in Section 5. That, of course, does not work comfortably for the exceptional groups, which must be approached by means of their root structure. The tool for this is a script for the use of the computer program LiE; it is described in Section 6 along with some examples of its application. Those examples have the interesting property that the complexifications $L_{\mathbb{C}}$ and $K_{\mathbb{C}}$ are conjugate in $G_{\mathbb{C}}$. They cover the delicate cases for $G$ exceptional and $\mathrm{rank}(L) = \mathrm{rank}(G)$, and the

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remaining exceptional equal rank cases are settled in Section 7. Finally, the few cases of $\mathrm{rank}(L) < \mathrm{rank}(G)$ are worked out in Section 8.

A possible extension of the theory is mentioned in Section 9.

After this paper was written, I learned that quite a lot was published on totally complex submanifolds of quaternionic symmetric spaces from the viewpoint of differential geometry. See, for example, [ADM], [AM1], [AM2], [F], [JKS], [L1], [L2], [Ma], [Mo], [Ts] and [X], but especially the first three. I also learned that M. Takeuchi [Ta] had studied the maximal totally complex submanifolds of quaternionic symmetric spaces, reducing their classification to that of certain Satake diagrams and writing out the classification in the classical group cases. *A priori* that is not quite the same as the classification of complex forms of quaternionic symmetric spaces, but it is very close. On the other hand, it seems to me that the method given here is more efficient and more direct, and more explicit in the exceptional group cases. I thank Dmitry Alekseevsky for calling the above-cited papers to my attention.

## 2 Quaternionic symmetric spaces

We recall the structure of quaternionic symmetric spaces [W]. A quaternionic structure on a connected Riemannian manifold $M$ is a parallel field $A$ of quaternion algebras $A_x$ on the real tangent spaces $T_x(M)$, such that every unimodular element of every $A_x$ is an orthogonal linear transformation. Thus, $A$ gives every tangent space the structure of quaternionic vector space, such that the Riemannian metric at $x$ is Hermitian relative to the elements of $A_x$ of square $-I$. If $n = \dim M$, then a quaternionic structure is the same as a reduction of the structure group of the tangent bundle from $O(n)$ to $Sp(n/4) \cdot Sp(1)$. Let $K_x$ denote the holonomy group of $M$ at $x$ (we will see in a minute that this is appropriate notation for symmetric spaces with no Euclidean factor). Suppose that $M$ is simply connected, so that the $K_x$ are connected. Let $A = \{A_x\}$ be a quaternionic structure on $M$. Then $A_x$ is stable under the action of $K_x$, so $K_x \cap A_x$ is a closed normal subgroup of $K_x$. Now, $K_x = K_x^{lin} \cdot K_x^{sca}$, where $K_x^{lin}$ is the quaternion-linear part, centralizer of $A_x$ in $K_x$, and $K_x^{sca} = K_x \cap A_x$ is the scalar part. We say that $K_x$ has real scalar part if $K_x^{sca}$ consists of real scalars, i.e., $K_x^{sca}$ is $\{1\}$ or $\{\pm 1\}$. We say that $K_x$ has complex scalar part if $K_x^{sca}$ is contained in a complex subfield of $A_x$ but not in the real subfield, and we say that $K_x$ has quaternion scalar part if $K_x^{sca}$ is not contained in a complex subfield of $A_x$. A Riemannian 4-manifold $M$ with holonomy $U(2)$ has a dual role: it has a quaternionic structure $A_1$ generated by the $SU(2)$-factor in the holonomy; that has quaternionic scalar part, the same $SU(2)$, $M$; it has a second quaternionic structure $A_2$ where $A_{2,x}$ is the centralizer of $A_{1,x}$ in the algebra of $\mathbb{R}$-linear transformations of $T_x(M)$; it has complex scalar part, generated by the circle center of the holonomy $U(2)$. Thus, we have an interesting dual picture. The holonomy of $M$ has quaternionic scalar part for $A_1$ and has complex scalar part for $A_2$.

**Proposition 2.1.** *The connected simply connected Riemannian symmetric spaces with quaternionic structure are the following.*

(i) *The Euclidean spaces of dimension divisible by* 4. *Here, the holonomy has real scalar part.*

(ii) *Products* $M = M_1 \times \cdots \times M_\ell$*, where each* $M_i$ *is* (a) *the complex projective or hyperbolic plane with the quaternionic structure of complex scalar part, or* (b) *a product* $M_i' \times M_i''$ *where each factor is a complex projective line and a complex hyperbolic line. Here* $M = G/K$*,* $K$ *is the holonomy, and the holonomy has complex scalar part.*

(iii) *Irreducible connected simply connected Riemannian symmetric spaces* $M = G/K$*, where* $K$ *has an* $Sp(1)$ *factor that generates quaternion algebras on the tangent spaces of* $M$*. Here* $K$ *is the holonomy, and the holonomy has quaternion scalar part.*

There is a structure theory for the spaces of Proposition 2.1(iii). There are two, a compact one and its non-compact dual, for each complex simple Lie algebra, and they are constructed from the highest root [W]. These spaces are listed in the Table 1 below. Here, we use the notation that $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$ denote the compact connected simply connected groups of those Cartan classification types, and their non-compact forms listed in the Table are connected real forms contained as analytic subgroups in the corresponding complex simply connected groups. All known examples of compact connected simply connected quaternionic manifolds with holonomy of quaternionic scalar type are Riemannian symmetric spaces.

**Table 1.**

| Irreducible Quaternionic Symmetric Spaces, Scalar Part of Holonomy Quaternionic | | | |
|---|---|---|---|
| compact $M = G/K$ | non-compact $M' = G'/K$ | Rank | Dimension/$\mathbb{H}$ |
| $SU(r+2)/S(U(r) \times U(2))$ | $SU(r,2)/S(U(r) \times U(2))$ | $\min(r,2)$ | $r$ |
| $SO(r+4)/[SO(r) \times SO(4)]$ | $SO(r,4)/[SO(r) \times SO(4)]$ | $\min(r,4)$ | $r$ |
| $Sp(n+1)/[Sp(n) \times Sp(1)]$ | $Sp(n,1)/[Sp(n) \times Sp(1)]$ | 1 | $n$ |
| $G_2/SO(4)$ | $G_{2,A_1A_1}/SO(4)$ | 2 | 2 |
| $F_4/[Sp(3) \cdot Sp(1)]$ | $F_{4,C_3C_1}/[Sp(3) \cdot Sp(1)]$ | 4 | 7 |
| $E_6/[SU(6) \cdot Sp(1)]$ | $E_{6,A_5C_1}/[SU(6) \cdot Sp(1)]$ | 4 | 10 |
| $E_7/[Spin(12) \cdot Sp(1)]$ | $E_{7,D_6C_1}/[Spin(12) \cdot Sp(1)]$ | 4 | 16 |
| $E_8/[E_7 \cdot Sp(1)]$ | $E_{8,E_7C_1}/[E_7 \cdot Sp(1)]$ | 4 | 28 |

Thus, irreducible quaternionic symmetric spaces have rank 1, 2, 3 or 4. Curiously, quaternionic symmetric spaces for $F_4$, $E_6$, $E_7$, and $E_8$ all have restricted root systems of type $F_4$.

## 3 Complex forms of quaternionic manifolds

Let $S$ be a smooth submanifold of a Riemannian manifold $M$. Let $A = \{A_x \mid x \in M\}$ denote a quaternionic structure on $M$. If $x \in S$, let $A_x^S$ denote the subalgebra of all elements in $A_x$ that preserve the real tangent space $T_x(S)$. We say that $S$ is totally complex, if $A_x^S \cong \mathbb{C}$ and $T_x(S) \cap q(T_x(S)) = 0$ for all $q \in A_x \setminus A_x^s$, for all $x \in S$. If $S$ is totally complex in $M$, then $A^S = \{A_x^S \mid x \in S\}$ restricts to a well-defined almost complex structure on $S$, parallel along $S$ because $A$ is parallel on $M$, so (see

[KN, Cor. 3.5, p. 145]) $(S, A^S|_S)$ is Kähler. If in addition, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, then we say that $S$ is a maximal totally complex submanifold of $M$.

Let $S$ be a maximal totally complex submanifold of $M$. Suppose that $S$ is a topological component of the fixed point set of an involutive isometry $\sigma$ of $M$. Then, we say that $S$ is a complex form of $M$ and that $\sigma$ is the quaternion conjugation of $M$ over $S$. The following is immediate.

**Lemma 3.1.** *Let $(M, A)$ be a quaternionic symmetric space. If $S$ is a complex form of $M$, then $S$ is a totally geodesic submanifold. If $S$ is a totally geodesic, totally complex submanifold of $M$, then $S$ is an Hermitian symmetric space.*

Let $M = G/K$, irreducible quaternionic symmetric space, with base point $x_0 = 1K$, where $K = K' \cdot Sp(1)$ as in Proposition 2.1(iii) and Table 1. Let $\theta$ denote the involutive automorphism of $G$ that is conjugation by the symmetry (say $t$) at $x_0$. Let $S \subset M$ be a totally geodesic submanifold through $x_0$. Then, $S$ is a Riemannian symmetric space with symmetry $t|_S$ at $x_0$. Express $S = L(x_0) \cong L/V$, where $L$ is the identity component of $\{g \in G \mid g(S) = S\}$ and $V = L \cap K$. Then $\theta(L) = L$.

The following three results are our basic tools for finding the complex forms $S = L/V$ of $M = G/K$, where $\operatorname{rank}(L) = \operatorname{rank}(G)$. Proposition 3.2 gives criteria for $L/V$ to be an appropriate submanifold of $G/K$. Proposition 3.3 tells us that when $L/V$ is identified abstractly, it in fact exists well positioned in $G/K$, and Proposition 3.4 is a uniqueness theorem showing when two complex forms are $G$-equivalent.

**Proposition 3.2.** *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let $\sigma$ be an involutive inner automorphism of $G$ that commutes with $\theta$. Let $L$ be the identity component of the fixed point set $G^\sigma$. Set $V = L \cap K$. Denote $S = L(x_0) \cong L/V$.*

1. *If $V \cap Sp(1)$ is a circle group, then $S$ is a totally complex submanifold of $M$.*
2. *$S$ is a complex form of $M$ if and only if* (i) *$V \cap Sp(1)$ is a circle group, and* (ii) $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$.
3. *If $S$ is a complex form of $M$, then $\sigma = Ad(s)$ where $s \in V$.*

**Proposition 3.3.** *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let $L$ be a symmetric subgroup of equal rank in $G$ that has an Hermitian symmetric quotient $L/V$, such that $V$ is isomorphic to a symmetric subgroup $V' \subset K$. Then, $L$ is conjugate to a $\theta$–stable subgroup $L' \subset G$ such that $L' \cap K = V'$.*

**Proposition 3.4.** *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let $S_i = L_i(x_0) \cong K_i/V_i$ be two complex forms of $M$. If $S_1$ and $S_2$ are isometric, then some element of $K$ carries $S_1$ onto $S_2$.*

*Proof of Proposition* 3.2. We can pass to the compact dual if necessary, so we may (and do) assume $M$ compact. Decompose the Lie algebra $\mathfrak{g}$ of $G$ under $d\theta$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{m}$ represents the real tangent space of $M$. Then $Sp(1)$ gives $\mathfrak{m}$ a quaternionic vector space structure, so any circle subgroup gives $\mathfrak{m}$ a complex vector space structure. If that circle is $V \cap Sp(1)$, it defines an $L$-invariant almost complex structure on $S$, and that is integrable because $S$ is a Riemannian symmetric space. We have proved Statement 1.

For Statement 2, first suppose that $S$ is a complex form of $M$. Since $\sigma$ is inner by hypothesis, $\mathrm{rank}(L) = \mathrm{rank}(G)$. Since $S$ is an Hermitian symmetric space, $\mathrm{rank}(V) = \mathrm{rank}(L)$. Now, $V$ contains a Cartan subgroup $T$ of $G$. Thus, $V \cap Sp(1)$ contains a circle group $T_1 := T \cap Sp(1)$. Now the only possibilities for $V \cap Sp(1)$ are (a) $T_1$, (b) the normalizer of $T_1$ in $Sp(1)$, and (c) all of $Sp(1)$. Here, (b) is excluded because it would prevent $S$ from having an $L$-invariant almost complex structure, and (c) is excluded because it would prevent $S$ from being totally complex, so $V \cap Sp(1)$ is a circle group. Finally, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$ because $S$ is a maximal totally complex submanifold of $M$.

Conversely, suppose that $V \cap Sp(1)$ is a circle group and $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$. By Statement 1, $S$ is a totally complex submanifold of $M$. By $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, it is a maximal totally complex submanifold. And we started with the symmetry $\sigma$, so $S$ is a complex form of $M$.

For Statement 3 note, as above, that $s \in L$ because $\mathrm{rank}(L) = \mathrm{rank}(G)$, and now $s \in V$ because $\mathrm{rank}(V) = \mathrm{rank}(L)$. □

*Proof of Proposition* 3.3. All our groups have equal rank, so $V'$ is the $K$-centralizer of some $v' \in V'$ with $v'^2$ central in $K$. Here, $K$ contains the center of $G$, and those centers satisfy $Z_K/Z_G = \{1, z\}Z_G$ cyclic order 2. Let $\sigma' = \mathrm{Ad}(v')$. If $v'^2 \in zZ_G$, then $\sigma'^2 = \theta$, so $d\sigma$ has eigenvalues $\pm\sqrt{-1}$ on $\mathfrak{m}$, and $L' = G^{\sigma'}$ has the property that $S' = L'(x_0) \cong L'/V'$ is Hermitian symmetric. Since $V \in L$ and $V' \in L'$ are symmetric subgroups of $G$, and their Hermitian symmetric subgroups are isomorphic, it follows from Table 1 and the classification of Riemannian symmetric spaces that $L \cong L'$. Now, $L$ and $L'$ are conjugate in $G$, so we may assume $L = L'$. Then, $V$ and $V'$ are isomorphic symmetric subgroups in $L$, so they are $L$-conjugate. This completes the proof. □

*Proof of Proposition* 3.4. Suppose that $S_1$ and $S_2$ are isometric, say $g : S_1 \cong S_2$ for some isometric map $g$. We can assume $g(x_0) = x_0$, so $dg$ gives a Lie triple system isomorphism of $\mathfrak{l}_1 \cap \mathfrak{m}$ onto $\mathfrak{l}_2 \cap \mathfrak{m}$. Write $\mathfrak{l}_i = \mathfrak{l}_i' \oplus \mathfrak{z}_i$, where $\mathfrak{l}_i'$ is generated by $\mathfrak{l}_i \cap \mathfrak{m}$ and $\mathfrak{z}_i \subset \mathfrak{v}_i$ is a complementary ideal. Then $dg$ gives a Lie algebra isomorphism of $\mathfrak{l}_1'$ onto $\mathfrak{l}_2'$. Let $\mathbf{j}_i \in \mathfrak{sp}(1)$ be orthogonal to the Lie algebra of the circle group $V_i \cap Sp(1)$. Then, $\mathbf{j}_i$ centralizes $\mathfrak{z}_i$ and $\mathfrak{m}$ is the real vector space direct sum of $\mathfrak{l}_i \cap \mathfrak{m}$ with $\mathrm{ad}(\mathbf{j}_i)(\mathfrak{l}_i \cap \mathfrak{m})$. Now, $\mathrm{ad}(\mathfrak{z}_i)|_{\mathfrak{m}} = 0$, so each $\mathfrak{z}_i = 0$, and $dg : \mathfrak{l}_1 \cong \mathfrak{l}_2$. Since $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are isomorphic symmetric subalgebras of $\mathfrak{g}$, they are $\mathrm{Ad}(G)$–conjugate. Thus we may assume $g \in G$. As $g(x_0) = x_0$ now $g \in K$. Thus some $g \in K$ carries $S_1$ onto $S_2$. □

Propositions 3.2 and 3.4 will let us do the classification of complex forms $S = L/V$ of quaternionic symmetric spaces $M = G/K$ in case $\mathrm{rank}(L) = \mathrm{rank}(G)$. There are only a few cases where $\mathrm{rank}(L) < \mathrm{rank}(G)$, and we will handle them individually. That is not very elegant, but it is very efficient.

## 4 The classification of complex forms

In this section, we state the classification of complex forms $S = L/V$ of quaternionic symmetric spaces $M = G/K$ and $M' = G'/K$ whose holonomy has quaternion scalar part. The proofs are given in Sections 5, 7 and 8. We state the results separately for the compact and the non-compact cases.

**Theorem 4.1.** *Let $M = G/K$ be a compact simply connected irreducible quaternionic Riemannian symmetric space. Then the complex forms $S = L/V$ of $M$ are exactly the following, and each is unique up to the action of $G$.*

1. $M = SU(r+2)/S(U(r)\times U(2))$. *Then* (1a) $S = \frac{SO(r+2)}{SO(r)\times SO(2)}$, *or*

   (1b) $S = P^u(\mathbb{C}) \times P^{r-u}(\mathbb{C}) = \frac{SU(u+1)}{S(U(u)\times U(1))} \times \frac{SU(r-u+1)}{S(U(r-u)\times U(1))}$, $0 \leqq u \leqq r$.
2. $M = SO(r+4)/[SO(r)\times SO(4)]$. *Then* (2a) $S = \frac{SU(r'+2)}{S(U(r')\times U(2))}$, $r = 2r'$ *even, or* (2b) $S = \frac{SO(u+2)}{[SO(u)\times SO(2)} \times \frac{SO(r-u+2)}{SO(r-u)\times SO(2)}$, $0 \leqq u \leqq r$.
3. $M = Sp(n+1)/[Sp(n)\times Sp(1)] = P^n(\mathbb{H})$. *Then* $S = P^n(\mathbb{C}) = \frac{U(n+1)}{[U(n)\times U(1)}$.
4. $M = G_2/SO(4)$. *Then* $S = P^1(\mathbb{C}) \times P^1(\mathbb{C}) = \frac{SO(4)}{SO(2)\times SO(2)}$.
5. $M = F_4/[Sp(3)\cdot Sp(1)]$. *Then* $S = \frac{Sp(3)}{U(3)} \times P^1(\mathbb{C})$.
6. $M = E_6/[SU(6)\cdot Sp(1)]$. *Then* (6a) $S = \frac{SU(6)}{S(U(3)\times U(3))} \times P^1(\mathbb{C})$, *or* (6b) $S = \frac{Sp(4)}{U(4)}$, *or* (6c) $S = \frac{SO(10)}{U(5)}$.
7. $M = E_7/[Spin(12)\cdot Sp(1)]$. *Then* (7a) $S = \frac{E_6}{Spin(10)\cdot U(1)}$, *or* (7b) $S = \frac{SU(8)}{S(U(4)\times U(4))}$, *or* (7c) $S = \frac{SO(12)}{U(6)} \times P^1(\mathbb{C})$.
8. $M = E_8/[E_7\cdot Sp(1)]$. *Then* (8a) $S = \frac{E_7}{E_6 T_1} \times P^1(\mathbb{C})$ *or* (8b) $S = \frac{SO(16)}{U(8)}$.

**Theorem 4.2.** *Let $M = G/K$ be a non-compact irreducible quaternionic Riemannian symmetric space. Then, the complex forms $S = L/V$ of $M$ are exactly the following, and each is unique up to the action of $G$.*

1. $M = SU(r,2)/S(U(r)\times U(2))$. *Then* (1a) $S = \frac{SO(r,2)}{SO(r)\times SO(2)}$, *or*

   (1b) $S = H^u(\mathbb{C}) \times H^{r-u}(\mathbb{C}) = \frac{SU(u,1)}{S(U(u)\times U(1))} \times \frac{SU(r-u,1)}{S(U(r-u)\times U(1))}$, $0 \leqq u \leqq r$.
2. $M = SO(r,4)/[SO(r)\times SO(4)]$. *Then* (2a) $S = \frac{SU(r',2)}{S(U(r')\times U(2))}$, $r = 2r'$ *even, or* (2b) $S = \frac{SO(u,2)}{[SO(u)\times SO(2)} \times \frac{SO(r-u,2)}{SO(r-u)\times SO(2)}$, $0 \leqq u \leqq r$.
3. $M = Sp(n,1)/[Sp(n)\times Sp(1)] = H^n(\mathbb{H})$. *Then* $S = H^n(\mathbb{C}) = \frac{U(n,1)}{[U(n)\times U(1)}$.
4. $M = G_{2,A_1A_1}/SO(4)$. *Then* $S = H^1(\mathbb{C}) \times H^1(\mathbb{C}) = \frac{SO(2,2)}{SO(2)\times SO(2)}$.
5. $M = F_{4,C_3C_1}/[Sp(3)\cdot Sp(1)]$. *Then* $S = \frac{Sp(3;\mathbb{R})}{U(3)} \times H^1(\mathbb{C})$.

6. $M = E_{6,A_5C_1}/[SU(6) \cdot Sp(1)]$. *Then* (6a) $S = \frac{SU(3,3)}{S(U(3)\times U(3))} \times H^1(\mathbb{C})$, *or* (6b) $S = \frac{Sp(4;\mathbb{R})}{U(4)}$, *or* (6c) $S = \frac{SO^*(10)}{U(5)}$.

7. $M = E_{7,D_6C_1}/[Spin(12) \cdot Sp(1)]$. *Then* (7a) $S = \frac{E_{6,D_5T_1}}{Spin(10)\cdot U(1)}$, *or* (7b) $S = \frac{SU(4,4)}{S(U(4)\times U(4))}$, *or* (7c) $S = \frac{SO^*(12)}{U(6)} \times P^1(\mathbb{C})$.

8. $M = E_{8,E_7C_1}/[E_7 \cdot Sp(1)]$. *Then* (8a) $S = \frac{E_{7,E_6T_1}}{E_6T_1} \times P^1(\mathbb{C})$ *or* (8b) $S = \frac{SO^*(16)}{U(8)}$.

Of course, Theorem 4.2 is immediate from Theorem 4.1 by passage to the non-compact dual symmetric spaces. So, we need only prove Theorem 4.1. The proof of Theorem 4.1 consists of consolidating the results of Sections 5, 7 and 8.

## 5 The equal rank classification — classical cases

We run through the list of compact irreducible quaternionic symmetric spaces $M = G/K$ from Table 1, for the cases where $G$ is a classical group. For each of them, we look at the possible symmetric subgroups $L$, that correspond to an Hermitian symmetric space $S = L/V$, such that $\mathrm{rank}(L) = \mathrm{rank}(G)$, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, $\mathrm{rank}(S) \leqq \mathrm{rank}(M)$, and $V$ is isomorphic to a symmetric subgroup of $K$ properly placed as in Proposition 3.2. The equal rank classification will follow using Proposition 3.4. We retain the notation used in Propositions 3.2 and 3.4. Fix $s \in K$, such that $L$ is the identity component of $\sigma = \mathrm{Ad}(s)$.

CASE $M = SU(r+2)/S(U(r) \times U(2))$. First, suppose $r \geqq 2$. We may take $s$ to be diagonal. It has only two distinct eigenvalues, and its component in the $U(2)$-factor of $K$ must have both eigenvalues. Now $L \cong S(U(u+1) \times U(v+1))$, $V \cong S([U(u) \times U(1)] \times [U(v) \times U(1)])$, and $S$ is the product $P^u(\mathbb{C}) \times P^v(C)$ of complex projective spaces. Here, $\dim_{\mathbb{H}} M = r = u + v = \dim_{\mathbb{C}} S$. If $u, v \geqq 1$, then $\mathrm{rank}(M) = 2 = \mathrm{rank}(S)$. If $u = 0$, then the factor $P^u(\mathbb{C})$ is reduced to a point, $S \cong P^v(C)$, and $\mathrm{rank}(S) = 1$. The analog holds, of course, if $v = 0$.

Now, consider the degenerate case $r = 1$. Then $M = P^2(\mathbb{C})$ and fits the dual pattern described in the paragraph just before the statement of Proposition 2.1. Relative to the quaternionic structure denoted $A_1$ there, the one with with quaternion scalar part, the matrix considerations above show that $M$ has a complex form $S = P^1(\mathbb{C})$.

CASE $M = SO(r+4)/[SO(r) \times SO(4)]$. As before, the matrix $s$ has just two distinct eigenvalues, and each one must appear with multiplicity 2 in the $SO(4)$-factor of $K$. If $s^2 = I$, then $L = SO(u+2) \times SO(v+2)$ with $u + v = r$, where $V = L \cap K = [SO(u) \times SO(2)] \times [SO(v) \times SO(2)]$. Here the $SO(2)$-factors in $V$ are the intersection with the $SO(4)$-factor of $K$. That gives us the forms $S = (SO(u+2)/[SO(u) \times SO(2)]) \times (SO(v+2)/[SO(v) \times SO(2)])$ of $M$.

Now, suppose $s^2 = -I$. Then, $r = 2r'$ even, $L \cong U(r'+2)$, $V \cong U(r') \times U(2)$, and we have the complex form $S \cong SU(r'+2)/S(U(r') \times U(2))$ of $M$.

CASE $M = Sp(n+1)/[Sp(n) \times Sp(1)] = P^n(\mathbb{H})$. The symmetric subgroups of $Sp(n+1)$ are the $Sp(u) \times Sp(v)$, $u + v = n + 1$, and $U(n+1)$. The first case,

$L = Sp(u) \times Sp(v)$, would give $V = Sp(u) \times Sp(v-1) \times Sp(1)$, so $S = Sp(v)/[Sp(v-1) \times Sp(1)]$, which is not Hermitian symmetric. That leaves the case $L = U(n+1)$ and $V = U(n) \times U(1)$, where $S = P^n(\mathbb{C})$. It satisfies the conditions of Proposition 3.2 and thus is a complex form of $M$.

## 6 The LiE program

While the matrix computation methods of Section 5 work well for the classical group cases, it is more convenient to make use of the root structure in the exceptional group cases. In this section, we indicate just how we used the LiE program [L] to do that. We illustrate it for $E_8$, but it is the same for any simple group structure. Here, node refers to the simple root at which the negative of the maximal root is attached in the extended Dynkin diagram.

STEP 0: INITIALIZE.

```
> setdefault(E8) > rank = 8
> diagram       ; prints out the Dynkin diagram and numbers the simple roots.
> node = 8      ; the number of the simple root that defines K.
```

STEP 1: POSITIVE ROOTS OF $\mathfrak{g}$.

```
> pos = pos_roots
> max_root = pos[n_rows(pos)]
```

STEP 2: POSITIVE ROOTS OF $\mathfrak{k}$.

```
> kkk = pos > for i = 1 to n_rows(kkk) do
  if kkk[i,node] == 1 then kkk[i] = null(rank) fi od    ; zeroes rows m-roots
> kk = unique(kkk)                                      ; eliminates duplicate rows
> k = null(n_rows(kk)-1,rank) > for i = 1 to n_rows(k) do k[i] = kk[i+1] od
                              ; eliminates last zero row
> Cartan_type(k)              ; verifies correct Cartan type for k, in this case E7A1
```

STEP 3: POSITIVE ROOTS OF $\mathfrak{m}$.

```
> mmm = pos > for i = 1 to n_rows(mmm) do
  if mmm[i,node] != 1 then mmm[i] = null(rank) fi od    ; zeroes rows for k–roots
> mm = unique(mmm)                                      ; eliminates duplicate rows
> m = null(n_rows(mm)-1,rank)
> for i = 1 to n_rows(m) do m[i] = mm[i+1] od           ; eliminates last zero row
```

STEP 4: CHOICE OF *sym* WHERE $\sigma = \mathrm{Ad}(sym)$; DEFINITION OF $\mathfrak{l} = \mathfrak{g}^\sigma$.

```
> sym = null(rank + 1)     ; initializes sym as row vector
> sym[node] = 1            ; one possibility for nonzero element of sym
> sym[rank+1] = 2          ; normalizes 1–parameter group containing symm
> l = cent_roots(sym)      ; defines l as centralizer of sym
> Cartan_type(l)           ; Cartan type of l, in this case E7A1
```

STEP 5: POSITIVE ROOTS OF $\mathfrak{s} := \mathfrak{l} \cap \mathfrak{m}$ AND OF $\mathfrak{v} := \mathfrak{l} \cap \mathfrak{k}$.

```
> sss = l > for i = 1 to n_rows(sss) do
  if sss[i,node] != 1 then sss[i] = null(rank) fi od
> ss = unique(sss)
> s = null(n_rows(ss)-1,rank)
> for i = 1 to n_rows(s) do s[i] = ss[i+1] od
> vvv = l
> for i = 1 to n_rows(vvv) do if vvv[i,2] == 1 then vvv[i] = null(rank) fi od
> vv = unique(vvv)
> v = null(n_rows(vv)-1,rank)
> for i = 1 to n_rows(v) do v[i] = vv[i+1] od
> Cartan_type(v)  ; Cartan type of v, in this case E6T1T1
                  ; At this point we know that S = L/V ≅ (E7/[E6 × T1]) × (T1/T1),
                  ; so it is an hermitian symmetric subspace of G/K.
```

STEP 6: VERIFY THAT $S$ IS A MAXIMAL TOTALLY COMPLEX IN $M$.

```
> t = null(n_rows(s)-1,rank)
> for i=1 to n_rows(t) do t[i] = max_root - s[i] od
> u = null(n_rows(s) + n_rows(t) + n_rows(m), rank)
> for i = 1 to n_rows(s) do u[i] = s[i] od
> for i = 1 to n_rows(t) do u[n_rows(s) + i] = t[i] od
> for i = 1 to n_rows(m) do u[n_rows(s) + n_rows(t) + i] = m[i] od

  ; now the rows of u are: positive roots of s,
  ; maximal root minus positive roots of s,
  ; positive roots of m
> w = unique(u)     ; the rows of w are the positive roots of m and non–root
                    ; linear functionals ( max root minus positive root of s )
> n_rows(w) - n_rows(m)    ; number of non–root linear functionals in w,
                           ; measures failure of S to be maximal totally complex;
                           ; OK here because it returns 0
```

We carry out the routine in some key cases. These are cases where $K$ and $L$ are conjugate in $G$.

CASE $G = B_7$. Here, node $= 2$, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = B_5A_1A_1$ and $V = B_4T_1T_1T_1$, thus to the complex form $S = SO(11)/[SO(9) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(15)/[SO(11) \times SO(4)$. More generally, for $B_n$ with $n \geqq 3$, node $= 2$, and sym = $[0, 1, 0, \dots, 0, 2]$ gives the complex form $S = SO(2n-3)/[SO(2n-5) \times SO(4)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(2n+1)/[SO(2n-3) \times SO(4)]$. This is the case $v = 2$, $u = r - 2$ considered for $G = SO(r + 4)$, $r$ odd, in Section 5.

CASE $G = D_7$. Here, node $= 2$, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = D_5A_1A_1$ and $V = D_4T_1T_1T_1$, thus to the complex form $S = SO(10)/[SO(8) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(14)/[SO(10) \times SO(4)]$. More generally, for $D_n$ with $n \geqq 3$, node $= 2$, and sym = $[0, 1, 0, \dots, 0, 2]$ gives the complex form $S = SO(2n-4)/[SO(2n-6) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(2n)/[SO(2n-4) \times SO(4)]$. This is the case $v = 2$, $u = r - 2$ considered for $G = SO(r + 4)$, $r$ even, in Section 5.

CASE $G = G_2$. Here, node = 2, and sym = [0, 1, 2] leads to $L = A_1 A_1$ and $V = T_1 T_1$, thus to the complex form $S = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = G_2/SO(4)$.

CASE $G = F_4$. Here, node = 1, and sym = [1, 0, 0, 0, 2] leads to $L = C_3 C_1$ and $V = A_2 T_1 T_1$, thus to the complex form $S = [Sp(3)/U(3)] \times P^1(\mathbb{C})$ of $G/K = F_4/C_3 C_1$.

CASE $G = E_6$. Here, node = 2, and sym = [0, 1, 0, 0, 0, 0, 2] leads to $L = A_5 A_1$ and $V = A_2 T_1 A_2 T_1$, thus to the complex form $S = [SU(6)/S(U(3) \times U(3))] \times P^1(\mathbb{C})$ of $G/K = E_6/A_5 A_1$.

CASE $G = E_7$. Here, node = 2, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = D_6 A_1$ and $V = A_5 T_1 T_1$, and thus to the complex form $S = [SO(12)/U(6)] \times P^1(\mathbb{C})$ of $G/K = E_7/D_6 A_1$.

CASE $G = E_8$. As we saw, sym = [0, 0, 0, 0, 0, 0, 0, 1, 2] leads to $L = E_7 A_1$ and $V = E_6 T_1 T_1$, and thus to the complex form $S = (E_7/[E_6 \times T_1]) \times P^1(\mathbb{C})$ of $G/K = E_8/E_7 A_1$.

CASES $A_7$ and $C_7$. Here, the computation using LiE has not yet produced complex forms $S$ of $M$. In other words, I have not yet guessed the appropriate vectors sym to define toral elements of $G$ whose centralizers are appropriate subgroups $L \subset G$.

## 7 The equal rank classification – exceptional cases

In this section, we complete the classification for the equal rank exceptional group cases.

CASE $G = G_2$. The only symmetric subgroup of $G_2$ is $SO(4)$, so here the only complex form of $M = G_2/SO(4)$ is $S = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ as described in Section 6.

CASE $G = F_4$. The only symmetric subgroups of $F_4$ are $Sp(3) \cdot Sp(1)$ and $Spin(9)$. If $L = Spin(9)$, then the Hermitian symmetric space $L/V = Spin(9)/[Spin(7) \times Spin(2)]$. That would place the $Spin(7)$-factor of $V$ in the $Sp(3)$-factor of $K$; but $Sp(3) \subset SU(6)$ while $Spin(7)$ has no non-trivial representation of degree $< 7$. Thus, $L \neq Spin(9)$, so, here the only complex form of $M = F_4/C_3 C_1$ is $S = [Sp(3)/U(3)] \times P^1(\mathbb{C})$ as described in Section 6.

CASE $G = E_6$. The only symmetric subgroups of maximal rank in $E_6$ are $A_5 A_1$ and $D_5 T_1$.

If $L = D_5 T_1$, then the Hermitian symmetric space $S = L/V$ must be either $SO(10)/[SO(8) \times SO(2)]$ with $V = [SO(8) \times SO(2)] \cdot SO(2)$, or $[SO(10)/U(5)]$ with $V = U(5) \cdot SO(2)$. The first is excluded because $\dim_{\mathbb{C}} SO(10)/[SO(8) \times SO(2)] = 8 < 10 = \dim_{\mathbb{H}} M$. The second of these is a complex form of $M = E_6/A_5 A_1$ by Propositions 3.2 and 3.3.

$L = A_5 A_1$ gives another complex form $S = [SU(6)/S(U(3) \times U(3))] \times P^1(\mathbb{C})$ of $M = E_6/A_5 A_1$ as described in Section 6.

CASE $G = E_7$. The only symmetric subgroups of $E_7$ are $D_6 A_1$, $A_7$ and $E_6 T_1$.

If $L \cong E_6T_1$, then the Hermitian symmetric space $S = L/V$ must be $E_6/D_5T_1$ with $V = D_5T_1T_1$. It is a complex form of $M = E_7/D_6A_1$ by Propositions 3.2 and 3.3.

If $L = A_7$, then the Hermitian symmetric space $S = L/V$ must be $SU(8)/S(U(u)\times U(v))$ with $u + v = 8$. Here $\dim_{\mathbb{C}} L/V = UV$ while $\dim_{\mathbb{H}} M = 16$, so $u = v = 4$. That would place the $[SU(4)\times SU(4)]$-factor of $V$ in the $Spin(12)$-factor of $K$. It could only sit there as $Spin(6) \times Spin(6)$, which is the identity component of its $Spin(12)$-normalizer because it is a symmetric subgroup of $Spin(12)$, so the circle center of $V$ is contained in the $Sp(1)$-factor of $K$. Thus, $S$ is a complex form of $M = E_7/D_6A_1$ by Propositions 3.2 and 3.3.

$L = D_6A_1$ gives another complex form $S = [SO(12)/U(6)] \times P^1(\mathbb{C})$ of $M = E_7/D_6A_1$ as as described in Section 6.

Case $G = E_8$. The only symmetric subgroups of $E_8$ are $E_7A_1$ and $D_8$.

If $L = D_8$, then the Hermitian symmetric space $S = L/V$ either must be $SO(16)/[SO(14) \times SO(2)]$ with $V = [SO(14) \times SO(2)]$, or $SO(16)/U(8)$ with $V = U(8)$. The first of these is excluded because $\dim_{\mathbb{C}} SO(16)/[SO(14)\times SO(2)] = 14 < 28 = \dim_{\mathbb{H}} M$. The second of these is a complex form of $M = E_8/E_7A_1$ by Propositions 3.2 and 3.3.

$L \cong E_7A_1$ gives another complex form $S = (E_7/[E_6 \times T_1]) \times P^1(\mathbb{C})$ of $M = E_8/E_7A_1$ as described in Section 6.

## 8 The unequal rank classification

In this section, we deal with the cases $\text{rank}(L) < \text{rank}(G)$. Here, $G$ is of type $A_n$, $D_n$ or $E_6$.

Case $M = SU(r+2)/S(U(r) \times U(2))$. The only symmetric subgroups of lower rank in $SU(r + 2)$ are $SO(r + 2)$ and, for $r = 2r'$ even, $Sp(r' + 1)$.

If $L = Sp(r'+1)$, $r = 2r'$ even, then $S = Sp(r'+1)/U(r'+1)$ with $V = U(r'+1)$. Here, $\dim_{\mathbb{H}} M = 2r'$ and $\dim_{\mathbb{C}} S = \frac{1}{2}(r' + 2)(r' + 1)$, so those dimensions are equal just when $r'^2 - r' + 2 = 0$. That equation has no integral solution. Thus, $L \neq Sp(r'+1)$.

If $L = SO(r + 2)$, then $S = SO(r + 2)/[SO(r) \times SO(2)]$ with $V = [SO(r) \times SO(2)]$. The $SO(2)$–factor of $V$ is contained in the derived group $SU(2)$ of the $U(2)$-factor of $K$, and $\dim_{\mathbb{C}} S = r = \dim_{\mathbb{H}} M$, so Proposition 3.2 shows that $S$ is a complex form of $M$.

Case $M = SO(2n+4)/[SO(2n) \times SO(4)]$. The only symmetric subgroups of lower rank in $SO(2n + 4)$ are $SO(2u + 1) \times SO(2v + 1)$, where $u + v = n + 1$. If $L = SO(2u+1)\times SO(2v+1)$ then $V = SO(2u-1)\times SO(2)\times SO(2v-1)\times SO(2)$ and $S = \{SO(2u+1)/[SO(2u-1) \times SO(2)]\} \times \{SO(2v+1)/[SO(2v-1) \times SO(2)]\}$, where the product of the two $SO(2)$-factors is contained in the $SO(4)$-factor of $K$. Since $\dim_{\mathbb{C}} S = (2u-1)+(2v-1) = n = \dim_{\mathbb{H}} M$, the argument of Proposition 3.2 shows that $S$ is a complex form of $M$.

Case $M = E_6/A_5A_1$. The only symmetric subgroups of lower rank in $E_6$ are $F_4$ and $C_4$, and $L \neq F_4$ because $F_4$ has no Hermitian symmetric quotient space. If $L = Sp(4)$, then $S = Sp(4)/U(4)$ with $V = U(4)$. Here, $V$ sits in $K$ as follows. The semi-simple part $[V, V] = U(4)/\{\pm I\} = SO(6) \subset SU(6) = A_5$. $[V, V]$ is a connected symmetric subgroup of the connected simple group $A_5$, so it is equal to the identity component of its normalizer in $A_5$. Thus, the projection $K = A_5A_1 \to A_5$ annihilates the circle center of $V$. In other words, $V \cap Sp(1)$ is a circle group central in $V$. It follows as in Proposition 3.2(1) that $S$ is a totally complex submanifold of $M$. Since $\dim_{\mathbb{C}} S = 10 = \dim_{\mathbb{H}} M$, it is a maximal totally complex submanifold, and being a symmetric submanifold it is a complex form.

This completes the proof of Theorems 4.1 and 4.2, the main results of this note.

## 9 Quaternionic forms

In this section, we look at the idea of quaternionic forms of symmetric spaces as suggested by the examples of projective planes $P^2(\mathbb{H}) \subset P^2(\mathbb{O})$ and hyperbolic planes $H^2(\mathbb{H}) \subset H^2(\mathbb{O})$. The meaning of Cayley structure is not entirely clear because of non-associativity, so we do not have a good definition for Cayley symmetric space. Here, we offer a tentative definition of quaternionic form and a number of examples, some interesting and some too artificial to be interesting.

Let $M$ be a Riemannian symmetric, let $\sigma$ be an involutive isometry of $M$, let $S$ be a totally geodesic submanifold of $M$, and suppose that (i) $S$ is a topological component of the fixed point set of $\sigma$, (ii) $\dim_{\mathbb{R}} S = \frac{1}{2} \dim_{\mathbb{R}} M$, and (iii) $S$ has quaternionic structure for which its holonomy has quaternion scalar part. Then we will say that $S$ is a quaternionic form of $M$.

Suppose that $M = G/K$ with base point $x_0 = 1K$ and $S = L(x_0) = L/V$, where $L$ is the identity component of the group generated by transvections of $S$. Following Proposition 2.1, $S = L/V$ is one of the spaces listed in Table 1. That gives us interesting examples

$\frac{SU(r+2)}{S(U(r)\times U(2))} = \frac{U(r+2)}{U(r)\times U(2)}$ in $Sp(r+2)/[Sp(r) \times Sp(2)]$;

$\frac{SO(r+4)}{SO(r)\times SO(4)}$ in $U(r+4)/[U(r) \times U(4)] = SU(r+4)/S(U(r) \times SU(4))$;

$\frac{Sp(r+1)}{Sp(r)\times Sp(1)}$ in $U(2r+2)/[U(2r) \times U(2)] = SU(2r+2)/S(U(2r) \times U(2))$;

$\frac{SU(r+2)}{S(U(r)\times U(2))} = \frac{U(r+2)}{U(r)\times U(2)}$ in $SO(2r+4)/[SO(2r) \times SO(4)]$; $\frac{Sp(3)}{Sp(2)\times Sp(1)} = P^2(\mathbb{H})$ in $P^2(\mathbb{O}) = F_4/Spin(9)$ (computing with LiE);

$\frac{E_7}{Spin(12)\cdot Sp(1)}$ in $E_8/SO(16)$ (using Proposition 3.3 with $L = E_7A_1$, as in §7).

It also gives us some other examples $S$ in $S \times S$ as a factor or as the diagonal;

$\frac{SU(r+2)}{S(U(r)\times U(2))}$ in $SU(r+4)/S(U(r) \times U(4))$ or $SU(2r+2)/S(U(2r) \times U(2))$;

$\frac{SO(r+4)}{SO(r)\times SO(4)}$ in $SO(r+8)/[SO(r) \times SO(8)]$ or $SO(2r+4)/[SO(2r) \times SO(4)]$;

$\frac{Sp(r+1)}{Sp(r)\times Sp(1)}$ in $Sp(r+2)/[Sp(r) \times Sp(2)]$ or $Sp(2r+1)/[Sp(2r) \times Sp(1)]$.

Those other examples somehow seem too formal to be interesting. Of course with any of these compact examples $S \subset M$, we also have the non-compact duals $S' \subset M'$.

These examples indicate that a reasonable theory for quaternionic forms $S$ of symmetric spaces $M$ will require some additional structure on the normal bundle of $S$ in $M$.

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