Schubert Varieties and Cycle Spaces

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Abstract

Let G_0 be a real semisimple Lie group. It acts naturally on every complex flag manifold $Z = G/Q$ of its complexification. Given an Iwasawa decomposition $G_0 = K_0A_0N_0$, a G_0–orbit $\gamma \subset Z$, and the dual K–orbit $\kappa \subset Z$, Schubert varieties are studied and a theory of Schubert slices for arbitrary G_0–orbits is developed. For this, certain geometric properties of dual pairs (γ, κ) are underlined. Canonical complex analytic slices contained in a given G_0-orbit γ which are transversal to the dual K_0-orbit $\gamma \cap \kappa$ are constructed and analyzed. Associated algebraic incidence divisors are used to study complex analytic properties of certain cycle domains. In particular, it is shown that the linear cycle space $\Omega_W(D)$ is a Stein domain that contains the universally defined Iwasawa domain Ω_I . This is one of the main ingredients in the proof that $\Omega_W(D) = \Omega_{AG}$ for all but a few hermitian exceptions. In the hermitian case, $\Omega_W(D)$ is concretely described in terms of the associated bounded symmetric domain.

0 Introduction

Let G be a connected complex semisimple Lie group and Q a parabolic subgroup. We refer to $Z = G/Q$ as a complex flag manifold. Write $\mathfrak{g}$ and $\mathfrak{q}$ for the respective Lie algebras of G and Q. Then Q is the G–normalizer of $\mathfrak{q}$. Thus we can view Z as the set of G–conjugates of $\mathfrak{q}$. The correspondence is $z \leftrightarrow \mathfrak{q}_z$ where $\mathfrak{q}_z$ is the Lie algebra of the isotropy subgroup Q_z of G at z.

Let G_0 be a real form of G, and let $\mathfrak{g}_0$ denote its Lie algebra. Thus there is a homomorphism $\varphi : G_0 \to G$ such that $\varphi(G_0)$ is closed in G and $d\varphi : \mathfrak{g}_0 \to \mathfrak{g}$ is an isomorphism onto a real form of $\mathfrak{g}$. This gives the action of G_0 on Z.

It is well known [21] that there are only finitely many G_0–orbits on Z. Therefore at least one of them must be open.

Consider a Cartan involution θ of G_0 and extend it as usual to G, $\mathfrak{g}_0$ and $\mathfrak{g}$. Thus the fixed point set $K_0 = G_0^\theta$ is a maximal compactly embedded subgroup of G_0 and $K = G^\theta$ is its complexification. This leads to Iwasawa decompositions $G_0 = K_0A_0N_0$.

By *Iwasawa–Borel subgroup* of G we mean a Borel subgroup $B \subset G$ such that $\varphi(A_0N_0) \subset B$ for some Iwasawa $G_0 = K_0A_0N_0$. Those are the Borel subgroups of the form $B = B_MAN$, where N is the complexification of N_0, A is the complexification of A_0, $M = Z_K(A)$ is the complexification of M_0 , and B_M is a Borel subgroup of M. Since any two Iwasawa decompositions of G_0 are G_0–conjugate, and any two Borel subgroups of M are M_0–conjugate because $\varphi(M_0)$ is compact, it follows that any two Iwasawa–Borel subgroups of G are G_0–conjugate.

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Given an Iwasawa–Borel subgroup $B \subset G$, we study the Schubert varieties $S = c\ell(\mathcal{O}) \subset Z$, where $\mathcal{O}$ is a B–orbit on Z. We extend the theory [15] of Schubert slices from the open G_0–orbits to arbitrary G_0–orbits. As a main application we show that the corresponding Schubert domain $\Omega_S(D)$ for an open G_0–orbit $D \subset Z$ is equal to the linear cycle space $\Omega_W(D)$ considered in [22]. That yields a direct proof that the $\Omega_W(D)$ are Stein manifolds. Another consequence is one of the two key containments for the complete description of linear cycle spaces when G_0 is of hermitian type (Section 8 and [25]). The identification $\Omega_W(D) = \Omega_S(D)$ also plays an essential role in the subsequently proved identification of $\Omega_W(D)$ with the universally defined domain Ω_{AG} [10].

Our main technical results (Theorem 3.1 and Corollary 3.4), which can be viewed as being of a complex analytic nature, provide detailed information on the q-convexity of D and the Cauchy-Riemann geometry of the lower-dimensional G_0-orbits.

We thank the referee for pointing out an error in our original argument for the existence of supporting incidence hypersurfaces at boundary points of the cycle space. This is rectified in Section 4 below.

1 Duality

We will need a refinement of Matsuki's (G_0, K)–orbit duality [19]. Write $Orb(G_0)$ for the set of G_0–orbits in Z, and similarly write $Orb(K)$ for the set of all K–orbits. A pair $(\gamma, \kappa) \in Orb(G_0) \times Orb(K)$ is *dual* or *satisfies duality* if $\gamma \cap \kappa$ contains an isolated K_0–orbit. The duality theorem states that

(1.1) if $\gamma \in Orb(G_0)$, or if $\kappa \in Orb(K)$, there is a unique dual $(\gamma, \kappa) \in Orb(G_0) \times Orb(K)$.

Furthermore,

(1.2) if (γ, κ) is dual, then $\gamma \cap \kappa$ is a single K_0–orbit.

Moreover, if (γ, κ) is dual, then the intersection $\gamma \cap \kappa$ is transversal: if $z \in \gamma \cap \kappa$, then the real tangent spaces satisfy

(1.3) $$T_z(\gamma) + T_z(\kappa) = T_z(Z) \text{ and } T_z(\gamma \cap \kappa) = T_z(\gamma) \cap T_z(\kappa) = T_z(K_0(z)).$$

We will also need a certain "non–isolation" property:

(1.4) Suppose that (γ, κ) is not dual but $\gamma \cap \kappa \neq \emptyset$.
If $p \in \gamma \cap \kappa$, there exists a locally closed K_0–invariant submanifold $M \subset \gamma \cap \kappa$ such that $p \in M$ and $\dim M = \dim K_0(p) + 1$.

The basic duality (1.1) is in [19], and the refinements (1.2) and (1.3) are given by the moment map approach ([20], [7]). See [7, Corollary 7.2 and §9].

The non–isolation property (1.4) is implicitly contained in the moment map considerations of [20] and [7]. Following [7], the two essential ingredients are the following.

1. Endow Z with a G_u–invariant Kähler metric, e.g., from the negative of the Killing form of $\mathfrak{g}_u$. Here G_u is the compact real form of G denoted U in [7]. The K_0-invariant gradient field ∇f^+ of the norm function $f^+ := \|\mu_{K_0}\|^2$ of the moment map for the K_0-action on Z, determined by the G_u–invariant metric, is tangent to both the G_0– and K–orbits.
2. A pair (γ, κ) satisfies duality if and only if their intersection is non–empty and contains a point of $\{\nabla f^+ = 0\}$.

If the pair does not satisfy duality, $p \in \gamma \cap \kappa$, $g = g(t)$ is the 1-parameter group associated to ∇f^+ and $\epsilon > 0$ is sufficiently small, then

$$M := \bigcup_{|t|<\epsilon} g(t)(K_0(p))$$

is the desired submanifold. For this it is only important to note that ∇f^+ does not vanish along, and is nowhere tangent to, $K_0(p)$. That completes the argument.

Dual pairs have a retraction property, which we prove using the moment map approach.

Theorem 1.5 *Let (γ, κ) be a dual pair. Fix $z_0 \in \gamma \cap \kappa = K_0(z_0)$. Then the intersection $\gamma \cap \kappa$ is a K_0-equivariant strong deformation retract of γ.*

Proof. We use the notation of [7, §9]. If $\phi(t, x)$ is the flow of ∇f^+, then it follows that, if $x \in \gamma$ and $t > 0$, then $\phi(t, x) \in \gamma$. Furthermore, the limiting set $\pi^+(x) = \lim_{t\to\infty} \phi(t, x)$ is contained in the intersection $K_0(z_0)$.

Let U be a K_0–slice neighborhood of $K_0(z_0)$ in γ. In other words, if $(K_0)_{z_0}$ denotes the isotropy subgroup of K_0 at z_0, then there is a $(K_0)_{z_0}$–invariant open ball B in the normal space $N_{z_0}(K_0(z_0))$ such that U is the $(K_0)_{z_0}$–homogeneous fiber space $K_0 \times_{(K_0)_{z_0}} B$ over B. The isotropy group $(K_0)_{z_0}$ is minimal over U in that, given $z \in U$, it is K_0–conjugate to a subgroup of $(K_0)_z$.

The flow $\phi(t, \cdot)$ is K_0–equivariant. Thus, since $\pi^+(x) \subset K_0.z_0$ for every $x \in \gamma$, every orbit $K_0.z$ in γ is equivariantly diffeomorphic, via some $\phi(t_0, \cdot)$, to a K_0-orbit in U. Consequently, $K_0(z_0)$ is minimal in γ.

The Mostow fibration of γ is a K_0–equivariant vector bundle with total space γ and base space that is a minimal K_0–orbit in γ. In other words the base space is $K_0(z_0)$. Any such vector bundle is K_0–equivariantly retractable to its 0–section. We may take that 0–section to be $K_0(z_0)$. $\square$

Corollary 1.6 *Every open G_0-orbit in Z is simply-connected. In particular the isotropy groups of G_0 on an open orbit are connected.*

Corollary 1.6 was proved by other methods in [21, Theorem 5.4]. In that open orbit case of Theorem 1.5, κ is the base cycle, maximal compact subvariety of γ.

2 Incidence divisors associated to Schubert varieties

Fix an open G_0–orbit $D \subset Z$. Its dual is the unique closed K–orbit C_0 contained in D. Denote $q = \dim_{\mathbb{C}} C_0$. Write $\mathcal{C}^q(Z)$ for the variety of q–dimensional cycles in Z. As a subset of Z, the complex group orbit $G \cdot C_0$ is Zariski open in its closure.

At this point, for simplicity of exposition we assume that $\mathfrak{g}_0$ is simple. This entails no loss of generality because all our flags, groups, orbits, cycles, etc. decompose as products according to the decomposition of $\mathfrak{g}_0$ as a direct sum of simple ideals.

In two isolated instances of (G_0, Z) (see [23]), $C_0 = Z$ and the orbit $G \cdot C_0$ consists of a single point. If G_0 is of hermitian type and D is an open G_0–orbit of "holomorphic type" in the terminology of [24], then $G \cdot C_0$ is the compact hermitian symmetric space dual to the bounded symmetric domain $\mathcal{B}$. This case is completely understood ([22], [24]). In these two cases we set $\Omega := G \cdot C_0$. Here Ω is canonically identified as a coset space of G, because the G–stabilizer of C_0 is its own normalizer in G.

Except in the two cases just mentioned, the G–stabilizer $\widetilde{K}$ of C_0 has identity component K, and there is a canonical finite equivariant map $\pi : G/K \to G \cdot C_0 \cong G/\widetilde{K}$. Here we set $\Omega = G/K$. Its base point is the coset K.

Suppose that Y is a complex analytic subset of Z. Then $A_Y := \{C \in \pi(\Omega) \mid \mathbb{C} \cap Y \neq \emptyset\}$ is a closed complex variety in Ω [5], called the incidence variety associated to Y. For purposes of comparison we work with the preimage $\pi^{-1}(A_Y)$ in Ω. From now on we abuse notation: we write $A_Y := \{C \in \Omega \mid C \cap Y \neq \emptyset\}$. If A_Y is purely of codimension 1 then we refer to it as the incidence divisor associated to Y and denote it by H_Y.

Now suppose that the complex analytic subset Y is a Schubert variety defined by an Iwasawa-Borel subgroup $B \subset G$. Thus Y is the closure of one or more orbits of B on Z. Then the incidence variety A_Y is B–invariant, because Ω and Y are B–invariant. Define $\mathcal{Y}(D)$ to be the set of all Iwasawa-Schubert varieties $Y \subset Z$ such that $Y \subset Z \setminus D$ and A_Y is a hypersurface H_Y. Then we define the *Schubert domain* $\Omega_S(D)$:

$$\Omega_S(D) \text{ is the connected component of } C_0 \text{ in } \Omega \setminus \Big(\bigcup_{Y \in \mathcal{Y}(D)} H_Y\Big). \tag{2.1}$$

See [16, §6] and [15]. Note that any two Iwasawa-Borel subgroups are conjugate by an element of K_0. Thus

$$\bigcup_{Y \in \mathcal{Y}(D)} H_Y = \bigcup_{k \in K_0} k(H),$$

where $H := H_1 \cup \ldots \cup H_m$ is the union of the incidence hypersurfaces defined by the Schubert varieties in the complement of D of a fixed Iwasawa-Borel subgroup. Thus $\Omega_S(D)$ is an open subset of Ω, and of course it is G_0-invariant by construction.

In Corollary 4.7 we will show that the cycle space $\Omega_W(D)$ (see (4.1)) agrees with $\Omega_S(D)$. Consequently, it has the same analytic properties. For example we now check that $\Omega_S(D)$ is a Stein domain.

In order to prove that $\Omega_S(D)$ is Stein, it suffices to show that it is contained in a Stein subdomain $\widetilde{\Omega}$ of Ω. For then, given a boundary point $p \in \mathrm{bd}(\Omega_S(D))$ in $\widetilde{\Omega}$, it will be contained in a complex hypersurface H that is equal to or a limit of incidence divisors H_Y. Now $H \cap \widetilde{\Omega}$ is in the complement of D and will be the polar set of a meromorphic function on $\widetilde{\Omega}$. So $\Omega_S(D)$ will be a domain of holomorphy in the Stein subdomain $\widetilde{\Omega}$, and will therefore be Stein.

As mentioned above, there are three possibilities for Ω. If $C_0 = Z$, then Ω is reduced to a point, and $\Omega_S(D)$ is Stein in a trivial way. Now suppose $D \subsetneqq Z$. Then either Ω is a compact hermitian symmetric space G/KP_- or it is the affine variety G/K. In the latter case Ω is Stein, so $\Omega_S(D)$ is Stein. Now we are down to the case where $\Omega = G/KP_-$ is an irreducible compact hermitian symmetric space. In particular the second Betti number $b_2(\Omega) = 1$. Therefore the divisor of every complex hypersurface in Ω is ample. For $Y \in \mathcal{Y}(D)$ this implies that $\Omega \setminus H_Y$ is affine. Since $\mathcal{Y}(D) \neq \emptyset$ and $\Omega \setminus H_Y \supset \Omega_S(D)$, this implies that $\Omega_S(D)$ is Stein in this case as well. Therefore we have proved

Proposition 2.2 *If D is an open G_0–orbit in the complex flag manifold Z, then the associated Schubert domain $\Omega_S(D)$ is Stein.*

3 Schubert varieties associated to dual pairs

Fix an Iwasawa decomposition $G_0 = K_0 A_0 N_0$. Let B be a corresponding Iwasawa–Borel subgroup of G; in other words $A_0 N_0 \subset B$. Fix a K–orbit κ on Z and let $\mathcal{S}_\kappa$ denote the set of all Schubert varieties S defined by B (that is, S is the closure of a B–orbit on Z) such that $\dim S + \dim \kappa = \dim Z$ and $S \cap c\ell(\kappa) \neq \emptyset$. The Schubert varieties generate the integral homology of Z. Hence $\mathcal{S}_\kappa$ is determined by the topological class of $c\ell(\kappa)$.

Theorem 3.1 (Schubert Slices) *Let $(\gamma, \kappa) \in Orb(G_0) \times Orb(K)$ satisfy duality. Then the following hold for every $S \in \mathcal{S}_\kappa$.*

1. *$S \cap \mathrm{c}\ell(\kappa)$ is contained in $\gamma \cap \kappa$ and is finite. If $w \in S \cap \kappa$, then $(AN)(w) = B(w) = \mathcal{O}$ where $S = \mathrm{c}\ell(\mathcal{O})$, and S is transversal to κ at w in the sense that the real tangent spaces satisfy $T_w(S) \oplus T_w(\kappa) = T_w(Z)$.*
2. *The set $\Sigma = \Sigma(\gamma, S, w) := A_0N_0(w)$ is open in S and closed in γ.*
3. *Let $\mathrm{c}\ell(\Sigma)$ and $\mathrm{c}\ell(\gamma)$ denote closures in Z. Then the map $K_0 \times \mathrm{c}\ell(\Sigma) \to \mathrm{c}\ell(\gamma)$, given by $(k, z) \mapsto k(z)$, is surjective.*

Proof. Let $w \in S \cap \mathrm{c}\ell(\kappa)$. Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, complexification of the Lie algebra version $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ of $G_0 = K_0A_0N_0$, we have $T_w(AN(w)) + T_w(K(w)) = T_w(Z)$. As $w \in S = \mathrm{c}\ell(\mathcal{O})$ and $AN \subset B$, we have $\dim AN(w) \leqq \dim B(w) \leqq \dim \mathcal{O} = \dim S$. Furthermore $w \in \mathrm{c}\ell(\kappa)$. Thus $\dim K(w) \leqq \dim \kappa$. If w were not in κ, this inequality would be strict, in violation of the above additivity of the dimensions of the tangent spaces. Thus $w \in \kappa$ and $T_w(S) + T_w(\kappa) = T_w(Z)$. Since $\dim S + \dim \kappa = \dim Z$ this sum is direct, i.e., $T_w(S) \oplus T_w(\kappa) = T_w(Z)$. Now also $\dim AN(w) = \dim S$ and $\dim K(w) = \dim \kappa$. Thus $AN(w)$ is open in S, forcing $AN(w) = B(w) = \mathcal{O}$. We have already seen that $K(w)$ is open in κ, forcing $K(w) = \kappa$. For assertion 1 it remains only to show that $S \cap \kappa$ is contained in γ and is finite.

Denote $\widehat{\gamma} = G_0(w)$. If $\widehat{\gamma} \neq \gamma$, then $(\widehat{\gamma}, \kappa)$ is not dual, but $\widehat{\gamma} \cap \kappa$ is nonempty because it contains w. By the non–isolation property (1.4), we have a locally closed K_0–invariant manifold $M \subset \widehat{\gamma} \cap \kappa$ such that $\dim M = \dim K_0(w) + 1$. We know $T_w(S) \oplus T_w(\kappa) = T_w(Z)$, and $K(w) = \kappa$, so $T_w(A_0N_0(w)) \cap T_w(M) = 0$. Thus $T_w(A_0N_0(w)) + T_w(K_0(w))$ has codimension 1 in the subspace $T_w(A_0N_0(w)) + T_w(M)$ of $T_w(\widehat{\gamma})$, which contradicts $G_0 = K_0A_0N_0$. We have proved that $(S \cap \mathrm{c}\ell(\kappa)) \subset \gamma$. Since that intersection is transversal at w, it is finite. This completes the proof of assertion 1.

We have seen that $T_w(AN(w)) \oplus T_w(K(w)) = T_w(Z)$, so $T_w(A_0N_0(w)) \oplus T_w(K_0(w)) = T_w(\gamma)$, and $G_0(w) = \gamma$. With the characterization (1.2) and the transversality conditions (1.3) for duality we have $\dim A_0N_0(w) = \dim T_w(\gamma) - \dim T_w(\kappa \cap \gamma) = \dim T_w(Z) - \dim T_w(\kappa) = \dim AN(w) = \dim S$. Now $A_0N_0(w)$ is open in S, .

Every A_0N_0–orbit in γ meets $K_0(w)$, because $\gamma = G_0(w) = A_0N_0K_0(w)$. Using (1.3), every such A_0N_0–orbit has dimension at least that of $\Sigma = A_0N_0(w)$. Since the orbits on the boundary of Σ in γ would necessarily be smaller, it follows that Σ is closed in γ. This completes the proof of assertion 2.

The map $K_0 \times \Sigma \to \gamma$, by $(k, z) \mapsto k(z)$, is surjective because $K_0A_0N_0(w) = \gamma$. Since K_0 is compact and γ is dense in $\mathrm{c}\ell(\gamma)$, assertion 3 follows. □

We now apply Theorem 3.1 to construct an Iwasawa-Schubert variety Y of codimension $q+1$, $q = \dim C_0$, which contains a given point $p \in \mathrm{bd}(D)$ and which is contained in $Z \setminus D$. Due to the presence of the large family of q–dimensional cycles in D, one could not hope to construct larger varieties with these properties.

Before going into the construction, let us introduce some convenient notation and mention several preliminary facts.

We say that a point $p \in \mathrm{bd}(D)$ is *generic*, written $p \in \mathrm{bd}(D)_{gen}$, if $\gamma_p := G_0(p)$ is open in $\mathrm{bd}(D)$. This is equivalent to γ_p being an isolated orbit in $\mathrm{bd}(D)$ in the sense that no other G_0-orbit in $\mathrm{bd}(D)$ has γ_p in its closure. Clearly $\mathrm{bd}(D)_{gen}$ is open and dense in $\mathrm{bd}(D)$.

Given $p \in \mathrm{bd}(D)_{gen}$ the orbit $\gamma = \gamma_p$ need not be a real hypersurface in Z. For example, $G_0 = SL_{n+1}(\mathbb{R})$ has exactly two orbits in $\mathbb{P}_n(\mathbb{C})$, an open orbit and its complement $\mathbb{P}_n(\mathbb{R})$. Nevertheless, for any z in such an orbit γ it follows that $c\ell(D) \cap \mathrm{bd}(D) = \gamma$ near z. If κ is dual to γ, then, since the intersection $\kappa \cap \gamma$ is transversal in Z, it follows that $\kappa \cap D \neq \emptyset$.

We summarize this as follows.

Lemma 3.2 *For $p \in \mathrm{bd}(D)_{gen}$, $\gamma = \gamma_p = G_0(p)$ and κ dual to γ, it follows that $\kappa \cap D \neq \emptyset$. Furthermore, if C_0 is the base cycle in D, then*

$$q = \dim\ C_0 < \dim\ \kappa.$$

Proof. The property $\kappa \cap D \neq \emptyset$ has been verified above. For the dimension estimate note that C_0 is dimension-theoretically a minimal K_0–orbit in D, e.g., the K_0–orbits in $\kappa \cap D$ are at least of its dimension. Since κ is not compact, it follows that $\dim\ \kappa > \dim\ C_0$. $\square$

We will also make use of the following basic fact about Schubert varieties.

Lemma 3.3 *Let B be a Borel subgroup of G, let S be a k-dimensional B–Schubert variety in Z, and suppose that $\dim Z \geqq \ell \geqq k$. Then there exists a B-Schubert variety S' with $\dim S' = \ell$ and $S' \supset S$.*

Proof. We may assume that $S \neq Z$. Let $\mathcal{O}$ be the open B–orbit in S and $\mathcal{O}'$ be a B–orbit of minimal dimension among those orbits with $\mathrm{c}\ell(\mathcal{O}') \supsetneqq \mathcal{O}$.

For $p \in \mathcal{O}$ it follows that $\mathrm{c}\ell(\mathcal{O}') \setminus \mathcal{O} = \mathcal{O}'$ near p. Since $\mathcal{O}'$ is affine, it then follows that $\dim\ \mathcal{O}' = (\dim\ \mathcal{O}) + 1$. Applying this argument recursively, we find Schubert varieties $S' := \mathrm{c}\ell(\mathcal{O}')$ of every intermediate dimension ℓ. $\square$

We now come to our main application of Theorem 3.1.

Corollary 3.4 *Let D be an open G_0–orbit on Z and fix a boundary point $p \in \mathrm{bd}\, D$. Then there exist an Iwasawa decomposition $G_0 = K_0 A_0 N_0$, an Iwasawa–Borel subgroup $B \supset A_0 N_0$, and a B–Schubert variety Y, such that* (1) *$p \in Y \subset Z \setminus D$,* (2) *$\mathrm{codim}_Z Y = q + 1$, and* (3) *$A_Y$ is a B–invariant analytic subvariety of Ω.*

Proof. Let $p \in \mathrm{bd}(D)_{gen}$, let $\gamma = \gamma_p$, and let κ be dual to γ. First consider the case where $p \in \gamma \cap \kappa$. From Lemma 3.2, $\mathrm{codim} S \geqq q + 1$ for every $S \in \mathcal{S}_\kappa$.

Now, given $S \in \mathcal{S}_\kappa$, further specify p to be in $S \cap \kappa$ and let Y be a $(q+1)$-codimensional Schubert variety containing S (see Lemma 3.3).

Recall that Y is $A_0 N_0$–invariant, that $D = G_0(z_0) = A_0 N_0 K_0(z_0)$, and that $C_0 = K_0(z_0)$. If $Y \cap D \neq \emptyset$ it follows that there is an intersection point $z \in Y \cap C_0$. Then $A_0 N_0(z) \subset Y$. Since $\dim_{\mathbb{C}} C_0 = q$ and $\mathrm{codim}_{\mathbb{C}} Y = q + 1$, it would follow that

$$q = \dim_{\mathbb{C}} C_0 = \ \mathrm{codim}_{\mathbb{C}}\ A_0 N_0(z) \geqq \ \mathrm{codim}_{\mathbb{C}}\ Y = q + 1.$$

Thus Y does not meet D. On the other hand, using Theorem 3.1, it meets every G_0-orbit in $\mathrm{c}\ell(\gamma)$. Thus, by conjugating appropriately, we have the desired result for any point in the closure of γ. Since γ was chosen to be an arbitrary isolated orbit in $\mathrm{bd}(D)$, the result follows for every point of $\mathrm{bd}(D)$. $\square$

4 Supporting hypersurfaces at the cycle space boundary

Let $D = G_0(z_0)$ be an open G_0–orbit on Z. Let C_0 denote the base cycle $K_0(z_0) = K(z_0)$ in D, i.e., the dual K-orbit κ to the open G_0-orbit $\gamma = D$. Then the cycle space of D is given by

$$\Omega_W(D) := \ \text{component of } C_0 \text{ in } \{gC_0 \mid g \in G \text{ and } gC_0 \subset D\}. \tag{4.1}$$

Since D is open and C_0 is compact, the cycle space $\Omega_W(D)$ initially sits as an open submanifold of the complex homogeneous space $G/\widetilde{K}$, where $\widetilde{K}$ is the isotropy subgroup of G at C_0. In the Appendix, Section 9, we will see (with the few Hermitian exceptions which have already been mentioned) that the finite covering $\pi : \Omega := G/K \to \widetilde{\Omega} := G/\widetilde{K}$ restricts to an equivariant biholomorphic diffeomorphism of the lifted cycle space

$$(4.2) \qquad \widetilde{\Omega_W(D)} := \text{ component of } gK \text{ in } \{gK \mid g \in G \text{ and } gC_0 \subset D\} \subset \Omega$$

onto $\Omega_W(D)$.

The main goal of the present section is, given $C \in \mathrm{bd}(\Omega_W(D))$, to determine a particular point $p \in C$ which is contained in a Iwasawa–Schubert variety Y with $\mathrm{codim}_Z Y = q+1$, so that $Y \cap D = \emptyset$, and $A_Y = H_Y$ is of pure codimension 1. It is then an immediate consequence (see Corollary 4.7) that $\Omega_W(D) = \Omega_S(D)$.

Given $p \in C \cap \mathrm{bd}(D)$, we consider Iwasawa–Schubert varieties $S = c\ell(\mathcal{O})$ of minimal possible dimension that satisfy the following conditions:

1. $p \in S \setminus \mathcal{O} := E$
2. $S \cap D \neq \emptyset$
3. The union of the irreducible components of E that contain p is itself contained in $Z \setminus D$.

Notation: Let A denote the union of all the irreducible components of E contained in $Z \setminus D$ and let B denote the union of the remaining components of E. In particular $E = A \cup B$.

Note that by starting with the Schubert variety $S_0 := Y$ as in the proof of Corollary 3.4, and by considering a chain $S_0 \subset S_1 \subset \dots$ with $\dim\, S_{i+1} = \dim\, S_i + 1$, we eventually come to a Schubert variety $S = S_k$ with these properties. Of course, given p, the Schubert variety S may not be unique, but $\dim\, S =: n - q + \delta \geqq n - q$.

The following Proposition gives a constructive method for determining an Iwasawa–Borel invariant incidence hypersurface that contains C and is itself contained in the complement $\Omega \setminus \Omega_W(D)$. Here S is constructed as above.

Proposition 4.3 *If $\delta > 0$, then $C \cap A \cap B \neq \emptyset$.*

Given Proposition 4.3, take a point $p_1 \in C \cap A \cap B$ and replace S by a component S_1 of B that contains p_1. Possibly there are components of $E_1 := S_1 \setminus \mathcal{O}_1$ that contain p_1 and also have non-empty intersection with D. If that is the case, we replace S_1 by any such component. Since this S_1 still has non-empty intersection with the Iwasawa–Borel invariant A, at least some of the components of its E_1 do not intersect in this way. Continuing in this way, we eventually determine an S_1 that satisfies all of the above conditions at p_1. The procedure stops because Schubert varieties of dimension less than $n - q$ have empty intersection with D.

Corollary 4.4 *If S_0 satisfies the above conditions at p_0, then there exist $p_1 \in \mathrm{bd}(D)$ and a Schubert subvariety $S_1 \subset S_0$ that satisfies these conditions at p_1 and has dimension $n - q$.*

Proof. We recursively apply the procedure indicated above until $\delta = 0$. □

Corollary 4.5 *If $C \in \mathrm{bd}(\Omega_W(D))$, there exists an Iwasawa–Schubert variety S of dimension $n - q$ such that $E := S \setminus \mathcal{O}$ has non-empty intersection with C.*

Corollary 4.6 *Let $C \in \mathrm{bd}(\Omega_W(D))$. Then there exists an Iwasawa–Borel subgroup $B \subset G$, and a component H_E of a B–invariant incidence variety A_E, where $E = S \setminus \mathcal{O}$ as above, such that $H_E \subset \Omega \setminus \Omega_W(D)$, $C \in H_E$, and $\mathrm{codim}_\Omega H_E = 1$, i.e., H_E is an incidence divisor.*

Proof. The hypersurface E in S is the support of an ample divisor [16]. Thus the trace-transform method ([4], see also [16, Appendix]) produces a meromorphic function on Ω with a pole at C and polar set contained in A_E. Hence A_E has a component H_E as required. $\square$

In the language of Section 3 this shows that for every $C \in \mathrm{bd}(\Omega_W(D))$ there exists $Y \in \mathcal{Y}(D)$ such that $C \in H_Y$. In other words, every such boundary point is contained in the complement of the Schubert domain $\Omega_S(D)$. By definition $\Omega_W(D) \subset \Omega_S(D)$. Using that, the equality of these domains follows immediately:

Corollary 4.7 $\Omega_W(D) = \Omega_S(D)$.

Let us now turn to certain technical preparations for the proof of Proposition 4.3. For S as in its statement, let $\mathcal{U}_S$ be its preimage in the universal family $\mathcal{U}$ parameterized by Ω. The mapping $\pi : \mathcal{U}_S \to \Omega$ is proper and surjective and the fiber $\pi^{-1}(C)$ over a point $C \in \Omega$ can be identified with $C \cap S$. All orbits of the Iwasawa–Borel group that defines S are transversal to the base cycle C_0; in particular, $C_0 \cap S$ is pure–dimensional with $\dim C_0 \cap S = \delta$. Thus the generic cycle in Ω has this property.

Choose a 1-dimensional (local) disk Δ in Ω with C corresponding to its origin, such that $I_z := \pi^{-1}(z)$ is δ–dimensional for $z \neq 0$. Define $\mathcal{X}$ to be the closure of $\pi^{-1}(\Delta \setminus \{0\})$ in $\mathcal{U}_S$. The map $\pi_{\mathcal{X}} := \pi|_{\mathcal{X}} : \mathcal{X} \to \Delta$ is proper and its fibers are purely δ–dimensional.

In the sequel we use the standard moving lemma of intersection theory and argue using a desingularization $\widetilde{\pi} : \widetilde{S} \to S$, where only points E are blown up. Let $\widetilde{E}$, $\widetilde{A}$ and $\widetilde{B}$ denote the corresponding $\widetilde{\pi}$-preimages. By taking Δ in generic position we may assume that for $z \neq 0$ no component of I_z is contained in E. Hence we may lift the family $\mathcal{X} \to \Delta$ to a family $\widetilde{\mathcal{X}} \to \Delta$ of δ-dimensional varieties such that $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite to one outside of the fiber over $0 \in \Delta$. Let $\widetilde{I}_z$ denote the fiber of $\widetilde{\mathcal{X}} \to \Delta$ at $z \in \Delta$, and shrink $\widetilde{\mathcal{X}}$ so that $\widetilde{I} := \widetilde{I}_0$ is connected. Since $\widetilde{I}_z \cap \widetilde{A} = \emptyset$ for $z \neq 0$, it follows that the intersection class $\widetilde{I}.\widetilde{A}$ in the homology of $\widetilde{S}$ is zero.

An irreducible component of $\widetilde{I}$ is one of the following types: it intersects $\widetilde{A}$ but not $\widetilde{B}$, or it intersects both $\widetilde{A}$ and $\widetilde{B}$, or it intersects $\widetilde{B}$ but not $\widetilde{A}$. Write $\widetilde{I} = \widetilde{I_A} \cup \widetilde{I_{AB}} \cup \widetilde{I_B}$ correspondingly.

Lemma 4.8 $\widetilde{I_{AB}} \neq \emptyset$.

Proof. Since $\widetilde{I_A} \cup \widetilde{I_{AB}} \neq \emptyset$, it is enough to consider the case where $\widetilde{I_A} \neq \emptyset$. Let H be a hyperplane section in Z with $H \cap S = E$ (see e.g. [16]) and put H in a continuous family H_t of hyperplanes with $H_0 = H$ such that $H_t \cap I_A$ is $(\delta - 1)$–dimensional for $t \neq 0$ and such that the lift $\widetilde{E}_t$ of $E_t := H_t \cap S$ contains no irreducible component of $\widetilde{I_A}$. In particular, $\widetilde{E}_t.\widetilde{I_A} \neq 0$ for $t \neq 0$. Since $\widetilde{I_A}.\widetilde{A} = \widetilde{I_A}.\widetilde{E} = \widetilde{I_A}.\widetilde{E}_t$, it follows that $\widetilde{I_A}.\widetilde{A} \neq 0$. But $0 = \widetilde{I}.\widetilde{A} = \widetilde{I_A}.\widetilde{A} + \widetilde{I_{AB}}.\widetilde{A}$ and therefore $\widetilde{I_{AB}} \neq \emptyset$. $\square$

Proof of Proposition 4.3. We first consider the case where $\delta \geqq 2$. Since $\widetilde{I_{AB}} \neq \emptyset$, it follows that some irreducible component I' of I has non-empty intersection with both A and B. Of course $I' \cap E = (I' \cap A) \cup (I' \cap B)$. But E is the support of a hyperplane section, and since $\dim I' \geqq 2$, it follows that $(I' \cap E)$ is connected. In particular $(I' \cap A)$ meets $(I' \cap B)$. Therefore $I' \cap A \cap B \neq \emptyset$ and consequently $C \cap A \cap B \neq \emptyset$.

Now suppose that $\delta = 1$, i.e., that $\widetilde{I}$ is 1-dimensional. Since $\widetilde{I}.\widetilde{A} = 0$, the (non-empty) intersection $\widetilde{I} \cap \widetilde{A}$ is not discrete. We will show that some component of $\widetilde{I_{AB}}$ is contained in $\widetilde{A}$. It will follow immediately that $C \cap A \cap B \neq \emptyset$. For this we assume to the contrary that every component of $\widetilde{I}$ which is contained in $\widetilde{A}$ is in $\widetilde{I_A}$. We decompose $\widetilde{I} = \widetilde{I}_1 \cup \widetilde{I}_2$, where $\widetilde{I}_1$ consists of those components of $\widetilde{I}$ which are contained in $\widetilde{A}$ and $\widetilde{I}_2$ of those which have discrete or empty intersection with $\widetilde{A}$.

Now $\widetilde{I}_1.\widetilde{A} = \widetilde{I}_1.\widetilde{E}$. Choosing H_t as above, we have $\widetilde{I}_1.\widetilde{E} = \widetilde{I}_1.\widetilde{E}_t \geqq 0$ for $t \neq 0$. If $\widetilde{I}_2 \neq \widetilde{I}_B$, then $\widetilde{I}_2.\widetilde{A} > 0$. This would contradict $0 = \widetilde{I}.\widetilde{A} = \widetilde{I}_1\widetilde{A} + \widetilde{I}_2.\widetilde{A}$. Thus $\widetilde{I}_2 = \widetilde{I}_B$ and $\widetilde{I}_1 = \widetilde{I}_A$. But

$\widetilde{I}_A$ and $\widetilde{I}_B$ are disjoint, contrary to $\widetilde{I}$ being connected. Thus it follows that $\widetilde{I_{AB}}$ does indeed contain a component that is contained in $\widetilde{A}$. The proof is complete. $\square$

Remark 4.9 In the non–hermitian case, the main result of [10] leads to a non–constructive, but very short, proof of the existence of an incidence hypersurface $H \subset G/\widetilde{K}$ containing a given boundary point $C \in \mathrm{bd}(\Omega_W(D))$. (The analogous construction in the Hermitian case is somewhat easier; see Section 8 below.) For this, note that if S is a q–codimensional Schubert variety with $S \cap C_0 \neq \emptyset$, then, using [4] as above, for $Y := S \setminus \mathcal{O}$, it follows that $H := H_Y$ is indeed a hypersurface. Now let Ω_H be the connected component containing the base point of $\Omega \setminus \bigcup_{k\in K} k(H)$. It is shown in [10] that Ω_H agrees with the Iwasawa domain Ω_I which can be defined as the intersection of all Ω_H, where H is a hypersurface in Ω which is invariant under some Iwasawa–Borel subgroup of G (see Section 6). It follows that $\Omega_S(D) = \Omega_H$, because by definition $\Omega_I \subset \Omega_S(D) \subset \Omega_I$.

By Corollary 3.4 there is an incidence variety A_Y in $Z \setminus D$ which contains C and is invariant by some Iwasawa–Borel subgroup B. Since the open B–orbit in Ω is affine, there exists a B–invariant hypersurface H which contains A_Y. Since $\Omega_S(D) = \Omega_I$, it is also contained in Z/D and thus it has the desired properties. That completes the short proof. In fact it follows that $\Omega_W(D) = \Omega_I = \Omega_S(D) = \Omega_H$. $\square$

5 Intersection properties of Schubert slices

Let (γ, κ) be a dual pair and $z_0 \in \gamma\cap\kappa$. Let Σ be the Schubert slice at z_0 , i.e., $\Sigma = A_0N_0(z_0) \subset \gamma$. In particular $z_0 \in \Sigma \cap \kappa$. We take a close look at the intersection set $\Sigma \cap \kappa$.

Let L_0 denote the isotropy subgroup $(G_0)_{z_0}$, and therefore $(K_0)_{z_0} = K_0 \cap L_0$ and $(A_0N_0)_{z_0} = (A_0N_0) \cap L_0$. Define $\alpha : (K_0)_{z_0} \times (A_0N_0)_{z_0} \to L_0$ by group multiplication.

Lemma 5.1 *The map $\alpha : (K_0)_{z_0} \times (A_0N_0)_{z_0} \to L_0$ is a diffeomorphism onto an open subgroup of L_0.*

Proof. Since $\dim K_0(z_0) + \dim (A_0N_0)(z_0) = \dim G_0(z_0)$ and $\dim K_0 + \dim (A_0N_0) = \dim G_0$ we have $\dim (K_0)_{z_0} + \dim (A_0N_0)_{z_0} = \dim L_0$. Thus the orbit of the neutral point under the action of the compact group $(K_0)_{z_0}$ is the union of certain components of $L_0/(A_0N_0)_{z_0}$, i.e., $\mathrm{Image}\,(\alpha) = (K_0)_{z_0} \cdot (A_0N_0)_{z_0}$ is an open subgroup of L_0 .

The injectivity of α follows from the fact that G_0 is the topological product $G_0 = K_0 \times (A_0N_0)$. This product structure also yields the fact that $(K_0)_{z_0}(m)$ is transversal to $(A_0N_0)_{z_0}$ at every $m \in (A_0N_0)_{z_0}$. Thus, α is a local diffeomorphism along $(A_0N_0)_{z_0}$ and by equivariance is therefore a diffeomorphism onto its image. $\square$

Corollary 5.2 *If L_0 is connected, in particular if γ is simply connected, then $\Sigma \cap \kappa = \{z_0\}$.*

Proof. If $z_1 \in \Sigma \cap \kappa$, then there exists $k \in K_0$ and $an \in A_0N_0$ so that $k^{-1}(z_0) = (an)(z_0)$, i.e., $kan \in L_0$. Therefore $k \in (K_0)_{z_0}$, $an \in (A_0N_0)_{z_0}$, and $z_1 = z_0$. $\square$

Theorem 5.3 *Let D be an open G_0-orbit in Z, $C_0 \subset D$ the base cycle, $z_0 \in C_0$, and $\Sigma = A_0N_0(z_0)$ a Schubert slice at z_0 . If $C \in \Omega_W(D)$, then $\Sigma \cap C$ consists of a single point, and the intersection $\Sigma \cap C$ at that point is transversal.*

Proof. Let $S := c\ell B(z_0)$ be the Schubert variety containing Σ, and let k denote the intersection number $[S] \cdot [C_0]$. We know from Theorem 3.1 that intersection points occur only in the open A_0N_0–orbits in S. The open G_0–orbit D is simply connected, and therefore Corollary 5.2 applies. Thus $\Sigma \cap C_0 = \{z_0\}$. From Theorem 3.1(1) it follows that intersection this transversal.

Hence it contributes exactly 1 to $[S] \cdot [C_0]$. Now we have k different open A_0N_0–orbits in S, each of which contains exactly one (transversal) intersection point.

Cycles $C \in \Omega_W(D)$ are homotopic to C_0. Thus $[S] \cdot [C] = k$. As we homotopy C_0 to C staying in D, the intersection points of course move around, but each stays in its original open A_0N_0–orbits in S. Since Σ is one of those open A_0N_0–orbits, it follows that $\Sigma \cap C$ consists of a single point, and the intersection there is transversal, as asserted. □

Remark 5.4 One might hope that the orbit D would be equivariantly identifiable with a bundle of type $K \times_{(K_0)_{z_0}} \Sigma$, but the following example shows that this is not the case. Let $Z = \mathbb{P}_2(\mathbb{C})$ be equipped with the standard $SU(2,1)$-action. Let D be the open $SU(2,1)$–orbit consisting of positive lines, i.e, the complement of the closure of the unit ball B in its usual embedding. The Schubert slice Σ for D is contained in a projective line tangent to $\mathrm{bd}(B)$; see [17]. If $z_0 \in C_0 \subset D$, the only $(K_0)_{z_0}$–invariant line in $\mathbb{P}_2(\mathbb{C})$ that contains z_0 and is not contained in C_0 is the line determined by z_0 and the K_0-fixed point in B.

6 The domains Ω_I and Ω_{AG}

The Schubert domain $\Omega_S(D)$ is defined as a certain subspace of the cycle space Ω. When G_0 is of hermitian type and Ω is the associated compact hermitian symmetric space, the situation is completely understood [22]: $\Omega_W(D)$ is the bounded symmetric domain dual to Ω in the sense of symmetric spaces. Now we put that case aside. Then $\Omega \cong G/K$, and we have

(6.1) $$\Omega_W(D) = \Omega_S(D) \subset \Omega = G/K.$$

Let B be an Iwasawa–Borel subgroup of G. It has only finitely many orbits on Ω, and those orbits are complex manifolds. The orbit $B(1K)$ is open, because ANK is open in G, and its complement $S \subset \Omega$ is a finite union $\bigcup H_i$ of B–invariant irreducible complex hypersurfaces. For any given open G_0–orbit, some of these H_i occur in the definition (2.1) of $\Omega_S(D)$. The *Iwasawa domain* Ω_I is defined as in (2.1) except that we use all the H_i:

(6.2) $$\Omega_I \text{ is the connected component of } C_0 \text{ in } = \Omega \setminus \Big(\bigcup_{g \in G_0} g(S)\Big).$$

This definition is independent of choice of B because any two Iwasawa–Borel subgroups of G are G_0–conjugate. Just as in the case of the Schubert domains, we note here that

$$\bigcup_{g \in G_0} g(S) = \bigcup_{k \in K_0} k(S)$$

is closed. By definition, $\Omega_I \subset \Omega_S(D)$ for every open G_0–orbit D in Z.

The argument for $\Omega_S(D)$ also shows that Ω_I is a Stein domain in Ω. See [15] for further properties of Ω_I

The Iwasawa domain has been studied by several authors from a completely different viewpoint and with completely different definitions. See [2], [3], [6], [12] and [18]. Here is the definition in [2]. Let X_0 be the closed G_0–orbit in G/B and let $\mathcal{O}_{max}$ be the open K–orbit there. The *polar* $\widehat{X_0}$ of X_0 is the connected component of $1.K$ in $\{gK \in \Omega \mid g \in G \text{ and } g^{-1}X_0 \subset \mathcal{O}_{max}\}$.

Proposition 6.3 [26] $\widehat{X_0} = \Omega_I$.

Proof. Let $\pi : G \to G/K = \Omega$ denote the projection. As S is the complement of $B \cdot K$ in Ω, $\pi^{-1}(\Omega_I)$ is the interior of $I := \bigcap_{g \in G_0} g(ANK)$. Note that $h \in I \Leftrightarrow g^{-1}h \in ANK$ for all $g \in G_0$ $\Leftrightarrow h^{-1}g \in KAN$ for all $g \in G_0 \Leftrightarrow h^{-1}G_0 \subset KAN$. Viewing $1B$ as the base point in X_0, the condition for hK being in $\widehat{X_0}$ is that $h^{-1}G_0B \subset KB = KAN$. Thus $h \in I \Leftrightarrow hK \in \widehat{X_0}$, in other words $\widehat{X_0} = \pi(I) = \Omega \setminus \bigcup_{g \in G_0} g(S)$. □

Corollary 6.4 *The polar $\widehat{X_0}$ to the closed G_0–orbit X_0 is a Stein subdomain of Ω.*

Now we turn to the domain Ω_{AG}. The Cartan involution θ of $\mathfrak{g}_0$ defines the usual Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ and the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\ \mathfrak{p}_0$ of $\mathfrak{g}$. Let G_u be the corresponding compact real form of G, real–analytic subgroup for $\mathfrak{g}_u$, acting on $\Omega = G/K$. Then

$$\Omega_{AG} := \{x \in \Omega \mid \text{ the isotropy subgroup } (G_0)_x \text{ is compactly embedded}\}^0,$$

the topological component of $x_0 = 1K$. It is important to note that the action of G_0 on the Akhiezer-Gindikin domain Ω_{AG} is proper [1].

In work related to automorphic forms ([6], [18]) it was shown that $\Omega_{AG} \subset \widehat{X_0}$, when G_0 is a classical group. Other related results were proved in [12].

By means of an identification of $\widehat{X_0}$ with a certain maximal domain Ω_{adpt} for the adapted complex structure inside the real tangent bundle of G_0/K_0, and using basic properties of plurisubharmonic functions, it was shown by the first author that $\Omega_{AG} \subset \Omega_I$ in complete generality [15]. Barchini proved the opposite inclusion in [2]. Thus $\Omega_{AG} = \Omega_I$. In view of Theorem 6.3, we now have

Theorem 6.5 $\Omega_I = \widehat{X_0} = \Omega_{AG}$.

Remark 6.6 In particular, this gives yet another proof that Ω_{AG} is Stein. That result was first proved in [8] where a plurisubharmonic exhaustion function was constructed.

Summary: In general, $\Omega_S(D) = \Omega_W(D)$ and $\Omega_I = \widehat{X_0} = \Omega_{AG}$.

7 Cycle spaces of lower-dimensional G_0-orbits

Let us recall the setting of [12]. For $Z = G/Q$, $\gamma \in Orb_Z(G_0)$ and $\kappa \in Orb_Z(K)$ its dual, let $G(\gamma)$ be the connected component of the identity of $\{g \in G : g(\kappa) \cap \gamma \text{ is non-empty and compact }\}$. Note that $G(\gamma)$ is an open K-invariant subset of G which contains the identity. Define $\mathcal{C}(\gamma) := G(\gamma)/K$. Finally, define $\mathcal{C}$ as the intersection of all such cycle spaces as γ ranges over $Orb_Z(G_0)$ and Q ranges over all parabolic subgroups of G.

Theorem 7.1 $\mathcal{C} = \Omega_{AG}$.

This result was checked in [12] for classical and hermitian exceptional groups by means of case by case computations, and the authors of [12] conjectured it in general. As will be shown here, it is a consequence of the statement

$$\Omega_W(D) = \Omega_S(D) \text{ when } D \text{ is an open } G_0\text{–orbit in } G/B,$$

and of the following general result [12, Proposition 8.1].

Proposition 7.2 $\left(\bigcap_{D \subset G/B\ open} \Omega_W(D)\right) \subset \mathcal{C}$.

Proof of Theorem. The polar $\widehat{X_0}$ in $Z = G/B$ coincides with the cycle space $\mathcal{C}_Z(\gamma_0)$, where γ_0 is the unique closed G_0-orbit in Z. As was shown above, this agrees with the Iwasawa domain Ω_I which in turn is contained in every Schubert domain $\Omega_S(D)$. Thus, for every open G_0-orbit D_0 in $Z = G/B$ we have the inclusions

$$\left(\bigcap_{D \subset G/B \text{ open}} \Omega_W(D)\right) \subset \mathcal{C} \subset \mathcal{C}_Z(\gamma_0) = \widehat{X_0} = \Omega_I \subset \Omega_S(D_0) = \Omega_W(D_0).$$

Intersecting over all open G_0-orbits D in G/B, the equalities

$$\left(\bigcap_{D\subset G/B \text{ open}} \Omega_W(D)\right) = \mathcal{C} = \Omega_I = \left(\bigcap_{D\subset G/B \text{ open}} \Omega_W(D)\right)$$

are forced, and $\mathcal{C} = \Omega_{AG}$ is a consequence of $\Omega_I = \Omega_{AG}$. □

As noted in our introductory remarks, using in particular the results of the present paper, it was shown in [10] that $\Omega_W(D) = \Omega_{AG}$ with the obvious exceptions in the well-understood hermitian cases. This is an essentially stronger result than the above theorem on intersections. On the other hand, it required a good deal of additional work and therefore it is perhaps of interest that the intersection result follows as above in a direct way from $\Omega_W(D) = \Omega_S(D)$. So, for example, in any particular case where this latter point was verified, the intersection theorem would be immediate (see e.g. [17] for the case of $SL(n,\mathbb{H})$).

8 Groups of hermitian type

Let G_0 be of hermitian type. Write $\mathcal{B}$ for the bounded symmetric domain G_0/K_0 with a fixed choice of invariant complex structure. Drop the colocation convention leading to (6.1), so that now the cycle space $\Omega_W(D)$ really consists of cycles as in [22] and [24]. It has been conjectured (see [24]) that, whenever D is an open G_0–orbit in a complex flag manifold $Z = G/Q$, there are just two possibilities:

1. A certain double fibration (see [24]) is holomorphic, and $\Omega_W(D)$ is biholomorphic either to $\mathcal{B}$ or to $\overline{\mathcal{B}}$, or
2. both $\Omega_W(D)$ and $\mathcal{B}\times\overline{\mathcal{B}}$ have natural biholomorphic embeddings into G/K, and there $\Omega_W(D) = \mathcal{B}\times\overline{\mathcal{B}}$.

The first case is known ([22], [24]), and the second case has already been checked [24] in the cases where G_0 is a classical group.

The inclusion $\Omega_W(D) \subset \mathcal{B}\times\overline{\mathcal{B}}$ was proved in general ([24]; or see [25]). It is also known [8] that $\mathcal{B}\times\overline{\mathcal{B}} = \Omega_{AG}$. Combine this with $\Omega_{AG} \subset \Omega_I$ ([15]; or see Theorem 6.5), with $\Omega_W(D) = \Omega_S(D)$ (Corollary 4.7), and with $\Omega_I \subset \Omega_S(D)$ (compare definitions (2.1) and (6.2)) to see that

$$\Omega_S(D) = \Omega_W(D) \subset (\mathcal{B}\times\overline{\mathcal{B}}) = \Omega_{AG} \subset \Omega_I \subset \Omega_S(D). \tag{8.1}$$

Now we have proved the following result. (Also see [25].)

Theorem 8.2 *Let G_0 be a simple noncompact group of hermitian type. Then either* (1) *a certain double fibration* (see [24]) *is holomorphic, and $\Omega_W(D)$ is biholomorphic to $\mathcal{B}$ or to $\overline{\mathcal{B}}$, or* (2) $\Omega = G/K$ *and* $\Omega_W(D) = \Omega_S(D) = \Omega_I = \Omega_{AG} = (\mathcal{B}\times\overline{\mathcal{B}})$.

Remark 8.3 Since the above argument already uses the inclusion $\Omega_W(D) \subset (\mathcal{B}\times\overline{\mathcal{B}})$ of [24], it should be noted that the construction for the proof of Corollary 4.7 can be replaced by the following treatment. Using Corollary 3.4, given $p \in \operatorname{bd}(\Omega_W(D))$, one has an Iwasawa–Borel subgroup $B \subset G$ and a B–invariant incidence variety A_Y such that $p \in A_Y \subset \Omega\setminus\Omega_W(D)$. Since the open B–orbit in Ω is affine, it follows that A_Y is contained in a B–invariant hypersurface H. But $\Omega_W(D) \subset \Omega_{AG} \subset \Omega_I$, and $H \subset \Omega\setminus\Omega_I$ by definition of the latter. Thus $\Omega_W(D) = \Omega_I = \Omega_{AG} = (\mathcal{B}\times\overline{\mathcal{B}})$.

9 Appendix: Lifting the cycle space to G/K

As mentioned in connection with the definitions (4.1) and (4.2), we can view the cycle space $\Omega_W(D)$ inside G/K because of

Theorem 9.1 *The projection $\pi : G/K \to G/\widetilde{K}$ restricts to a G_0–equivariant holomorphic cover $\pi : \widetilde{\Omega_W(D)} \to \Omega_W(D)$, and $\pi : \widetilde{\Omega_W(D)} \to \Omega_W(D)$ is one to one.*

We show that $\widetilde{\Omega_W(D)}$ is homeomorphic to a cell, and then we apply [11, Corollary 5.3].

Without loss of generality we may assume that G is simply connected. Let G_u denote the θ–stable compact real form of G such that $G_u \cap G_0 = K_0$, connected. Then G_u is simply connected because it is a maximal compact subgroup of the simply connected group G. It follows that G_u/K_0 is simply connected. We view G_u/K_0 as a riemannian symmetric space M_u, using the negative of the Killing form of G_u for metric and $\theta|_{G_u}$ for the symmetry at $1 \cdot K_0$. It is connected and simply connected.

Definition 9.2 Let x_0 denote the base point $1 \cdot K_0 \in G_u/K_0 = M_u$. Let $L \subset T_{x_0}(M_u)$ denote the conjugate locus at x_0, all tangent vectors ξ at x_0 such that $d\exp_{x_0}$ is nonsingular at $t\xi$ for $0 \leqq t < 1$ but singular at ξ. Then we define

$$\tfrac{1}{2}M_u := \left\{ \exp_{x_0}(t\xi) \,\middle|\, \xi \in L \text{ and } 0 \leqq t < \tfrac{1}{2} \right\}.$$

The conjugate locus L and the cut locus are the same for M_u [9], so $\frac{1}{2}M_u$ consists of the points in M_u at a distance from x_0 less than half way to $\exp_{x_0}(L)$. For example, if G_0/K_0 is a bounded symmetric domain $\mathcal{B}$, then a glance at the polysphere that sweeps out M_u under the action of K_0 shows that $\frac{1}{2}M_u = \mathcal{B}$.

Proposition 9.3 *The lifted cycle space $\widetilde{\Omega_W(D)} = G_0 \cdot \frac{1}{2}M_u \subset G/K$. It is G_0–equivariantly diffeomorphic to $(G_0/K_0) \times M_u$. In particular it is homeomorphic to a cell.*

Proof. According to [10, Theorem 5.2.6], the lifted cycle space $\widetilde{\Omega_W(D)} \subset G/K$ coincides with the Akhiezer–Gindikin domain Ω_{AG}. The restricted root description [1] of Ω_{AG} is (in our notation)

$$\Omega_{AG} = G_0 \cdot \exp(\{\xi \in \mathfrak{a}_u \mid |\alpha(\xi)| < \pi/2 \;\; \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{a})\})K/K,$$

where $\mathfrak{a}_u$ is a maximal abelian subspace of $\{\xi \in \mathfrak{g}_u \mid \theta(\xi) = -\xi\}$ and $\Delta(\mathfrak{g}, \mathfrak{a})$ is the resulting family of restricted roots. A glance at Definition 9.2 shows that

$$\tfrac{1}{2}M_u = K_0 \cdot \exp(\{\xi \in \mathfrak{a}_u \mid |\alpha(\xi)| < \pi/2 \;\; \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{a})\})K/K.$$

Thus $\widetilde{\Omega_W(D)} = \Omega_{AG} = G_0 \cdot \exp(\{\xi \in \mathfrak{a}_u \mid |\alpha(\xi)| < \pi/2 \;\; \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{a})\})K/K = G_0 \cdot \frac{1}{2}M_u$. That is the first assertion.

For the second assertion note that $\widetilde{\Omega_W(D)}$ fibers G_0–equivariantly over G_0/K_0 by $gx \mapsto gK_0$ for $g \in G_0$ and $x \in \frac{1}{2}M_u$. For the third assertion note that G_0/K_0 and $\frac{1}{2}M_u$ are homeomorphic to cells. $\square$

Proof of Theorem. As $\widetilde{\Omega_W(D)}$ is a cell, $H^q(\widetilde{\Omega_W(D)}; \mathbb{Z}) = 0$ for $q > 0$ and the Euler characteristic $\chi(\widetilde{\Omega_W(D)}) = 1$. Let γ be a covering transformation for $\pi : \widetilde{\Omega_W(D)} \to \Omega_W(D)$ and let n be its order. Then the cyclic group $\langle\gamma\rangle$ acts freely on $\widetilde{\Omega_W(D)}$ and the quotient manifold $\widetilde{\Omega_W(D)}/\langle\gamma\rangle$ has Euler characteristic $\chi(\widetilde{\Omega_W(D)})/n$ [11, Corollary 5.3], so $n = 1$. Now the covering group of $\pi : \widetilde{\Omega_W(D)} \to \Omega_W(D)$ is trivial, so π is one to one. $\square$

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