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Complex Geometry and Representations of Lie Groups

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Dedicated to the memory of my friend and colleague Alfred Gray

ABSTRACT. Let $Z = G/Q$ be a complex flag manifold and let G_0 be a real form of G. Then the representation theory of the real reductive Lie group G_0 is intimately connected with the geometry of G_0–orbits on Z. The open orbits correspond to the discrete series representations and their analytic continuations, closed orbits correspond to the principal series, and certain other orbits give the other series of tempered representations. Here I try to indicate some of that interplay between geometry and analysis, concentrating on the complex geometric aspects of the open orbits and the related representations.

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0. Introduction.

There is an mysterious intimate correlation between the theory of homogeneous complex manifolds and the theory of unitary representations of real Lie groups. This relation is becoming clear in the case of a real reductive Lie group G_0. Let G be the complexification of G_0 and suppose that G_0 has a compact Cartan subgroup. Then the open G_0–orbits on a complex flag manifold $Z = G/Q$ (where Q is a parabolic subgroup of G) can be used to construct the discrete series representations of G_0 as the action of G_0 on L^2 cohomologies of various sorts. If G_0 is compact, this is the famous Bott–Borel–Weil Theorem. If G_0 is not compact, one use the same idea, even though noncompactness introduces a number of nontrivial technical problems, but finally one does get good geometric realizations for discrete series representations. Whether G_0 has a compact Cartan subgroup or not, the other series of representations of G_0 that are involved in the Plancherel formula, all come out of discrete series representations of certain subgroups $M_0 \subset G_0$ by a straightforward construction, and those representations occur as cohomologies over certain partially complex G_0–orbits. Here I'll sketch some of the main points in this beautiful liaison of geometry with analysis.

This paper updates [**141**], with fewer details on the older material but with much more material on singular representations, cycle spaces, and double fibration transforms.

Part I addresses geometric aspects of complex flag manifolds $Z = G/Q$ and G_0–orbits on Z. In Sections 1 through 4, we review the basic facts on complex flags, real group orbits and convexity of real group orbits. This material is found in much more detail in [**133**], [**116**] and [**139**]. Then in Section 5 we indicate the current state of information on cycle spaces of open G_0–orbits, from [**128**], [**139**], [**149**], [**150**], [**88**], [**89**] and [**90**]. Finally, in Section 6, we indicate the current state of affairs on the double fibration transforms that come out of these cycle spaces; see [**97**], [**99**] and [**150**] for details.

Part II is concerned with the representation theory of real reductive (or semisimple) Lie groups G_0. It concentrates on the series of unitary representations that are involved in the Plancherel Theorem and Fourier Inversion Formula of G_0. As we will see in Part III, those are the representations that appear in a straightforward geometric setting on G_0–orbits in complex flag manifolds $Z = G/Q$. In Section 7 we illustrate the construction with the "principal series", where a certain subgroup $M_0 \subset G_0$ is compact and thus presents no technical complications. Then in Section 8 we indicate the main results from Harish–Chandra's theory of the "discrete series", which we need both for itself and for the cases where M_0 is noncompact. In Section 9 we fit this together for the construction of all the relevant series of unitary representations of G_0. In Section 10 we give a brief indication of just how these "tempered series" of representations give the Plancherel Theorem and Fourier Inversion Formula of G_0.

Part III indicates the geometric realization of tempered representations as square integrable cohomologies of certain partially holomorphic vector bundles over G_0–orbits in complex flag manifolds $Z = G/Q$. Much as in Part II, we start in Section 11 illustrating the construction for principal series representations where there essentially are no technical problems. In Section 12 we indicate the realization of discrete series representations on certain Hilbert spaces of L^2 harmonic $(0,q)$–forms with values in a homogeneous holomorphic vector bundle over an open G_0–orbit. Then in Section 13 we fit the two together to realize tempered representations in general on certain Hilbert spaces of L^2 partially harmonic partially $(0,q)$–forms with values in a homogeneous partially holomorphic vector bundle over an appropriate G_0–orbit. Finally, in Section 14, we look at some of the many approaches to geometric realization of non–tempered representations.

Part I. Geometry of Flag Manifolds.

Complex flag manifolds are of considerable interest in algebraic geometry, in differential geometry, and in harmonic analysis. The most familiar ones are the complex Grassmann manifolds and the other compact hermitian symmetric spaces. In this Part we discuss the theory of complex flag manifolds and of real group orbits on those flag manifolds. Our emphasis is on the geometry of open orbits, which are instrumental for the realization of discrete series representations for semisimple Lie groups.

1. Parabolic Subalgebras and Complex Flags.

We start with the definitions of Borel subgroups and subalgebras, parabolic subgroups and subalgebras, and complex flag manifolds. Parabolic subgroups and flag manifolds were invented by J. Tits [**121**], and independently Borel subgroups were invented by A. Borel [**19**]. Also see [**20**], [**21**], and [**22**].

Fix a complex reductive Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{g}$, and $\mathfrak{h} \cap \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$, and $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}') \oplus \mathfrak{z}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{h})$ denote the corresponding root system, and fix a positive subsystem $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$. The corresponding *Borel subalgebra*

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha} \subset \mathfrak{g} \tag{1.1}$$

has its nilradical[1] $\mathfrak{b}^{-n} = \sum \mathfrak{g}_{-\alpha}$ and a Levi complement $\mathfrak{h}$. In general a subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is called a *Borel subalgebra* if it is conjugate by an inner automorphism of $\mathfrak{g}$ to a subalgebra of the form (1.1), in other words if there exist choices of $\mathfrak{h}$ and $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{s}$ is given by (1.1).

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The Cartan subgroup of G corresponding to $\mathfrak{h}$ is $H = Z_G(\mathfrak{h})$, the centralizer of $\mathfrak{h}$ in G. It has Lie algebra $\mathfrak{h}$, and it is connected because G is connected, complex and reductive. The *Borel*

[1] Here we describe the nilradical as a sum of negative root spaces, rather than positive, so that, in applications, positive functionals on $\mathfrak{h}$ will correspond to positive bundles (instead of negative bundles), and holomorphic discrete series representations will be highest weight (instead of lowest weight) representations.

subgroup $B \subset G$ corresponding to a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is defined to be the G–normalizer of $\mathfrak{b}$, that is,

$$B = \{g \in G \mid \mathrm{Ad}\,(g)\mathfrak{b} = \mathfrak{b}\}\ . \tag{1.2}$$

The basic facts on Borel subgroups are:

LEMMA 1.3. *B has Lie algebra $\mathfrak{b}$, B is a closed connected subgroup of G, and B is its own normalizer in G.*

LEMMA 1.4. (See [**122**], [**144**] and [**145**]) *Let $G_u \subset G$ be a compact real form. Then G_u is transitive on $X = G/B$, and X has a G_u–invariant Kähler metric. In particular X has the structure of compact Kähler manifold.*

LEMMA 1.5. *There is a finite dimensional irreducible representation π of G with the property: let $[v]$ be the image of a lowest weight vector in the projective space $\mathbb{P}(V_\pi)$ corresponding to the representation space of π. Then the action of G on V_π induces a holomorphic action of G on $\mathbb{P}(V_\pi)$, and B is the G–stabilizer of $[v]$. In particular $X = G/B$ is a projective algebraic variety.*

LEMMA 1.6. *B is a maximal solvable subgroup of G.*

A subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is called *parabolic* if it contains a Borel subalgebra. For example, let Ψ be the simple root system corresponding to Σ^+ and let Φ be an arbitrary subset of Ψ. Every root $\alpha \in \Sigma$ has unique expression

$$\alpha = \sum_{\psi \in \Psi} n_\psi(\alpha)\psi \tag{1.7}$$

where the $n_\psi(\alpha)$ are integers, all $\geqq 0$ if $\alpha \in \Sigma^+$ and all $\leqq 0$ if $\alpha \in \Sigma^- = -\Sigma^+$. Set

$$\Phi^r = \{\alpha \in \Sigma \mid n_\psi(\alpha) = 0 \text{ whenever } \psi \notin \Phi\} \tag{1.8}$$

and

$$\Phi^n = \{\alpha \in \Sigma^+ \mid \alpha \notin \Phi^r\} = \{\alpha \in \Sigma \mid n_\psi(\alpha) > 0 \text{ for some } \psi \notin \Phi\}. \tag{1.9}$$

Now define

$$\mathfrak{q}_\Phi = \mathfrak{q}^r + \mathfrak{q}^{-n} \text{ with } \mathfrak{q}^r = \mathfrak{h} + \sum_{\alpha \in \Phi^r} \mathfrak{g}_\alpha \text{ and } \mathfrak{q}^{-n} = \sum_{\alpha \in \Phi^n} \mathfrak{g}_{-\alpha}\ . \tag{1.10}$$

Then $\mathfrak{q}_\Phi$ is a subalgebra of $\mathfrak{g}$ that contains the Borel subalgebra (1.1), so it is a parabolic subalgebra of $\mathfrak{g}$.

PROPOSITION 1.11. *Let $\mathfrak{q} \subset \mathfrak{g}$ be a subalgebra that contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$. Then there is a set Φ of simple roots such that $\mathfrak{q} = \mathfrak{q}_\Phi$.*

The *parabolic subgroup* $Q \subset G$ corresponding to a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is defined to be the G–normalizer of $\mathfrak{q}$, that is,

$$Q = \{g \in G \mid \mathrm{Ad}\,(g)\mathfrak{q} = \mathfrak{q}\}\ . \tag{1.12}$$

The basic facts on parabolic subgroups are most easily derived from the corresponding results for Borel subgroups. However, the two notions were developed separately, and from different viewpoints, in the 1950's.

LEMMA 1.13. *The parabolic subgroup $Q \subset G$ defined by* (1.12) *has Lie algebra $\mathfrak{q}$. That group Q is a closed connected complex subgroup of G, and Q is its own normalizer in G. In particular, a Lie subgroup of G is parabolic if and only if it contains a Borel subgroup.*

LEMMA 1.14. *Fix a standard parabolic subgroup $Q = Q_\Phi$ in G. Then there is a finite dimensional irreducible representation π of G with the property: let $[v]$ be the image of a lowest weight vector in the projective space $\mathbb{P}(V_\pi)$ corresponding to the representation space of π. Then the action of G on V_π induces a holomorphic action of G on $\mathbb{P}(V_\pi)$, and Q is the G–stabilizer of $[v]$. In particular $Z = G/Q$ is a projective algebraic variety.*

Since Q is its own normalizer in G,

LEMMA 1.15. *Let $Z = G/Q$ where Q is a parabolic subgroup of G. Then we can view Z as the space of all G–conjugates of $\mathfrak{q}$, by the correspondence $gQ \leftrightarrow Ad(g)\mathfrak{q}$. Write $\mathfrak{q}_z$ for the parabolic subalgebra of $\mathfrak{g}$ corresponding to $z \in Z$, and write Q_z for the corresponding parabolic subgroup of G. Then the usual action $g : g'Q \mapsto gg'Q$ of G on Z carries over to $g : \mathfrak{q}_z \mapsto \ \mathrm{Ad}(g)\mathfrak{q}_z = \mathfrak{q}_{g(z)}$.*

Let $B \subset Q \subset G$ consist of a Borel subgroup contained in a parabolic subgroup. Then we have complex homogeneous quotient spaces $X = G/B$ and $Z = G/Q$, and a G–equivariant holomorphic projection $X \to Z$ given by $gB \mapsto gQ$. In particular, transitivity of G_u on X gives

LEMMA 1.16. *Let $G_u \subset G$ be a compact real form. Then G_u is transitive on $Z = G/Q$.*

Combining Lemmas 1.14 and 1.16 we have

LEMMA 1.17. *Let $G_u \subset G$ be a compact real form. Then the isotropy subgroup $G_u \cap Q$ is the centralizer of a torus subgroup of G_u , and Z has a G_u–invariant Kähler metric. Thus Z has the structure of G_u–homogeneous compact simply connected Kähler manifold.*

At this point we summarize:

PROPOSITION 1.18. *Let Q be a complex Lie subgroup of G. Then the following conditions are equivalent.* (1) *G/Q is a compact complex manifold.* (2) *G/Q is a projective algebraic variety.* (3) *If G_u denotes a compact real form of G then G/Q is a G_u–homogeneous compact Kähler manifold.* (4) *G/Q is the projective space orbit of an extremal weight vector in an irreducible finite dimensional representation of G.* (5) *G/Q is a G–equivariant quotient manifold of G/B, for some Borel subgroup $B \subset G$.* (6) *Q is a parabolic subgroup of G.*

We will simply refer to these spaces $Z = G/Q$ as *complex flag manifolds.*

2. Real Group Orbits on Complex Flags.

We now consider the action of a real group on a complex flag manifold, developed in [**133**] and [**134**]. Also see [**146**], [**134**] and [**135**]. The most familiar case is the action of $SL(2;\mathbb{R})$ on the Riemann sphere $P^1(\mathbb{C})$ by linear fractional transformation.

Let G_0 be a real form of G. In other words, G_0 is a Lie group whose Lie algebra $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. Although G is connected, G_0 does not have to be connected, but we do need some control over the components. For this reason, and for some technical reasons that will come out in the representation theory, we assume that G_0 is of *Harish–Chandra class* developed in [**69**], [**70**] and [**71**]:

DEFINITION 2.1. *A real Lie group G_0 belongs to the* Harish–Chandra class *of reductive Lie groups if* (i) *the Lie algebra $\mathfrak{g}_0$ is reductive,* (ii) *the component group G_0/G_0^0 is finite,* (iii) *the derived group $[G_0^0, G_0^0]$ is closed in G_0, and* (iv) *if $g \in G_0$ then* $\mathrm{Ad}(g)$ *is an inner automorphism of the complex Lie algebra $\mathfrak{g}$.*

The results we describe in this report hold for a somewhat larger class of reductive Lie groups developed a bit earlier [**135**].

The group G_0 acts on the complex flag manifold $Z = G/Q$ by $g : \mathfrak{q}_z \mapsto \mathrm{Ad}(g)\mathfrak{q}_z$ as in Lemma 1.15, because of condition (iv) in Definition 2.1.

We write τ both for the complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$ and for the corresponding conjugation of G over its analytic subgroup $\exp(\mathfrak{g}_0)$.

The basic results on the G_0–orbit structure of $Z = G/Q$ depends on the consequence

LEMMA 2.2. *The intersection of two Borel subalgebras of $\mathfrak{g}$ contains a Cartan subalgebra of $\mathfrak{g}$.*

of the Bruhat decomposition ([**26**], [**62**]) of Z. From this,

LEMMA 2.3. *The intersection $\mathfrak{q} \cap \tau\mathfrak{q}$ contains a τ–stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.*

THEOREM 2.4. *Consider an orbit $G_0(z)$ on $Z = G/Q$. Then there exist a τ–stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{q}_z$ of $\mathfrak{g}$, a positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$, and a set Φ of simple roots, such that $\mathfrak{q}_z = \mathfrak{q}_\Phi$ and $Q_z = Q_\Phi$.*

There are only finitely many G_0–conjugacy classes of Cartan subalgebras $\mathfrak{h}_0$ in $\mathfrak{g}_0$, for each $\mathfrak{h}_0$ there are only finitely many positive root systems $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$, and for each such Σ^+ there are only finitely many subsets Φ of the simple root system. Thus

COROLLARY 2.5. *There are only finitely many G_0–orbits on Z. The maximal–dimensional orbits are open and the minimal–dimensional orbits are closed.*

COROLLARY 2.6. *In the notation of* Theorem 2.4, $\mathfrak{q}_z \cap \tau\mathfrak{q}_z$ *is the semidirect sum of its nilpotent radical* $(\mathfrak{q}_\Phi^{-n} \cap \tau\mathfrak{q}_\Phi^{-n}) + (\mathfrak{q}_\Phi^r \cap \tau\mathfrak{q}_\Phi^{-n}) + (\mathfrak{q}_\Phi^{-n} \cap \tau\mathfrak{q}_\Phi^r)$ *with the Levi complement* $\mathfrak{q}_\Phi^r \cap \tau\mathfrak{q}_\Phi^r = \mathfrak{h} + \sum_{\Phi^r \cap \tau\Phi^r} \mathfrak{g}_\alpha$. *In particular,* $\dim_{\mathbb{R}} \mathfrak{g}_0 \cap \mathfrak{q}_z = \dim_{\mathbb{C}} \mathfrak{q}_\Phi^r + |\Phi^n \cap \tau\Phi^n|$.

COROLLARY 2.7. *In that notation,* $\mathrm{codim}_{\mathbb{R}}(G_0(z) \subset Z) = |\Phi^n \cap \tau\Phi^n|$. *In particular, $G_0(z)$ is open in Z if and only if $\Phi^n \cap \tau\Phi^n$ is empty.*

3. The Closed Orbit.

We now look at the closed real group orbit. This material was developed in [**133**].

There is least one closed G_0–orbit on Z, for minimal dimensional orbits are closed. Consider the case where $G = SL(2;\mathbb{C}), G_0 = SU(1,1)$, and X is the Riemann sphere. G acts as usual by linear fractional transformations. Let $D =$

$G_0(0)$, the unit disk. There are three G_0–orbits:

$$G_0(0),\ \text{interior of } D:\ Q_0 = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix}\right\} \text{ and } H_0 = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} \,\middle|\, \theta \text{ real}\right\},$$

$$G_0(\infty),\ \text{exterior of } D:\ Q_\infty = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\} \text{ and } H_0 = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} \,\middle|\, \theta \text{ real}\right\},$$

$$G_0(1),\ \text{boundary of } D:\ Q_1 = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \,\middle|\, ad-bc=1, a+b=c+d\right\} \text{ and}$$

$$H_0 = \left\{\pm\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t)\end{pmatrix} \,\middle|\, t \text{ real}\right\}.$$

The first two give the open orbits, with H_0 compact, and the third gives the closed orbit, where H_0 is the T_0A_0 of an Iwasawa decomposition of G_0. That mirrors the general case for closed orbits:

THEOREM 3.1. *Let $X = G/Q$ be a complex flag manifold and let G_0 be a real form of G. Then there is a unique closed orbit $G_0(z) \subset Z$. Further, there is an Iwasawa decomposition $G_0 = K_0A_0N_0$ such that $G_0 \cap Q_z$ contains H_0N_0 whenever H_0 is a Cartan subgroup of G_0 that contains A_0.*

THEOREM 3.2. *Let $Z = G/Q$ be a complex flag manifold and let G_0 be a real form of G. Let $G_0(z)$ be the unique closed G_0–orbit on Z. Then $\dim_{\mathbb{R}} G_0(z) \geqq \dim_{\mathbb{C}} Z$, and the following conditions are equivalent.*

1. $\dim_{\mathbb{R}} G_0(z) = \dim_{\mathbb{C}} Z$.

2. $\tau\Phi^n = \Phi^n$.

3. *View G as the group of complex points, and G_0 as an open subgroup in the group of real points, of a linear algebraic group defined over $\mathbb{R}$. Then Q_z is the group of complex points in an algebraic subgroup defined over $\mathbb{R}$.*

4. *Z is the set of complex points in a projective variety defined over $\mathbb{R}$, and $G_0(z)$ is the set of real points.*

4. Open Orbits.

At the other extreme, in general we have several open real group orbits. The material here is from [**133**], except that the material on the exhaustion function is from [**116**].

Fix a Cartan involution θ of $\mathfrak{g}_0$ and G_0. θ is an automorphism of square 1 and the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0. Thus $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of K_0 and is the $(+1)$–eigenspace of θ on $\mathfrak{g}_0$, and $\mathfrak{s}_0$ is the (-1)–eigenspace. The Killing form of $\mathfrak{g}_0$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{s}_0$, and $\mathfrak{k}_0 \perp \mathfrak{s}_0$ under the Killing form.

Every Cartan subalgebra of $\mathfrak{g}_0$ is $\mathrm{Ad}(G_0)$–conjugate to a θ–stable Cartan subalgebra. A θ–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is called *fundamental* if it maximizes $\dim(\mathfrak{h}_0 \cap \mathfrak{k}_0)$, *compact* if it is contained in $\mathfrak{k}_0$ (a more stringent condition). More generally, a Cartan subalgebra of $\mathfrak{g}_0$ is called *fundamental* if it is conjugate to a θ–stable fundamental Cartan subalgebra, *compact* if it is conjugate to a θ–stable compact Cartan subalgebra.

THEOREM 4.1. *Let $Z = G/Q$ be a complex flag manifold, G a connected reductive complex Lie group, and let G_0 be a real form of G. The orbit $G_0(z)$ is open in Z if and only if $\mathfrak{q}_z = \mathfrak{q}_\Phi$ where*

(i) *$\mathfrak{q}_z \cap \mathfrak{g}_0$ contains a fundamental Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and*
(ii) *Φ is a set of simple roots for a positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ such that $\tau\Sigma^+ = \Sigma^-$.*

Fix $\mathfrak{h}_0 = \theta\mathfrak{h}_0$, $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ and Φ as above. Let $W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0}$ and $W(\mathfrak{q}_\Phi^r, \mathfrak{h})^{\mathfrak{h}_0}$ denote the respective subgroups of Weyl groups that stabilize $\mathfrak{h}_0$. Then the topological components of the open G_0-orbits on Z are parameterized by the double coset space $W(\mathfrak{k}, \mathfrak{h} \cap \mathfrak{k})\backslash W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0}/W(\mathfrak{q}_z, \mathfrak{h})^{\mathfrak{h}_0}$.

COROLLARY 4.2. *Suppose that G_0 has a compact Cartan subgroup, i.e. that $\mathfrak{k}_0$ contains a Cartan subalgebra of $\mathfrak{g}_0$. Then an orbit $G_0(z)$ is open in Z if and only if $\mathfrak{g}_0 \cap \mathfrak{q}_z$ contains a compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and then, in the notation of* Theorem 4.1, *the topological components of the open G_0-orbits on Z are parameterized by $W(\mathfrak{k}, \mathfrak{h})\backslash W(\mathfrak{g}, \mathfrak{h})/W(\mathfrak{q}_z^r, \mathfrak{h})$.*

A careful examination of the way $\mathfrak{k}_0$ sits in both $\mathfrak{k}$ and $\mathfrak{g}_0$, relative to the integrability condition [**45**] for homogeneous complex structures, gives us

THEOREM 4.3. *Let $Z = G/Q$ be a complex flag manifold, G a connected reductive complex Lie group, and let G_0 be a real form of G. Let $z \in Z$ such that $G_0(z)$ is open in Z, and let $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{q}_z$ be a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}_0$. Then $K_0(z)$ is a compact complex submanifold of $G_0(z)$. Let K be the complexification of K_0, analytic subgroup of G with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$. If G_0 (and thus K_0) is connected, then $K_0(z) = K(z) \cong K/(K \cap Q_z)$, complex flag manifold of K.*

If G_0 is not connected then of course one has essentially the same result, but one must be careful about topological components.

COROLLARY 4.4. *The compact subvariety $K_0(z)$ is a deformation retract of $G_0(z)$. In particular, $G_0(z)$ is simply connected and has connected isotropy subgroup $(Q_z \cap \tau Q_z)_0$ at z.*

Fix a complex flag manifold $Z = G/Q$. An open orbit $G_0(z) \subset Z$ is called *measurable* if it carries a G_0–invariant volume element. If that is the case, then the invariant volume element is the volume element of a G_0–invariant, possibly indefinite, Kähler metric on the orbit, and the isotropy subgroup $G_0 \cap Q_z$ is the centralizer in G_0 of a (compact) torus subgroup of G_0. In more detail, measurable open orbits are characterized by

PROPOSITION 4.5. *Let $D = G_0(z)$ be an open G_0-orbit on the complex flag manifold $Z = G/Q$. Then the following conditions are equivalent.*

1. *The orbit $G_0(z)$ is measurable.*

2. *$G_0 \cap Q_z$ is the G_0-centralizer of a (compact) torus subgroup of G_0.*

3. *D has a G_0-invariant possibly-indefinite Kähler metric, thus a G_0-invariant measure obtained from the volume form of that metric.*

4. *$\tau\Phi^r = \Phi^r$, and $\tau\Phi^n = -\Phi^n$ where $\mathfrak{q}_z = \mathfrak{q}_\Phi$.*

5. $\mathfrak{q}_z \cap \tau\mathfrak{q}_z$ *is reductive, i.e.* $\mathfrak{q}_z \cap \tau\mathfrak{q}_z = \mathfrak{q}_z^r \cap \tau\mathfrak{q}_z^r$.

6. $\mathfrak{q}_z \cap \tau\mathfrak{q}_z = \mathfrak{q}_z^r$.

7. $\tau\mathfrak{q}$ *is* Ad *(G)–conjugate to the parabolic subalgebra* $\mathfrak{q}^- = \mathfrak{q}^r + \mathfrak{q}^n$ *opposite to* $\mathfrak{q}$.

In particular, if one open G_0*–orbit on* Z *is measurable, then they all are measurable.*

Note that condition 4 of Proposition 4.5 is automatic if the Cartan subalgebra $\mathfrak{h}_0$, relative to which $\mathfrak{q}_z = \mathfrak{q}_\Phi$, is the Lie algebra of a compact Cartan subgroup of G_0 , for in that case $\tau\alpha = -\alpha$ for every $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{h})$. In particular, if G_0 has discrete series representations, so that by a result of Harish–Chandra it has a compact Cartan subgroup, then every open G_0–orbit on Z is measurable.

Condition 4 is also automatic if P is a Borel subgroup of G, and more generally Condition 7 provides a quick test for measurability.

Bounded symmetric domains $D \subset \mathbb{C}^n$ are convex, and thus Stein, so cohomologies $H^k(D; \mathcal{F}) = 0$ for $k > 0$ whenever $\mathcal{F} \to D$ is a coherent analytic sheaf. This is a key point in dealing with holomorphic discrete series representations. More generally, for general discrete series representations and their analytic continuations, one has

THEOREM 4.6. *Let* $Z = G/P$ *be a complex flag manifold,* G *a connected reductive complex Lie group, and let* G_0 *be a real form of* G*. Let* $D = G_0(z) \subset Z = G/P$ *be a measurable open orbit. Let* $Y = K_0(z)$*, maximal compact subvariety of* D*, and let* $s = \dim_{\mathbb{C}} Y$*. Then* D *is* $(s+1)$*–complete in the sense of Andreotti–Grauert* [**2**]*. In particular, if* $\mathcal{F} \to D$ *is a coherent analytic sheaf then* $H^k(D; \mathcal{F}) = 0$ *for* $k > s$*.*

Indication of Proof. Let $\mathbb{K}_Z \to Z$ and $\mathbb{K}_D = K_Z|_D \to D$ denote the canonical line bundles. Their dual bundles $\mathbb{L}_Z = \mathbb{K}_Z^* \to Z$ and $\mathbb{L}_D = \mathbb{K}_D^* \to D$ are the homogeneous holomorphic line bundles over Z associated to the character

$$e^\lambda : Q_z \to \mathbb{C} \text{ defined by } e^\lambda(q) = \text{ trace}\,\mathrm{Ad}(q)|_{\mathfrak{q}_z^n}\,. \tag{4.7}$$

Write $D = G_0/V_0$ where V_0 is the real form $G_0 \cap Q_z$ of Q_z^r. Write V for the complexification Q_z^r of V_0 , $\rho_{G/V}$ for half the sum of the roots that occur in $\mathfrak{q}_z^n$, and $\lambda = 2\rho_{G/V}$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{h})$ then (i) $\langle \alpha, \lambda \rangle = 0$ and $\alpha \in \Phi^r$, or (ii) $\langle \alpha, \lambda \rangle > 0$ and $\alpha \in \Phi^n$, or (iii) $\langle \alpha, \lambda \rangle < 0$ and $\alpha \in \Phi^{-n}$. Now $\tau\lambda = -\lambda$. Decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ under the Cartan involution with fixed point set $\mathfrak{k}_0$, thus decomposing the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{q}_z$ as $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ with $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{s}_0$. Then $\lambda(\mathfrak{a}_0) = 0$.

View $D = G_0/V_0$ and $Z = G_u/V_u$ where G_u is the analytic subgroup of G for the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$, and where V_u is its analytic subgroup for $\mathfrak{q}_u \cap \mathfrak{g}_z$, compact real form of Q_z^r. Then e^λ is a unitary character both on V_0 and on V_u , so

$$\begin{aligned} &\mathbb{L}_Z \to Z = G_u/V_0 \text{ has a } G_u\text{–invariant hermitian metric } h_u\,, \\ &\mathbb{L}_D \to D = G_0/V_0 \text{ has a } G_0\text{–invariant hermitian metric } h_0\,. \end{aligned} \tag{4.8}$$

With this information we obtain

LEMMA 4.9. *The hermitian form* $\sqrt{-1}\,\partial\bar{\partial}h_u$ *on the holomorphic tangent bundle of* Z *is negative definite. The hermitian form* $\sqrt{-1}\,\partial\bar{\partial}h_0$ *on the holomorphic tangent bundle of* D *has signature* $n - 2s$ *where* $n = \dim_{\mathbb{C}} D$*.*

COROLLARY 4.10. *Define $\phi : D \to \mathbb{R}$ by $\phi = \log(h_0/h_u)$. Then the Levi form $\mathcal{L}(\phi)$ has at least $n - s$ positive eigenvalues at every point of D.*

The next point is to show that ϕ is an exhaustion function for D, in other words that $\{z \in D \mid \phi(z) \leqq c\}$ is compact for every $c \in \mathbb{R}$. It suffices to show that $e^{-\phi}$ has a continuous extension from D to the compact manifold Z that vanishes on the topological boundary $\mathrm{bd}(D)$ of D in Z. For that, choose a G_u–invariant metric h_u^* on $\mathbb{L}_Z^* = \mathbb{K}_Z$ normalized by $h_u h_u^* = 1$ on Z, and a G_0–invariant metric h_0^* on $\mathbb{L}_D^* = \mathbb{K}_D$ normalized by $h_0 h_0^* = 1$ on D. Then $e^{-\phi} = h_0^*/h_u^*$. So it suffices to show that h_0^*/h_u^* has a continuous extension from D to Z that vanishes on $\mathrm{bd}(D)$.

The holomorphic cotangent bundle $\mathbb{T}_Z^* \to Z$ has fiber $\mathrm{Ad}(g)(\mathfrak{q}_z^n)^* = \mathrm{Ad}(g)(\mathfrak{q}_z^{-n})$ at $g(z)$, so its G_u–invariant hermitian metric is given on the fiber $\mathrm{Ad}(g)(\mathfrak{q}_z^{-n})$ at $g(z)$ by $F_u(\xi, \eta) = -\langle \xi, \tau\theta\eta \rangle$ where $\langle , \rangle$ is the Killing form. Similarly the G_0–invariant indefinite–hermitian metric on $\mathbb{T}_D^* \to D$ is given on the fiber $\mathrm{Ad}(g)(\mathfrak{q}_z^{-n})$ at $g(z)$ by $F_0(\xi, \eta) = -\langle \xi, \tau\eta \rangle$. But $\mathbb{K}_Z = \det \mathbb{T}_Z^*$ and $\mathbb{K}_D = \det \mathbb{T}_D^*$, so

$$h_0^*/h_u^* = c \cdot (\text{determinant of } F_0 \text{ with respect to } F_u) \tag{4.11}$$

for some nonzero real constant c. This extends from D to a C^∞ function on Z given by

$$f(g(z)) = c \cdot \det F_0|_{\mathrm{Ad}(g)(\mathfrak{q}_z^{-n})} . \tag{4.12}$$

It remains only to show that the function f of (4.12) vanishes on $\mathrm{bd}(D)$. If $g(z) \in \mathrm{bd}(D)$ then $G_0(g(z))$ is not open in Z, so
$$\mathrm{Ad}(g)(\mathfrak{q}_z) + \tau\mathrm{Ad}(g)(\mathfrak{q}_z) \neq \mathfrak{g}.$$
Thus $\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)(\mathfrak{q}_z^{-n})$ while there exists an $\alpha \in \Sigma(\mathfrak{g}, \mathrm{Ad}(g)\mathfrak{h})$ such that
$$\mathfrak{g}_{-\alpha} \not\subset \mathrm{Ad}(g)(\mathfrak{q}_z) + \tau\mathrm{Ad}(g)(\mathfrak{q}_z)\,.$$
If $\beta \in \Sigma(\mathfrak{g}, \mathrm{Ad}(g)\mathfrak{h})$ with $\mathfrak{g}_\beta \subset \mathrm{Ad}(g)(\mathfrak{q}_z^{-n})$ then $F_0(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$, so $f(g(z)) = 0$. Thus ϕ is an exhaustion function for D in Z. In view of Corollary 4.10 now D is $(s+1)$–complete, and the Theorem is proved.

In the non–measurable case, $(s+1)$–completeness is not completely settled.

5. Cycle Spaces

We now look at the linear cycle space associated to an open real group orbit. It is contained in a component of the Barlet cycle space ([**14**], [**15**], and also see [**30**]) for that open orbit. The general material here was developed in [**133**] and [**139**], and the explicit information comes from [**149**], [**88**], [**89**] and [**90**].

As above, $Z = G/Q$ is a complex flag manifold, G_0 is a real form of G, and $D = G_0(z) \subset Z$ is an open orbit. Theorem 4.3 says that
$$Y = K_0(z) \cong K_0/(K_0 \cap P_z) \cong K/(K \cap P_z)$$
is a complex submanifold of D. Y is not contained in any compact complex submanifold of D of greater dimension, so it is a maximal compact subvariety of D.

Let $L = \{g \in G \mid gY = Y\}$. Then L is a closed complex subgroup of G, so $M_Z = \{gY \mid g \in G\} \cong G/L$ has a natural structure of G–homogeneous complex manifold. Since Y is compact and D is open in Z, $\widetilde{M_D} = \{gY \mid g \in G \text{ and } gY \subset D\}$

is open in M_Z, and thus has a natural structure of complex manifold. The *linear cycle space* of D is

(5.1) $\quad M_D$: topological component of Y in $\widetilde{M_D} = \{gY \mid g \in G \text{ and } gY \subset D\}$.

Thus M_D has a natural structure of complex manifold. Its structure is given by

THEOREM 5.2. *If D is measurable then M_D is a Stein manifold.*

This is proved by taking the exhaustion function ϕ of Corollary 4.10 and "pushing it down" from D to M_D. In general, G, Q, Z, D, K and Y break up as direct products according to any decomposition of $\mathfrak{g}_0$ as a direct sum of ideals, equivalently any decomposition of G_0 as a local direct product. Here we may assume that G is connected and simply connected. So, for purposes of determining L we may, and do, assume that G_0 is noncompact and simple, in other words that G_0/K_0 is an irreducible riemannian symmetric space of noncompact type.

As usual we say that G_0 is of *hermitian type* if the irreducible riemannian symmetric space G_0/K_0 is an hermitian symmetric space.

Let θ be the Cartan involution of G_0 with fixed point set K_0 and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ under θ, as usual. By irreducibility of G_0/K_0, the adjoint action of K_0 on $\mathfrak{s}_0 = \mathfrak{g}_0 \cap \mathfrak{s}$ is irreducible. G_0 is of hermitian type if and only if this action fails to be absolutely irreducible. Then there is a positive root system $'\Sigma^+ = {}'\Sigma^+(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$ where $\mathfrak{s}_+$ is a sum of $'\Sigma^+$–positive root spaces and represents the holomorphic tangent space of G_0/K_0, and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ is a sum of $'\Sigma^+$–negative root spaces and represents the antiholomorphic tangent space. Write $S_\pm = \exp(\mathfrak{s}_\pm) \subset G$. Then G_0/K_0 is an open G_0–orbit on G/KS_-.

Now we have three cases, based on the possibilities for $L = \{g \in G \mid gY = Y\}$:

1. (Trivial Case) $K = G$, i.e. $D = Z$, so M_Z (and thus M_D) is reduced to a single point. These cases have been classified [**142**].

2. (Hermitian Case) $\mathfrak{l} = \mathfrak{k} + \mathfrak{s}_\pm$, i.e., $\mathfrak{k} \subsetneq \mathfrak{l} \subsetneq \mathfrak{g}$, so $M_Z = G/KS_-$ is a projective variety and $B = G_0/K_0$ is a bounded symmetric domain. This is the situation where the two maps of the double fibration

$$\begin{array}{ccc} & G_0/(K_0 \cap Q_z) & \\ \swarrow & & \searrow \\ D & & B \end{array}$$

are simultaneously holomorphic for some choice between B and $\overline{B}$ and some choice of invariant complex structure on $G_0/(L_0 \cap K_0)$. Then $M_D = B$.

3. (Generic Case) $\mathfrak{l} = \mathfrak{k}$, so K is the identity component of L and M_Z is a nontrivial affine variety. Here a number of cases are known:

- $D \cong SU(2p, q)/(SO(2p) \times U(q))$. Then M_D has an explicit description, and that description shows that it is a Stein manifold [**127**].
- $B = G_0/K_0$ is a bounded symmetric domain and G_0 is a classical group. Then $M_D = B \times \overline{B}$ [**149**]. Some special cases had been worked out in [**103**], in [**34**], in [**97**] and in [**98**].

- $B = G_0/K_0$ is a bounded symmetric domain and D is a pseudo–riemannian symmetric space. Then $M_D = B \times \overline{B}$ [**150**].
- $G_0 = SL(n;\mathbb{R})$, real special linear group. Then M_D has an explicit description, and that description shows that it is a Stein manifold [**88**].
- $G_0 = SL(n;\mathbb{H})$, quaternion special linear group. Then M_D has an explicit description, and that description shows that it is a Stein manifold [**90**].

The proof [**139**] of Theorem 5.2 is rather easy in Cases 1 and 3 above, but in Case 2 it involves detailed information [**146**] on the G_0–orbit structure of G/KS_-.

The case [**127**] where $D \cong SU(2p,q)/(SO(2p) \times U(q))$ was the first nontrivial case worked out. It was done in view of then–recent work on moduli spaces for compact Kähler manifolds in dimension > 1 ([**52**] and [**53**]). In [**128**] it was first shown that M_D is Stein in some degree of generality, but the emphasis was on automorphic cohomology in connection with [**52**] and [**53**]. This theory of automorphic cohomology has been extended in [**137**], [**125**], [**129**], [**130**], and [**131**].

The proof of Theorem 5.2 does not quite work in the non–measurable case, but there we have explicit descriptions (see just above), which in particular prove Stein, except when G_0 is a complex simple group or is of type E_6 with maximal compact subgroup of type F_4 or C_4. Theorem 5.2 would be shown in general, and probably with less effort, if we could show that M_D is closed in the Barlet cycle space of D. There is some progress in that direction ([**16**], [**143**]).

For other approaches, viewing M_D as a Stein neighborhood of G_0/K_0 in G/K, see [**1**], [**11**], [**24**], [**25**], [**45**], [**43**], [**44**], [**47**], [**87**], [**101**] and [**147**]. Also see [**28**], [**29**] and [**92**] for an approach based on Grauert tubes and the Monge–Ampère equations.

6. The Double Fibration Transform

Double fibrations are an old topic, starting with the Crofton formulae and Chern's intersection theory of two homogeneous spaces of the same group, the classical Radon and X–ray transforms, the Gelfand–Graev horocycle transforms [**48**] and Helgason's group–theoretic reformulation of the horocycle transform. Real–analytic double fibration transforms come up in [**12**], [**117**] and [**151**], with the Identity Theorem [**112**] as a degenerate early case, in the study of cohomology representations. Holomorphic double fibration transforms were first used in the study of automorphic cohomology [**128**]. Penrose' twistor theory (cf. [**3**], [**104**], [**105**], [**106**], [**107**] and [**86**]) took advantage of those transforms, and the special case $G_0 = SU(2,2)$ was reworked, with physical applications, in [**41**], as the *Penrose Transform.* The material here is taken from [**150**].

In general let $D = G_0(z)$ be an open orbit in the complex flag manifold $Z = G/P$, let Y be the maximal compact linear subvariety $K_0(z)$, and consider the linear cycle space M_D: component of Y in $\{gY \mid g \in G \text{ and } gY \subset D\}$. Then we have a double fibration

$$
\begin{tikzcd}
 & W_D \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
D & & M_D
\end{tikzcd}
\tag{6.1}
$$

where $W_D = \{(Y', y') \mid y' \in Y' \in M_D\}$ is the incidence space. The projections are given by $\mu(Y', y') = y'$ and $\nu(Y', y') = Y'$.

Let $n = \dim_{\mathbb{C}} D$ and $s = \dim_{\mathbb{C}} Y$ as before. Consider a *negative* homogeneous holomorphic vector bundle $\mathbb{E} \to D$. Then we can expect nonzero cohomology only in degree s. For many purposes, for example for making estimates of one sort or another, it is better to have representations of G_0 occur on spaces of functions rather than on cohomology spaces. Here we indicate a double fibration transform that carries $H^s(D; \mathcal{O}(\mathbb{E}))$ to a space of functions on M_D. Given a coherent analytic sheaf $\mathcal{E} \to D$ we construct a coherent sheaf $\mathcal{E}' \to M$ and a transform

$$P : H^s(D; \mathcal{E}) \to H^0(M; \mathcal{E}'). \tag{6.2}$$

We refer to the transform (6.2) as a *double fibration transform.* The Penrose transform is the case where $G_0 = SU(2,2)$ and Z is complex projective space $P^3(\mathbb{C})$.

One wants two things in (6.2): that P be injective, and that there be an explicit description of its image. Assuming (6.5) below, injectivity of P is equivalent to injectivity of $j^{(p)}$ in (6.4) below. There are several ways to ensure this. The most general is the collection of vanishing conditions in Theorem 6.6 below. Another, more specific to our situation, is that in many cases we know that the fibers of μ are Stein manifolds. Finally, in some cases one knows that $H^p(D; \mathcal{E})$ is an irreducible representation space for a group under which all our constructions are equivariant, so P is an intertwining operator, thus zero or injective.

The double fibration transform is constructed in several steps.

We first pull cohomology back from D to W_D in (6.1). Let $\mu^{-1}(\mathcal{E}) \to W_D$ denote the inverse image sheaf. As μ is open, it is the sheaf defined by the presheaf whose value at an open set $\widetilde{U} \subset W_D$ is $\Gamma(U, \mathcal{E})$ where $U = \mu(\widetilde{U})$. Here, as usual, Γ denotes the space of sections. For every integer $r \geqq 0$ there is a natural map $\mu^{(r)} : H^r(D; \mathcal{E}) \to H^r(W_D; \mu^{-1}(\mathcal{E}))$ given on the Cech cocycle level by $\mu^{(r)}(c)(\sigma) = c(\mu(\sigma))$ where $c \in Z^r(D; \mathcal{E})$ and where $\sigma = (w_0, \ldots, w_r)$ is a simplex.

PROPOSITION 6.3. [**27**] *Suppose that the fiber F of $\mu : W_D \to D$ is connected and that $H^r(F; \mathbb{C}) = 0$ for $1 \leqq r \leqq p-1$. Then the map $\mu^{(r)}$ is an isomorphism for $r \leqq p-1$ and is injective for $r = p$. In particular, if the fibers of μ are contactable then $\mu^{(r)}$ is an isomorphism for all r.*

To complete the pull–back, we change the inverse image sheaf $\mu^{-1}(\mathcal{E}) \to W_D$ into a coherent analytic sheaf over W_D. This is necessary for the push–down step.

As usual, if X is a complex manifold then $\mathcal{O}_X \to X$ denotes its structure sheaf, the sheaf of germs of holomorphic $\mathbb{C}$–valued functions on X. If $\mathbb{E} \to X$ is a holomorphic vector bundle then $\mathcal{O}(\mathbb{E}) \to X$ is its sheaf of germs of holomorphic sections.

Denote $\mu^*(\mathcal{E}) = \mu^{-1}(\mathcal{E}) \otimes_{\mu^{-1}(\mathcal{O}_D)} \mathcal{O}_{W_D}$. It is a sheaf of $\mathcal{O}_{W_D}$–modules. If it happens that $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$, then $\mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E}))$, where $\mu^*(\mathbb{E})$ is the pull–back bundle. In any case, $[\sigma] \mapsto [\sigma] \otimes 1$ defines a map $i : \mu^{-1}(\mathcal{E}) \to \mu^*(\mathcal{E})$ which in turn specifies the coefficient morphisms $i_p : H^p(W_D; \mu^{-1}(\mathcal{E})) \to H^p(W_D; \mu^*(\mathcal{E}))$ for $p \geqq 0$. Our natural pull–back maps are the compositions

$$j^{(p)} : H^p(D; \mathcal{E}) \to H^p(W_D; \mu^*(\mathcal{E})) \quad \text{for } p \geqq 0, \tag{6.4}$$

that is, $j^{(p)} = i_p \cdot \mu^{(p)}$.

Consider the case $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$. Then $\mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E}))$, we realize these sheaf cohomologies as Dolbeault cohomologies, and the pull–back maps are given by pulling back $[\omega] \mapsto [\mu^*(\omega)]$ on the level of differential forms.

Proposition 6.3 is not quite enough to show that the maps $j^{(p)}$ of (6.4) are isomorphism. That involves some study of relative Dolbeault cohomology and the vanishing hypothesis of Theorem 6.6 below.

The second step is to push cohomology down from W_D to M_D. This requires

$$\nu : W_D \to M \text{ is a proper map and } M \text{ is a Stein manifold.} \tag{6.5}$$

THEOREM 6.6. *Suppose that the fiber F of $\mu : W_D \to D$ is connected and, for some fixed integer $s \geqq 0$, that $H^r(F; \mathbb{C}) = 0$ for $1 \leqq r < s$. Assume* (6.5) *that $\nu : W_D \to M$ is a proper map and that M_D is a Stein manifold. Suppose further that $H^p(\nu^{-1}(Y'); \Omega^q_\mu(\mathbb{E})|_{\nu^{-1}(Y')}) = 0$ for all $Y' \in M_D$, all $p < s$, and $1 \leqq q \leqq m$. Then $P : H^s(D; \mathcal{E}) \to H^0(M_D; \mathcal{R}^s(\mu^*(\mathcal{E})))$ is injective.*

The argument is a little bit technical. Write $\mathcal{R}^p$ gives the pth Leray derived sheaf in the The Leray spectral sequence for $\mu : W_D \to M_D$. Write Ω^q_μ for relative holomorphic q–forms on W_D. The assumption on F ensures, that $\mu^{(s)} : H^s(D; \mathcal{E}) \to H^s(W_D; \mu^{-1}(\mathcal{E}))$ is injective. The Leray spectral sequence for $\mu : W_D \to M_D$ and $\Omega^q_\mu(\mathcal{E}) \to W_D$, and the Stein condition on M_D, give $H^p(W_D; \Omega^q_\mu(\mathcal{E})) \cong H^0(M_D; \mathcal{R}^p(\Omega^q_\mu(\mathcal{E})))$. The vanishing assumption for certain $H^p(\nu^{-1}(Y'); \Omega^q_\mu(\mathcal{E})|_{\nu^{-1}(Y')})$ says

$$\mathcal{R}^p(\Omega^q_\mu(\mathcal{E})) = 0 \text{ for } p < s \text{ and } 1 \leqq q \leqq m.$$

So

$$H^p(W_D; \Omega^q_\mu(\mathcal{E})) = 0 \text{ for } p < s \text{ and } 1 \leqq q \leqq m,$$

and

$$H^s(W_D; \Omega^q_\mu(\mathcal{E})) \cong H^0(M_D; \mathcal{R}^s(\Omega^q_\mu(\mathcal{E}))) \text{ for } 1 \leqq q \leqq m. \quad .$$

Now $i_s : H^s(W_D; \mu^{-1}\mathcal{E}) \to H^s(W_D; \mu^*\mathcal{E})$ is injective, and Proposition 6.3 shows that $j^{(s)} : H^s(D; \mathcal{E}) \to H^s(W_D; \mu^*(\mathcal{E}))$ is injective. We conclude that the double fibration transform $P : H^s(D; \mathcal{E}) \to H^0(M_D; \mathcal{R}^s(\mu^*(\mathcal{E})))$ is injective.

In the cases of interest to us, $\mathbb{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$, and P has an explicit formula. Let ω be an $\mathbb{E}$–valued $(0, s)$–form on D representing a Dolbeault cohomology class $[\omega] \in H^s_{\bar\partial}(D, \mathbb{E})$. Note $\mathcal{R}^s(\mu^*(\mathcal{E})) = \mathcal{O}(\mathbb{H}^s(\mu^*(\mathbb{E})|_{\mu^{-1}(Y)})$ where the latter bundle has fiber $H^s(Y'; \mu^*(\mathbb{E})|_{\mu^{-1}(Y')})$ over

$Y' \in M_D$. Thus

$$P([\omega])(Y') \text{ is the section of } \mathcal{R}^s(\mu^*(\mathcal{E})) \to M_D \text{ whose value at } Y' \in M_D \text{ is } \mu^*(\omega)|_{\nu^{-1}(Y')}. \tag{6.7}$$

In other words,

$$P([\omega])(Y') = [\mu^*(\omega)|_{\nu^{-1}(Y')}] \in H^0_{\overline{\partial}}(M_D; \mathbb{H}^s(\mu^*(\mathbb{E})|_{\nu^{-1}(Y)})). \tag{6.8}$$

This is most conveniently interpreted by viewing $P([\omega])(Y')$ as the Dolbeault class of $\omega|_{Y'}$, and by viewing $Y' \mapsto [\omega|_{Y'}]$ as a holomorphic section of the holomorphic vector bundle over M_D whose fiber at Y' is $H^s(Y'; \mu^*(\mathbb{E})|_{\nu^{-1}(Y')})$.

In order to make the double fibration transform explicit one needs to know

- the exact structure of M_D and
- the differential equations that pick out image of P

The structure of M_D was addressed at the end of the previous section.

There has been a lot of work on the image of double fibration transforms. Almost all of this has been on the image of the Radon transform [**108**], the X–ray transform (see [**5**] and [**38**]), the Funk transform [**46**], and the Penrose transform ([**104**], [**105**], [**106**], [**107**], [**37**], [**39**],). The Funk and Radon transforms are essentially the same, and they can be viewed as Penrose transforms [**6**]. For the image of the Penrose transform in twistor theory see, for example, [**3**], [**4**], [**7**], [**17**], [**35**], [**36**], [**41**], [**86**], [**126**], and some of the references in [**49**] and in [**37**]. See [**73**], [**74**], [**75**], [**76**], [**77**], [**78**], [**79**], [**80**] and [**81**] for connections between Radon transforms and analysis on Riemannian symmetric spaces of noncompact type, with consequences for principal series representations. There also is a $\mathcal{D}$–module approach ([**31**]and [**32**]) which, I expect, will soon have consequences in semisimple representation theory through connections with [**72**]. See [**112**], [**128**], [**109**] and [**151**] for the image of the first double fibration transforms related to semisimple representation theory (rather than particular examples). See [**37**] and [**40**] for a more geometric viewpoint, and, finally, see [**118**], [**99**] (and eventually, [**100**]) for the image of the double fibration transform in our setting.

Part II. Representations of Reductive Lie Groups.

In this Part we describe those unitary representations of the reductive Lie group that come into the Plancherel formula, and we indicate the Plancherel Plancherel formula based on their characters. For the most part, these are the representations that have the cleanest geometric realization in real group orbits on complex flag manifolds. See [**69**], [**70**] and [**71**] for Harish–Chandra's treatment of the Plancherel formula, [**135**] and then [**84**] and [**85**] for another approach.

7. The Principal Series.

The "principal series" of unitary representations of a semisimple or reductive Lie group G_0 was the first series to be constructed in some degree of generality. We consider it separately from the other tempered series in order to demonstrate the general construction without the technical considerations of the discrete series.

A subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ is a (*real*) *parabolic subalgebra* of $\mathfrak{g}_0$ if it is a real form of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. A subgroup $P_0 \subset G_0$ is a *parabolic subgroup* of G_0 if it is the G_0–normalizer of a parabolic subalgebra, say $P_0 = \{g \in G_0 \mid \mathrm{Ad}(g)\mathfrak{p}_0 = \mathfrak{p}_0\}$ where $\mathfrak{p}_0$ is a parabolic subalgebra of $\mathfrak{g}_0$. In that case $\mathfrak{p}_0$ is the Lie algebra of P_0.

For example, let $G_0 = K_0A_0N_0$ is an Iwasawa decomposition. Let M_0 be the centralizer of A_0 in K_0. Then $M_0A_0N_0$ is a minimal parabolic subgroup of G_0. Any two minimal parabolic subgroups of G_0 are conjugate. Now fix a minimal parabolic subgroup $P_0 = M_0A_0N_0$.

Whenever E is a topological group we write $\widehat{E}$ for its unitary dual. Thus $\widehat{E}$ consists of the unitary equivalence classes $[\eta]$ of (strongly continuous) topologically irreducible unitary representations η of E. Now $[\eta] \in \widehat{M_0}$ and $\sigma \in \mathfrak{a}_0^*$ determine $[\alpha_{\eta,\sigma}] \in \widehat{P_0}$ by

$$\alpha_{\eta,\sigma}(man) = \eta(m)e^{i\sigma(\log a)}. \tag{7.1}$$

The corresponding *principal series representation* of G_0 is

$$\pi_{\eta,\sigma} = \mathrm{Ind}_{P_0}^{G_0}(\alpha_{\eta,\sigma}), \quad \text{unitarily induced representation.} \tag{7.2}$$

The *principal series* of G_0 consists of the unitary equivalence classes of these representations. A famous result of Bruhat says that if σ satisfies a certain nonsingularity condition then $\pi_{\eta,\sigma}$ is irreducible.

8. The Discrete Series.

The representations of the "discrete series" are the basic building blocks for the representations involved in the Plancherel formula. We recall the definition and Harish–Chandra parameterization of the discrete series for reductive Lie groups. For application we have to make G_0 more general so that the description applies to certain subgroups of G_0 as well as to G_0. Later we will show how discrete series representations can be realized over certain open orbits.

The *discrete series* of a unimodular locally compact group G_0 is the subset $\widehat{G_{0,d}} \subset \widehat{G_0}$ consisting of all classes $[\pi]$ for which π is equivalent to a subrepresentation of the left regular representation of G_0. These are equivalent: (i) π is a discrete series representation of G_0, (ii) every coefficient $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ belongs to $L^2(G_0)$, (iii) for some nonzero u, v in the representation space H_π, the coefficient $f_{u,v} \in L^2(G_0)$. Then one has orthogonality relations much as in the case of finite groups: there is a real number $\deg(\pi) > 0$ such that the $L^2(G_0)$-inner product of coefficients of π is given by

$$\langle f_{u,v}, f_{s,t}\rangle = \frac{1}{\deg(\pi)}\langle u, s\rangle\overline{\langle v, t\rangle} \text{ for } s, t, u, v \in H_\pi. \tag{8.1}$$

Furthermore, if π' is a discrete series representation not equivalent to π, then

$$\langle f_{u,v}, f_{u',v'}\rangle = 0 \text{ for } u, v \in H_\pi \text{ and } u', v' \in H_{\pi'}. \tag{8.2}$$

These orthogonality relations come out of convolution formulae. With the usual $f * h(x) = [L(f)h](x) = \int_G f(y)h(y^{-1}x)dy$ we have $f_{u,v} * f_{s,t} = \frac{1}{\deg(\pi)}\langle u, t\rangle f_{s,v}$ for

$s, t, u, v \in H_\pi$. Also $f_{u,v} * f_{u',v'} = 0$ for $u, v \in H_\pi$ and $u', v' \in H_{\pi'}$ whenever π and π' are inequivalent discrete series representations of G_0.

If G_0 is compact, then every class in $\widehat{G_0}$ belongs to the discrete series, and if Haar measure is normalized as usual to total volume 1 then $\deg(\pi)$ has the usual meaning, the dimension of H_π. The orthogonality relations for irreducible unitary representations of compact groups are more or less equivalent to the Peter–Weyl Theorem. More generally, if G_0 is a unimodular locally compact group then $L^2(G_0) = {}^0L^2(G_0) \oplus {}'L^2(G_0)$, orthogonal direct sum, where ${}^0L^2(G_0) = \sum_{[\pi]\in\widehat{G_{0,d}}} H_\pi \otimes H_\pi^*$, the "discrete" part, and ${}'L^2(G_0) = {}^0L^2(G_0)^\perp$, the "continuous" part. If, further, G_0 is a group of type I then ${}'L^2(G_0)$ is a continuous direct sum (direct integral) over $\widehat{G_0} \setminus \widehat{G_{0,d}}$ of the Hilbert spaces $H_\pi \otimes H_\pi^*$.

Recall our assumption that G_0 belongs to the Harish–Chandra class of reductive Lie groups (2.1): the Lie algebra $\mathfrak{g}_0$ is reductive, the component group G_0/G_0^0 is finite, the derived group $[G_0^0, G_0^0]$ is closed in G_0, and if $g \in G_0$ then $\mathrm{Ad}(g)$ is an inner automorphism of the complex Lie algebra $\mathfrak{g}$. The results we describe hold somewhat more generally, but that is not the point here.

If $[\pi] \in \widehat{G_0}$ and $f \in \mathbb{C}_c^\infty(G_0)$ then $\pi(f) = \int_G f(x)\pi(x)dx$ is a trace class operator on H_π, and the map

$$\Theta_\pi : C_c^\infty(G_0) \to \mathbb{C} \text{ defined by } \Theta_\pi(f) = \text{ trace } \pi(f) \tag{8.3}$$

is a distribution on G_0. Θ_π is called the *character*, the *distribution character* or the *global character* of π. The equivalence class $[\pi]$ determines Θ_π and, conversely, Θ_π determines $[\pi]$.

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. If we interpret $\mathcal{U}(\mathfrak{g})$ as the algebra of all left-invariant differential operators on G_0 then $\mathcal{Z}(\mathfrak{g})$ is the subalgebra of those which are also invariant under right translations. If π is irreducible then $d\pi|_{\mathcal{Z}(\mathfrak{g})}$ is an associative algebra homomorphism $\chi_\pi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ called the *infinitesimal character* of π. We say that π is *quasi-simple* if it has an infinitesimal character, i.e. if it is a direct sum of irreducible representations that have the same infinitesimal character.

Let π be quasi-simple. Then the distribution character Θ_π satisfies a system of differential equations

$$z \cdot \Theta_\pi = \chi_\pi(z)\Theta_\pi \text{ for all } z \in \mathcal{Z}(\mathfrak{g}). \tag{8.4}$$

Now let π be irreducible. A serious study of these equations shows that Θ_π is integration against a locally L^1 function T_π that is real analytic on a dense open subset G_0' of G_0,

$$\Theta_\pi(f) = \int_{G_0} f(x)T_\pi(x)dx \text{ for all } f \in C_c^\infty(G_0). \tag{8.5}$$

So we identify Θ_π with the function T_π, and it makes sense to talk about *a priori* estimates on characters and coefficients as well as explicit character formulae.

Fix a Cartan involution θ of G_0: $\theta \in \mathrm{Aut}(G_0)$, $\theta^2 = 1$, and the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0. The choice is essentially unique because any two are conjugate in $\mathrm{Aut}(G_0)$. If $G_0 = U(p, q)$ then $\theta(x) = {}^tx^{-1}$ and $K_0 = U(p) \times U(q)$.

Every Cartan subgroup of G_0 is $\mathrm{Ad}(G_0^0)$-conjugate to a θ-stable Cartan subgroup. In particular, G_0 has compact Cartan subgroups if and only if K_0 contains a Cartan subgroup of G_0.

Harish–Chandra proved that G_0 has discrete series representations if and only if it has a compact Cartan subgroup. Suppose that this is the case and fix a compact Cartan subgroup $T_0 \subset K_0$ of G_0. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{t})$ be the root system, $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{t})$ a choice of positive root system, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$. If $\xi \in \mathfrak{t}$ then $\rho(\xi)$ is half the trace of $ad(\xi)$ on $\sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$.

If π is a discrete series representation of G_0 and Θ_π is its distribution character, then the equivalence class of π is determined by the restriction of Θ_π to $T_0 \cap G_0'$. Harish–Chandra parameterized the discrete series of G_0 by parameterizing those restrictions.

Let $G_0^\dagger$ denote the finite index subgroup $T_0G_0^0 = Z_{G_0}(G_0^0)G_0^0$ of G_0. Here $T_0 = Z_{G_0}(G_0^0)T_0^0$, so $T_0 = T_0^\dagger$. Lemma 11.2 says that the group M_0 of a minimal parabolic subgroup of G_0 satisfies $M_0 = M_0^\dagger$, and similarly in that context we have $U_{\Phi,0} = U_{\Phi,0}^\dagger$. In general, where M_0 may be noncompact, this need not hold. In any case, the Weyl group $W^\dagger = W(G_0^\dagger, T_0)$ coincides with $W^0 = W(G_0^0, T_0^0)$ and is a normal subgroup of $W = W(G_0, T_0)$.

$\widehat{T_0} = (Z_{G_0}(G_0^0)T_0^0)^\frown$ consists of the $\chi \otimes e^{i(\lambda-\rho)}$ where $\lambda \in i\mathfrak{t}_0^*$ and $\lambda - \rho$ satisfies an integrality condition, where $\chi \in Z_{G_0}(G_0^0)^\frown$, and where χ and $e^{i(\lambda-\rho)}$ restrict to (multiples of) the same unitary character on the center of G_0^0.

Given $\chi \otimes e^{i(\lambda-\rho)} \in \widehat{T_0}$ as above, with λ is regular, i.e., there are unique discrete series representations π_λ^0 of G_0^0 and $\pi_{\chi,\lambda}^\dagger = \chi \otimes \pi_\lambda^0$ of $G_0^\dagger$, whose distribution characters satisfy

(8.6)

$$\Theta_{\pi_\lambda^0}(x) = (-1)^{q(\lambda)} \frac{\sum_{w\in W^0} \mathrm{sign}(w) e^{w(\lambda)}}{\prod_{\alpha\in\Sigma^+} (e^{\alpha/2} - e^{-\alpha/2})} \text{ and } \Theta_{\pi_{\chi,\lambda}^\dagger}(zx) = \mathrm{trace}\, \chi(z) \Theta_{\pi_\lambda^0}(x)$$

for $z \in Z_{G_0}(G_0^0)$ and $x \in T_0^0 \cap G_0'$, where $q(\lambda)$ is the cardinality

$$q(\lambda) = |\{\alpha \in \Sigma^+(\mathfrak{k}, \mathfrak{t}) \mid \langle\alpha, \lambda\rangle < 0\}| + |\{\beta \in \Sigma^+(\mathfrak{g}, \mathfrak{t}) \setminus \Sigma^+(\mathfrak{k}, \mathfrak{t}) \mid \langle\beta, \lambda\rangle > 0\}|.$$

The same datum (χ, λ) specifies a discrete series representation $\pi_{\chi,\lambda}$ of G_0, by the formula $\pi_{\chi,\lambda} = \mathrm{Ind}_{G_0^\dagger}^{G_0}(\pi_{\chi,\lambda}^\dagger)$. This induced representation is irreducible because its conjugates by elements of $G_0/G_0^\dagger$ are mutually inequivalent, consequence of regularity of λ. $\pi_{\chi,\lambda}$ is characterized by the fact that its distribution character is supported in $G_0^\dagger$ and is given on $G_0^\dagger$ by

$$\Theta_{\pi_{\chi,\lambda}} = \sum_{1\leqq i\leqq r} \Theta_{\pi_{\chi,\lambda}^\dagger} \cdot \gamma_i^{-1}, \tag{8.7}$$

with $\gamma_i = \mathrm{Ad}(g_i)|_{G_0^\dagger}$ where $\{g_1, \ldots, g_r\}$ is any system of coset representatives of G_0 modulo $G_0^\dagger$. To combine these into a single formula one chooses the g_i so that they normalize T_0, i.e. chooses the γ_i to be a system of coset representatives of W modulo $W^\dagger$.

Every discrete series representation of G_0 is equivalent to a representation $\pi_{\chi,\lambda}$ as just described. Discrete series representations $\pi_{\chi,\lambda}$ and $\pi_{\chi',\lambda'}$ are equivalent if and only if $\chi'\otimes e^{i\lambda'} = (\chi\otimes e^{i\lambda})\cdot w^{-1}$ for some $w \in W$. And λ is both the infinitesimal character and the Harish–Chandra parameter for the discrete series representation $\pi_{\chi,\lambda}$.

9. The Various Tempered Series.

A representation of G_0 is called *tempered* if its character is a tempered distribution in a suitable sense. This means that it is weakly contained in the left regular representation or, roughly, that it is involved in the Plancherel formula for G_0. Tempered representations are constructed from a class of real parabolic subgroups of G_0 called *cuspidal parabolic subgroups*, constructed using minimal parabolic subgroups, but using a discrete series representation on the part of the parabolic corresponding to M_0.

A (real) parabolic subgroup $P_0 \subset G_0$ is called *cuspidal* if the commutator subgroup of the Levi component (reductive part) has a compact Cartan subgroup.

Let H_0 be a Cartan subgroup of G_0. Fix a Cartan involution θ of G_0 such that $\theta(H_0) = H_0$. Write K_0 for the fixed point set G_0^θ, maximal compact subgroup of G_0. Decompose

$$\text{(9.1)} \qquad \begin{aligned} &\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ and } H_0 = T_0 \times A_0 \text{ where} \\ &\qquad T_0 = H_0 \cap K_0, \theta = -1 \text{ on } \mathfrak{a}_0, \text{ and } A_0 = \exp_G(\mathfrak{a}_0). \end{aligned}$$

Then the centralizer $Z_{G_0}(A_0)$ of A_0 in G_0 has form $M_0 \times A_0$ where $\theta(M_0) = M_0$. The group M_0 is a reductive Lie group of Harish–Chandra class. T_0 is a compact Cartan subgroup of M_0, so M_0 has discrete series representations. Now suppose that $\Sigma^+ = \Sigma^+(\mathfrak{g},\mathfrak{h})$ is defined by positive root systems $\Sigma^+(\mathfrak{m},\mathfrak{t})$ and $\Sigma^+(\mathfrak{g}_0,\mathfrak{a}_0)$ as in (11.1).

The Cartan subgroup $H_0 \subset G_0$ and the positive system $\Sigma^+ = \Sigma^+(\mathfrak{g},\mathfrak{h})$ define a cuspidal parabolic subgroup $P_0 = M_0A_0N_0$ of G_0 as follows. The Lie algebra of N_0 is $\mathfrak{n}_0 = \sum_{\alpha\in\Sigma^+(\mathfrak{g}_0,\mathfrak{a}_0)}(\mathfrak{g}_0)_{-\alpha}$, M_0 and A_0 are as above, and $M_0A_0 = M_0 \times A_0$ is the Levi component of P_0. One extreme is the case where $\dim \mathfrak{a}_0$ is maximal; then P_0 is a minimal parabolic subgroup of G_0. The other extreme is where $\dim \mathfrak{a}_0$ is minimal; if $\mathfrak{a}_0 = 0$ then $P_0 = G_0$.

Every cuspidal parabolic subgroup of G_0 is produced by the construction just described, as H_0 varies. Two cuspidal parabolic subgroups of G_0 are *associated* if they are constructed as above from G_0–conjugate Cartan subgroups; then we say that the G_0–conjugacy class of Cartan subgroups is *associated* to the G_0–association class of cuspidal parabolic subgroups.

In the same manner as the construction of the principal series representations,

$$\text{(9.2)} \quad [\eta] \in \widehat{M_0} \text{ and } \sigma \in \mathfrak{a}_0^* \text{ determine } [\alpha_{\eta,\sigma}] \in \widehat{Q_0} \text{ by } \alpha_{\eta,\sigma}(man) = \eta(m)e^{i\sigma(\log a)}.$$

Then we have

$$\text{(9.3)} \qquad \pi_{\eta,\sigma} = \operatorname{Ind}_{P_0}^{G_0}(\alpha_{\eta,\sigma}), \qquad \text{unitarily induced representation.}$$

The H_0*–series* or *principal* H_0*–series* of G_0 consists of the unitary equivalence classes of the representations (9.3) for which η is a discrete series representation of M_0.

The character formula for the $\pi_{\eta,\sigma}$ is a bit complicated, and we just refer to [**135**] and [**84**]. We mention that the formula is independent of choice of $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$ so, as the terminology indicates, $\pi_{\eta,\sigma} = \operatorname{Ind}_{P_0}^{G_0}(\alpha_{\eta,\sigma})$ is independent of choice of $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$. In fact this is the case even if η does not belong to the discrete series of M_0.

10. The Plancherel Formula.

There are two approaches to the Plancherel formula. Harish–Chandra starts with an analysis of his Schwartz space $\mathcal{C}(G_0)$ and construction of functions on G_0 as wave packets of Eisenstein integrals ([**69**], [**70**], [**71**]). He knew that it is better to step through the conjugacy classes of Cartan subgroups from maximally compact to minimally compact, using explicit character formulae [**54**]. This was first carried out in real rank 1 [**111**], and then carried out in general in [**84**] and [**85**]. That is the argument we indicate here.

We start with Kostant's "cascade construction" for the conjugacy classes of Cartan subgroups of G_0. Suppose first that G_0 has a compact Cartan subgroup T_0. Fix a Cartan involution θ of G_0 such that $\theta(T_0) = T_0$ and the corresponding ± 1 eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of the maximal compact subgroup $K_0 = \{g \in G_0 \mid \theta(g) = g\}$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{t})$ then either $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and we say that α is *compact*, or $\mathfrak{g}_\alpha \subset \mathfrak{s}$ and we say that α is *noncompact*.

Let $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{t})$ be noncompact. Let $\mathfrak{g}[\alpha] = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}(2;\mathbb{C})$, let $G[\alpha]$ denote the corresponding analytic subgroup of G, and consider the corresponding real forms $\mathfrak{g}_0[\alpha] = \mathfrak{g}_0 \cap \mathfrak{g}[\alpha] \cong \mathfrak{sl}(2;\mathbb{R})$ and $G_0[\alpha] = G_0 \cap G[\alpha]$. Then $G_0[\alpha] \cap T_0$ is a compact Cartan subgroup, and we can simply replace it by the noncompact Cartan subgroup of $G_0[\alpha]$. Let $\mathfrak{a}_0[\alpha]$ denote the Lie algebra of that noncompact Cartan subgroup. Then we have a new Cartan subalgebra and a new Cartan subgroup

$$\mathfrak{h}_0\{\alpha\} = \left(\mathfrak{t}_0 \cap (\mathfrak{g}_0[\alpha] \cap \mathfrak{t}_0)^\perp\right) + \mathfrak{g}_0[\alpha] \cap \mathfrak{a}_0 \text{ and } H_0\{\alpha\} = Z_{G_0}(\mathfrak{h}_0\{\alpha\}),$$

where Z_{G_0} denotes the centralizer in G_0. The point is that $H_0\{\alpha\}$ has one compact dimension less than that of T_0 and one noncompact dimension more.

Let $\alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{t})$ be noncompact. We can carry out the above construction for α and β independently, one after the other, if α and β are *strongly orthogonal* in the sense that neither of $\alpha \pm \beta$ are roots. (We write this relation as $\alpha \underline{\perp} \beta$. It implies $\alpha \perp \beta$.) If $\alpha \underline{\perp} \beta$ then we have the new Cartan subalgebra and a new Cartan subgroup given by $\mathfrak{h}_0\{\alpha,\beta\} = \left(\mathfrak{t}_0 \cap ((\mathfrak{g}_0[\alpha] \oplus \mathfrak{g}_0[\beta]) \cap \mathfrak{t}_0)^\perp\right) + \left(\mathfrak{a}_0[\alpha] \oplus \mathfrak{a}_0[\beta]\right)$ and $H_0\{\alpha,\beta\} = Z_{G_0}(\mathfrak{h}_0\{\alpha,\beta\})$. Here $H_0\{\alpha,\beta\}$ has two compact dimensions less than that of T_0 and two noncompact dimensions more.

We say that a set S of noncompact roots is *strongly orthogonal* if its elements are mutually strongly orthogonal. Then as above we have a Cartan subalgebra by

$$\mathfrak{h}_0\{S\} = \left(\mathfrak{t}_0 \cap \left(\left(\sum\nolimits_{\alpha \in S} \mathfrak{g}_0[\alpha]\right) \cap \mathfrak{t}_0\right)^\perp\right) + \left(\sum\nolimits_{\alpha\in S} \mathfrak{a}_0[\alpha]\right), \tag{10.1}$$

and the corresponding Cartan subgroup is $H_0\{S\} = Z_{G_0}(\mathfrak{h}_0\{S\})$. Here $H_0\{S\}$ has $|S|$ compact dimensions fewer than T_0 has, and $H_0\{S\}$ has $|S|$ noncompact dimensions more than T_0 has.

Cartan subgroups $H_0\{S_1\}$ and $H_0\{S_2\}$ are G_0–conjugate just when some element $w \in W(G_0, T_0)$ sends S_1 to S_2. Every Cartan subgroup of G_0 is conjugate to $\mathfrak{h}_0\{S\}$ for some set S of strongly orthogonal noncompact roots. This sets up a hierarchy among the conjugacy classes of Cartan subgroups of $G_0 : H_0\{S_1\} \leqq H_0\{S_2\}$ if and only if $w(S_2) \subset S_1$ for some $w \in W(G_0, T_0)$. That in turn sets up a hierarchy among parts of the regular set G_0'. If H_0 is any Cartan subgroup of G_0 we denote $G'_{H_0} = G'_0 \cap \mathrm{Ad}(G)H_0$, the set of all regular elements G'_0 that are conjugate to an element of H_0. Now $G'_{H_0\{S_1\}} \leqq G'_{H_0\{S_2\}}$ if and only if some Weyl group element $w \in W(G_0, T_0)$ sends S_2 to a subset of S_1. Here G'_{T_0} sits at the top, the $G'_{H_0\{\alpha\}}$ sit just below, the $G'_{H_0\{\alpha,\beta\}}$ are on the next level down, and finally the part of G'_0 corresponding to the Cartan subgroup of the minimal parabolic subgroups sit at the bottom.

If G_0 does not have a compact Cartan subgroup, let $H_0 = T_0 \times A_0$ be a fundamental (maximally compact) Cartan subgroup, so T_0 is a Cartan subgroup of a maximal compact subgroup $K_0 \subset G_0$. Let $P_0 = M_0A_0N_0$ be an associated cuspidal parabolic subgroup. Then just do the cascade construction for M_0, obtaining a family of Cartan subgroups $H_{M,0}\{S\} \subset M_0$ as S runs over the $W(M_0, T_0)$–conjugacy classes of strongly orthogonal sets $S \subset \Sigma(\mathfrak{m}, \mathfrak{t})$ of noncompact roots of $\mathfrak{m}$. Then the $H_0\{S\} = H_{M,0}\{S\} \times A_0$ give the conjugacy classes of Cartan subgroups of G_0.

The character formulae for the various tempered series exhaust enough of $\widehat{G_0}$ for a decomposition of $L_2(G_0)$ essentially as

$$\sum_{[H_0]\in Car(G_0)} \sum_{\chi\otimes e^{\nu-\rho_{\mathfrak{m}}}\in\widehat{T_0}} \int_{\mathfrak{a}_0^*} H_{\pi_{\chi,\nu,\sigma}} \otimes H^*_{\pi_{\chi,\nu,\sigma}} m(H_0 : \chi : \nu : \sigma) d\sigma. \tag{10.2}$$

Here $Car(G_0)$ denotes the set of G_0–conjugacy classes $[H_0]$ of Cartan subgroups H_0 and the Borel measure $m(H_0 : \chi : \nu : \sigma)d\sigma$ is the *Plancherel measure* on $\widehat{G_0}$. In general the Plancherel density $m(H_0 : \chi : \nu : \sigma)$ has a formula that varies with the component of the regular set. This was worked out by Harish–Chandra for groups of Harish–Chandra class, and somewhat more generally by Herb and myself. Harish–Chandra's approach is based on an analysis of the structure of the Schwartz space, while Herb and I use explicit character formulae. These explicit formulae allow us to prove (10.2), as follows.

Start with G'_{H_0} where H_0 represents the conjugacy class of Cartan subgroups of G_0 that are as compact as possible. The H_0–series representations suffice to expand functions $f \in C_0^\infty(G'_{H_0})$. That expansion formula gives us a map $C_0^\infty(G_0) \to C^\infty(G_0 \setminus G'_{H_0})$ by $f \mapsto f_1$ where r_x denotes right translation by $x \in G_0$ and $f_1(x) = f(x) - \sum_{\chi\otimes e^{\nu-\rho_{\mathfrak{m}}}\in\widehat{T_0}} \int_{\mathfrak{a}_0^*} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0 : \chi : \nu : \sigma) d\sigma$. This map requires an exact knowledge of the characters of the H_0–series. Now let $\{H_0\{\alpha_1\}, \ldots, H_0\{\alpha_{m_1}\}$ be a set of representatives of the conjugacy classes of Cartan subgroups just below H_0. The $H_0\{\alpha_i\}$–series representations suffice to expand functions $f \in C_0^\infty(G'_{H_0\{\alpha_i\}})$. Those expansions do not interact at this level, nor do they introduce nonzero values in G'_{H_0}, so they give us a map

$$C^\infty\left(G_0 \setminus G'_{H_0}\right) \to C^\infty\left(G_0 \setminus \left(G'_{H_0} \cup \bigcup G'_{H_0\{\alpha_i\}}\right)\right) \text{ by } f_1 \mapsto f_2 ,$$

where

$$f_2(x) - f_1(x) = \sum_{1 \leqq i \leqq m_1} \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0\{\alpha_i\}}} \int_{\mathfrak{a}_0^*\{\alpha_i\}} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0\{\alpha_i\} : \chi : \nu : \sigma) d\sigma.$$

Now simply proceed down one level at a time. The tricky point here is to know the character formulae completely, so that one knows f_j well enough to compute f_{j+1}. Thus one obtains exact information on the Plancherel densities $m(H_0 : \chi : \nu : \sigma)$ and the final form

$$f(x) = \sum_{H_0 \in Car(G_0)} \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0}} \int_{\mathfrak{a}_0^*} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0 : \chi : \nu : \sigma) d\sigma \tag{10.3}$$

of the Plancherel formula.

PART III. GEOMETRIC REALIZATIONS OF REPRESENTATIONS.

Now we show how the tempered representations $\pi_{\eta,\sigma}$ are realized on appropriate real G_0–orbits on complex flags $Z = G/Q$. The material on L^2 realizations is taken from [**135**].

11. L^2 Realizations of the Principal Series.

In order to realize the principal series of G_0 on closed orbits, we need the Bott–Borel–Weil Theorem for M_0, where $P_0 = M_0 A_0 N_0$ is a minimal parabolic subgroup of G_0. We have to be careful here because the compact group M_0 need not be connected. We will first decompose M_0 as the product $Z_{M_0}(M_0^0)M_0^0$ where M_0^0 is its identity component, then indicate the analog of the Cartan highest weight description for $\widehat{M_0}$. That done, the standard Bott–Borel–Weil Theorem for M_0^0 will carry over to M_0.

A Cartan subgroup $T_0 \subset M_0$ specifies a Cartan subgroup $H_0 = T_0 A_0 \cong T_0 \times A_0$ in G_0. Our choice of P_0 specifies a choice of positive restricted root system $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$: the Lie algebra of N_0 is given by $\mathfrak{n}_0 = \sum_{\alpha \in \Sigma^+(\mathfrak{g}_0,\mathfrak{a}_0)} (\mathfrak{g}_0)_{-\alpha}$. Now any positive root system $\Sigma^+(\mathfrak{m}, \mathfrak{t})$ specifies a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ by

$$\begin{aligned} \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{h}) \text{ if and only if either } \alpha|_{\mathfrak{a}_0} = 0 \text{ and } \alpha|_{\mathfrak{t}} \in \Sigma^+(\mathfrak{m}, \mathfrak{t}) \\ \text{or } \alpha|_{\mathfrak{a}_0} \neq 0 \text{ and } \alpha|_{g\mathfrak{a}_0} \in \Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0). \end{aligned} \tag{11.1}$$

LEMMA 11.2. $M_0 = Z_{M_0}(M_0^0)M_0^0$. *Given a representation class* $[\eta] \in \widehat{M_0}$, *there exist unique classes* $[\chi] \in \widehat{Z_{M_0}(M_0^0)}$ *and* $[\eta^0] \in \widehat{M_0^0}$ *such that* $[\eta] = [\chi \otimes \eta^0]$, *and* $[\chi]$ *and* $[\eta^0]$ *restrict to multiples of the same unitary character on the center of* M_0^0.

Let $\Psi_{\mathfrak{m}}$ denote the set of simple roots in $\Sigma^+(\mathfrak{m},\mathfrak{t})$. Every subset $\Phi \subset \Psi_{\mathfrak{m}}$ defines

$$(11.3)\quad \begin{cases} \mathfrak{z}_\Phi = \{\xi \in \mathfrak{t} \mid \phi(\xi) = 0 \text{ for all } \phi \in \Phi\} \text{ with real form } \mathfrak{z}_{\Phi,0} = \mathfrak{m}_0 \cap \mathfrak{z}_\Phi \ , \\ U_\Phi = Z_M(\mathfrak{z}_\Phi), U_{\Phi,0} = M_0 \cap U_\Phi, \text{ and their Lie algebras } \mathfrak{u}_\Phi \text{ and } \mathfrak{u}_{\Phi,0} \ , \\ \mathfrak{r}_\Phi = \mathfrak{u}_\Phi + \sum_{\gamma \in \Sigma^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_{-\gamma}, \text{ parabolic subalgebra of } \mathfrak{m} \ , \\ R_\Phi = N_M(\mathfrak{r}_\Phi), \text{ corresponding parabolic subgroup of } M \ , \text{ and} \\ S_\Phi = M/R_\Phi, \text{ associated complex flag manifold.} \end{cases}$$

Lemma 11.2 holds for $U_{\Phi,0}$, M_0 acts transitively on S_Φ , and $M_0 \cap R_\Phi = U_{\Phi,0}$, so

LEMMA 11.4. *S_Φ is a compact homogeneous Kähler manifold under the action of M_0, and $S_\Phi = M_0/U_{\Phi,0}$ as coset space. Furthermore $U_{\Phi,0} = Z_{M_0}(M_0^0)U_{\Phi,0}^0$, so $\widehat{U_{\Phi,0}}$ decomposes as does $\widehat{M_0}$ in* Lemma 11.2.

An irreducible unitary representation μ of $U_{\Phi,0}$, say with representation space V_μ , gives us

$$(11.5)\quad \begin{cases} \mathbb{V}_\mu \to S_\Phi : U_{\Phi,0}\text{–homogeneous, hermitian, holomorphic vector bundle,} \\ A^{p,q}(S_\Phi;\mathbb{V}_\mu) : \text{ space of } C^\infty \ \mathbb{V}_\mu\text{–valued } (p,q)\text{–forms on } S_\Phi \ , \\ \mathcal{O}(\mathbb{V}_\mu) : \text{ sheaf of germs of holomorphic sections of } \mathbb{V}_\mu \to S_\Phi \ . \end{cases}$$

$A^{p,q}(S_\Phi;\mathbb{V}_\mu)$ is the space of C^∞ sections of $\mathbb{V}_\mu^{p,q} = \mathbb{V}_\mu \otimes \Lambda^p(\mathbb{T}^*) \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to S_\Phi$. $\mathbb{V}_\mu^{p,q}$ has an M_0–invariant hermitian metric, so we also have the Hodge–Kodaira orthocomplementation operators

$$(11.6)\quad \begin{aligned} &\sharp : A^{p,q}(S_\Phi;\mathbb{V}_\mu) \to A^{n-p,n-q}(S_\Phi;\mathbb{V}_\mu^*) \\ &\qquad\qquad \text{and } \tilde{\sharp} : A^{n-p,n-q}(S_\Phi;\mathbb{V}_\mu^*) \to A^{p,q}(S_\Phi;\mathbb{V}_\mu), \end{aligned}$$

where $n = \dim_{\mathbb{C}} S_\Phi$. The global M_0–invariant inner product on $A^{p,q}(S_\Phi;\mathbb{V}_\mu)$ is given by taking the inner product in each fiber of $\mathbb{V}_\mu^{p,q}$ and integrating over S_Φ ,

$$(11.7)\qquad \langle F_1, F_2\rangle_{S_\Phi} = \int_{M_0} \langle F_1, F_2\rangle_{mU_{\Phi,0}} d(mU_{\Phi,0}) = \int_{S_\Phi} F_1 \bar{\wedge} \sharp F_2 \ .$$

In the usual way that gives us the Kodaira–Hodge–Laplace operator $\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $A^{p,q}(S_\Phi;\mathbb{V}_\mu)$, where $\bar{\wedge}$ means exterior product followed by contraction of V_μ against V_μ^* . So we arrive at the Hilbert space

$$(11.8)\quad \mathcal{H}_2^{p,q}(S_\Phi;\mathbb{V}_\mu) : \mathbb{V}_\mu\text{–valued square integrable harmonic}(p,q)\text{–forms on } S_\Phi$$

Everything is invariant under the action of M_0, and the natural action of the group M_0 on $\mathcal{H}_2^{p,q}(S_\Phi;\mathbb{V}_\mu)$ is a unitary representation. Denote $\mathcal{H}_2^{q}(S_\Phi;\mathbb{V}_\mu) = \mathcal{H}_2^{0,q}(S_\Phi;\mathbb{V}_\mu)$.

Let $\rho_{\mathfrak{m}}$ denote half the sum of the roots in $\Sigma^+(\mathfrak{m},\mathfrak{t})$, and let η_ν^0 denote the irreducible representation of M_0^0 of highest weight $\nu - \rho_{\mathfrak{m}}$ (corresponding to infinitesimal character ν). With these conventions, the Bott–Borel–Weil Theorem for M_0 is

THEOREM 11.9. *Let* $[\mu] = [\chi \otimes \mu^0_\beta] \in \widehat{U_{\Phi,0}}$ *and fix an integer* $q \geqq 0$.

1. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha\rangle = 0$ *for some* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$ *then* $\mathcal{H}^q_2(S_\Phi;\mathbb{V}_\mu) = 0$.

2. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha\rangle \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$, *let* w *be the unique element in* $W(\mathfrak{m},\mathfrak{t})$ *such that* $\langle w(\beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}), \alpha\rangle > 0$ *for all* $\alpha \in \Sigma^+(\mathfrak{m},\mathfrak{t})$. *Then* $\mathcal{H}^q_2(S_\Phi;\mathbb{V}_\mu) = 0$ *for* $q \neq q_0$, *and* M_0 *acts irreducibly on* $\mathcal{H}^{q_0}_2(S_\Phi;\mathbb{V}_\mu)$ *by* $[\chi \otimes \eta^0_\nu]$.

Fix $[\mu] = [\chi\otimes\mu^0_\beta] \in \widehat{U_{\Phi,0}}$ and $\sigma \in \mathfrak{a}_0^*$. We will use the Bott–Borel–Weil Theorem 11.9 to find the principal series representation $\pi_{\chi\otimes\eta^0_\nu,\sigma}$ of G_0 on a cohomology space related to the closed orbit in the complex flag manifold $Z_\Phi = G/Q_\Phi$. Here the simple root system $\Psi_{\mathfrak{m}} \subset \Psi$ by the coherence in our choice of $\Sigma^+(\mathfrak{g},\mathfrak{h})$, so $\Phi \subset \Psi$ and Φ defines a parabolic subgroup $Q_\Phi \subset G$.

Let $z_\Phi = 1Q_\Phi \in G/Q_\Phi = Z_\Phi$. As $A_0N_0 \subset G_0 \cap Q_\Phi$ we have $G_0\cap Q_\Phi$ (in the sense of the action on Z_Φ) equal to $U_{\Phi,0}A_0N_0$. Thus $Y_\Phi = G_0(z_\Phi)$ is the closed G_0–orbit on Z_Φ , and S_Φ sits in Y_Φ as the orbit $M_0(z_\Phi)$. Here $P_0 = M_0A_0N_0 = \{g \in G_0 \mid gS_\Phi = S_\Phi\}$.

LEMMA 11.10. *The map* $Y_\Phi \to G_0/P_0$, *given by* $g(z_\Phi) \mapsto gP_0$, *defines a* G_0–*equivariant fiber bundle with structure group* M_0 *and whose fibers* gS_Φ *are the maximal complex analytic submanifolds of* Y_Φ .

The data (μ,σ) defines a representation $\gamma_{\mu,\sigma}(uan) = e^{(\rho_{\mathfrak{g}}+i\sigma)(\log a)}\mu(u)$ of $U_{\Phi,0}A_0N_0$ where $\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. That defines a G_0–homogeneous complex vector bundle

$$\mathbb{V}_{\mu,\sigma} \to G_0/U_{\Phi,0}A_0N_0 = Y_\Phi \text{ such that } \mathbb{V}_{\mu,\sigma}|_{S_\Phi} = \mathbb{V}_\mu \ . \tag{11.11}$$

Each $\mathbb{V}_{\mu,\sigma}|_{gS_\Phi}$ is an $\mathrm{Ad}(g)P_0$–homogeneous holomorphic vector bundle. Also, as $[\mu]$ is unitary we have a K_0–invariant hermitian metric on $\mathbb{V}_{\mu,\sigma}$.

The complexified tangent bundle of Y_Φ has a subbundle $\mathbb{T} \to Y_\Phi$ defined by

$$\mathbb{T}|_{gS_\Phi} \to gS_\Phi \text{ is the holomorphic tangent bundle of } gS_\Phi \ .$$

It defines

$$\begin{cases} \mathbb{V}^{p,q}_{\mu,\sigma} = \mathbb{V}_{\mu,\sigma}\otimes\Lambda^p(\mathbb{T}^*)\otimes\Lambda^q(\overline{\mathbb{T}}^*) \to Y_\Phi \ , \\ A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) : C^\infty \text{ sections of } \mathbb{V}^{p,q}_{\mu,\sigma} \to Y_\Phi \ , \text{ and} \\ \mathcal{O}(\mathbb{V}_{\mu,\sigma}) : \text{ sheaf of } C^\infty \text{ sections of } \mathbb{V}_{\mu,\sigma} \text{ holomorphic over every } gS_\Phi \ . \end{cases} \tag{11.12}$$

$A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ is the space of $\mathbb{V}_{\mu,\sigma}$–valued partially (p,q)–forms on Y_Φ .

We take the positive definite $U_{\Phi,0}$–invariant hermitian inner product on fiber V_μ of $\mathbb{V}_\mu \to S_\Phi$, and translate it around by K_0 to obtain a K_0–invariant hermitian structure on $\mathbb{V}^{p,q}_{\mu,\sigma} \to Y_\Phi$. In the same way we have a K_0–invariant hermitian metric on $\mathbb{T} \to Y_\Phi$. So we have K_0–invariant Hodge–Kodaira orthocomplementation operators $\sharp : A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) \to A^{n-p,n-q}(Y_\Phi;\mathbb{V}^*_{\mu,\sigma})$ and $\tilde{\sharp} : A^{n-p,n-q}(Y_\Phi;\mathbb{V}^*_{\mu,\sigma}) \to A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$. where $n = \dim_{\mathbb{C}} S_\Phi$.

We obtain the global G_0–invariant hermitian inner product on $A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ from the inner product along each fiber of $Y_\Phi \to G_0/P_0$ by integrating over G_0/P_0 ,

$$\langle F_1,F_2\rangle_{Y_\Phi} = \int_{K_0/M_0}\left(\int_{kS_\Phi} F_1\bar{\wedge}\sharp F_2\right) d(kM_0). \tag{11.13}$$

where $\bar{\wedge}$ means exterior product followed by contraction of V_μ against V_μ^*.

The $\overline{\partial}$ operator of Z_Φ induces the $\overline{\partial}$ operators on each of the gS_Φ, so they fit together to give us an operator $\overline{\partial} : A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) \to A^{p,q+1}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$. It has formal adjoint $\overline{\partial}^* : A^{p,q+1}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) \to A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ given by $\overline{\partial}^* = -\tilde{\sharp}\overline{\partial}\sharp$. That defines the "partial Kodaira–Hodge–Laplace operator"

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) \to A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}). \tag{11.14}$$

$A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ is a pre Hilbert space with the global inner product (11.13). Denote

$$L_2^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) : \text{ Hilbert space completion of } A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}). \tag{11.15}$$

Essentially as in the case of the Bott–Borel–Weil Theorem, we apply Weyl's Lemma along each gS_Φ to see that the closure of $\widetilde{\square}$ of $\square$, as a densely defined operator on $L_2^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ from the domain $A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$, is essentially self–adjoint. Its kernel

$$\mathcal{H}_2^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) = \{\omega \in \text{ Domain}(\widetilde{\square}) \mid \widetilde{\square}\omega = 0\} \tag{11.16}$$

is the space of $\mathbb{V}_{\mu,\sigma}$*–valued square integrable partially harmonic* (p,q)–forms on Y_Φ.

The factor $e^{\rho_\mathfrak{g}}$ in the representation $\gamma_{\mu,\sigma}$ that defines $\mathbb{V}_{\mu,\sigma}$ insures that the global inner product on $A^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ is invariant under the action of G_0. It corresponds to the $\Delta(G_0/P_0)$ in the definition (7.2) of (unitarily induced) principal series representation.

The other ingredients in the construction of $\mathcal{H}_2^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ are invariant as well, so G_0 acts naturally on $\mathcal{H}_2^{p,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ by isometries. This action is a unitary representation of G_0.

Write $\mathcal{H}_2^{q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ for $\mathcal{H}_2^{0,q}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$. We combine the Bott–Borel–Weil Theorem 11.9 with the definition (7.2) of the principal series, obtaining a geometric realization of the principal series of G_0 based on the closed G_0–orbit in the complex flag $Z_\Phi = G/Q_\Phi$, as follows.

THEOREM 11.17. *Let* $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ *and* $\sigma \in \mathfrak{a}_0^*$, *and fix* $q \geqq 0$.

1. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_\mathfrak{m}, \alpha\rangle = 0$ *for some* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$ *then every* $\mathcal{H}_2^{q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) = 0$.

2. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_\mathfrak{m}, \alpha\rangle \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$, *let w be the unique element in* $W(\mathfrak{m},\mathfrak{t})$ *such that* $\langle w(\beta - \rho_{\mathfrak{u}_\Phi} + \rho_\mathfrak{m}), \alpha\rangle > 0$ *for all* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$. *Then* $\mathcal{H}_2^{q}(Y_\Phi;\mathbb{V}_{\mu,\sigma}) = 0$ *for* $q \neq q_0$, *and the natural action of* G_0 *on* $\mathcal{H}_2^{q_0}(Y_\Phi;\mathbb{V}_{\mu,\sigma})$ *is the principal series representation* $\pi_{\chi\otimes\eta_\nu^0,\sigma}$.

12. L^2 Realizations of the Discrete Series.

Suppose that G_0 has a compact Cartan subgroup $T_0 \subset K_0$. Let $Z = G/Q$ be a complex flag manifold, let $z \in Z$, set $D = G_0(z)$, and suppose that

(12.1) $\quad D$ is open in Z and G_0 has compact isotropy subgroup U_0 at z.

Passing to a conjugate, equivalently moving z within D, we may suppose $T_0 \subset U_0$.

Let $\mu \in \widehat{U_0}$, let E_μ denote the representation space, and let $\mathbb{E}_\mu \to D \cong G_0/U_0$ denote the associated holomorphic homogeneous vector bundle. Then μ is finite dimensional and is constructed as follows. First, $U_0 \cap G_0^0$ is the identity component

U_0^0, and $U_0 = Z_{G_0}(G_0^0)U_0^0$. Second there are irreducible unitary representations $[\chi] \in \widehat{Z_{G_0}(G_0^0)}$ and $[\mu^0] \in \widehat{U_0^0}$ that agree on Z_{G_0} such that $[\mu] = [\chi \otimes \mu^0]$.

Let $\beta - \rho_{\mathfrak{u}}$ denote the highest weight of μ^0, corresponding to infinitesimal character β, and suppose that

$$\lambda = \beta - \rho_{\mathfrak{u}} + \rho_{\mathfrak{g}} \text{ is regular.} \tag{12.2}$$

Then G_0 has a discrete series representation $\pi_{\chi,\lambda}$, whose infinitesimal character has Harish–Chandra parameter λ.

Since μ is unitary, the bundle $\mathbb{E}_\mu \to D$ has a G_0–invariant hermitian metric. Essentially as in the compact case, we have the spaces

$$A_0^{(p,q)}(D;\mathbb{E}_\mu) : C^\infty \text{ compactly supported } \mathbb{E}_\mu\text{–valued } (p,q)\text{–forms on } D, \tag{12.3}$$

the Kodaira–Hodge orthocomplementation operators

$$\begin{aligned} \sharp : & A_0^{(p,q)}(D;\mathbb{E}_\mu) \to A_0^{(n-p,n-q)}(D;\mathbb{E}_\mu^*) \\ \text{and } \tilde{\sharp} : & A_0^{(n-p,n-q)}(D;\mathbb{E}_\mu^*) \to A_0^{(p,q)}(D;\mathbb{E}_\mu) \end{aligned} \tag{12.4}$$

where $n = \dim_{\mathbb{C}} D$. Thus we have a positive definite inner product on $A_0^{(p,q)}(D;\mathbb{E}_\mu)$ give by

$$\langle F_1, F_2\rangle_D = \int_{G_0} \langle F_1, F_2\rangle_{gU_0} d(gU_0) = \int_D F_1 \overline{\wedge} \sharp F_2 \tag{12.5}$$

and thus

$$L_2^{(p,q)}(D;\mathbb{E}_\mu) : \text{ Hilbert space completion of } \left(A_0^{(p,q)}(D;\mathbb{E}_\mu), \langle\cdot,\cdot\rangle_D\right). \tag{12.6}$$

Let $\square$ denote the Kodaira–Hodge–Laplace operator $\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ of $\mathbb{E}_\mu$. Then $\square$ is a hermitian–symmetric elliptic operator on $L_2^{(0,q)}(D;\mathbb{E}_\mu)$ with domain $A_0^{(p,q)}(D;\mathbb{E}_\mu)$, and a result of Andreotti and Vesentini allows one to conclude that its closure $\widetilde{\square}$ is self–adjoint. Accordingly, we have the Hilbert spaces

$$\mathcal{H}_2^{(p,q)}(D;\mathbb{E}_\mu) = \{\omega \in \text{ Domain}\,(\widetilde{\square}) \mid \widetilde{\square}(\omega) = 0\} \tag{12.7}$$

of L^2 harmonic $\mathbb{E}_\mu$–valued $(0,q)$–forms on D. G_0 acts on $\mathcal{H}_2^{(p,q)}(D;\mathbb{E}_\mu)$ by a unitary representation.

We write $\mathcal{H}_2^q(D;\mathbb{E}_\mu)$ for $\mathcal{H}_2^{(0,q)}(D;\mathbb{E}_\mu)$ and we write π_μ^q for the unitary representation of G_0 on $\mathcal{H}_2^q(D;\mathbb{E}_\mu)$.

THEOREM 12.8. *Let $[\mu] = [\chi \otimes \mu^0] \in \widehat{U_0}$ where μ^0 has highest weight $\beta - \rho_{\mathfrak{u}}$ and thus has infinitesimal character β. If $\lambda + \rho$ (as in (12.2)) is $\Sigma(\mathfrak{g},\mathfrak{t})$–singular then every $\mathcal{H}_2^q(D;\mathbb{E}_\mu) = 0$. Now suppose that $\lambda = \beta - \rho_{\mathfrak{u}} + \rho_{\mathfrak{g}}$ is $\Sigma(\mathfrak{g},\mathfrak{t})$–regular and define*

$$\begin{aligned} q_{\mathfrak{u}}(\lambda) = & |\{\alpha \in \Sigma^+(\mathfrak{k},\mathfrak{t}) \setminus \Sigma^+(\mathfrak{u},\mathfrak{t}) \mid \langle\lambda,\alpha\rangle < 0\}| \\ & + |\{\beta \in \Sigma^+(\mathfrak{g},\mathfrak{t}) \setminus \Sigma^+(\mathfrak{k},\mathfrak{t}) \mid \langle\lambda,\beta\rangle > 0\}|. \end{aligned} \tag{12.9}$$

Then $\mathcal{H}_2^q(D;\mathbb{E}_\mu) = 0$ for $q \neq q_{\mathfrak{u}}(\lambda)$, and G_0 acts irreducibly on $\mathcal{H}_2^{q_{\mathfrak{u}}(\lambda)}(D;\mathbb{E}_\mu)$ by the discrete series representation $\pi_{\chi,\lambda}$ of infinitesimal character λ.

One can also realize the discrete series on spaces of L_2 bundle–valued harmonic spinors [**136**].

13. L^2 Realizations of the Various Tempered Series.

Choose a Cartan subgroup $H_0 \subset G_0$. We are going to realize the H_0–series representations of G_0 in a way analogous to the way we realized the principal series in §11, with Theorem 12.8 in place of the Bott–Borel–Weil Theorem 11.9.

Let θ be the Cartan involution of G_0 that stabilizes H_0, split $H_0 = T_0 \times A_0$ and let $Z_{G_0}(A_0) = M_0 \times A_0$ as before. Fix a positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$ defined by positive root systems $\Sigma^+(\mathfrak{m}, \mathfrak{t})$ and $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$ as in (11.1). Let $P_0 = M_0 A_0 N_0$ be the corresponding cuspidal parabolic subgroup of G_0 associated to H_0.

Following the idea of the geometric realization of the principal series, we fix a set $\Phi \subset \Psi_\mathfrak{m}$ where $\Psi_\mathfrak{m}$ is the simple root system for $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. Then as in (11.2) we have

$$
(13.1) \quad \begin{cases}
\mathfrak{z}_\Phi = \{\xi \in \mathfrak{t} \mid \phi(\xi) = 0 \ \forall \phi \in \Phi\} \text{ and its real form } \mathfrak{z}_{\Phi,0} = \mathfrak{m}_0 \cap \mathfrak{r}_\Phi \mathfrak{z}_\Phi \ , \\
U_\Phi = Z_M(\mathfrak{z}_\Phi), U_{\Phi,0} = M_0 \cap U_\Phi, \text{ and Lie algebras } \mathfrak{u}_\Phi \text{ and } \mathfrak{u}_{\Phi,0} \ , \\
\mathfrak{r}_\Phi = \mathfrak{u}_\Phi + \sum_{\gamma \in \Sigma^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_{-\gamma}, \text{ parabolic subalgebra of } \mathfrak{m} \ , \\
R_\Phi = N_M(\mathfrak{r}_\Phi), \text{ corresponding parabolic subgroup of } M \ , \text{ and} \\
S_\Phi = M/R_\Phi, \text{ associated complex flag manifold.}
\end{cases}
$$

Let r_Φ denote the base point, $r_\Phi = 1R_\Phi \in R_\Phi$. As T_0 is a compact Cartan subgroup of M_0 contained in $U_{\Phi,0}$, $D_\Phi = M_0(r_\Phi) \subset S_\Phi$ is a measurable open M_0–orbit on R_Φ. We now assume that $U_{\Phi,0}$ is compact, so the considerations of §11 apply to $D_\Phi \subset S_\Phi$.

Fix $[\mu] = [\chi \otimes \mu^0_\beta] \in \widehat{U_{\Phi,0}}$ as before. Given $\sigma \in \mathfrak{a}_0^*$ we will use the Theorem 12.8 to find the H_0–series representation $\pi_{\chi \otimes \eta^0_\nu, \sigma}$ on a cohomology space related to a particular orbit in the complex flag manifold $Z_\Phi = G/Q_\Phi$. Here as before, the simple root system $\Psi_\mathfrak{m} \subset \Psi$ by the coherence in our choice of $\Sigma^+(\mathfrak{g}, \mathfrak{h})$, so $\Phi \subset \Psi$ and Φ defines a parabolic subgroup $Q_\Phi \subset G$.

Let $z_\Phi = 1Q_\Phi \in G/Q_\Phi = Z_\Phi$. As $A_0 N_0 \subset G_0 \cap Q_\Phi$ we have $G_0 \cap Q_\Phi = U_{\Phi,0} A_0 N_0$. Thus $Y_\Phi = G_0(z_\Phi)$ is a G_0–orbit on Z_Φ, and D_Φ sits in Y_Φ as the orbit $M_0(z_\Phi)$. Here note that $P_0 = M_0 A_0 N_0 = \{g \in G_0 \mid gD_\Phi = D_\Phi\}$.

LEMMA 13.2. *The map $Y_\Phi \to G_0/P_0$, given by $g(z_\Phi) \mapsto gP_0$, defines a G_0–equivariant fiber bundle with structure group M_0 and whose fibers gD_Φ are the maximal complex analytic submanifolds of Y_Φ.*

The data (μ, σ) defines a representation $\gamma_{\mu,\sigma}(uan) = e^{(\rho_\mathfrak{g} + i\sigma)(\log a)} \mu(u)$ of $U_{\Phi,0} A_0 N_0$ where $\rho_\mathfrak{g} = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. That defines a G_0–homogeneous vector bundle

$$
(13.3) \qquad \mathbb{E}_{\mu,\sigma} \to G_0/U_{\Phi,0} A_0 N_0 = Y_\Phi \text{ such that } \mathbb{E}_{\mu,\sigma}|_{D_\Phi} = \mathbb{E}_\mu \ .
$$

Each $\mathbb{E}_{\mu,\sigma}|_{gD_\Phi}$ is an $\mathrm{Ad}(g)P_0$–homogeneous holomorphic vector bundle.

Since $[\mu]$ is unitary and K_0 acts transitively on G_0/P_0 we have a K_0–invariant hermitian metric on $\mathbb{E}_{\mu,\sigma}$. We will use it without explicit reference.

Consider the subbundle $\mathbb{T} \to Y_\Phi$ of the complexified tangent bundle to Y_Φ defined by

$$\mathbb{T}|_{gD_\Phi} \to gD_\Phi \text{ is the holomorphic tangent bundle of } gD_\Phi \,. \tag{13.4}$$

It defines

$$\begin{cases} \mathbb{E}^{p,q}_{\mu,\sigma} = \mathbb{E}_{\mu,\sigma} \otimes \Lambda^p(\mathbb{T}^*) \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to D_\Phi \,, \\ A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) : C^\infty \text{ compactly supported sections of } \mathbb{E}_{\mu,\sigma} \to Y_\Phi \,, \\ \mathcal{O}(\mathbb{E}_{\mu,\sigma}) : \text{ sheaf of } C^\infty \text{ sections of } \mathbb{E}_{\mu,\sigma} \text{ holomorphic over every } gD_\Phi \,. \end{cases} \tag{13.5}$$

$A^{p,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is the space consisting of all $\mathbb{E}_{\mu,\sigma}$–valued partially (p,q)–forms on Y_Φ, and $A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is the subspace of compactly supported forms.

The fiber E_μ of $\mathbb{E}_\mu \to D_\Phi$ has a positive definite $U_{\Phi,0}$–invariant hermitian inner product because μ is unitary; we translate this around by K_0 to obtain a K_0–invariant hermitian structure on the vector bundle $\mathbb{E}^{p,q}_{\mu,\sigma} \to Y_\Phi$. Similarly $\mathbb{T} \to Y_\Phi$ carries a K_0–invariant hermitian metric. Using these hermitian metrics we have K_0–invariant Hodge–Kodaira orthocomplementation operators

$$\begin{aligned} \sharp : A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) &\to A^{n-p,n-q}_0(Y_\Phi;\mathbb{E}^*_{\mu,\sigma}) \\ \text{and } \tilde{\sharp} : A^{n-p,n-q}_0(Y_\Phi;\mathbb{E}^*_{\mu,\sigma}) &\to A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}), \end{aligned} \tag{13.6}$$

where $n = \dim_\mathbb{C} D_\Phi$. The global G_0–invariant inner product on $A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is given by taking the M_0–invariant inner product along each fiber of $Y_\Phi \to G_0/P_0$ and integrating over G_0/P_0,

$$\langle F_1, F_2\rangle_{Y_\Phi} = \int_{K_0/(K_0\cap M_0)} \left(\int_{kD_\Phi} F_1 \bar{\wedge} \sharp F_2 \right) d(k(K_0 \cap M_0)), \tag{13.7}$$

where $\bar{\wedge}$ means exterior product followed by contraction of E_μ against E^*_μ.

The $\overline{\partial}$ operator of Z_Φ induces the $\overline{\partial}$ operators on each of the gD_Φ, so they fit together to give us an operator

$$\overline{\partial} : A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{p,q+1}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \tag{13.8}$$

that has formal adjoint

$$\overline{\partial}^* : A^{p,q+1}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \text{ given by } \overline{\partial}^* = -\tilde{\sharp}\overline{\partial}\sharp \,. \tag{13.9}$$

That in turn defines an elliptic operator, the "partial Kodaira–Hodge–Laplace operator"

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}). \tag{13.10}$$

$A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is a pre Hilbert space with the global inner product (13.10). Denote

$$L^{p,q}_2(Y_\Phi;\mathbb{E}_{\mu,\sigma}) : \text{Hilbert space completion of } A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}). \tag{13.11}$$

Apply Andreotti–Vesentini along each gD_Φ to see that the closure of $\widetilde{\square}$ of $\square$, as a densely defined operator on $L^{p,q}_2(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ from the domain $A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$, is essentially self–adjoint. Its kernel

$$\mathcal{H}^{p,q}_2(Y_\Phi;\mathbb{E}_{\mu,\sigma}) = \{\omega \in \text{ Domain}(\widetilde{\square}) \mid \widetilde{\square}\omega = 0\} \tag{13.12}$$

is the space of *square integrable partially harmonic* (p,q)–forms on Y_Φ with values in $\mathbb{E}_{\mu,\sigma}$.

The factor $e^{\rho_{\mathfrak{g}}}$ in the representation $\gamma_{\mu,\sigma}$ that defines $\mathbb{E}_{\mu,\sigma}$ insures that the global inner product on $A_0^{p,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is invariant under the action of G_0. The other ingredients in the construction of $\mathcal{H}_2^{p,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ are invariant as well, so G_0 acts naturally on $\mathcal{H}_2^{p,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ by isometries. This action is a unitary representation of G_0.

As before we write $\mathcal{H}_2^{q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ for $\mathcal{H}_2^{0,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$.

We can now combine Theorem 12.8 with the definition (9.3) of the H_0–series, obtaining

THEOREM 13.13. *Let* $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ *where* μ^0 *has highest weight* $\beta - \rho_{\mathfrak{u}}$ *and thus has infinitesimal character* β. *Let*

$$\nu = \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \tag{13.14}$$

suppose $\sigma \in \mathfrak{a}_0^*$, *and fix an integer* $q \geqq 0$.

1. *If* $\langle \nu, \alpha\rangle = 0$ *for some* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$ *then* $\mathcal{H}_2^{q}(Y_\Phi;\mathbb{E}_{\mu,\sigma}) = 0$.
2. *If* $\langle \nu, \alpha\rangle \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{m},\mathfrak{t})$, *define*

$$\begin{aligned} q_{\mathfrak{u}_\Phi}(\nu) &= |\{\alpha \in \Sigma^+((\mathfrak{k}\cap\mathfrak{m}),\mathfrak{t}) \setminus \Sigma^+(\mathfrak{u},\mathfrak{t}) \mid \langle \nu,\alpha\rangle < 0\}| \\ &\quad + |\{\beta \in \Sigma^+(\mathfrak{m},\mathfrak{t}) \setminus \Sigma^+((\mathfrak{k}\cap\mathfrak{m}),\mathfrak{t}) \mid \langle \nu,\beta\rangle > 0\}|. \end{aligned} \tag{13.15}$$

Then $\mathcal{H}^q(D;\mathbb{E}_{\mu,\sigma}) = 0$ *for* $q \neq q_{\mathfrak{u}_\Phi}(\nu)$, *and the action of* G_0 *on* $\mathcal{H}^{q_{\mathfrak{u}_\Phi}(\nu)}(D;\mathbb{E}_{\mu,\sigma})$ *is the* H_0*–series representation* $\pi_{\chi,\nu,\sigma}$ *of infinitesimal character* $\nu + i\sigma$.

A variation on this theorem [**136**] realizes the tempered series on spaces of L_2 bundle-valued partially harmonic spinors.

14. Approaches to Non–Tempered Representations.

There are a number of approaches to the problem of finding good geometric realizations of representations that need not be tempered — or even need not be unitary. There is no general agreement here on which approaches are "best", so I'll just give thumbnail descriptions of the four that I find most interesting. I apologize in advance to the many researchers in this area whose approach is not mentioned or is only mentioned in passing, and whose papers are not referenced.

Geometric quantization. In the Kostant–Souriau theory of geometric quantization, one considers a linear functional $\lambda \in \mathfrak{g}_0^*$ on the Lie algebra $\mathfrak{g}_0$ of G_0 and the coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*(G_0)(\lambda)$. Let L_0 denote the G_0–stabilizer of λ, so $\mathcal{O} \cong G_0/L_0$ as homogeneous space, and let $\mathfrak{l}_0$ be the Lie algebra of L_0. For complexification we drop the subscript 0, as before. A *polarization* for $\mathcal{O}$ is a (complex) subalgebra $\mathfrak{q} \subset \mathfrak{g}$ that (i) contains $\mathfrak{l}$ with $\dim\mathfrak{g} - \dim\mathfrak{q} = \dim\mathfrak{q} - \dim\mathfrak{l}$ and (ii) is $\mathrm{Ad}^*(L_0)$–invariant. The coadjoint orbit $\mathcal{O}$ is *integral* if $\exp(2\pi\sqrt{-1}\lambda) : L_0^0 \to \mathbb{C}$ is well defined. Then every extension χ of $\exp(2\pi\sqrt{-1}\lambda)$ to an irreducible representation of L_0, say on a vector space E_χ, leads to a G_0–homogeneous vector bundle $\mathbb{E}_\chi \to \mathcal{O}$ with typical fiber E_χ.

By $\mathrm{Ad}(L_0)$–invariance, $\mathfrak{q}$ acts on E_χ in a manner consistent with the action of L_0. This can happen in several different ways; choose one. Denote this action just by $d\chi$. Also, $\mathfrak{q}$ acts by differential operators on local sections, $s(x;\xi) =$

$\frac{d}{dt}|_{t=0}(s(x\exp(t\xi_1))) + \sqrt{-1}\frac{d}{dt}|_{t=0}(s(x\exp(t\xi_2)))$ for $\xi = \xi_1 + \sqrt{-1}\xi_2 \in \mathfrak{q}$ with $\xi_1, \xi_2 \in \mathfrak{g}_0$. Let $\mathcal{E}_\chi$ denote the sheaf of local sections s of $\mathbb{E}_\chi$ such that $s(x;\xi) + d\chi(\xi)s(x) = 0$ for all $\xi \in \mathfrak{q}$. Then one studies the representations of G_0 on the cohomologies $H^q(\mathcal{O};\mathcal{E}_\chi)$.

The most familiar example is where $\mathcal{O}$ has an invariant complex structure, $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ in such a way that $\mathfrak{n}$ represents the antiholomorphic tangent space, $d\chi(\mathfrak{n}) = 0$ so that $s(x;\xi) + d\chi(\xi)s(x) = 0$ simplifies to $s(x;\xi) = 0$, and $s(x;\xi) = 0$ is the Cauchy–Riemann equation, so $\mathcal{E}_\chi$ is the sheaf of germs of holomorphic sections of $\mathbb{E}_\chi \to \mathcal{O}$. Our L^2 realizations of tempered representations were done by an L^2 partially harmonic form variation on this pattern. Essentially this works because we are dealing with semisimple coadjoint orbits there, and their polarizations are parabolic subalgebras.

Several things can go wrong here. It can happen that there is no invariant polarization. A method of moving polarizations was developed to deal with the case where the polarization is invariant only under a certain subgroup of L_0. Another approach to this problem is given by the theory of metalinear structures. It can happen that one is looking for representations in some topological category, so one looks for the $H^q(\mathcal{O};\mathcal{E}_\chi)$ as cohomologies of some complex of topological vector spaces, and one needs a closed range theorem for the differentials of the complex in order that the cohomologies inherit a topology. This can be very delicate, but it has been settled in a few cases by using hyperfunction (instead of C^∞) coefficients; see [**115**], [**117**], [**138**] and [**151**]. It can happen that one looks to unitarize these representations, but the positive definite hermitian metrics of the previous three sections are not available. This has been carried out in a few cases; see the subsection "Indefinite metric harmonic theory" below. It can happen that one tries to realize representations that cannot occur on semisimple coadjoint orbits; there one goes in principle to the nascent technical theory of unipotent representations. And finally there is the problem (which the theory of unipotent representations tries to address) that, for most real semisimple Lie groups, the unitary dual is not completely known.

Double fibration transforms. This is a tool for investigating growth properties of Fréchet space representations on Dolbeault cohomology, with an eye toward questions of unitarity and Lebesgue class, and as such is essentially a tool for use in geometric quantization. It hasn't really been applied yet, but I think that it will be extremely useful for analysis of non–tempered representations. Historically this sort of thing was done the other way around, starting with differential equations and then getting a representation on Dolbeault (or some other) cohomology whose double fibration transform was the set of solutions to the differential equations.

Let $D = G_0(z) \subset Z = G/Q$ be a measurable open orbit and let $\mathbb{E} \to D$ be a *negative* G_0–homogeneous holomorphic vector bundle. Let s denote the complex dimension of the maximal compact subvariety $Y_0 = K_0(z)$ of D. Theorem 4.6 says that $H^q(D;\mathcal{O}(\mathbb{E})) = 0$ for $q > s$ and the negativity ensures that $H^q(D;\mathcal{O}(\mathbb{E})) = 0$ for $q < s$. We have (6.5). In many cases we have verified that the fiber F of $\mu : W_D \to D$ is contractible, so Theorem 6.6 says that the double fibration transform map $P : H^s(D;\mathcal{O}(\mathbb{E})) \to H^0(M_D;\mathcal{R}^s(\mu^*(\mathcal{O}(\mathbb{E}))))$ is injective. Now the point is to characterize its image by a set of PDE which will allow us to make *a priori* growth estimates on the coefficients of the representation of G_0 on $H^s(D;\mathcal{O}(\mathbb{E}))$.

Dual reductive pairs and the ϑ correspondence. A pair of subgroups $G_0, G_0' \subset Sp(n;\mathbb{R})$ is a *dual reductive pair* if they are reductive in the real symplectic group $Sp(n;\mathbb{R})$ and each is the centralizer of the other in $Sp(n;\mathbb{R})$. There are just a few such pairs:

$$\begin{aligned}&(O(p,q), Sp(k;\mathbb{R})) \text{ in } Sp((p+q)k;\mathbb{R})),\\ &(U(p,q), U(k,\ell)) \text{ in } Sp((p+q)(k+\ell);\mathbb{R}), \text{ and}\\ &(Sp(p,q), SO^*(2k)) \text{ in } Sp((p+q)k;\mathbb{R}),\end{aligned}$$

which Howe calls Type I, and the Type II pairs

$$(GL(u,\mathbb{F}), GL(v,\mathbb{F})) \text{ in } Sp(uvd;\mathbb{R}), d = \dim_{\mathbb{R}} \mathbb{F}, \text{ for } \mathbb{F} = \mathbb{R}, \mathbb{C}, \text{ and } \mathbb{H}.$$

The point, for us, is Howe's theorem that the metaplectic representation μ of (the double cover of) $Sp(n;\mathbb{R})$ has restriction to G_0G_0' of the form $\int \alpha_i \otimes \alpha_i'\, dm(i)$ with $\alpha_i \in \widehat{G_0}$ and $\alpha_i' \in \widehat{G_0'}$, and such that α_i determines α_i' and α_i' determines α_i, a.e. $m(i)$. The correspondence $\widehat{G_0} \ni \alpha_i \leftrightarrow \alpha_i' \in \widehat{G_0'}$ is the ϑ *correspondence.* If one knows one of $\widehat{G_0}, \widehat{G_0'}$ it gives a lot of useful information about the other. This is especially useful when one of G_0, G_0' is compact. Also, representations that occur on dual reductive pairs are automatically unitary, and they are sufficiently concrete so that one can often check questions of growth and irreducibility.

Indefinite metric harmonic theory. Consider a variation on the L^2 realizations of the discrete series in Section 12, in which the isotropy subgroup U_0 of G_0 may be noncompact. Thus the open orbit $D = G_0(z) \subset Z = G/Q$ carries an invariant pseudo–kähler metric from the Killing form of $\mathfrak{g}$, but perhaps does not carry an invariant positive definite hermitian metric. In the language of geometric quantization, D is an elliptic coadjoint orbit.

Let $[\mu] \in \widehat{U_0}$ and let E_μ be the representation space. The associated vector bundle $\mathbb{E}_\mu \to D \cong G_0/U_0$ is a G_0–homogeneous hermitian holomorphic vector bundle. As in (12.3) one has the space $A_0^{(p,q)}(D;\mathbb{E}_\mu)$ of compactly supported $\mathbb{E}_\mu$–valued C^∞ (p,q)–forms on D, and, from the invariant pseudo–kähler metric, one has the Kodaira–Hodge orthocomplementation operators (12.4). The problem is that the inner product $\langle F_1, F_2\rangle_D = \int_D F_1 \overline{\wedge} \sharp F_2$ on $A_0^{(p,q)}(D;\mathbb{E}_\mu)$ will in general be indefinite rather than positive definite, so the definition of $L_2^{(p,q)}(D;\mathbb{E}_\mu)$ is problematical. The approach of [**109**] is to use an auxiliary hermitian metric which is not G_0–invariant, but for which $L_2^{(p,q)}(D;\mathbb{E}_\mu)$ is defined and is a space on which G_0 acts by a bounded representation. Then, relative to the invariant pseudo–kähler metric one still has $\overline{\partial}^*$ and the indefinite metric analog $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ of the Kodaira–Hodge–Laplace operator. We say that a form $\omega \in L_2^{(p,q)}(D;\mathbb{E}_\mu)$ is *harmonic* if it is annihilated by the closure $\widetilde{\square}$ of $\square$, *strongly harmonic* if $\overline{\partial}\omega = 0 = \overline{\partial}^*\omega$.

Let $\widetilde{\mathcal{H}}_2^q(D;\mathbb{E}_\mu)_{K_0}$ denote the space of K_0–finite L^2 strongly harmonic $\mathbb{E}_\mu$–valued $(0,q)$-forms on D. It has a natural map $\omega \mapsto [\omega]$ to K_0–finite Dolbeault cohomology $H^q(D;\mathbb{E}_\mu)_{K_0}$ of $\mathbb{E}_\mu \to D$. Let $\mathcal{H}_2^q(D;\mathbb{E}_\mu)_{K_0}$ denote the image. Under certain circumstances, in [**109**] it is shown that every Dolbeault class $[\omega] \in \widetilde{\mathcal{H}}_2^q(D;\mathbb{E}_\mu)_{K_0}$

has a "best" representative ω, that $\langle \omega, \omega \rangle_D = \int_D \omega \overline{\wedge} \sharp \omega > 0$ for $[\omega] \neq 0$, and that $\langle \cdot, \cdot \rangle_D$ is null on the kernel of $\widetilde{\mathcal{H}}_2^q(D; \mathbb{E}_\mu)_{K_0} \to H^q(D; \mathbb{E}_\mu)_{K_0}$. That unitarizes the action of G_0 on certain Dolbeault cohomologies as the Hilbert space completions of the $\mathcal{H}_2^q(D; \mathbb{E}_\mu)_{K_0}$. Although it moves rather slowly, this program remains quite active and has since been carried much further. See **[8]**, **[9]**, **[10]**, **[13]**, **[18]**, **[138]**, **[152]**, **[153]**, **[154]** and **[155]**.

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