Cycle Spaces of Real Forms of $\mathrm{SL}_n(\mathbb{C})$

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1 Background

Let us introduce the notation and goals of this paper in the context of an example. For this let the complex Lie group $G_\mathbb{C} = \mathrm{SL}_3(\mathbb{C})$ act on the complex projective space $Z = \mathbb{P}_2(\mathbb{C})$ in the usual way and consider the induced action of the real form $G_\mathbb{R} = \mathrm{SL}_3(\mathbb{R})$. The latter has only two orbits on Z, the set $M = \mathbb{P}_2(\mathbb{R})$ of real points, and its complement D.

This situation leads one to consider representations of $G_\mathbb{R}$ on linear spaces that are defined by the complex geometry at hand. We focus our attention on the open orbit D. Here M is totally real and has a basis of Stein neighborhoods. It follows that D is pseudoconcave. Consequently $\mathcal{O}(D) \cong \mathbb{C}$ and we must look further for appropriate linear spaces.

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If $\mathbb{E} \to D$ is a holomorphic $G_{\mathbb{R}}$-homogeneous vector bundle then, taking the pseudoconcavity into consideration, the theorem of Andreotti and Grauert [1] suggests that we consider the linear space $H^1(D; \mathcal{O}(\mathbb{E}))$. For simplicity consider the case where $\mathbb{E} \to D$ is the holomorphic cotangent bundle, dual to the holomorphic tangent bundle, and identify $H^1(D; \mathcal{O}(\mathbb{E}))$ with the space $H^{1,1}(D)$ of $\overline{\partial}$-closed $(1,1)$-forms modulo those that are exact.

The Andreotti–Norguet transform [2] allows one to represent such a Dolbeault cohomology space as a space of holomorphic functions on a space of cycles of appropriate dimension. For this, recall that a (compact) q-cycle C in a complex space X is a linear combination $C = n_1 C_1 + \cdots + n_k C_k$ where the n_j are positive integers and each C_j is an irreducible q-dimensional compact subvariety of X. Equipped with the topology of Hausdorff convergence, the space $\mathcal{C}^q(X)$ of all such cycles has a natural structure of complex space [3].

If X is a complex manifold, then the theorem of Andreotti and Norguet [2] states that the map defined by integration has values in $\mathcal{O}(\mathcal{C}^q(X))$:

$$AN : H^{q,q}(X) \to \mathcal{O}(\mathcal{C}^q(X)) \quad \text{where} \quad AN(\alpha)(C) = \int_C \alpha \, .$$

In general it is not a simple matter to explicitly describe either side of this correspondence. However, in the special case where D is an open $G_{\mathbb{R}}$-orbit such as the one in our example, there is at least a very natural space of cycles.

In the case of $D = \mathbb{P}_2(\mathbb{C}) \setminus \mathbb{P}_2(\mathbb{R})$ as above, choose $K_{\mathbb{R}} = SO_3(\mathbb{R})$ as a maximal compact subgroup of $G_{\mathbb{R}}$. Observe that $K_{\mathbb{R}}$ has a unique orbit in D that is a complex submanifold, namely the quadric curve $C_0 = \{z \in \mathbb{P}_2(\mathbb{C}) \mid z_0^2 + z_1^2 + z_2^2 = 0\}$. We may regard C_0 as a point in $\mathcal{C}^1(Z)$ and consider its orbit $\Omega := G_{\mathbb{C}} \cdot C_0$. The isotropy subgroup of $G_{\mathbb{C}}$ at C_0 is $K_{\mathbb{C}}\mathbb{Z}_3$, where $K_{\mathbb{C}}$ is the complexification $SO_3(\mathbb{C})$ of $K_{\mathbb{R}}$, and $\mathbb{Z}_3 = \{\omega I \mid \omega^3 = 1\}$ is the center of $G_{\mathbb{C}}$. So Ω may be regarded as the homogeneous space $G_{\mathbb{C}}/K_{\mathbb{C}}\mathbb{Z}_3$. That in turn can be identified with the complex symmetric 3×3 matrices of determinant 1, modulo $\mathbb{Z}_3$. So Ω is a very concrete, very familiar object.

It is another matter to give a concrete description of the space $\Omega(D) := \{C \in \Omega \mid C \subset D\}$, which is naturally associated to D. However we at least see that it is a $G_{\mathbb{R}}$-invariant open set in Ω and that, by restriction, we have the Andreotti–Norguet transform $AN : H^{1,1}(D) \to \mathcal{O}(\Omega(D))$.

Before looking more closely at the example, we introduce an appropriate general setting. For proofs and other basic facts we refer the reader to [14].

Let $Z = G_{\mathbb{C}}/Q$ be a projective algebraic variety, necessarily compact, viewed as a homogeneous manifold of a complex semisimple group $G_{\mathbb{C}}$. (Other terminology: Q is a parabolic subgroup of $G_{\mathbb{C}}$, or, equivalently, Z is a complex flag manifold.) Let $G_{\mathbb{R}}$ be a noncompact real form of $G_{\mathbb{C}}$. It can be shown that $G_{\mathbb{R}}$ has only finitely many orbits in Z; in particular at least one of them is open.

If D is such an open $G_{\mathbb{R}}$-orbit on Z, and $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$, then $K_{\mathbb{R}}$ has exactly one orbit in D that is a complex submanifold. We refer to it as the "base cycle" C_0 and regard it as a point $C_0 \in \mathcal{C}^q(Z)$ where $q = \dim_{\mathbb{C}} C_0$.

Since the action of $G_{\mathbb{C}}$ on $\mathcal{C}^q(Z)$ is algebraic, the orbit $\Omega := G_{\mathbb{C}} \cdot C_0$ is Zariski open in its closure. Define $\Omega(D)$ to be the connected component of $\{C \in \Omega \mid C \subset D\}$ that contains C_0.

In certain hermitian symmetric cases Ω is compact [15], but in those cases $\Omega(D)$ is just the associated bounded symmetric domain. There are also a few strange cases where $G_{\mathbb{R}}$ is transitive on Z [18]; in those cases $\Omega(D) = \Omega$ and it is reduced to a single point. In general, however, the isotropy subgroup of $G_{\mathbb{C}}$ at C_0 is a finite extension of the complexification of $K_{\mathbb{R}}$. By abuse of notation we write that finite extension as $K_{\mathbb{C}}$, so $\Omega = G_{\mathbb{C}}/K_{\mathbb{C}}$, and Ω is affine [15].

Just as in the example we have $AN : H^{q,q}(D) \to \mathcal{O}(\Omega(D))$, and it is of interest to understand the complex geometry of $\Omega(D)$, in particular with respect to functions in the image $AN(H^{q,q}(D))$. Recently Barlet and Magnusson developed some general methods involving "incidence varieties" [5], and Barlet and Koziarz developed a general "trace method" or "trace transform" for constructing holomorphic functions on cycle spaces [4]. These general results can be applied to our concrete situation; see [9, Appendix]. Here we discuss this only from the perspective of the trace transform, which produces functions in the image $\mathrm{Im}(AN)$ in a simple and elegant way.

In the example $Z = \mathbb{P}_2(\mathbb{C})$ above, let $S = \{z \in Z \mid z_2 = 0\}$. Note that $S \cap D$ can be regarded as the union of the upper and lower hemispheres in $S \cong \mathbb{P}_1(\mathbb{C})$, which are separated by $S \cap M \cong \mathbb{P}_1(\mathbb{R})$. The intersection $C_0 \cap S$ consists of exactly one point in each component, say p_0 and q_0. In fact, whenever $C \in \Omega(D)$ the intersection $C \cap S$ consists of two points $p(C)$ and $q(C)$, one in each component of $D \cap S$. Here the trace transform

$$\mathcal{T} : \mathcal{O}(S \cap D) \to \mathcal{O}(\Omega(D))$$

is given by $\mathcal{T}(f)(C) = f(p(C)) + f(q(C))$ whenever $f \in \mathcal{O}(S \cap D)$ and $C \in \Omega(D)$. In this case it is easy to see that $\mathcal{T}$ does have image in $\mathcal{O}(\Omega(D))$. It is also easy to see that, given C_∞ in the boundary $\mathrm{bd}\Omega(D)$ with $z \in C_\infty \cap \mathrm{bd}D$, there exists $f \in \mathcal{O}(S \cap D)$ with a pole at z and such that

$$\lim_{n\to\infty} |\mathcal{T}(f)(C_n)| = \infty \quad \text{whenever } \{C_n\} \text{ is a sequence in } \Omega(D) \text{ that converges to } C_\infty .$$

This shows that $\Omega(D)$ is Stein. More precisely, given $z \in C_\infty$ as above, there are functions in the image of the trace transform $\mathcal{T}$ which display the holomorphic convexity of $\Omega(D)$ by having poles in the incidence variety $\{C \in \Omega \mid z \in C\}$.

In general this trace transform method, applied to a subvariety $S \subset Z$ of codimension q, transforms a function $f \in \mathcal{O}(S \cap D)$ to a certain function $\mathcal{T}(f) \in \mathcal{O}(\Omega(D))$. Here $C \cap S$ is finite for every $C \in \Omega(D)$, and $\mathcal{T}(f)$ is defined by $\mathcal{T}(f)(C) = \sum_{p \in C \cap S} f(p)$, counting intersection multiplicities. The trace transform method is an essential tool for the proof of the following result.

Theorem 1.1. *Let $G_{\mathbb{R}}$ be a real form of $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$. Let D be an open $G_{\mathbb{R}}$-orbit in a complex flag manifold $Z = G_{\mathbb{C}}/Q$. Then either $G_{\mathbb{R}}/K_{\mathbb{R}}$ is a bounded symmetric domain and $\Omega(D) = G_{\mathbb{R}}/K_{\mathbb{R}}$, or $\Omega(D)$ is a Stein domain in $\Omega = G_{\mathbb{C}}/K_{\mathbb{C}}$.*

This is proved for $G_{\mathbb{R}} = \mathrm{SL}_n(\mathbb{R})$ in [9] using *ad hoc* generalizations of the transversal Schubert variety $S = \{z \in \mathbb{P}_2(\mathbb{C}) \mid z_2 = 0\}$ above. The method of transversal Schubert varieties has since been systematized [8] and may very well lead to a general proof, without conditions on Z or $G_{\mathbb{R}}$, that $\Omega(D)$ is Stein. This approach is reviewed in Sect. 2. There we also show that it suffices to understand Schubert variety intersections $\Sigma = S \cap D$ in the measurable case, where it is known that $\Omega(D)$ is Stein [15].

In Sect. 3 we go to the case case $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$, obtaining an immense simplification of the combinatorial aspects of [9] and a relatively elementary proof of the theorem stated above.

In [17] transversal varieties T are constructed in open $G_{\mathbb{R}}$-orbits D in a hermitian symmetric flag manifold $Z = G_{\mathbb{C}}/Q$. Thus $D = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a bounded symmetric domain and Z, as a homogeneous space of the compact real form of $G_{\mathbb{C}}$, is the compact dual hermitian symmetric space. This is done with the partial Cayley transforms intrinsic to the $G_{\mathbb{R}}$-orbit structure of Z, and for each open orbit D it produces a precisely described bounded symmetric domain $\Sigma_T = T \cap D$ of Z-codimension equal to the dimension of the base cycle C_0 in D, and such that T intersects C_0 transversally. Just as in the case of the Schubert slices above, the closure $c\ell(D)$ meets every $G_{\mathbb{R}}$-orbit in $c\ell(D)$. The trace transform method can therefore be used to transfer the Stein property from Σ_T to $\Omega(D)$, thus proving the above theorem in the hermitian symmetric space case.

The transversal varieties T of [17] have the advantage that they are constructed at an explicit base point in C_0. This leads to a concrete description of the slice Σ_T mentioned above. The Schubert slices Σ_S have the advantage that they exist in general, but the disadvantage that no distinguished base point is given in the construction. (So far, this has meant that the Stein property must be proved by *ad hoc* considerations.) At the end of Sect. 3 we give an example which shows that Σ_T and Σ_S can be very different: $Z = \mathbb{P}_n(\mathbb{C})$, D is the complement of the closure of the unit ball $B_n \subset \mathbb{C}^n \subset Z$, Σ_T is a disk whose closure $c\ell(\Sigma_T)$ is transversal to the boundary $\mathrm{bd}(B_n)$, and $\Sigma_S \cong \mathbb{C}$ in such a way that its closure $c\ell(\Sigma_S) \cong \mathbb{P}_1(\mathbb{C})$ is the projective tangent line to a point on $\mathrm{bd}(B_n)$.

2 Schubert Slices

As in Sect. 1, $G_{\mathbb{C}}$ denotes a complex connected semisimple linear algebraic group with a given noncompact real form $G_{\mathbb{R}}$. Let Q be a (complex) parabolic subgroup of $G_{\mathbb{C}}$, so the homogeneous space $G_{\mathbb{C}}/Q$ is a projective algebraic variety. Let D denote an open $G_{\mathbb{R}}$-orbit on Z, and fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Let $C_0 = K_{\mathbb{R}}(z_0)$ denote the unique $K_{\mathbb{R}}$-orbit in D that is a complex submanifold, so of course $D = G_{\mathbb{R}}(z_0)$, and define $q := \dim_{\mathbb{C}} C_0$.

We write $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{q}$ and $\mathfrak{k}_{\mathbb{R}}$ for the respective Lie algebras of $G_{\mathbb{C}}$, $G_{\mathbb{R}}$, Q and $K_{\mathbb{R}}$, and we write $\mathfrak{k}_{\mathbb{C}}$ for the complexification of $\mathfrak{k}_{\mathbb{R}}$.

The action of $G_{\mathbb{C}}$ on the cycle space $\mathcal{C}^q(Z)$ is algebraic, so the orbit $\Omega := G_{\mathbb{C}}\cdot C_0$ is Zariski open in its closure. If $G_{\mathbb{R}}$ is of hermitian type, then for certain special orbits D (called "holomorphic type" in [19]) it is in fact closed. There are also a few strange cases where $D = Z$, so $C_0 = Z$ and $\mathcal{C}^q(Z)$ is reduced to a point; see [18]. But in general the $G_{\mathbb{C}}$-stabilizer of C_0 is a finite extension of the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}}$; see [15]. By abuse of notation, we write $K_{\mathbb{C}}$ for that stabilizer, so $\Omega = G_{\mathbb{C}}/K_{\mathbb{C}}$ and Ω is an affine, spherical homogeneous space.

Let $\Omega(D)$ denote the connected component of $\{C \in \Omega \mid C \subset D\}$ that contains C_0. It contains the Riemannian symmetric space $G_{\mathbb{R}} \cdot C_0 = G_{\mathbb{R}}/K_{\mathbb{R}}$ as a closed totally real submanifold of real dimension equal to $\dim_{\mathbb{C}} \Omega(D)$. In the special hermitian cases mentioned above, where Ω is compact, our $\Omega(D)$ is the bounded symmetric domain $G_{\mathbb{R}}/K_{\mathbb{R}}$ [15].

2.1 The Slice Theorem

A Borel subgroup of $G_{\mathbb{C}}$ is, by definition, a maximal connected solvable subgroup. Borel subgroups are complex algebraic subgroups, and any two are conjugate in $G_{\mathbb{C}}$. Let B be a Borel subgroup of $G_{\mathbb{C}}$. If $Z = G_{\mathbb{C}}/Q$ as above, then B has only finitely many orbits on Z, and each orbit O is algebraic–geometrically equivalent to a complex affine space $\mathbb{A}^{\dim O}$. The covering of Z by the closures $S = c\ell(O)$ of B-orbits realizes Z as a CW complex. In fact the "Schubert varieties" form a free set of generators of the integral homology $H_*(Z;\mathbb{Z})$. See [6] for these and other basic facts.

If $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ is an Iwasawa decomposition (see, for example, [7], [13] or [12]), then we refer to the connected solvable group $A_{\mathbb{R}}N_{\mathbb{R}}$ as an *Iwasawa component.* The important Borel subgroups, for our considerations of Schubert slices transversal to the base cycle C_0, will be those that contain an Iwasawa component.

In most cases there is no Borel subgroup defined over $\mathbb{R}$. See [14] for an analysis of this. So instead one considers "minimal parabolic subgroups." They are the $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ where $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ is an Iwasawa decomposition and $M_{\mathbb{R}}$ is the centralizer of $A_{\mathbb{R}}$ in $K_{\mathbb{R}}$. Any two minimal parabolic subgroups of $G_{\mathbb{R}}$ are conjugate. If $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ is a minimal parabolic subgroup of $G_{\mathbb{R}}$, then its complexification $P_{\mathbb{C}} = M_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ specifies the class of Borel subgroups that contain the Iwasawa component $A_{\mathbb{R}}N_{\mathbb{R}}$. Those are the $B = B_M A_{\mathbb{C}} N_{\mathbb{C}}$ where B_M is a Borel subgroup of $M_{\mathbb{C}}$.

Choose a Borel subgroup that contains an Iwasawa component, as above. Let $S = c\ell(O)$ be a B-Schubert variety with $\mathrm{codim}_Z(S) = q$, and let Σ

be a connected component of $S \cap D$. We refer to such a component Σ as a "Schubert slice." The main properties of Schubert slices, formulated and proved in [8], can be summarized as follows.

Theorem. *The intersection $\Sigma \cap C_0$ is non-empty and is transversal at each of its points. Suppose $z_0 \in \Sigma \cap C_0$. Then B_M fixes z_0, $\Sigma = A_{\mathbb{R}}N_{\mathbb{R}}(z_0)$, and $K_{\mathbb{R}} \cdot c\ell(\Sigma) = c\ell(D)$.*

Remarks. (1) $c\ell(\Sigma)$ meets every $G_{\mathbb{R}}$-orbit in $c\ell(D)$ because $K_{\mathbb{R}} \cdot c\ell(\Sigma) = c\ell(D)$.

(2) It would be extremely interesting to explicitly compute homology class $[C_0] \in H_{2q}(Z;\mathbb{Z})$.

2.2 The Trace Transform Method

Let $\Sigma = S \cap D$ as above and $f \in \mathcal{O}(\Sigma)$. If $C \in \Omega(D)$, then $\Sigma \cap C$ is finite because it is a compact subvariety of the Stein manifold Σ. Define $\mathcal{T}(f)(C) := \sum_{p \in \Sigma \cap C} f(p)$, counting multiplicities. This defines a holomorphic function on $\Omega(D)$, and as a result we have the *trace transform*

$$\mathcal{T} = \mathcal{T}_\Sigma : \mathcal{O}(\Sigma) \to \mathcal{O}(\Omega(D))\,. \tag{2.1}$$

See [4] and [9, Appendix].

Corollary 2.2. *The cycle space $\Omega(D)$ is holomorphically separable. More precisely, if $C_1 \neq C_2$ in $\Omega(D)$ then there exist a Schubert slice Σ and a function $f \in \mathcal{O}(\Sigma)$ with $\mathcal{T}_\Sigma(f)(C_1) \neq \mathcal{T}_\Sigma(f)(C_2)$.*

Proof. Suppose that we have $C_1, C_2 \in \Omega(D)$ such that $\mathcal{T}_\Sigma(f)(C_1) = \mathcal{T}_\Sigma(f)(C_2)$ for every Schubert slice Σ and every $f \in \mathcal{O}(\Sigma)$. As the orbit O is affine, holomorphic functions separate points on Σ. Now, as we vary Σ, its generic intersections with C_1 and C_2 coincide. It follows that $C_1 \cap C_2$ contains interior points of C_1 or C_2, and each is the algebraic hull of that set of interior points. Thus $C_1 = C_2$. □

Theorem 2.3. *If Σ is a Stein manifold, then so is $\Omega(D)$.*

Proof. Let $\{C_n\}$ be a sequence in $\Omega(D)$, $\{C_n\} \to C_\infty \in \mathrm{bd}(\Sigma(D))$. Each $\Sigma \cap C_n$ is finite. Choose $p \in C_\infty \cap \mathrm{bd}(D)$. Since $c\ell(D) = K_{\mathbb{R}} \cdot c\ell(\Sigma)$ for any choice of $K_{\mathbb{R}}$, we may choose the Schubert slice Σ, in other words choose the Iwasawa component $A_{\mathbb{R}}N_{\mathbb{R}}$, so that $p \in c\ell(\Sigma)$. Choose $p_n \in \Sigma \cap C_n$ such that $\{p_n\} \to p$ and choose $f \in \mathcal{O}(\Sigma)$ such that (i) $\limsup |f(p_n)| = \infty$ and (ii) $f(q_n) = 0$ for all other intersection points $q_n \in \Sigma \cap C_n$. For (ii) use finiteness of the $\Sigma \cap C_n$. Thus the trace transform satisfies $\limsup |\mathcal{T}(f)(C_n)| = \infty$. Consequently $\Omega(D)$ is holomorphically convex. As it is holomorphically separable, it is Stein. □

Remarks. (1) See [4] for trace transform proofs of more general results on holomorphic convexity of cycle spaces.

(2) We emphasize that, if some Schubert slice Σ is Stein, then we display the Stein property for $\Omega(D)$ using only functions on $\Omega(D)$ that are in the image of the trace transform. In many cases those functions can be chosen so that their polar sets are contained in incidence divisors $\mathcal{H}_Y$, where Y is a B–invariant divisor on S. Such functions are in fact rational functions on the closure $X = c\ell(\Omega)$ in the cycle space. See [9].

2.3 Measurable Orbits

An open $G_{\mathbb{R}}$-orbit on Z is called *measurable* if it carries a $G_{\mathbb{R}}$-invariant pseudo–Kähler structure. There are a number of equivalent conditions, e.g. that the isotropy subgroups of $G_{\mathbb{R}}$ in D are reductive. This is always the case for $Z = G_{\mathbb{C}}/B$ where B is a Borel subgroup of $G_{\mathbb{C}}$. Also, if one open $G_{\mathbb{R}}$-orbit on Z is measurable, then every open $G_{\mathbb{R}}$-orbit on Z is measurable. In other words, measurability of open orbits is a property of the pair $(G_{\mathbb{R}}, Z)$. See [14] for details.

It is known that if D is measurable then $\Omega(D)$ is Stein [15]. The proof is not constructive in the sense that it goes via the solution to the Levi Problem. In particular it is not at all clear whether the functions on $\Omega(D)$ that display the Stein property have anything to do with the cohomology of D. Thus, even in the measurable case, constructive methods such as those used in our Corollaries 2.2 and 2.3 are of interest.

Let us discuss the real form $G_{\mathbb{R}} = \mathrm{SL}_n(\mathbb{R})$ of $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$ with respect to the concept of measurability. The manifolds $Z = G_{\mathbb{C}}/Q$ are the classical flag manifolds Z_δ. Here $\delta = (d_1, \ldots, d_k)$ is a "dimension symbol" of integers with $0 < d_1 < \ldots < d_k < n$, and Z_δ consists of the flags

$$z = (\{0\} \subset \Lambda_{d_1} \subset \cdots \subset \Lambda_{d_k} \subset \mathbb{C}^n)\ ,$$

where Λ_d is a d-dimensional linear subspace of $\mathbb{C}^n$.

Let τ be the standard antiholomorphic involution of $\mathbb{C}^n$, complex conjugation of $\mathbb{C}^n$ over $\mathbb{R}^n$. A flag $z \in Z_\delta$ is called *τ-generic* if, for each $\{i,j\}$, $\dim(\Lambda_{d_i} \cap \tau(\Lambda_{d_j}))$ is minimal, i.e. is equal to $\max\{d_i + d_j - n, 0\}$. See [9] for a proof that z is τ-generic if and only if $G_{\mathbb{R}}(z)$ is open in Z_δ.

Note that if $n = 2m$ and $\Lambda = \mathrm{Span}(v_1, v_2, \ldots, v_m)$ in such a way that $\Lambda \oplus \tau(\Lambda) = \mathbb{C}^n$, then the ordered basis $\{\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_m), \mathrm{Im}(v_m)\}$ defines an orientation on $\mathbb{R}^n$ that depends only on Λ. Comparing this to the standard orientation we may speak of Λ as being positively or negatively oriented.

If $\delta = (d_1, \ldots, d_k)$ is a dimension symbol with some $d_i = m$, then we say that a τ-generic flag $z = (\{0\} \subset \Lambda_{d_1} \subset \cdots \subset \Lambda_{d_k} \subset \mathbb{C}^n) \in Z_\delta$ is *positively* (resp. *negatively*) oriented if Λ_m is positively (resp. negatively) oriented. Since $G_{\mathbb{R}}$ preserves this notion of orientation it has at least two open orbits in Z_δ.

In fact $G_{\mathbb{R}}$ has exactly two open orbits on Z_δ in this case, and otherwise has only one open orbit; see [9, §2.2].

We say that $\delta = (d_1, \ldots, d_k)$ is *symmetric* if $\delta = \delta'$ where $\delta' = (n - d_k, \ldots, n - d_1)$.

Proposition 2.4. *An open* $\mathrm{SL}_n(\mathbb{R})$*-orbit on* Z_δ *is measurable if and only if* δ *is symmetric.*

Proof. Consider the open orbit $D = G_{\mathbb{R}}(z_0)$. We may assume that Q is the $G_{\mathbb{C}}$-stabilizer of z_0. The involution τ extends from $\mathbb{C}^n$ to $G_{\mathbb{C}}$ by $\tau(g)(v) = \tau(g(\tau(v)))$. Now $L = Q \cap \tau(Q)$ is the complexification of the $G_{\mathbb{R}}$-stabilizer of z_0, and [14] D is measurable if and only if L is a Levi factor of Q, which, by a dimension count, is equivalent to L being reductive. To check that this condition is equivalent to δ being symmetric it is convenient to use a particular basis. Also, for notational simplicity, we only describe the even–dimensional case.

Let $\{e_1, \ldots, e_{2m}\}$ be the standard basis of $\mathbb{C}^{2m}$, define $h_j = e_j + \sqrt{-1}\, e_{m+j}$ for $1 \leqq j \leqq m$, and use the basis $\mathbf{h} = \{h_1, \ldots, h_m; \tau(h_1), \ldots, \tau(h_m)\}$ to define a base point $z_0 \in Z_\delta$ as follows. If $d_i \leqq m$ then $\Lambda_{d_i} = \mathrm{Span}(h_1, \ldots, h_{d_i})$; if $d_i > m$ then $\Lambda_{d_i} = \mathrm{Span}(h_1, \ldots, h_m, \tau(h_m), \ldots, \tau(h_{m+(m-d_i+1)}))$.

The action of τ on $G_{\mathbb{C}}$ is given, in matrices relative to the basis $\mathbf{h}$, in $m \times m$ blocks, by $\tau(g) = \overline{gJ}$ where $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ in $m \times m$ blocks and the bar is complex conjugation of matrix entries.

In the basis $\mathbf{h}$, Q consists of the block form upper triangular matrices with block sizes given by δ, and $\tau(Q)$ consists of the lower triangular matrices with block sizes given by δ'. Now $\delta = \delta'$ if and only if L consists of all block diagonal matrices with block size given by δ, and the latter is equivalent to reductivity of L. □

Remarks. (1) The correspondence $\delta \mapsto \delta'$ implements an instance of the flag duality of [10].

(2) The condition $\delta = \delta'$ of Proposition 2.4 is analogous to the tube domain criterion of [11].

The following result, along with Proposition 2.4, will lead to a description of the measurable flag manifolds Z_δ for any real form of $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$.

Proposition 2.5. *Let* $G^1_{\mathbb{R}}$ *and* $G^2_{\mathbb{R}}$ *be two real forms of a connected complex semisimple Lie group* $G_{\mathbb{C}}$. *Let* τ_1 *and* τ_2 *be the antiholomorphic involutions of* $G_{\mathbb{C}}$ *with respective fixed point sets* $G^1_{\mathbb{R}}$ *and* $G^2_{\mathbb{R}}$, *and suppose that* $\beta := \tau_1 \cdot \tau_2^{-1}$ *is an inner automorphism of* $G_{\mathbb{C}}$. *Fix a complex flag manifold* $Z = G_{\mathbb{C}}/Q$. *Then the open* $G^1_{\mathbb{R}}$*-orbits on* Z *are measurable if and only if the open* $G^2_{\mathbb{R}}$*-orbits on* Z *are measurable.*

Proof. Let $\mathfrak{h} \subset \mathfrak{q}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Here lower case Gothic letters denote Lie algebras of groups denoted by the corresponding upper case Roman letters. Then the nilradical $\mathfrak{q}^{nil}$ of $\mathfrak{q}$ is a sum $\sum_{\alpha\in R}\mathfrak{g}_{\mathbb{C}}^{\alpha}$ of $\mathfrak{h}$-root spaces, and $\mathfrak{h}$ determines a choice $\mathfrak{q}^{red} = \mathfrak{h} + \sum_{\alpha\in L}\mathfrak{g}_{\mathbb{C}}^{\alpha}$ of Levi component of $\mathfrak{q}$. The *opposite* (of the $G_{\mathbb{C}}$-conjugacy class) of $\mathfrak{q}$ is the ($G_{\mathbb{C}}$-conjugacy class of the) parabolic subalgebra $\mathfrak{q}^- = \mathfrak{q}^{red} + \sum_{\alpha\in R}\mathfrak{g}_{\mathbb{C}}^{-\alpha}$. This "opposition" is a well defined relation between conjugacy classes of parabolic subalgebras. The point here, for us, is the fact [14, Theorem 6.7] that the $G_{\mathbb{R}}^i$-orbits on Z are measurable if and only if $\tau_i(\mathfrak{q})$ is opposite to $\mathfrak{q}$, i.e. is $G_{\mathbb{C}}$-conjugate to $\mathfrak{q}^-$.

Let $\mathrm{Int}(G_{\mathbb{C}})$ denote the group of inner automorphisms of $G_{\mathbb{C}}$. Now the $G_{\mathbb{R}}^1$-orbits on Z are measurable, if and only if $\mathfrak{q}^- = \alpha\tau_1\mathfrak{q}$ for some $\alpha \in \mathrm{Int}(G_{\mathbb{C}})$, if and only if $\mathfrak{q}^- = \gamma\tau_2\mathfrak{q}$ for some $\gamma \in \mathrm{Int}(G_{\mathbb{C}})$ where $\gamma = \alpha\beta$, if and only if the $G_{\mathbb{R}}^2$-orbits on Z are measurable. □

The real forms of $SL_n(\mathbb{C})$ are the real special linear group $SL_n(\mathbb{R})$, the quaternion special linear group $SL_m(\mathbb{H})$ defined for $n = 2m$, and the special unitary groups $SU(p,q)$ with $p+q=n$. The quaternion special linear group is defined as follows. We have $\mathbb{R}$-linear transformations of $\mathbb{C}^{2m}$ given by

$$\mathbf{i}: v \mapsto \sqrt{-1}v, \quad \mathbf{j}: v \mapsto \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}\overline{v}, \quad \mathbf{k} = \mathbf{ij}: v \mapsto \sqrt{-1}\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}\overline{v} \quad (2.6)$$

where $v \mapsto \overline{v}$ is complex conjugation of $\mathbb{C}^{2m}$ over $\mathbb{R}^{2m}$. Then $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -I_{2m}$, and any different ones of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ anticommute. So they generate a quaternion algebra $\mathbb{H}$ of linear transformations of $\mathbb{C}^{2m}$, and we have

$$\mathbb{H}^m : \text{ quaternionic vector space structure on } \mathbb{C}^{2m} \text{ defined by (2.6).} \quad (2.7)$$

An $\mathbb{R}$-linear transformation of $\mathbb{C}^{2m}$ is *quaternion–linear* if it commutes with every element of $\mathbb{H}$. The group $SL_m(\mathbb{H})$ is defined to be the group of all quaternion–linear transformations of $\mathbb{C}^{2m}$ of determinant 1, in other words all volume preserving linear transformations of $\mathbb{H}^n$. Thus $SL_m(\mathbb{H})$ is the centralizer of $\mathbf{j}$ in $SL_{2m}(\mathbb{C})$, and we have

Lemma 2.8. $SL_m(\mathbb{H})$ *is the real form of* $SL_{2m}(\mathbb{C})$ *such that* $\tau(g) = \mathbf{j}g\mathbf{j}^{-1}$ *is complex conjugation of* $SL_{2m}(\mathbb{C})$ *over* $SL_m(\mathbb{H})$.

Combining Lemma 2.8 with Propositions 2.4 and 2.5 we have

Corollary 2.9. *Consider a flag manifold* Z_δ *of* $SL_{2m}(\mathbb{C})$. *Then the following are equivalent.*
(i) *The open* $SL_{2m}(\mathbb{R})$*-orbits on* Z_δ *are measurable.*
(ii) *The open* $SL_m(\mathbb{H})$*-orbits on* Z_δ *are measurable.*
(iii) *The dimension symbol* δ *is symmetric.*

Since $SU(p,q)$ has a compact Cartan subgroup T, complex conjugation of $SL_{p+q}(\mathbb{C})$ over $SU(p,q)$ sends every $\mathfrak{t}_{\mathbb{C}}$-root to its negative, so the discussion of opposition in the proof of Proposition 2.5 shows that complex conjugation of $SL_{p+q}(\mathbb{C})$ over $SU(p,q)$ sends $\mathfrak{q}$ to its opposite. Thus every open $SU(p,q)$-orbit on every Z_δ is measurable. Now Corollary 2.9 gives us

Corollary 2.10. *Let Z_δ be a flag manifold of $G_\mathbb{C} = \mathrm{SL}_n(\mathbb{C})$. Let $G_\mathbb{R}$ be a real form of $G_\mathbb{C}$. Then the open $G_\mathbb{R}$-orbits on Z_δ are measurable except when* (i) *δ is not symmetric and* (ii) *$G_\mathbb{R} = \mathrm{SL}_n(\mathbb{R})$, or $G_\mathbb{R} = \mathrm{SL}_m(\mathbb{H})$ with $n = 2m$ even.*

2.4 The Transfer Lemma

As we already indicated, one of our main goals here is to show that, whenever D is an open orbit of a real form of $G_\mathbb{C} = \mathrm{SL}_n(\mathbb{C})$ in an arbitrary flag manifold Z_δ, the cycle space $\Omega(D)$ is a Stein domain. Since this is known for measurable orbits, by Corollary 2.10 it is enough to consider Z_δ for δ non-symmetric and either $G_\mathbb{R} = \mathrm{SL}_n(\mathbb{R})$ or $G_\mathbb{R} = \mathrm{SL}_m(\mathbb{H})$ with $2m = n$. We carry this out by our Schubert slice method. For any given Z this is related to an analysis of a certain associated measurable flag manifold $\widetilde{Z}$.

Given a complex flag manifold $Z = G_\mathbb{C}/Q$ and a real form $G_\mathbb{R} \subset G_\mathbb{C}$, there exists a root–theoretically canonically associated parabolic group $\widetilde{Q} \subset Q$ such that (i) the open $G_\mathbb{R}$-orbits in $\widetilde{Z} = G_\mathbb{C}/\widetilde{Q}$ are measurable and (ii) $\widetilde{Q}$ is maximal for this. Let $\pi : \widetilde{Z} \to Z$ be the holomorphic bundle defined by $g\widetilde{Q} \mapsto gQ$, using $\widetilde{Q} \subset Q$. Its (k-dimensional) typical fiber is $F = Q/\widetilde{Q}$. If z_0 is the neutral point in Z associated to Q and $D = G_\mathbb{R}(z_0)$ is open, then there is a unique open orbit $\widetilde{D} = G_\mathbb{R}(\widetilde{z}_0)$ in $\pi^{-1}(D)$. The fiber of the $G_\mathbb{R}$-homogeneous fibration $\pi|_{\widetilde{D}} : \widetilde{D} \to D$ is Zariski open in F and isomorphic to an affine space $\mathbb{A}^k$. See [16] or [10] for the details and for related information.

Now K has unique orbits in D and $\widetilde{D}$ which are complex submanifolds, i.e. the base cycles C_0 and $\widetilde{C}_0$ in the respective spaces. Since $\mathbb{A}^k$ is affine, $\pi|_{\widetilde{C}_0} : \widetilde{C}_0 \to C_0$ has finite fibers and, since the base is simply connected, it is in fact biholomorphic. Thus in a very natural way we have an induced $G_\mathbb{R}$-equivariant open immersion $\pi_* : \Omega(\widetilde{D}) \to \Omega(D)$.

It can happen that the isotropy subgroup of $G_\mathbb{C}$ at $C_0 \in \Omega$ is a finite extension of the isotropy subgroup of $G_\mathbb{C}$ and $\widetilde{C}_0 \in \widetilde{\Omega}$, but in general this is not the case. Nevertheless, we may think of $\Omega(\widetilde{D})$ as an open subset of $\Omega(D)$. In certain cases it has been shown that $\Omega(\widetilde{D}) = \Omega(D)$, for example when $G_\mathbb{R} = \mathrm{SL}_n(\mathbb{R})$ (see [9]). In general this a very interesting open problem.

We now compare the Schubert slices in D and $\widetilde{D}$. For this we fix an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$, let B be a Borel subgroup of $G_\mathbb{C}$ which contains the Iwasawa component $A_\mathbb{R} N_\mathbb{R}$ and let S be a q-codimensional Schubert variety in Z. We may assume that the base point $z_0 \in S \cap C_0$. Recall that $(S \cap C_0) \subset (O \cap C_0)$, where $O = B(z_0)$ is the open B-orbit in S.

The cycle $\widetilde{C}_0$ is likewise q-dimensional. Thus we restrict our attention to the sets $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ of q-codimensional B-Schubert varieties in the respective spaces, and we note that the projection induces a natural injective map $\pi^* : \mathcal{S} \to \widetilde{\mathcal{S}}$.

Proposition 2.11. *Let $S \in \mathcal{S}$ and $\widetilde{S} = \pi^*(S)$. Choose the neutral point $\widetilde{z}_0 \in \widetilde{S} \cap \widetilde{C}_0$, let $z_0 := \pi(\widetilde{z}_0)$, let $\widetilde{\Sigma}$ denote the Schubert slice $A_{\mathbb{R}}N_{\mathbb{R}}(\widetilde{z}_0)$ in $\widetilde{D}$ and let $\Sigma = \pi(\widetilde{\Sigma})$. Then $\Sigma = A_{\mathbb{R}}N_{\mathbb{R}}(z_0)$ is a Schubert slice in D, and the map $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ has the same fibers as $\pi|_{\widetilde{D}}$.*

Proof. Equivariance and the basic properties of Schubert slices immediately imply all but the last statement. A dimension count shows that the fibers of $\pi|_{\widetilde{\Sigma}}$ are open in those of $\pi|_{\widetilde{D}}$. Since $\widetilde{\Sigma}$ is closed in $\widetilde{D}$, they must therefore agree. Thus we have the last assertion. □

We refer to the following as the "Transfer Lemma". The interesting aspect of the transfer is given, of course, by combining it with Corollary 2.3.

Lemma 2.12. *The Schubert slice Σ is a Stein manifold if and only if $\widetilde{\Sigma}$ is a Stein manifold.*

Proof. The fiber of $\pi|_{\widetilde{D}}$ is a Zariski open orbit of a solvable group in the π-fiber F, so it is isomorphic to $\mathbb{A}^k$. It is therefore an open Schubert cell in F; see [9]. If $\widetilde{\Sigma} = A_{\mathbb{R}}N_{\mathbb{R}}(\widetilde{z}_0)$ as above, which is open in $\widetilde{O} = B(\widetilde{z}_0)$, now Proposition 2.11 shows that $\pi|_{\widetilde{\Sigma}}$ and $\pi|_{\widetilde{O}}$ have the same fibers. Since O is equivalent to an affine space, the holomorphic bundle $\pi|_{\widetilde{O}} : \widetilde{O} \to O$ is trivial. Therefore $\widetilde{\Sigma} \cong \Sigma \times \mathbb{A}^k$ and the assertion follows. □

3 Cycle Spaces of Open Orbits of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{H})$

In this section we consider cycle spaces $\Omega(D)$ of open orbits D of $\mathrm{SL}_n(\mathbb{R})$, and of $\mathrm{SL}_m(\mathbb{H})$ where $n = 2m$, on flag manifolds Z_δ of $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$. In particular, using Schubert slices and the trace transform method, we prove that $\Omega(D)$ is a Stein domain in the affine homogeneous space $\Omega = G_{\mathbb{C}}/K_{\mathbb{C}}$. This was shown for $\mathrm{SL}_n(\mathbb{R})$ in [9], but the proof here, which relies on Theorem 2.3, is essentially simpler.

3.1 The Case of the Real Form $G_{\mathbb{R}} = \mathrm{SL}_n(\mathbb{R})$

For notational simplicity we restrict our attention to the even dimensional case, $n = 2m$, and, if the choice arises, to the open orbit D of positively oriented flags in Z_δ. Recall that the dimension symbol $\delta = (d_1, \ldots, d_k)$ is called symmetric if $\delta = \delta'$ where $\delta' = (n - d_k, \ldots, n - d_1)$. Proposition 2.4 says that D is measurable if and only if δ is symmetric. In that measurable case it is known that $\Omega(D)$ is Stein [15]. The proof in [15] is not constructive: functions displaying the Stein properties are not given. Thus the independent constructive proof given here can be of use even in the measurable case.

Let $V = \mathbb{C}^n$, let $\{e_1, \ldots, e_{2m}\}$ denote its standard basis, define $f_j = e_{2j-1} + \sqrt{-1}\, e_{2j}$ for $1 \leqq j \leqq m$, and consider the basis $\mathbf{b} = \{f_1, \ldots, f_m; \tau(f_m), \ldots, \tau(f_1)\}$ of V where τ is the complex conjugation $v \mapsto \overline{v}$ of V that leaves the

e_i fixed. We refer to **b** as the standard ordered basis of isotropic vectors with respect to the complex bilinear form $b(v,w) = {}^t v \cdot w$. As usual $K_{\mathbb{C}} = SO_n(\mathbb{C})$ and $K_{\mathbb{R}} = SO_n(\mathbb{R})$ relative to b. Let B denote the Borel subgroup of $G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$ that fixes the full flag defined by the standard basis $\{e_1, \dots, e_{2m}\}$. Then $B = A_{\mathbb{C}} N_{\mathbb{C}}$ where $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ is a fixed Iwasawa decomposition such that $A_{\mathbb{R}} N_{\mathbb{R}} = (B \cap G_{\mathbb{R}})^0$.

If $z = (\{0\} \subset \Lambda_{d_1} \subset \cdots \subset \Lambda_{d_k} \subset \mathbb{C}^n) \in Z_\delta$, then we define the *lower part* $L(z)$ to be the flag $(\{0\} \subset \Lambda_{d_1} \subset \cdots \subset \Lambda_{d_\ell} \subset \mathbb{C}^n) \in Z_{L(\delta)}$, defined by the conditions $d_\ell \leqq m$ and $d_{\ell+1} > m$ and by $L(\delta) = (d_1, \dots, d_\ell)$. (The *upper part* is the flag $(\{0\} \subset \Lambda_{d_{\ell+1}} \subset \cdots \subset \Lambda_{d_k} \subset \mathbb{C}^n)$.) Let $L : Z_\delta \to Z_{L(\delta)}$ denote the associated map of flag manifolds.

The open orbit $D \subset Z_\delta$ under discussion is the set of all (positively oriented in case some $d_i = m$) τ*-generic* flags. Those are the ones such that, for each $\{i,j\}$, $\dim(\Lambda_{d_i} \cap \tau(\Lambda_{d_j}))$ is minimal, i.e. is equal to $\max\{d_i + d_j - n, 0\}$. See [9, §2.2]

Lemma 3.1. *The base cycle C_0 is the set of all maximally isotropic flags in D. In other words C_0 consists of the flags*

$$z = (\{0\} \subset \Lambda_{d_1} \subset \cdots \subset \Lambda_{d_k} \subset \mathbb{C}^n) \in Z_\delta$$

in D such that, for each $\{i,j\}$, $\dim(\Lambda_{d_i} \cap \Lambda_{d_j}^\perp)$ is maximal. If δ is symmetric, then the flag $z \in C_0$ if and only if all the subspaces Λ_d in $L(z)$ are isotropic. In that case, $\Lambda_d^\perp = \Lambda_{n-d}$ for all Λ_d in $L(z)$.

Remark. This result is contained in [9, §2.3], but our argument is more direct.

Proof. It suffices to prove Lemma 3.1 when Q is a Borel subgroup of $G_{\mathbb{C}}$. That is the case where $\delta = (1, 2, \dots, n)$. Then the set C_0' of maximally isotropic flags $z \in Z_\delta$ is given by: the subspaces Λ_d in $L(z)$ are isotropic and satisfy $\Lambda_d^\perp = \Lambda_{n-d}$. The orthogonal groups $O_n(\mathbb{R})$ and $O_n(\mathbb{C})$ act transitively on C_0', with action on $k(z)$ determined by $k(L(z))$ for $z \in C_0'$. The action of $O_n(\mathbb{C})$ is transitive by Witt's Theorem, and C_0' is a complex flag manifold of $O_n(\mathbb{C})$. Thus also the maximal compact subgroup $O_n(\mathbb{R})$ is transitive on C_0'. Passing to identity components, now $C_0' \cap D$ is an orbit of $K_{\mathbb{R}} = SO_n(\mathbb{R})$ that is a complex flag manifold of $K_{\mathbb{C}} = O_n(\mathbb{C})$, and thus is the base cycle C_0. □

The standard ordered basis $\{f_1, \dots, f_m; \tau(f_m), \dots, \tau(f_1)\}$ of isotropic vectors will be used to determine base points of Schubert slices, i.e. points of intersection of Σ with C_0. For δ nonsymmetric we will re-order this basis in a simple way.

If $g \in B$ then, up to a nonzero scalar multiplication, which has no effect on Z_δ,

$$g(f_j) = f_j + z_j \tau(f_j) + \sum\nolimits_{i<j} (\xi_{i,j} f_i + \eta_{i,j} \tau(f_i)) \,. \tag{3.2}$$

Here z_j, the $\xi_{i,j}$ and the $\eta_{i,j}$ are arbitrary complex numbers as g ranges over B.

Lemma 3.3. *If $g \in A_{\mathbb{R}}N_{\mathbb{R}}$, then (again up to a nonzero scalar multiplication) $g(f_j)$ has the same form (3.2) where z_j ranges over the unit disk ($|z| < 1$), and where the $\xi_{i,j}$ and the $\eta_{i,j}$ are arbitrary complex numbers, as g ranges over B.*

Proof.

$$g(f_j) = g(e_{2j-1}) + \sqrt{-1}\, g(e_{2j}) = \left(\lambda_{2j-1}\, e_{2j-1} + \sum\nolimits_{i<2j-1} \beta_{i,2j-1}\, e_i\right) + \sqrt{-1}(\lambda_{2j}\, e_{2j} + \alpha_j\, e_{2j-1}) + \sum\nolimits_{i<2j-1} (\beta_{i,2j}\, e_i)$$

with λ_i positive real and $\alpha_j, \beta_{i,k}$ arbitrary real. It follows immediately that the $\xi_{i,j}$ and the $\eta_{i,j}$ range over all of $\mathbb{C}$ as g ranges over B.

We can't normalize the leading coefficient to 1 in (3.2) without losing track of the fact $g \in A_{\mathbb{R}}N_{\mathbb{R}}$, but without that normalization we use (3.2) to express

$$\begin{aligned} g(f_j) &= x_j f_j + y_j \tau(f_j) + x_j \sum\nolimits_{i<j} (\xi_{i,j} f_i + \eta_{i,j} \tau(f_i)) \\ &= (x_j + y_j) e_{2j-1} + \sqrt{-1}(x_j - y_j) e_{2j} + x_j \sum\nolimits_{i<j} (\xi_{i,j} f_i + \eta_{i,j} \tau(f_i)) \,. \end{aligned}$$

Equating coefficients of e_{2j-1} we have $x_j + y_j = \lambda_{2j-1} + \sqrt{-1}\, \alpha_j$. Equating coefficients of e_{2j} we have $x_j - y_j = \lambda_{2j}$. In other words $x_j = 1/2(\lambda_{2j-1} + \lambda_{2j} + \sqrt{-1}\, \alpha_j)$ and $y_j = 1/2(\lambda_{2j-1} - \lambda_{2j} + \sqrt{-1}\, \alpha_j)$. Now

$$z_j = y_j / z_j = (\lambda_{2j-1} - \lambda_{2j} + \sqrt{-1}\, \alpha_j)/(\lambda_{2j-1} + \lambda_{2j} + \sqrt{-1}\, \alpha_j) \,.$$

As the λ_i are positive real and α_j is arbitrary real, the only restriction as g varies over $A_{\mathbb{R}}N_{\mathbb{R}}$ is $|z_j| < 1$. □

Since $g \in A_{\mathbb{R}}N_{\mathbb{R}}$, and in particular $\tau(g) = g$, the descriptions of Lemma 3.3 also describe the $g(\tau(f_j))$.

If δ is symmetric with $L(\delta) = (d_1, \ldots, d_\ell)$ we define our base point

$$z_0 = (\{0\} \subset \Lambda_{d_1} \subset \ldots \subset \Lambda_{d_\ell} \subset \Lambda_{n-d_\ell} \subset \ldots \subset \Lambda_{n-d_1} \subset V) \tag{3.4}$$

by

$$\Lambda_{d_i} = \text{Span}(f_1, \ldots f_{d_i}) \text{ and } \Lambda_{n-d_i} = \Lambda_{d_i}^{\perp} \text{ for } i \leqq \ell \,. \tag{3.5}$$

This is the flag associated with the standard ordered basis. Each Λ_{d_i} is isotropic, so $z_0 \in D$.

Lemma 3.6. *Suppose that δ is symmetric and that z_0 is defined as in (3.4) and (3.5). Then $B(z_0) \cap C_0 = \{z_0\}$. In particular $(A_{\mathbb{R}}N_{\mathbb{R}})(z_0) = \Sigma$ is a Schubert slice,* *Proof.* Let $g \in B$. According to (3.2), $g(f_1)$ is isotropic if and only if $z_1 = 0$, in other words if and only if it is a multiple of f_1. Recursively in k one uses (3.2) to see that $g(\mathrm{Span}(f_1, \ldots, f_k))$ is isotropic if and only if g fixes $\mathrm{Span}(f_1, \ldots, f_k)$. Thus $z \in (B(z_0) \cap C_0)$ implies $L(z) = L(z_0)$. As $z \in C_0$, and by symmetry of δ, this implies $z = z_0$. □

In dealing with nonsymmetric δ it is necessary to change the ordering of the basis. Keep in mind here that, in the standard ordered basis $\mathbf{b}$, the only difference between the f_j and the $\tau(f_j)$ is that the f_j are in increasing order and the $\tau(f_j)$ are in decreasing order. If δ is not symmetric it is necessary to exchange the roles of certain of the f_j with the corresponding $\tau(f_j)$. This will amount to changing the ordering of certain matrix blocks.

The ordering must be changed if there is a "gap" in $L(\delta)$ in the sense that, for some $d_a < d_e$ adjacent in $L(\delta)$, we have $n - d_e$, $n - d$ and $n - d_a$ in δ where $d_a < d < d_e$. In the gaps, here for indices $d_a + 1$ through $d_e - 1$, we change the order of the f_j to be decreasing with j and the order of the $\tau(f_j)$ to be increasing with j. Thus in the lower part $L(z_0)$ of the base point we will have

$$\Lambda_{d_a} = \mathrm{Span}(f_1, \ldots, f_{d_a}) \subset \mathrm{Span}(f_1, \ldots, f_{d_e}) = \Lambda_{d_e}$$

as usual, and in the upper part of the base point we will have

$$\begin{aligned} \Lambda_{n-d_e} &= \mathrm{Span}\left(f_1, \ldots, f_m, \tau(f_m), \ldots, \tau(f_{d_e+1})\right) \\ &\subset \mathrm{Span}\left(f_1, \ldots, f_m, \tau(f_m), \ldots, \tau(f_{d_a+1})\right) = \Lambda_{n-d_a} \end{aligned}$$

as usual, but

$$\begin{aligned} \Lambda_{n-d} = \mathrm{Span}\big(&f_1, \ldots, f_m, \tau(f_m), \\ &\ldots, \tau(f_{d_e+1}), \tau(f_{d_a+1}), \tau(f_{d_a+2}), \ldots, \tau(f_{d_a+(d_e-d)})\big) . \end{aligned}$$

Gaps may occur in the upper part of δ as well, but they will not require any reordering.

Lemma 3.7. *Let δ be any dimension symbol, and z_0 the base point in Z_δ defined by the basis reordered as above. Then $z_0 \in C_0$, $B(z_0) \cap C_0 = \{z_0\}$, and $\Sigma := (A_{\mathbb{R}} N_{\mathbb{R}})(z_0)$ is a Schubert slice.*

Proof. The spaces $\Lambda_{d_j} = \mathrm{Span}(f_1, \ldots, f_{d_j})$ in $L(z_0)$ are isotropic, and if $n - d_j \in \delta$ then $\Lambda_{n-d_j} = \Lambda_{d_j}^{\perp}$ as before. The intermediate spaces Λ_{n-d} satisfy $\Lambda_{d_a} \subset \Lambda_{n-d}^{\perp} \subset \Lambda_{d_e}$. Thus z_0 is maximally isotropic, so $z_0 \in C_0$.

Fix $g \in B$ so that $g(z_0)$ is maximally isotropic. As in Lemma 3.6, the fact that $g(\mathrm{Span}(f_1, \ldots, f_j))$ is isotropic implies

$$g(\mathrm{Span}(f_1, \ldots, f_j)) = \mathrm{Span}(f_1, \ldots, f_j) .$$

If in addition $n - d_j \in \delta$, then, since $g(z_0)$ is maximally isotropic, $g(\Lambda_{n-d_j}) = \Lambda_{n-d_j}$. Thus, to show that $g(z_0) = z_0$ we need only discuss $g(\Lambda_{n-d})$ for the intermediate spaces in a gap, i.e. for $d_a < d < d_e$ as in the discussion just before the statement of Lemma 3.7.

As in (3.2) we express (up to a scalar that fixes the base point in Z_δ)

$$g(\tau(f_{d_a+1})) = \tau(f_{d_a+1}) + z f_{d_a+1} + \sum\nolimits_{i \leqq d_a} (\xi_i f_i + \eta_i \tau(f_i)). \tag{3.8}$$

Since g fixes $\mathrm{Span}(f_1, \ldots, f_{d_a})$, we have $\mathrm{Span}(f_1, \ldots, f_{d_a}) \subset g(\Lambda_{n-d})$. As $g(\Lambda_{n-d})$ is maximally isotropic we also have $\mathrm{Span}(f_1, \ldots, f_{d_a}) \subset g(\Lambda_{n-d})^{\perp}$. Therefore all the η_i vanish in (3.8). In other words,

$$g(\tau(f_{d_a+1})) \in \ \mathrm{Span}(\tau(f_{d_a+1}), f_1, \ldots, f_m)\,.$$

Proceeding recursively in k for the $g(\tau(f_{d_a+k}))$ we obtain $g(\Lambda_{n-d}) = \Lambda_{n-d}$. Now $g \in B$ with $g(z_0) \in C_0$ implies $g(z_0) = z_0$. □

Our final goal in this section is to give an explicit description of the Schubert slice Σ determined by the base point z_0. For this let Δ denote the open unit disc in $\mathbb{C}$. The result for real special linear groups is

Proposition 3.9. *Let δ be any dimension symbol, let $z_0 \in C_0 \subset D \subset Z_\delta$ denote the base point defined above, and consider the associated Schubert slice $\Sigma = (A_{\mathbb{R}} N_{\mathbb{R}})(z_0)$. Then Σ is biholomorphic to $\Delta^p \times \mathbb{C}^q$ where $p = d_\ell$ if $\delta = L(\delta)$, $p = m$ if $\delta \neq L(\delta)$, and $p + q = \mathrm{codim}_{\mathbb{C}}(C_0)$.*

Since $\Delta^p \times \mathbb{C}^q$ is Stein, Theorem 2.3 gives us

Corollary 3.10. *The cycle space $\Omega(D)$ of an open $\mathrm{SL}_n(\mathbb{R})$-orbit D in a flag manifold Z_δ is a Stein domain.*

Proof of Proposition 3.9. Let $g \in B$. Applying (3.2) one has

$$g(f_j) = f_j + z_j \tau(f_j) + \sum\nolimits_{i<j} \eta_{i,j} \tau(f_i) \text{ modulo } \mathrm{Span}(g(f_1), \ldots, g(f_{j-1}))\,. \tag{3.11}$$

As mentioned above, the only restriction imposed if $g \in A_{\mathbb{R}} N_{\mathbb{R}}$ is $|z_j| < 1$. Also,

$$g(\tau(f_j)) = \tau(f_j) + \sum\nolimits_{i<j} \xi_{i,j} \tau(f_i) \text{ modulo } \mathrm{Span}(g(f_1), \ldots, g(f_m))\,, \tag{3.12}$$

where the $\xi_{i,j}$ can be chosen without restriction, even if $g \in A_{\mathbb{R}} N_{\mathbb{R}}$. Compare with Lemma 3.3.

Now let z_0 be the base point in Z_δ defined by the re–ordered basis of isotropic vectors. We may suppose that Q is the isotropy subgroup Q_{z_0} of $G_{\mathbb{C}}$ at z_0. Let $\mathfrak{q}^{\mathrm{nil}}$ denote the nilradical $\mathfrak{q}^{\mathrm{nil}} = \sum_{\alpha \in R} \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ of $\mathfrak{q}$, where R is the appropriate set of positive roots, let $\mathfrak{u}^+ = \sum_{\alpha \in R} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, opposite to $\mathfrak{q}^{\mathrm{nil}}$, which represents the holomorphic tangent space to Z_δ at z_0, and let U^+ denote the corresponding unipotent subgroup of $G_{\mathbb{C}}$. We will describe $B(z_0)$ in the coordinate chart $U^+(z_0)$.

Consider the most complicated case, where the upper part of δ is not empty. Given $1 < j \leqq m$, let I_j denote the set of indices $i < j$ such that the 1-parameter group

$$g_{i,j}(t) : \tau(f_j) \mapsto \tau(f_j) + t\tau(f_i),\ \tau(f_k) \mapsto \tau(f_k) \text{ for } k \neq j,$$
$$f_k \mapsto f_k \text{ for } 1 \leqq k \leqq m$$

belongs to U^+. Then the $g(z_0), g \in B$, are described by (3.11) and by

$$g(\tau(f_j)) = \tau(f_j) + \sum\nolimits_{i \in I_j} \xi_{i,j}\tau(f_i)\,. \tag{3.13}$$

Here, as g runs over B all coefficients run over $\mathbb{C}$ independently, and if g is constrained to run over $A_{\mathbb{R}}N_{\mathbb{R}}$ the only restrictions are $|z_j| < 1$ for $1 \leqq j \leqq m$. In this case $p = m$.

Finally consider the case where the upper part of δ is empty. Then the considerations of (3.12) and (3.13) are not needed, and the $g(z_0),\ g \in A_{\mathbb{R}}N_{\mathbb{R}}$, are described by

$$g(f_j) = f_j + \tau(f_j) + \sum\nolimits_{i<j} \eta_{i,j}\tau(f_i),\ 1 \leqq j \leqq d_\ell\,,$$

where, as g runs over $A_{\mathbb{R}}N_{\mathbb{R}}$, the $\eta_{i,j}$ run over $\mathbb{C}$, and the z_j run over Δ, independently. In this case $p = d_\ell$. □

3.2 The Case of the Real Form $G_{\mathbb{R}} = \mathrm{SL}_m(\mathbb{H})$

As in (2.7), $n = 2m$ and $V = \mathbb{C}^{2m}$ carries a quaternionic vector space structure $\mathbb{H}^m$ defined by $\mathbf{j} : v \mapsto J\overline{v}$ where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. Lemma 2.8 exhibits the quaternion linear group $G_{\mathbb{R}} = \mathrm{SL}_m(\mathbb{H})$ as the real form of $G_{\mathbb{C}} = \mathrm{SL}_{2m}(\mathbb{C})$ that is the centralizer of $\mathbf{j}$ in $\mathrm{SL}_{2m}(\mathbb{C})$. The Cartan involution $\theta : g \mapsto {}^t\overline{g}^{-1}$ of $\mathrm{SL}_{2m}(\mathbb{C})$ commutes with the complex conjugation $\tau : g \mapsto \mathbf{j}g\mathbf{j}^{-1}$ of $\mathrm{SL}_{2m}(\mathbb{C})$ over $\mathrm{SL}_m(\mathbb{H})$, so it restricts to a Cartan involution (which we also call θ) of $\mathrm{SL}_m(\mathbb{H})$. Thus $K_{\mathbb{R}} := G_{\mathbb{R}} \cap SU_{2m} = Sp_m$, the unitary symplectic group, whose complexification $K_{\mathbb{C}}$ is the complex symplectic group $Sp_m(\mathbb{C})$. Now $K_{\mathbb{C}} = \{g \in G_{\mathbb{C}} \mid g^*\omega = \omega\}$ where ω is the standard complex symplectic structure on V defined by $\omega(v, w) = {}^tv \cdot Jw$.

We refer to a flag $z = (\Lambda_j) \in Z_\delta$ as *$\mathbf{j}$-generic* if the dimensions $\dim(\Lambda_i \cap \mathbf{j}(\Lambda_j))$ are minimal for all Λ_i, Λ_j in z. A flag $z \in Z_\delta$ is *maximally ω-isotropic* if the $\dim(\Lambda_i \cap \Lambda_j^\perp)$ are maximal for all Λ_i, Λ_j in z. Here $\perp$ refers to ω. Maximally ω-isotropic flags are $\mathbf{j}$-generic.

Proposition 3.14. *There is just one open $G_{\mathbb{R}}$-orbit D in Z_δ; it is the set of all $\mathbf{j}$-generic flags. The base cycle C_0 in D, in other words the unique closed $K_{\mathbb{C}}$-orbit in D, is the set of all maximally ω-isotropic flags in Z_δ.*

Proof. Define D to be the set of all $\mathbf{j}$-generic flags in Z_δ. To show that D is the unique open $G_\mathbb{R}$-orbit in Z_δ it suffices to show that $G_\mathbb{R}$ is transitive on D. Let Z_ϵ denote the manifold of full flags, corresponding to the dimension symbol $\epsilon = (1, 2, \dots, 2m-1)$ Every $\mathbf{j}$-generic flag $z \in Z_\delta$ can be filled out to a $\mathbf{j}$-generic flag $\widetilde{z} \in Z_\epsilon$. To show that $G_\mathbb{R}$ is transitive on D, now it suffices to prove transitivity on the set D_ϵ of all $\mathbf{j}$-generic flags in Z_ϵ. We proceed to do that.

Let z_ϵ be the base point in Z_ϵ associated to the ordered basis

$$\{e_1, \dots, e_m, \mathbf{j}(e_m), \dots, \mathbf{j}(e_1)\}$$

where $\{e_1, \dots, e_{2m}\}$ is the standard basis of V. Let z be any $\mathbf{j}$-generic flag in Z_ϵ. Then z is defined by an ordered basis $\{v_1, \dots, v_m, w_m, \dots, w_1\}$. By $\mathbf{j}$-genericity,

$$\mathrm{Span}(v_1, \dots, v_m) \cap \mathbf{j}(\mathrm{Span}(v_1, \dots, v_m)) = 0 ,$$

so the set $\{v_1, \dots, v_m\}$ is linearly independent over $\mathbb{H}$, and we can define $g \in G_\mathbb{R}$ by $g(v_j) = e_j$ for $1 \leqq j \leqq m$. It follows that $g(\mathbf{j}(v_j)) = \mathbf{j}(e_j)$ for $1 \leqq j \leqq m$. In other words we may assume that z is defined by

$$\{e_1, \dots, e_m, w_m, \dots, w_1\} .$$

Since we may redefine each w_j modulo $\mathrm{Span}(e_1, \dots, e_m)$ we may also assume that each $w_j \in \mathbf{j}(\mathrm{Span}(e_1, \dots, e_m))$. By $\mathbf{j}$-genericity,

$$w_m \notin \mathbf{j}(\mathrm{Span}(e_1, \dots, e_{m-1})) ,$$

so we may suppose $w_m = \mathbf{j}(e_m) + \sum_{i<m} a_i \, \mathbf{j}(e_i)$. Now define $g \in G_\mathbb{R}$ by $g(e_i) = e_i$ for $i < m$ and $g(e_m) = e_m - \sum_{i<m} a_i \, \mathbf{j}(e_i)$. In other words, we may assume $w_m = \mathbf{j}(e_m)$. Continuing this procedure we may assume that each $w_j = \mathbf{j}(e_j)$. Thus we have $g \in G_\mathbb{R}$ with $g(z) = z_\epsilon$. This completes the proof that the set of all $\mathbf{j}$-generic flags in Z_δ forms the unique open $G_\mathbb{R}$-orbit D there.

For the second statement, assume first that the dimension symbol δ is symmetric. Let C_1 denote the set of all maximally ω-isotropic flags in Z_δ. Then C_1 is closed in Z_δ because it is defined by equations, and it contains the base point of $z_0 \in C_0 \subset Z_\delta$ defined by the ordered basis

$$\{e_1, \dots, e_m, \mathbf{j}(e_m), \dots, \mathbf{j}(e_1)\} .$$

As C_0 is the unique closed $K_\mathbb{C}$-orbit in D, we need only check that $K_\mathbb{C}$ acts transitively on C_1.

Let $z \in C_1$ and denote

$$L(z) = (0 \subset \ \mathrm{Span}(v_1, \dots, v_{d_1}) \subset \dots \subset \ \mathrm{Span}(v_1, \dots, v_{d_\ell}) \subset V) .$$

We may recursively normalize so that $\omega(v_i, \mathbf{j}(v_j)) = \delta_{i,j}$ for $1 \leqq i,j \leqq d_\ell$. By Witt's Theorem, the map $v_1 \mapsto e_i$, $\mathbf{j}(v_i) \mapsto \mathbf{j}(e_i)$, for $1 \leqq i \leqq d_\ell$, extends to an element $k \in K_{\mathbb{C}}$. Since $L(k(z)) = L(z_0)$, δ is symmetric, and $k(z)$ and z_0 are maximally ω-isotropic, it follows that $k(z) = z_0$. The second statement is therefore proved for symmetric δ.

To handle the nonsymmetric case, let $\widetilde{\delta}$ denote the symmetrized dimension symbol $\delta \cup \delta'$ consisting of all the d_j and all the $n - d_j$. The set $\widetilde{C}_1$ of all maximally isotropic flags in $Z_{\widetilde{\delta}}$ maps onto C_1 under the natural projection $Z_{\widetilde{\delta}} \to Z_\delta$. As $\widetilde{C}_1$ is a closed $K_{\mathbb{C}}$-orbit, C_1 also is a closed $K_{\mathbb{C}}$-orbit. That completes the proof of the second statement. □

Exactly as in Sect. 3.1 the main goal here is to give a concrete description of a Schubert slice for a given open $G_{\mathbb{R}}$-orbit D in the flag manifold Z_δ for an arbitrary dimension symbol δ. For this it is first necessary to determine an appropriate Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$.

Regard $V = \mathbb{H}^m$, the direct sum of quaternionic lines $\mathbb{H}(e_j)$, $1 \leqq j \leqq m$. Our $A_{\mathbb{R}}$ consists of the hermitian operators $v = \sum v_j \mapsto \sum a_j v_j$ where $v_j \in \mathbb{H}(e_j)$ and $a_j > 0$ with $a_1 a_2 \dots a_m = 1$. Evidently this group is commutative and is contained in $G_{\mathbb{R}}$, and its elements are semisimple with all eigenvalues positive real. It is maximal for this: any such group containing the group $A_{\mathbb{R}}$ just defined, preserves each line $\mathbb{H}(e_i)$, acts on $\mathbb{H}(e_i)$ by positive real scalars, and preserves volume, hence is equal to $A_{\mathbb{R}}$. Thus our $A_{\mathbb{R}}$ is the split component of an Iwasawa decomposition of $G_{\mathbb{R}}$.

Consider $\delta = (2, 4, \dots, 2m-2)$ and let $z_0 \in Z_\delta$ be the flag associated to the ordered $\mathbb{C}$-basis $\{e_1, \mathbf{j}(e_1), e_2, \mathbf{j}(e_2), \dots, e_m, \mathbf{j}(e_m)\}$. The isotropy subgroup $P_{\mathbb{C}}$ of $G_{\mathbb{C}}$ at z_0 is normalized by $\mathbf{j}$, in other words invariant under complex conjugation of $G_{\mathbb{C}}$ over $G_{\mathbb{R}}$, so $P_{\mathbb{C}}$ is the complexification of $P_{\mathbb{R}} = P_{\mathbb{C}} \cap G_{\mathbb{R}}$, and a moment's thought shows that $P_{\mathbb{R}} = M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ where (i) $M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(A_{\mathbb{R}}) = (\mathrm{SL}_1(\mathbb{H}))^m$, the product of the quaternion special linear groups of the $\mathbb{H}(e_i)$, (ii) $A_{\mathbb{R}}$ is the group we defined above, and (iii) $N_{\mathbb{R}}$ is a real form of the unipotent radical of $P_{\mathbb{C}}$. So $P_{\mathbb{R}}$ is a minimal parabolic subgroup of $G_{\mathbb{R}}$ and we have the Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$.

B will be the Borel subgroup of $G_{\mathbb{C}}$ defined by the ordered $\mathbb{C}$-basis $\{e_1, \mathbf{j}(e_1), e_2, \mathbf{j}(e_2), \dots, e_m, \mathbf{j}(e_m)\}$ of V. It contains $A_{\mathbb{R}} N_{\mathbb{R}}$ and is contained in $P_{\mathbb{C}}$.

We proceed as in Sect. 3.1. The main points are (i) to determine a base point $z_0 \in C_0 \subset D \subset Z_\delta$ such that $B(z_0) \cap C_0 = \{z_0\}$ and (ii) to explicitly compute $\Sigma = (A_{\mathbb{R}} N_{\mathbb{R}})(z_0)$. We summarize this as follows.

Proposition 3.15. *There exists $z_0 \in C_0$ such that $B(z_0) \cap C_0 = \{z_0\}$ and such that $(A_{\mathbb{R}} N_{\mathbb{R}})(z_0) = B(z_0) \cong \mathbb{C}^p$ where $p = \mathrm{codim}_{\mathbb{C}}(C_0)$.*

Proof. First consider the case of a symmetric dimension symbol δ. Here let $z_0 \in Z_\delta$ be associated to the standard ordered basis $\{e_1, \dots, e_m, \mathbf{j}(e_m), \dots, \mathbf{j}(e_1)\}$. Normalizing the leading coefficients, for $g \in B$ we have

$$g(e_j) = e_j + \sum\nolimits_{i<j} (z_{i,j} e_i + w_{i,j}\, \mathbf{j}(e_i)) \text{ for } 1 \leqq j \leqq m\,. \tag{3.16}$$

By induction on k now

$$g(\mathrm{Span}(e_1,\dots,e_k)) \text{ is } \omega\text{-isotropic if and only if}$$
$$g(\mathrm{Span}(e_1,\dots,e_k)) = \mathrm{Span}(e_1,\dots,e_k)\,.$$

Thus, if $g(z_0) \in C_0$ then $L(g(z_0)) = L(z_0)$. Since δ is symmetric, $g(z_0) \in C_0$ further implies $g(z_0) = z_0$. Now $B(z_0) \cap C_0 = \{z_0\}$ and $\Sigma = (A_{\mathbb{R}}N_{\mathbb{R}})(z_0)$ is a Schubert slice.

We parameterize $B(z_0)$ in the case of symmetric δ. As in Sect. 3.1 we use the coordinate chart $U^+(z_0)$ where U^+ is the unipotent subgroup of $G_{\mathbb{C}}$ whose Lie algebra $\mathfrak{u}^+$ is opposite to the nilradical $\mathfrak{q}_{z_0}^{\mathrm{nil}}$ of the isotropy subalgebra of $\mathfrak{g}_{\mathbb{C}}$ at z_0. We have $\delta = (d_1,\dots,d_k)$ and symmetry implies $d_k = n - d_1$. Let $L(\delta) = (d_1,\dots,d_\ell)$. If $g \in B$ and $j \leqq m$ then, modulo linear combinations of the $g(e_i)$ for $1 \leqq i < j$, and normalizing the leading coefficients,

$$g(e_j) = e_j + \sum\nolimits_{i<j} z_{i,j}\, \mathbf{j}(e_i)\,. \tag{3.17}$$

Similarly, modulo linear combinations of the $g(e_i)$ for $1 \leqq i \leqq m$, normalizing leading coefficients,

$$g(\mathbf{j}(e_b)) = \mathbf{j}(e_b) + \sum\nolimits_{a<b} w_{a,b}\, \mathbf{j}(e_a)\,. \tag{3.18}$$

Thus $B(z_0)$ is parameterized by (3.17) and (3.18) for $1 \leqq j \leqq d_\ell$ and $d_{\ell+1} \leqq b \leqq m$. There are no restrictions on the coefficients $z_{i,j}, w_{a,b} \in \mathbb{C}$.

If $g \in A_{\mathbb{R}}N_{\mathbb{R}}$ then we have, again normalizing leading coefficients,

$$g(e_j) = e_j + \sum\nolimits_{i<j} (\xi_{i,j}e_i + \eta_{i,j}(j)(e_i)) \text{ for } 1 \leqq j \leqq m \tag{3.19}$$

where there are no restrictions on the $\xi_{i,j}, \eta_{i,j} \in \mathbb{C}$. If we set $\xi_{a,b} = \overline{w}_{a,b}$ and $\eta_{i,j} = z_{i,j}$ and compare (3.18) with (3.19) we see that $(A_{\mathbb{R}}N_{\mathbb{R}})(z_0) = B(z_0)$.

We just proved Proposition 3.15 in the case of symmetric dimension symbol, using the base point $z_0 \in Z_\delta$ associated to the standard ordered basis $\{e_1,\dots,e_m,\mathbf{j}(e_m),\dots,\mathbf{j}(e_1)\}$. For the general case we must re–order this basis in order to manage gaps in $L(\delta)$. This goes essentially as in the case of $SL_n(\mathbb{R})$.

As in Sect. 3.1, suppose that there is a gap in $L(\delta)$ between adjacent entries $d_a < d_e$ there. We reorder the standard basis so that, in the range of the gap, the ordered subset

$$\{\mathbf{j}(e_{d_e}), \mathbf{j}(e_{d_e-1}), \dots, \mathbf{j}(e_{d_a+2}), \mathbf{j}(e_{d_a+1})\}$$

is reversed to

$$\{\mathbf{j}(e_{d_a+1}), \mathbf{j}(e_{d_a+2}), \dots, \mathbf{j}(e_{d_e-1}), \mathbf{j}(e_{d_e})\}\,.$$

The base point $z_0 \in Z_\delta$ is the flag associated to this reordered basis.

As in the symmetric case, if $g \in B$ and $g(z_0) \in C_0$, then $g(\Lambda_{d_i}) = \Lambda_{d_i}$ for all d_i except possibly when $d_i = n - d$ and d is in a gap, $d_a < d < d_e$ as above. In that case the reordered basis yields

$$\begin{aligned}\Lambda_{n-d} = \; & \mathrm{Span}(e_1, \dots, e_m, \mathbf{j}(e_m), \dots \\ & \dots, \mathbf{j}(e_{d_e+1}), \mathbf{j}(e_{d_a+1}), \mathbf{j}(e_{d_a+2}), \dots, \mathbf{j}(e_{d_a+(d_e-d)})) \,.\end{aligned}$$

Now

$$g(\mathbf{j}(e_{d_a+j})) = \mathbf{j}(e_{d_a+j}) + \sum\nolimits_{u \leqq d_a+j} z_{u,j} e_u + \sum\nolimits_{u<d_a+j} w_{u,j} \mathbf{j}(e_u) \,.$$

But

$$\begin{aligned}& g(\mathrm{Span}(e_1, \dots, e_m, \mathbf{j}(e_m), \dots, \mathbf{j}(e_{d_e+1}))) \\ & \quad = \mathrm{Span}(e_1, \dots, e_m, \mathbf{j}(e_m), \dots, \mathbf{j}(e_{d_e+1})) \,.\end{aligned}$$

In particular $\mathrm{Span}(e_1, \dots, e_m) \subset g(\Lambda_{n-d})$. Thus, modulo elements of $g(\Lambda_{n-d})$,

$$g(\mathbf{j}(e_{d_a+j})) = \mathbf{j}(e_{d_a+j}) + \sum\nolimits_{u<d_a+j} w_{u,j} \mathbf{j}(e_u) \,. \tag{3.20}$$

Because of the "maximally isotropic" condition,

$$\mathrm{Span}(e_1, \dots, e_{d_a}) \subset g(\Lambda_{n-d})^{\perp} \,,$$

and that implies vanishing of the $w_{u,j}$ for all u, j. Thus $g(\Lambda_{n-d}) = \Lambda_{n-d}$ for gap flag entries as well. We have proved $B(z_0) \cap C_0 = \{z_0\}$.

Finally, we show that $B(z_0) = (A_{\mathbb{R}} N_{\mathbb{R}})(z_0)$ as in the symmetric case. For this note that there was no change in the ordering in $L(z_0)$ and, as we saw in (3.20), in the upper part of z_0 the point is that only terms involving $\mathbf{j}(e_u)$ appear. So the choice $\xi_{u,j} = \overline{w_{u,j}}$ is possible as in the symmetric case. This completes the proof of Proposition 3.15. □

Corollary 3.21. *For an arbitrary flag manifold Z_δ of $G_{\mathbb{C}} = \mathrm{SL}_{2m}(\mathbb{C})$, and an open orbit $D \subset Z_\delta$ of $G_{\mathbb{R}} = \mathrm{SL}_m(\mathbb{H})$, the cycle space $\Omega(D)$ is a Stein domain in the affine homogeneous space $\Omega = G_{\mathbb{C}}/K_{\mathbb{C}}$.*

Proof. This is immediate from Theorem 2.3 and the fact that $\Sigma \cong \mathbb{C}^p$ is Stein. □

Corollaries 3.10 and 3.21, and the fact that open $SU(k, \ell)$-orbits are measurable, combine to prove Theorem 1.1.

3.3 Example with $G_{\mathbb{R}} = U(n, 1)$: Comparison of Transversal Varieties

In this section $G_{\mathbb{R}}$ is the unitary group $U(n, 1)$ acting on the complex projective space $Z = \mathbb{P}_n(\mathbb{C})$, and $G_{\mathbb{C}}$ is its complexification $GL_{n+1}(\mathbb{C})$. We use the unitary group $U(n,1)$ rather than special unitary group $SU(n,1)$ for notational convenience; this has no effect on the spaces we consider. Fix a $G_{\mathbb{R}}$-orthonormal basis: the hermitian form h that defines $G_{\mathbb{R}}$ is given by $h(e_i, e_j) = \delta_{i,j}$ if $1 \leqq i \leqq n$, by $-\delta_{i,j}$ if $i = n+1$. There are 3 orbits: the open ball $D_0 = G_{\mathbb{R}}(z_0), z_0 = e_{n+1}\mathbb{C}$ consisting of negative definite lines; its boundary $S = G_{\mathbb{R}}(cz_0), cz_0 = (e_n + e_{n+1})\mathbb{C}$, consisting of isotropic lines; and the complement $D_1 = G_{\mathbb{R}}(z_1), z_1 = c^2 z_0 = e_n\mathbb{C}$ of $D_0 \cup S$ consisting of positive definite lines. Here c is a certain Cayley transform.

Let $P_{\mathbb{R}}$ denote the $G_{\mathbb{R}}$-stabilizer of the point $cz_0 \in S$. Its Lie algebra $\mathfrak{p}_{\mathbb{R}}$ is the sum of the non–positive eigenspaces of $\mathrm{ad}(x_0)$, so $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p}_{\mathbb{R}}^0 + \mathfrak{p}_{\mathbb{R}}^{-1} + \mathfrak{p}_{\mathbb{R}}^{-2}$ where $\mathfrak{p}_{\mathbb{R}}^s$ is the s-eigenspace. Calculate

$$\begin{aligned} \mathfrak{p}_{\mathbb{R}}^0 &= \left\{ \begin{pmatrix} \mathfrak{u}(n-1) & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \right\}, \quad \mathfrak{p}_{\mathbb{R}}^{-1} = \left\{ \begin{pmatrix} 0_{n-1} & u & u \\ v & 0 & 0 \\ -v & 0 & 0 \end{pmatrix} \right\}, \\ \mathfrak{p}_{\mathbb{R}}^{-2} &= \left\{ \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & d & d \\ 0 & -d & -d \end{pmatrix} \right\} \end{aligned} \tag{3.22}$$

where $a, b, d \in \mathbb{R}$, $u \in \mathbb{R}^{(n-1)\times 1}$ and $v \in \mathbb{R}^{1\times(n-1)}$. The real parabolic subalgebra $\mathfrak{p}_{\mathbb{R}}$ and its complexification $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^0 + \mathfrak{p}_{\mathbb{C}}^{-1} + \mathfrak{p}_{\mathbb{C}}^{-2}$ are given by

$$\mathfrak{p}_{\mathbb{R}} = \left\{ \begin{pmatrix} \mathfrak{u}(n-1) & u & u \\ v & a+d & b+d \\ -v & b-d & a-d \end{pmatrix} \right\} \text{ and} \tag{3.23}$$

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} \mathfrak{gl}(n-1;\mathbb{C}) & u & u \\ v & a+d & b+d \\ -v & b-d & a-d \end{pmatrix} \right\} .$$

The base cycle in D_1 is

$$\begin{aligned} C_1 = K_0(z_1) &= (U(n) \times U(1))(z_1) \\ &= \{\mathrm{Span}(v) \in Z \mid v \in \mathrm{Span}(e_1 \wedge \cdots \wedge e_n)\} \cong \mathbb{P}_{n-1}(\mathbb{C}) . \end{aligned} \tag{3.24}$$

The complexification $P_{\mathbb{C}}$ of $P_{\mathbb{R}}$ has Levi factor $P_{\mathbb{C}}^0$ with Lie algebra $\mathfrak{p}_{\mathbb{C}}^0$. The Borel subalgebras of $\mathfrak{p}_{\mathbb{C}}^0$ are just the

$$\mathfrak{b}_1 = \left\{ \begin{pmatrix} p_1' & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \middle| \, p_1' \in \mathfrak{b}_1' \text{ and } a, b \in \mathbb{C} \right\} \tag{3.25}$$

where $\mathfrak{b}_1'$ is a Borel subalgebra of $\mathfrak{gl}_{n-1}(\mathbb{C})$. Let B_1 denote the Borel subgroup of P^0 with Lie algebra $\mathfrak{b}_1$. Then the fixed points of B_1 on Z are $z_1 = (e_n + e_{n+1})\mathbb{C}$, $(e_n - e_{n+1})\mathbb{C}$, and a unique point on Y_1. We choose $\mathfrak{b}_1'$ to be the lower triangular matrices in $\mathfrak{gl}_{(n-1}(\mathbb{C})$. Then $e_{n-1}\mathbb{C}$ is the unique B_1-fixed point in Y_1.

The unipotent radical $P_{\mathbb{C}}^{\text{unip}}$ of $P_{\mathbb{C}}$ has Lie algebra $\mathfrak{p}_{\mathbb{C}}^{\text{nil}} = \mathfrak{p}_{\mathbb{C}}^{-1} + \mathfrak{p}_{\mathbb{C}}^{-2}$. The Borel subgroup $B_1 \in P_{\mathbb{C}}^0$ determines the Borel subgroup $B = B_1 P_{\mathbb{C}}^{\text{unip}} \subset G_{\mathbb{C}}$ with Lie algebra $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{p}_{\mathbb{C}}^{-1} + \mathfrak{p}_{\mathbb{C}}^{-2}$,

$$\mathfrak{b} = \left\{ \begin{pmatrix} p_1' & u & u \\ v & a+d & b+d \\ -v & b-d & a-d \end{pmatrix} \middle| \begin{array}{l} p_1' \text{ is lower triangular }, \\ u \in \mathbb{C}^{(n-1)\times 1}, v \in \mathbb{C}^{1\times(n-1)}, \\ \text{and } a, b, d \in \mathbb{C}. \end{array} \right\}. \tag{3.26}$$

Compute

$$\exp \begin{pmatrix} 0 & u & u \\ v & d & d \\ -v & -d & -d \end{pmatrix} = \begin{pmatrix} I & u & u \\ v & 1+d+\frac{1}{2}vu & d+\frac{1}{2}vu \\ -v & -d+\frac{1}{2}vu & 1-d+\frac{1}{2}vu \end{pmatrix}$$

to see that the orbit $B(e_{n-1}\mathbb{C})$ satisfies

$$B(e_{n-1}\mathbb{C}) = P^-(e_{n-1}\mathbb{C}) = \{(e_{n-1} + te_n - te_{n+1})\mathbb{C} \mid t \in \mathbb{C}\} . \tag{3.27}$$

Here in effect t is the last entry of the row vector v in the matrix exponential just above. That comes from $\mathfrak{p}_{\mathbb{C}}^{-1}$. It comes from $\mathfrak{p}_{\mathbb{R}}^{-1}$ if and only if $u^* + v = 0$, which does not effect the range of possibilities for v. Thus $B(e_{n-1}\mathbb{C}) = P_{\mathbb{R}}^{\text{unip}}(e_{n-1}\mathbb{C})$.

We have an Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}} = N_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$ given by $K_{\mathbb{R}} = U(n) \times U(1)$, $A_{\mathbb{R}} = \exp(\mathfrak{a}_{\mathbb{R}})$, and $\mathfrak{n}_{\mathbb{R}} = (\mathfrak{n}_{\mathbb{R}} \cap \mathfrak{b}_1) + \mathfrak{p}_{\mathbb{R}}^{\text{unip}}$. Since $B(e_{n-1}\mathbb{C}) = P_{\mathbb{R}}^{\text{unip}}(e_{n-1}\mathbb{C})$ we have the slice

$$\Sigma_S := (A_{\mathbb{R}} N_{\mathbb{R}})(e_{n-1}\mathbb{C}) = B(e_{n-1}\mathbb{C}) . \tag{3.28}$$

This explicitly shows that the slice (3.28) is biholomorphic to $\mathbb{C}$, with closure in Z that is the projective line based on the subspace $\text{Span}(e_{n-1}, (e_n - e_{n+1}))$ of $\mathbb{C}^{n+1}$. Now it is transversal to C_1, with complementary dimension, so it meets every cycle C transversally and at just one point, and thus is a Schubert slice. In sharp contrast, the transversal variety Σ_T constructed in [17] is, as described at the end of Sect. 1, holomorphically equivalent to the unit disk Δ in $\mathbb{C}$.

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