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Results in Mathematics

Isospectrality for Spherical Space Forms

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To my teacher, S.-S. Chern, on his 90^{th} birthday

Abstract

Consider the set $\mathcal{S}(q,\Gamma)$ of isometry classes of q–dimensional spherical space forms whose fundamental groups are isomorphic to a fixed group Γ. We define a certain group $\mathcal{A}(q,\Gamma)$ of transformations on the finite set $\mathcal{S}(q,\Gamma)$, prove that any two elements in the same $\mathcal{A}(q,\Gamma)$–orbit are strongly isospectral, and study some consequences. Then a number of the results are carried over to riemannian quotients of oriented real Grassmann manifolds.

Some of these results were first obtained by Ikeda, mostly for the special case (Γ cyclic) of lens spaces, and by Gilkey and Ikeda for the case where every Sylow subgroup of Γ is cyclic.

1 Introduction

Let $\mathcal{S}(q,\Gamma)$ denote the set of all isometry classes of q–dimensional spherical space forms with fundamental groups that are isomorphic to a fixed group Γ. In this paper we define a certain group $\mathcal{A}(q,\Gamma)$ of transformations on the finite set $\mathcal{S}(q,\Gamma)$, we prove that any two elements in the same $\mathcal{A}(q,\Gamma)$–orbit are strongly isospectral, and we study some of the consequences of that isospectrality. If q is minimal for Γ, in other words if the actions of Γ on the sphere S^q are $\mathbb{R}$–irreducible on the ambient $\mathbb{R}^{q+1}$, we show that $\mathcal{A}(q,\Gamma)$ is transitive on $\mathcal{S}(q,\Gamma)$, so any two space forms $M_i = S^q/\rho_i(\Gamma) \in \mathcal{S}(q,\Gamma)$ are isospectral. But $\mathcal{A}(q,\Gamma)$ is generically nontrivial, and that provides a plethora of new examples of isospectral non–isometric riemannian manifolds. There are a number of other applications to low–dimensional spherical space forms and to homogeneous spherical space forms. Then a number of the results are carried over to riemannian quotients of oriented real Grassmann manifolds.

The idea is to combine my classification of spherical space forms with the "Generalized Sunada Theorem" of DeTurck, Gordon and Bérard. In some cases, those in which one has to count elements of order 3 or order 4, it is better to combine the spherical space form theory with the classical formula of Molien for the Hilbert series of the ring of $\rho_i(\Gamma)$–invariants in $L^2(S^q)$, as reformulated by Ikeda. Our considerations, at least until now, require very little of the more delicate results of Bérard and Pesce ([1], [2], [3], [20], [21], [22], [23]).

Several of our results were first obtained by Ikeda, mostly for lens spaces (the case where Γ is cyclic), and by Gilkey and Ikeda for the case where every Sylow subgroup of Γ is cyclic. The new tool here is a better understanding of the classification [31] of spherical space forms and [30] of space forms of Grassmann manifolds.

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In Section 2 we recall some basic definitions and tools for isospectrality. As an example we recall the spectrum of the Laplace–Beltrami operator for the sphere. If S^q/G is a spherical space form we recall Molien's formula for the generating function of the space of G–invariant polynomial functions on $\mathbb{R}^{q+1}$, graded by degree, and we derive Ikeda's generating function from that of Molien. The result, Proposition 2.5, is a direct proof of Ikeda's isospectrality criterion. We then recall the Generalized Sunada Theorem of DeTurck and Gordon [9], as sharpened by Bérard [2], and specialize it to the case we will use, usually just called Sunada's Theorem, for considerations of strong isospectrality. This is based on some unpublished lectures of Gordon.

In Sections 3, 4 and 5 we recall some elements of the classification of spherical space forms from [31]. We need these details in order to define our isospectral groups.

In Section 6 we define the isospectral groups $\mathcal{A}(\Gamma)$ and $\mathcal{A}(q,\Gamma)$. The formulation [31] of the classification of spherical space forms, recalled in Sections 3, 4 and 5, is exactly what we need for the definition of $\mathcal{A}(\Gamma)$. The action of $\mathcal{A}(\Gamma)$ on $\mathcal{S}(q,\Gamma)$ defines the group $\mathcal{A}(q,\Gamma)$. Our main results are Proposition 6.3, which tells us that $\mathcal{A}(q,\Gamma)$ has fairly large orbits on $\mathcal{S}(q,\Gamma)$, and Theorem 6.4, which says that any two elements in the same $\mathcal{A}(q,\Gamma)$–orbit on $\mathcal{S}(q,\Gamma)$ are strongly isospectral.

In Section 7 we apply Proposition 6.3 and Theorem 6.4 to the case where S^q/G is irreducible in a certain sense. If $\frac{q+1}{2}$ is prime and G is not cyclic, it follows in Corollary 7.3 that S^q/G and S^q/G' are isospectral iff they are strongly isospectral iff $G \cong G'$. If $q = 3$, or if $q = 5$ but G is not cyclic, it follows in Corollaries 7.4 and 7.5 that S^q/G and S^q/G' are isospectral if and only if they are isometric.

In Section 8 we use Ikeda's generating function and my work ([27], [28]) on homogeneous spherical space forms to show that if S^q/G and S^q/G' are isospectral, and one of them is homogeneous, then they are isometric.

Finally in Section 9 we extend a number of our results from spherical space forms to space forms of Grassmann manifolds. The main result is Theorem 9.4, which relates isospectrality for quotients of Grassmann manifolds with isospectrality for quotients of spheres as indicated in [30]. There are several open questions on this relationship, especially concerning the orientation reversing isometry and the orthocomplementation isometry. They will be studied in [32].

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2 Generating Functions and Conjugacy Conditions

Two compact riemannian manifolds M and M' are *isospectral* if their respective Laplace–Beltrami operators, Δ on $L^2(M)$ and Δ' on $L^2(M')$, have the same spectra with the same multiplicities. Let $T \to M$ and $T' \to M'$ denote the tangent bundles. Given $p \geqq 0$ let $\bigwedge^p(T^*) \to M$ and $\bigwedge^p(T'^*) \to M'$ denote the bundles whose local sections are p–forms, and let $L^2(M, \bigwedge^p(T^*))$ and $L^2(M', \bigwedge^p(T'^*))$ denote the respective spaces of square integrable p–forms. Then M and M' are *p–isospectral* if Δ on $L^2(M, \bigwedge^p(T^*))$ and Δ' on $L^2(M', \bigwedge^p(T'^*))$ have the same spectra with the same multiplicities. Thus 0–isospectral is the same as isospectral. More generally, if D on $L^2(M,V)$ and D' on $L^2(M',V')$ have the same spectra with the same multiplicities whenever

- $V \to M$ and $V' \to M'$ are corresponding natural (see [10]) vector bundles with inner products on the fibers derived from the riemannian metrics on M and M', and
- D and D' are corresponding natural (see [24]) strongly elliptic differential operators on the respective spaces $L^2(M,V)$ and $L^2(M',V')$ of square integrable sections,

then we say that M and M' are *strongly isospectral*. Without going into details we mention that the Δ on $L^2(M, \bigwedge^p(T^*))$ are natural differential operators. Thus strongly isospectral implies

p–isospectral for every p. I don't know a counterexample to the converse.

In this Section we discuss two rather different isospectrality criteria that are especially appropriate for spherical space forms.

By *spherical space form* we mean a complete connected riemannian manifold of constant sectional curvature $+1$. The reason, of course, is that a q–dimensional spherical space form M is isometric to a riemannian quotient S^q/G, where S^q is the unit sphere in $\mathbb{R}^{q+1}$ and G is an appropriate finite group of isometries of S^q. Here G is a subgroup of the orthogonal group $O(q+1)$ and we identify functions on M with G–invariant functions on S^q. We will discuss the precise meaning of "appropriate" in Section 5.

Let $\{\lambda_m\}_{m=0,1,\dots}$ denote the distinct eigenvalues of the Laplace–Beltrami operator Δ on S^q, in strictly increasing order $0 = \lambda_0 < \lambda_1 < \dots$ Let $A(S^q, \lambda_m)$ denote the λ_m–eigenspace, and let d_m denote the multiplicity $\dim A(S^q, \lambda_m)$. With a certain normalization of the riemannian metric on S^q (from the negative of the Killing form of $O(q+1)$), they are given as follows (see, for example, [8] or [7]).

(2.1)

q	eigenvalues λ_m	$d_m = \dim A(S^q, \lambda_m)$	for
$2\ell - 1$	$\frac{m^2+2m(\ell-1)}{4(\ell-1)}$	$\frac{m+\ell-1}{\ell-1}\prod_{k=1}^{2\ell-3}\frac{m+k}{k}$	$\ell \geqq 2, m \geqq 0$
2ℓ	$\frac{m^2+m(2\ell-1)}{4\ell-2}$	$\frac{2m+2\ell-1}{2\ell-1}\prod_{k=1}^{2\ell-2}\frac{m+k}{k}$	$\ell \geqq 1, m \geqq 0$

Now $L^2(S^q) = \sum_m A(S^q, \lambda_m)$, Hilbert space direct sum of the eigenspaces of Δ as in (2.1). The key point here is that $A(S^q, \lambda_m)$ consists of the functions on S^q that are restrictions of homogeneous $\mathbb{C}$–valued polynomial functions of degree m on the ambient $\mathbb{R}^{q+1}$. It is usual to identify $A(S^q, \lambda_m)$ with the space of harmonic polynomials of degree m on $\mathbb{R}^{q+1}$.

Write $Pol^m(\mathbb{R}^{q+1})$ for the space of homogeneous $\mathbb{C}$–valued polynomial functions of degree m on $\mathbb{R}^{q+1}$. Write $Pol(\mathbb{R}^{q+1})$ for the space of all $\mathbb{C}$–valued polynomial functions on $\mathbb{R}^{q+1}$. So $Pol(\mathbb{R}^{q+1})$ is the algebraic direct sum of the $Pol^m(\mathbb{R}^{q+1})$. Write $A(S^q)$ for the algebraic direct sum of the eigenspaces $A(S^q, \lambda_m)$. Then $A(S^q) = Pol(\mathbb{R}^{q+1})/\mathcal{I}$, quotient of the ring $Pol(\mathbb{R}^{q+1})$ by the ideal $\mathcal{I}$ generated by $(\sum x_i^2) - 1$.

Consider a spherical space form $M = S^q/G$. Then $L^2(M) = L^2(S^q)^G$, space of G–invariant L^2 functions on S^q. G preserves $(\sum x_i^2) - 1$, and $Pol^m(\mathbb{R}^{q+1})$ is G–invariant for each m, so $A(S^q, \lambda_m)$ is G–invariant for each m, and the $A(S^q, \lambda_m)^G$ are well defined. $A(S^q)^G$ is their algebraic direct sum, and $A(S^q)^G$ is dense in $L^2(M)$. It follows that the Hilbert series

$$H(Pol(\mathbb{R}^{q+1})^G, z) = \sum\nolimits_{m \geqq 0} \dim Pol^m(\mathbb{R}^{q+1})^G \cdot z^m \tag{2.2}$$

for G–invariant polynomials graded by degree, determines the Hilbert series

$$H(A(S^q)^G, z) = \sum\nolimits_{m \geqq 0} \dim A(S^q, \lambda_m)^G \cdot z^m \tag{2.3}$$

for the spectrum of Δ. Here we have Molien's formula ([19]; or see [6, Sec. 227]; or see [18])

$$H(Pol(\mathbb{R}^{q+1})^G, z) = \frac{1}{|G|}\sum_{g \in G}\frac{1}{\det(I - zg)} \tag{2.4}$$

for the Hilbert series of $Pol(\mathbb{R}^{q+1})^G$. Note that $\det(zI - g) = w^{-(q+1)}\det(I - wg)$ where $w = 1/z$, so $\det(zI - g)$ determines $\det(I - zg)$. Putting all this together, we have the famous result [12, Proposition 2.1 and Theorem 2.2] of Ikeda:

Proposition 2.5 *Let $M = S^q/G$ and $M' = S^q/G'$ be spherical space forms (of the same constant positive curvature). Then the following are equivalent:*

1. *M and M' are isospectral.*
2. *The (meromorphic) Molien function $\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(I-zg)} = \frac{1}{|G'|}\sum_{g'\in G'}\frac{1}{\det(I-zg')}$.*
3. *The (meromorphic) Ikeda function $\frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(zI-g)} = \frac{1}{|G'|}\sum_{g'\in G'}\frac{1-z^2}{\det(zI-g')}$.*

In particular, if the sets $\{\det(zI-g)\}_{g\in G} = \{\det(zI-g')\}_{g'\in G'}$ (counting repetitions) then M and M' are isospectral, and if M and M' are isospectral then $|G| = |G'|$.

For the last comment, $F_G(z) = \frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(zI-g)}$ has a pole of order q at $z = 1$, and $\lim_{z\to 1}(1-z)^q F_G(z) = 2/|G|$.

Proposition 2.5 provides a generating function F_G that is appropriate for questions of 0–isospectrality. There are other generating functions F_G^k appropriate for the study of k–isospectrality in a range $k \leqq p$ [17, Proposition 2.1], but they are rather difficult to apply, even when G is a cyclic group. We rely instead on the Generalized Sunada Theorem of DeTurck and Gordon [9, Theorem 1.16 and Remark 1.18]; here also see Bérard [2].

Let J be a Lie group of isometries of a riemannian manifold M. In most of our applications we will have $M = S^q$ and $J = O(q+1)$, but J will involve orthocomplementation in some cases where M is an oriented real Grassmann manifold. Let G_1 and G_2 be subgroups of J that act freely and properly discontinuously on M such that the riemannian quotients $G_i\backslash M$ are compact. One says that G_1 and G_2 are *representation equivalent in J* if the induced representations

$$\rho_{G_i} = \mathrm{Ind}_{G_i}^J(1_{G_i}) : \text{ quasiregular representation of } J \text{ on } L^2(G_i\backslash J) \tag{2.6}$$

are equivalent. If G is a cocompact discrete subgroup in J, and if $a \in J$, we denote

$$r(J,G;a) = \sum_{[g]_G\subset[a]_J} \mathrm{vol}(C(g,G)\backslash C(g,J)) \tag{2.7}$$

with the following conventions. First, $[a]_J$ denotes the conjugacy class of a in J, and we are summing over the conjugacy classes $[g]_G$ in G contained in $[a]_J$, and the right hand side of (2.7) is zero in case $[a]_J$ does not meet G. Second, $C(g,J)$ and $C(g,G)$ denote the respective centralizers of g in J and in G. As $G\backslash J$ is compact, $C(g,G)\backslash C(g,J)$ is compact for every $g \in G$, so each $C(g,J)$ is unimodular. Now let $a \in J$ such that $C(a,J)$ meets G. Choose a Haar measure μ_a on $C(a,J)$, which is unimodular since it is J–conjugate to $C(g,J)$ for some $g \in G$. If $b \in J$ is J–conjugate to a now that conjugation carries μ_a to a well defined measure μ_b on $C(b,J)$. The volumes in (2.7) are the μ_g–volumes of fundamental domains on the $C(g,G)\backslash C(g,J)$. Thus, if G_1 and G_2 are cocompact discrete subgroups of J, and with any choices of measures μ_g, comparisons of the sort

$$r(J,G_1;a) = r(J,G_2;a) \text{ for all } a \in J \tag{2.8}$$

are meaningful. Then, using an appropriate variation on the Selberg Trace Formula (see [9, Proposition 1.14], [21, Proposition 6], or [23, Corollaire 3(a)]),

(2.9) G_1 and G_2 are representation equivalent in $J \Leftrightarrow r(J,G_1;a) = r(J,G_2;a)$ for all $a \in J$.

The Generalized Sunada Theorem ([9]; also see [2]) is

Proposition 2.10 *Let J be a closed subgroup of the Lie group of isometries of a riemannian manifold M. Let G_1 and G_2 be subgroups of J that act freely and properly discontinuously on M such that the riemannian quotient manifolds $M_i = G_i\backslash M$ are compact. If G_1 and G_2 are representation equivalent in J, then M_1 and M_2 are strongly isospectral.*

Suppose that the G_i are finite. Then one says that G_1 and G_2 are *almost conjugate in* J if $|G_1 \cap [g]_J| = |G_2 \cap [g]_J|$ for every $g \in J$. If J is compact and the G_i are finite, then

$$
\begin{aligned}
r(J, G_i; a) &= \sum\nolimits_{[g]_{G_i} \subset [a]_J} \operatorname{vol}(C(g, G_i) \backslash C(g, J)) \\
&= \sum\nolimits_{[g]_{G_i} \subset [a]_J} \operatorname{vol}(C(g, J))/|C(g, G_i)| \\
&= \big(\operatorname{vol}(C(a, J))/|G_i|\big) \sum\nolimits_{[g]_{G_i} \subset [a]_J} |G_i|/|C(g, G_i)| \\
&= \big(\operatorname{vol}(C(a, J))/|G_i|\big) \sum\nolimits_{[g]_{G_i} \subset [a]_J} \big|[g]_{G_i}\big| = \big(\operatorname{vol}(C(a, J))/|G_i|\big) \cdot \big|[a]_J \cap G_i\big|.
\end{aligned} \tag{2.11}
$$

We formulate this as

Lemma 2.12 *Suppose that J is compact and the G_i are finite. Then G_1 and G_2 are representation equivalent in J if and only if they are almost conjugate in J.*

Proof. If the G_i are representation equivalent, then the $M_i = G_i \backslash M$ are isospectral, in particular $\operatorname{vol}(M_1) = \operatorname{vol}(M_2)$, forcing $|G_1| = |G_2|$. Thus (2.11) shows that they are almost conjugate in J. If the G_i are almost conjugate in J, then $|G_1| = \sum_{[a]_J} |G_1 \cap [a]_J| = \sum_{[a]_J} |G_2 \cap [a]_J| = |G_2|$, so (2.11) shows that they are representation equivalent in J. □

Combining Proposition 2.10 with Lemma 2.12 we have the version of Sunada's Theorem that we will use for spherical space forms in Section 6 and for riemannian quotients of Grassmann manifolds in Section 9. Compare[1] [16, Corollary 2.6].

Corollary 2.13 *Let M be a compact riemannian manifold. Let G_1 and G_2 be finite groups of isometries acting freely on M. Suppose that there is a one–one map ψ of G_1 onto G_2 such that, if $g \in G_1$ then $\psi(g)$ is conjugate to g in the isometry group $I(M)$. Then the riemannian quotient manifolds $G_1 \backslash M$ and $G_2 \backslash M$ are strongly isospectral.*

3 Fixed Point Free Groups

In the next few Sections we recall certain elements of my classification [31] of spherical space forms. We need that for considerations of isospectrality. In this Section we recall (Propositions 3.1 and 3.2) the classification of abstract finite groups that can occur as fundamental groups of spherical space forms. Then in Section 4 we recall (Proposition 4.19) the ways in which any such abstract finite group can occur as the fundamental group of a spherical space form. In Section 5 we recall (Proposition 5.3) the equivalence condition that is the last element in the classification of spherical space forms. Finally, in Section 6 we will get back to isospectral questions.

Let $M = S^q/G$ be a spherical space form. Then G is a discontinuous group of isometries of S^q and the action of G on S^q is free. In other words, G is a finite subgroup of the orthogonal group $O(q+1)$, and if $g \in G$ has $+1$ for an eigenvalue then $g = 1$. For a $(+1)$–eigenvector of length 1 is a fixed point on S^q. The classification of all such groups G, up to isomorphism, is the content of [31, Theorems 6.1.11 and 6.3.1]. In this Section we describe the result in detail because we will need the details.

Let Γ be a finite group and let π be a finite dimensional complex representation of Γ. If $\pi(\gamma)$ does not have $+1$ for an eigenvalue, whenever $1 \neq \gamma \in \Gamma$, then we say that π and Γ are *fixed point free*. If Γ is a fixed point free group, then every abelian subgroup of Γ is cyclic. The classification [31, Theorem 6.1.11] for the solvable case is given by

[1] The term "isospectral" in [16] means p–isospectral for all $p \geqq 0$.

Proposition 3.1 *Let Γ be a finite solvable group. Then every abelian subgroup of Γ is cyclic, if and only if Γ is isomorphic to one of the following groups.*

Type	*Generators*	*Relations*	*Conditions*	*Order*
I	A, B	$A^m = B^n = 1$ $BAB^{-1} = A^r$	$m \geqq 1,\ n \geqq 1$ $(n(r-1), m) = 1$ $r^n \equiv 1\ (m)$	mn
II	A, B, R	*As for Type I; also* $R^2 = B^{n/2}$ $RAR^{-1} = A^s,\ RBR^{-1} = B^t$	*As for Type I; also* $s^2 \equiv r^{t-1} \equiv 1\ (m)$, $n = 2^u n'',\ u \geqq 2,\ (n'', 2) = 1,$ $t \equiv -1\ (2^u),\ t^2 \equiv 1\ (n)$	$2mn$
III	A, B, P, Q	*As for Type I; also* $P^4 = 1,\ P^2 = Q^2 = (PQ)^2,$ $AP = PA,\ AQ = QA,$ $BPB^{-1} = Q,\ BQB^{-1} = PQ$	*As for Type I; also* $n \equiv 1\ (2),\ n \equiv 0\ (3),$	$8mn$
IV	A, B, P, Q, R	*As for Type III; also* $R^2 = P^2,\ RPR^{-1} = QP,$ $RQR^{-1} = Q^{-1},$ $RAR^{-1} = A^s,\ RBR^{-1} = B^t$	*As for Type III; also* $s^2 \equiv r^{t-1} \equiv 1\ (m)$, $t \equiv -1\ (3),\ t^2 \equiv 1\ (n),$	$16mn$

Furthermore the following conditions are equivalent.

1. *Γ has a fixed point free representation.*
2. *If p_1 and p_2 are primes then every subgroup of order $p_1 p_2$ in Γ is cyclic.*
3. *Γ is of type I, II, III or IV above, with the additional condition: if d is the order of r in the multiplicative group of residues modulo m of integers prime to m, then $n' = n/d$ is divisible by every prime divisor of d.*

Recall that the symmetry groups of the regular m–gon and the regular polyhedra are

- Regular m–gon: $\mathbb{D}_m = \langle A, B\rangle,\ A^m = B^2 = 1,\ BAB^{-1} = A^{-1}$, order $2m$,
- Tetrahedron: $\mathbb{T} = \langle B, P, Q\rangle,\ B^3 = P^2 = Q^2 = 1,\ PQ = QP,\ BPB^{-1} = Q,\ BQB^{-1} = PQ$, order 12,
- Octahedron: $\mathbb{O} = \langle B, P, Q, R\rangle,\ B^3 = P^2 = Q^2 = R^2 = 1,\ PQ = QP,\ RBR^{-1} = B^{-1},\ BPB^{-1} = Q,\ BQB^{-1} = PQ,\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1}$, order 24,
- Icosahedron: $\mathbb{I} = \langle A, B, C\rangle,\ A^3 = B^3 = C^5 = ABC = 1$, order 60.

Their inverse images under the 2–sheeted covering group $p : SU(2) \to SO(3)$ are

- Binary dihedral group $\mathbb{D}^*_m = p^{-1}(\mathbb{D}_m)$, order $4m$,
- Binary tetrahedral group $\mathbb{T}^* = p^{-1}(\mathbb{T})$, order 24,
- Binary octahedral group $\mathbb{O}^* = p^{-1}(\mathbb{O})$, order 48, and
- Binary icosahedral group $\mathbb{I}^* = p^{-1}(\mathbb{I})$, order 120.

Further, $\mathbb{T}^* = SL(2,3)$ and $I^* = SL(2,5)$ where $SL(2,q)$ denotes the 2×2 special linear group over the field of q elements. Denote

$$\theta : SL(2,5) \to SL(2,5), \text{ conjugation by } \left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right).$$

Then $\theta^2 = 1$ and θ generates the group of outer (mod inner) automorphisms of $SL(2,5)$.

The classification [31, Theorem 6.3.1] for the non–solvable case is given by

Proposition 3.2 *Let Γ be a non–solvable finite group. Then Γ has a fixed point free representation if and only if it is isomorphic to one of the following groups.*

Type V. $\Gamma = \Sigma \times SL(2,5)$ *where* Σ *is a fixed point free group of Type I with* $(|\Sigma|, 30) = 1$.

Type VI. $\Gamma = \langle \Phi, S\rangle$ *where* $\Phi = \Sigma \times SL(2,5)$ *is a normal subgroup of index* 2 *that is a fixed point free group of Type V, where* $S^2 = -I \in SL(2,5)$ *and* $SLS^{-1} = \theta(L)$ *for* $L \in SL(2,5)$. *(Here it is automatic that* S *normalizes both* Σ *and* $SL(2,5)$.)

Now we have listed the fundamental groups of spherical space forms, as abstract finite groups. The next step is to indicate how each of those abstract groups Γ can occur as a group G for a spherical space form S^q/G.

4 Fixed Point Free Representations

We recall the classification [31, Theorem 7.2.18] of irreducible complex fixed point free representations of fixed point free groups. Write $\mathcal{F}_{\mathbf{C}}(\Gamma)$ for the set of equivalence classes of irreducible fixed point free complex representations of Γ. When $m > 0$ is an integer, write K_m for the multiplicative group of residues modulo m of integers prime to m. The Euler function φ is defined by: $\phi(m)$ is the order $|K_m|$ of K_m.

Type I. Let $\Gamma = \langle A, B\rangle$ be a fixed point free group of Type I as in Proposition 3.1. Then [31, Theorem 5.5.6] then $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ (equivalence classes of) irreducible complex fixed point free representations $\pi_{k,\ell}$ of Γ, k and ℓ integers with $(k,m) = 1 = (\ell, n)$, of degree d, given by

$$(4.1) \qquad \pi_{k,\ell}(A) = \operatorname{diag}\{e^{2\pi ik/m}, e^{2\pi ikr/m}, \dots, e^{2\pi ikr^{d-1}/m}\} \text{ and } \pi_{k,\ell}(B) = \begin{pmatrix} 0 & I \\ e^{2\pi i\ell/n'} & 0 \end{pmatrix}$$

where the I in $\pi_{k,\ell}(B)$ is $(d-1)\times(d-1)$. In other words, $\pi_{k,\ell}$ is the representation $\operatorname{Ind}_{\Sigma}^{\Gamma}(\sigma'_k \otimes \sigma''_\ell)$ induced from $\Sigma = \langle A, B^d\rangle \cong \langle A\rangle \times \langle B^d\rangle$ where $\sigma'_k(A^s) = e^{2\pi iks/m}$ and $\sigma''_\ell((B^d)^t) = e^{2\pi i\ell t/n'}$. The representations $\pi_{k,\ell} \cong \pi_{k',\ell'}$ if, and only if, both $\ell \equiv \ell'$ (n') and $k \equiv k'r^c$ (m) for some integer c with $0 \leqq c < d$.

Type II. Let $\Gamma = \langle A, B, R\rangle$ be a fixed point free group of Type II as in Proposition 3.1. Denote

$$(4.2) \qquad \alpha_{k,\ell} = \operatorname{Ind}_{\Sigma}^{\Gamma}(\pi_{k,\ell}) \text{ for all } \pi_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\Sigma), \text{ where } \Sigma = \langle A, B\rangle.$$

Decompose

A : $\langle A\rangle = \langle A_1\rangle \times \langle A_2\rangle$ where $RA_1R^{-1} = A_1^{-1}$ and $RA_2R_{-1} = A_2$. Let m_i be the order of A_i. Thus $m = m_1m_2$ and $(m_1, m_2) = 1$.

B : $\langle B\rangle = \langle B_1\rangle \times \langle B_2\rangle$ where $RB_1R^{-1} = B_1^{-1}$ and $RB_2R_{-1} = B_2$. Let n_i be the order of B_i. Thus $n = n_1n_2$ and $(2n_1, b_2) = 1$.

Then $\alpha_{k,\ell} \cong \alpha_{k',\ell'}$ just when $k \equiv \epsilon k'r^u$ (m_2), $\ell d \equiv \epsilon\ell' d$ (n_1), and $\ell d \equiv \ell' d$ (n_2), with $\epsilon = \pm 1$ and $0 \leqq u < d$.

The $\alpha_{k,\ell}$ are irreducible, and $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(2mn)/(2mn)^2$ (equivalence classes of) representations $\alpha_{k,\ell}$ of degree $2d$, except when both $\ell \equiv t\ell$ (n') and $k \equiv skr^c$ (m) for some integer c with $0 \leqq c < d$. Now [31, pp. 204–205] $\alpha_{k,\ell}$ is irreducible except in the case where $n_1 = 1, d \equiv 2$ (4), and some RB^w commutes with A. In that case

$$(4.3) \qquad \beta_{k,\ell} \text{ is irreducible where } \alpha_{k,\ell} = \beta_{k,\ell} \oplus \beta_{k,\ell},$$

and $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ equivalence classes of representations $\beta_{k,\ell}$ of degree d.

Type III. Let $\Gamma = \langle A, B, P, Q\rangle$ be a fixed point free group of Type III as in Proposition 3.1. In order to describe $\mathcal{F}_{\mathbf{C}}(\Gamma)$ we will need the generalized quaternion groups

$$\mathbb{Q}2^a: \quad A^{2^{a-1}} = 1,\ B^2 = A^{2^{a-2}},\ BAB^{-1} = A^{-1},\ a \geqq 3, \text{ order } 2^a, \cong D^*_{2^{a-2}}$$

$\mathbb{Q}8$ is the most familiar example, $\mathbb{Q}8 \cong \{\pm 1, \pm i, \pm j, \pm k\}$.

Each odd integer k gives a faithful representation $\sigma_k(A^s) = e^{2\pi iks/2^{a-1}}$ of the cyclic subgroup $\langle A\rangle \subset \mathbb{Q}2^a$ and an irreducible fixed point free complex representation of $\mathbb{Q}2^a$:

$$(4.4) \qquad \alpha_k = \operatorname{Ind}^\Gamma_{\langle A\rangle}(\sigma_k), \text{ so } \alpha_k(A) = \begin{pmatrix} e^{2\pi iks/2^{a-1}} & 0 \\ 0 & e^{-2\pi iks/2^{a-1}} \end{pmatrix} \text{ and } \alpha_k(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\mathcal{F}_{\mathbb{C}}(\Gamma)$ consists of the (equivalence classes of the) α_k.

We will also need certain extensions of the binary tetrahedral group $\mathbb{T}^*$:

$$\mathbb{T}^*_v\colon\ X^{3^v} = P^4 = 1,\ P^2 = Q^2,\ PQP^{-1} = Q^{-1},\ XPX^{-1} = Q,\ XQX^{-1} = PQ,\ v \geqq 1.$$

Note that $\mathbb{T}^*_1 = \mathbb{T}^*$. One knows $\mathcal{F}_{\mathbb{C}}(\mathbb{T}^*_v)$ from [31, Lemma 7.1.3]:

Lemma 4.5 *$\mathbb{T}^*$ has just one irreducible fixed point free complex representation, $\mathcal{F}_{\mathbb{C}}(\mathbb{T}^*) = \{\tau\}$. If $v > 1$ then $\mathbb{T}^*_v$ has just $2\cdot 3^{v-1}$ inequivalent irreducible fixed point free complex representations,*

$$(4.6) \qquad \mathcal{F}_{\mathbb{C}}(\mathbb{T}^*_v) = \{\tau_k\},\ k \text{ determined mod } 3^v, (k,3) = 1,$$

where $\tau_k(X)$ has eigenvalues $e^{2\pi i(k\pm 3^{v-1})/3^v}$. Here $\tau_{-k} = \overline{\tau_k}$, and $\tau_k(X^3) = e^{2\pi ik/3^{v-1}}I$, and τ and the τ_k all have degree 2.

Lemma 4.5 will also play a key role in the calculations (6.6) of Theorem 6.4.

Case $n \not\equiv 0$ (9). Then $d \not\equiv 0$ (3) and $\Gamma = \langle A, B^3\rangle \times \langle B^{n''}, P, Q\rangle$ where $n'' = n/3$. Here $\langle A, B^3\rangle$ is a group of Type I with the same integer d as $\langle A, B\rangle$ and $\langle B^{n''}, P, Q\rangle \cong \mathbb{T}^*$. $\mathcal{F}_{\mathbb{C}}(\Gamma)$ consists of the $\varphi(mn)/2d^2$ (equivalence classes of) representations of degree $2d$

$$(4.7) \qquad \nu_{k,\ell} = \pi_{k,\ell} \otimes \tau \text{ where } \pi_{k,\ell} \in \mathcal{F}_{\mathbb{C}}(\langle A, B^3\rangle) \text{ and } \{\tau\} = \mathcal{F}_{\mathbb{C}}(\mathbb{T}^*) = \mathcal{F}_{\mathbb{C}}(\langle B^{n''}, P, Q\rangle).$$

Case $n \equiv 0$ (9) but $d \not\equiv 0$ (3). Then $n = 3^v n''$ with $(3, n'') = 1$ and $v > 1$, and $\Gamma = \langle A, B^{3^v}\rangle \times \langle B^{n''}, P, Q\rangle$. As before, $\langle A, B^{3^v}\rangle$ is a group of Type I with the same integer d as $\langle A, B\rangle$ and $\langle B^{n''}, P, Q\rangle \cong \mathbb{T}^*_v$, $\mathcal{F}_{\mathbb{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ (equivalence classes of) representations of degree $2d$

$$(4.8) \qquad \nu_{k,\ell,j} = \pi_{k,\ell} \otimes \tau_j \text{ where } \pi_{k,\ell} \in \mathcal{F}_{\mathbb{C}}(\langle A, B^{3^v}\rangle) \text{ and } \tau_j \in \mathcal{F}_{\mathbb{C}}(\mathbb{T}^*_v) = \mathcal{F}_{\mathbb{C}}(\langle B^{n''}, P, Q\rangle).$$

Case $d \equiv 0$ (3). The index d normal subgroup $\Sigma = \langle A, B^d, P, Q\rangle$ decomposes as $\langle A\rangle \times \langle B^d\rangle \times \langle P, Q\rangle$ and $\langle P, Q\rangle = \mathbb{Q}8$. Thus $\mathcal{F}_{\mathbb{C}}(\Sigma)$ consists of the $\sigma'_k \otimes \sigma''_\ell \otimes \alpha$ where $\sigma'_k(A^a) = e^{2\pi iak/m}$, where $\sigma''_\ell((B^d)^b) = e^{2\pi ibd/n'}$, and where $\{\alpha\} = \mathcal{F}_{\mathbb{C}}(\mathbb{Q}8)$. Here $\mathcal{F}_{\mathbb{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ (equivalence classes of) representations of degree $2d$

$$(4.9) \qquad \mu_{k,\ell} = \operatorname{Ind}^\Gamma_\Sigma(\sigma'_k \otimes \sigma''_\ell \otimes \alpha),\ (k,m) = 1 = (\ell, n) \text{ and } \{\alpha\} = \mathcal{F}_{\mathbb{C}}(\mathbb{Q}8) = \mathcal{F}_{\mathbb{C}}(\langle P, Q\rangle).$$

Here $\mu_{k,\ell} \cong \mu_{k',\ell'}$ if and only if $k \equiv k'r^u$ (m) and $\ell d \equiv \ell' d$ (n) with $0 \leqq u < d$.

Type IV. Let $\Gamma = \langle A, B, P, Q, R\rangle$ be a fixed point free group of Type IV as in Proposition 3.1. In order to describe $\mathcal{F}_{\mathbb{C}}(\Gamma)$ we will need certain extensions of the binary octahedral group $\mathbb{O}^*$:

$$\mathbb{O}^*_v\colon\ X^{3^v} = P^4 = 1,\ P^2 = Q^2 = R^2,\ PQP^{-1} = Q^{-1},\ XPX^{-1} = Q,\ XQX^{-1} = PQ,\ RXR^{-1} = X^{-1},\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1},\ v \geqq 1.$$

Note that $\mathbb{O}^*_1 = \mathbb{O}^*$. One knows $\mathcal{F}_{\mathbb{C}}(\mathbb{O}^*_v)$ from [31, Lemma 7.1.5]:

Lemma 4.10 $\mathbb{O}^*$ *has just two irreducible fixed point free complex representations* $o_{\pm 1}$*, each self-conjugate and degree* 2*, differing by an outer automorphism. If* $v > 1$ *then* $\mathbb{O}_v^*$ *has just* 3^{v-1} *inequivalent irreducible fixed point free complex representations,*

$$\mathcal{F}_{\mathbf{C}}(\mathbb{O}_v^*) = \{o_k\},\ o_k = \mathrm{Ind}_{\mathbf{T}_v^*}^{\mathbb{O}_v^*}(\tau_k) = o_{-k},\ \pm k \text{ determined modulo } 3^v, (k,3) = 1, \tag{4.11}$$

each self-conjugate and degree 4*, characterized by the fact that* $o_k(X)$ *has eigenvalues* $e^{2\pi i(\pm k \pm 3^{v-1})/3^v}$.

Lemma 4.10 will also play a key role in the calculations (6.6) of Theorem 6.4.

Case $n \not\equiv 0$ (9) and $\Gamma \cong \langle A, B^3\rangle \times \langle R, B^{n''}, P, Q\rangle$ with $n'' = n/3$. Then $\langle R, B^{n''}, P, Q\rangle \cong \mathbb{O}^*$ and we identify them. It is automatic that $d \not\equiv 0$ (3) in this case. Here $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ equivalence classes of representations of degree $2d$

$$\psi_{k,\ell,j} = \pi_{k,\ell} \otimes o_j \text{ where } \pi_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\langle A, B^3\rangle) \text{ and } \{o_1, o_{-1}\} = \mathcal{F}_{\mathbf{C}}(\mathbb{O}^*). \tag{4.12}$$

Case $n \not\equiv 0$ (9) and $\Gamma \not\cong \langle A, B^3\rangle \times \langle R, B^{n''}, P, Q\rangle$ with $n'' = n/3$. Then $\langle R, B^{n''}, P, Q\rangle \cong \mathbb{O}^*$ and we identify them. It is automatic that $d \not\equiv 0$ (3) in this case. If $\pi \in \mathcal{F}_{\mathbf{C}}(\langle A, B^3\rangle)$ then π is not equivalent to π' defined by $\pi'(g) = \pi(RgR^{-1})$, so $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/4d^2$ equivalence classes of induced representations of degree $4d$

$$\gamma_{k,\ell} = \mathrm{Ind}_{\Sigma}^{\Gamma}(\nu_{k,\ell}) \text{ where } \Sigma = \langle A, B, P, Q\rangle \text{ and } \nu_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\Sigma) \text{ as in (4.7).} \tag{4.13}$$

Here $\gamma_{k,\ell} \cong \gamma_{k',\ell'}$ if and only if $k \equiv \epsilon k' r^u$ (m_1), $k \equiv k' r^u$ (m_2), $3\ell d \equiv \epsilon 3\ell' d$ (n_1) and $3\ell d \equiv 3\ell' d$ (n_2) with $\epsilon = \pm 1$ and $0 \leqq u < d$.

Case $n \equiv 0$ (9), $d \not\equiv 0$ (3), and $\Gamma \cong \langle A, B^{3^v}\rangle \times \langle R, B^{n''}, P, Q\rangle$ with $n = 3^v n''$ where $(3, n'') = 1$. As before, $\langle R, B^{n''}, P, Q\rangle \cong \mathbb{O}_v^*$, and we identify them. $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/2d^2$ equivalence classes of representations of degree $4d$

$$\xi_{k,\ell,j} = \pi_{k,\ell} \otimes o_j \text{ where } \pi_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\langle A, B^{3^v}\rangle) \text{ and } o_j \in \mathcal{F}_{\mathbf{C}}(\mathbb{O}_v^*). \tag{4.14}$$

Case $n \equiv 0$ (9), $d \not\equiv 0$ (3), and $\Gamma \not\cong \langle A, B^{3^v}\rangle \times \langle R, B^{n''}, P, Q\rangle$ with $n = 3^v n''$ where $(3, n'') = 1$. As before, $\langle R, B^{n''}, P, Q\rangle \cong \mathbb{O}_v^*$, and we identify them. If $\pi \in \mathcal{F}_{\mathbf{C}}(\langle A, B^{3^v}\rangle)$ then π is not equivalent to π' defined by $\pi'(g) = \pi(RgR^{-1})$, so $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/2d^2$ equivalence classes of induced representations of degree $4d$

$$\gamma_{k,\ell,j} = \mathrm{Ind}_{\Sigma}^{\Gamma}(\nu_{k,\ell,j}) \text{ where } \Sigma = \langle A, B, P, Q\rangle \text{ and } \nu_{k,\ell,j} \in \mathcal{F}_{\mathbf{C}}(\Sigma) \text{ as in (4.8).} \tag{4.15}$$

Here $\gamma_{k,\ell,j} \cong \gamma_{k',\ell',j'}$ if and only if $k \equiv \epsilon k' r^u$ (m_1), $k \equiv k' r^u$ (m_2), $3^v \ell d \equiv \epsilon 3^v \ell' d$ (n_1), $3^v \ell d \equiv 3^v \ell' d$ (n_2) and $j = \epsilon j'$ where $\epsilon = \pm 1$ and $0 \leqq u < d$.

Case $d \equiv 0$ (3). Then 3 divides $n' = n/d$, $n \equiv 0$ (9), and $RB^dR^{-1} \neq B^d$. If $\mu \in \mathcal{F}_{\mathbf{C}}(\langle A, B, P, Q\rangle)$ and $\mu'(g) = \mu(RgR^{-1})$ now $\mu \not\cong \mu'$, so $\mathrm{Ind}_{\langle A,B,P,Q\rangle}^{\Gamma}(\mu)$ is irreducible. Thus $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/2d^2$ equivalence classes of induced representations of degree $4d$

$$\eta_{k,\ell} = \mathrm{Ind}_{\Sigma}^{\Gamma}(\mu_{k,\ell}) \text{ where } \Sigma = \langle A, B, P, Q\rangle \text{ and } \mu_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\Sigma) \text{ as in (4.9).} \tag{4.16}$$

Here $\eta_{k,\ell} \cong \eta_{k',\ell'}$ if and only if $k \equiv \epsilon k' r^u$ (m_1), $k \equiv k' r^u$ (m_2), $\ell d \equiv \epsilon \ell' d$ (n_1) and $\ell d \equiv \ell' d$ (n_2) where $\epsilon = \pm 1$ and $0 \leqq u < d$.

Type V. Let $\Gamma = \Sigma \times \mathbb{I}^*$ be a fixed point free group of Type V as in Proposition 3.2. $\mathcal{F}_{\mathbf{C}}(\mathbb{I}^*) = \{\iota_1, \iota_{-1}\}$ where the ι_j are degree 2, self-conjugate, and differ by the outer automorphism θ used

to describe groups of Type VI. Thus $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $2\varphi(mn)/d^2$ equivalence classes of representations of degree 2d given by

$$\iota_{k,\ell,j} = \pi_{k,\ell} \otimes \iota_j \text{ where } \pi_{k,\ell} \in \mathcal{F}_{\mathbf{C}}(\Sigma) \text{ and } \iota_j \in \mathcal{F}_{\mathbf{C}}(\mathbb{I}^*). \tag{4.17}$$

Type VI. Let $\Gamma = \langle \Phi S \rangle$ where $\Phi = \Sigma \times \mathbb{I}^*$ is a normal subgroup of index 2 that is a fixed point free group of Type V, and conjugation by S is the outer automorphism θ of $\mathbb{I}^*$. Then $\mathcal{F}_{\mathbf{C}}(\Gamma)$ consists of the $\varphi(mn)/d^2$ equivalence classes[2] of induced representations of degree $4d$

$$\kappa_{k,\ell} = \operatorname{Ind}_{\Phi}^{\Gamma}(\iota_{k,\ell,1}) \text{ where } \iota_{k,\ell,1} \in \mathcal{F}_{\mathbf{C}}(\Phi) \text{ as in (4.17).} \tag{4.18}$$

Here $\kappa_{k,\ell} \cong \kappa_{k',\ell'}$ if and only if $k \equiv \epsilon k' r^u \ (m_1)$, $k \equiv k' r^u \ (m_2)$, $\ell d \equiv \epsilon \ell' d \ (n_1)$ and $\ell d \equiv \ell' d \ (n_2)$ where $0 \leqq u < d$.

We summarize as in [31, Theorem 7.2.18].

Proposition 4.19 *Let Γ be a finite group that admits a fixed point free representation. Let $\mathcal{F}_{\mathbf{C}}(\Gamma)$ denote the set of all equivalence classes of irreducible complex fixed point free representations of Γ. Then the elements of $\mathcal{F}_{\mathbf{C}}(\Gamma)$ all have the same degree $\delta(\Gamma)$. Define $\zeta(\Gamma) = \varphi(|\Gamma|)/\delta(\Gamma)^2$. Then $\mathcal{F}_{\mathbf{C}}(\Gamma)$ is given as follows,[3] where d refers to the subgroup $\langle A, B \rangle$ and "Type" refers to Propositions 3.1 and 3.2.*

Type	Conditions	$\delta(\Gamma)$	$\lvert\mathcal{F}_{\mathbf{C}}(\Gamma)\rvert$	$\mathcal{F}_{\mathbf{C}}(\Gamma)$	See	self-conjugate condition	real
I	none	d	$\zeta(\Gamma)$	$\pi_{k,\ell}$	(4.1)	$n' \leqq 2$	$\lvert\Gamma\rvert \leqq 2$
II	$n_1 \not\equiv 4$ or d odd or all $RB^wA \neq ARB^w$	$2d$	$\zeta(\Gamma)$	$\alpha_{k,\ell}$	(4.2)	$RBR^{-1} = B^{-1}$, $d = 2$ if $m \neq 1$, $RAR^{-1} = A^{-1}$	never
	$n = 4$, $d \equiv 2$ (4) some $RB^wA = ARB^w$	d	$\zeta(\Gamma)/2$	$\beta_{k,\ell}$	(4.3)	$\Gamma = \mathbb{D}_{2m}^*$	never
III	$n \not\equiv 0$ (9)	$2d$	$\zeta(\Gamma)/2$	$\nu_{k,\ell}$	(4.7)	$\Gamma = \mathbb{T}^*$	never
	$n \equiv 0$ (9), $d \not\equiv 0$ (3)	$2d$	$\zeta(\Gamma)$	$\nu_{k,\ell,j}$	(4.8)	never	never
	$d \equiv 0$ (3)	$2d$	$\zeta(\Gamma)$	$\mu_{k,\ell}$	(4.9)	never	never
IV	$n \not\equiv 0$ (9) $\Gamma = \langle A, B^3 \rangle \times \mathbb{O}^*$	$2d$	$\zeta(\Gamma)/2$	$\psi_{k,\ell,j}$	(4.12)	$\Gamma = \mathbb{O}^*$	never
	$n \not\equiv 0$ (9) $\Gamma \neq \langle A, B^3 \rangle \times \mathbb{O}^*$	$4d$	$\zeta(\Gamma)/2$	$\gamma_{k,\ell}$	(4.13)	$A = 1$, $RBR^{-1} = B^{-1}$	never
	$n \equiv 0$ (9), $d \not\equiv 0$ (3) $\Gamma = \langle A, B^{3^v} \rangle \times \mathbb{O}_v^*$	$4d$	$\zeta(\Gamma)$	$\xi_{k,\ell,j}$	(4.14)	$\Gamma = \mathbb{O}_v^*$	never
	$n \equiv 0$ (9), $d \not\equiv 0$ (3) $\Gamma \neq \langle A, B^{3^v} \rangle \times \mathbb{O}_v^*$	$4d$	$\zeta(\Gamma)$	$\gamma_{k,\ell,j}$	(4.15)	$A = 1$, $RBR^{-1} = B^{-1}$	never
	$d \equiv 0$ (3)	$4d$	$\zeta(\Gamma)$	$\eta_{k,\ell}$	(4.16)	never	never
V	none	$2d$	$\zeta(\Gamma)/4$	$\iota_{k,\ell,j}$	(4.17)	$\Gamma = \mathbb{I}^*$	never
VI	none	$4d$	$\zeta(\Gamma)/4$	$\kappa_{k,\ell}$	(4.18)	$\Gamma = \langle S, \mathbb{I}^* \rangle$	never

5 The Classification of Spherical Space Forms

Let $M = S^q/G$ be a spherical space form of dimension q. We view $G = \rho(\Gamma)$, image of a fixed point free group Γ by a fixed point free real representation ρ. If $|\Gamma| = 1$ then $M = S^q$. If $|\Gamma| = 2$ then M is the real projective space $P^q(\mathbb{R})$. If $|\Gamma| > 2$ then [31, Theorem 7.2.18], quoted as Proposition 4.19 above, shows that none of the representations in $\mathcal{F}_{\mathbf{C}}(\Gamma)$ are real, and it follows that ρ is of the form $\sum(\phi_a \oplus \overline{\phi_a})$ with each $\phi_a \in \mathcal{F}_{\mathbf{C}}(\Gamma)$. From this point on we assume $|\Gamma| > 2$, though much of what we say applies in a trivial way when $|\Gamma|$ is 1 or 2.

[2]There are some misprints in [31, §7.2]. In some places one finds $\kappa_{k,\ell,j}$; that should always be $\kappa_{k,\ell}$. The point here is that, since ι_1 and ι_{-1} differ by θ, the representations $\operatorname{Ind}_{\Phi}^{\Gamma}(\iota_{k,\ell,1})$ and $\operatorname{Ind}_{\Phi}^{\Gamma}(\iota_{k,\ell,-1})$ are equivalent.

[3]This corrects some misprints in the second column of the table in [31, Theorem 7.2.18], where N should be n, and a misprint in the fifth column, where $\kappa_{k,\ell,j}$ should be $\kappa_{k,\ell}$.

Two q–dimensional spherical space forms $S^q/\rho_1(\Gamma)$ and $S^q/\rho_2(\Gamma)$ are isometric if and only if there is an automorphism A of Γ such that $\rho_1 \cong \rho_2 \cdot \mathrm{A}$. Thus the automorphism group $\mathrm{Aut}(\Gamma)$ acts on

(5.1) $\mathcal{F}_{\mathbf{R}}(q,\Gamma)$: real fixed point free representations of Γ of degree $q+1$.

Now recall the main point of [31, Chapter 7].

Proposition 5.2 *The quotient $\mathcal{S}(q,\Gamma) = \mathcal{F}_{\mathbf{R}}(q,\Gamma)/\mathrm{Aut}(\Gamma)$ is the set of isometry classes of q-dimensional spherical space forms with fundamental group isomorphic to Γ.*

The action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{F}_{\mathbf{R}}(q,\Gamma)$ is specified by the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{F}_{\mathbf{C}}(\Gamma)$. That is worked out in [31, Theorem 7.3.21] as follows.

Proposition 5.3 *Let Γ be a finite group that admits a fixed point free representation. Let $\mathcal{F}_{\mathbf{C}}(\Gamma)$ denote the set of all equivalence classes of irreducible complex fixed point free representations of Γ. Then the transformations of $\mathcal{F}_{\mathbf{C}}(\Gamma)$, given by composition with automorphisms of Γ, are given as follows, where we retain the notation of Proposition 4.19.*

Type	Case	Transformations on $\mathcal{F}_{\mathbf{C}}(\Gamma)$	Conditions
I	all	$\mathrm{A}_{a,b} : \pi_{k,\ell} \mapsto \pi_{ak,b\ell}$	$(a,m) = (b,n) = 1,\ b \equiv 1\ (d)$
II	all	$\mathrm{A}_{a,b} : \alpha_{k,\ell} \mapsto \alpha_{ak,b\ell}$ or $\mathrm{A}_{a,b} : \beta_{k,\ell} \mapsto \beta_{ak,b\ell}$	$(a,m) = (b,n) = 1,\ b \equiv 1\ (d)$
III	$n \not\equiv 0\ (9)$	$\mathrm{A}_{a,b} : \nu_{k,\ell} \mapsto \nu_{ak,b\ell}$	$(a,m) = (b,n/3) = 1,\ b \equiv 1\ (d)$
	$n \equiv 0\ (9),\ d \not\equiv 0\ (3)$	$\mathrm{A}_{a,b,c} : \nu_{k,\ell,j} \mapsto \nu_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1,$ $b \equiv 1\ (d)$
	$d \equiv 0\ (3)$	$\mathrm{A}_{a,b} : \mu_{k,\ell} \mapsto \mu_{ak,b\ell}$	$(a,m) = (b,n) = 1,\ b \equiv 1\ (d)$
IV	$n \not\equiv 0\ (9),$ $\Gamma = \langle A, B^3\rangle \times \mathbb{O}^*$	$\mathrm{A}_{a,b,c} : \psi_{k,\ell,j} \mapsto \psi_{ak,b\ell,cj}$	$(a,m) = (b,n/3) = 1,\ c = \pm 1,$ $b \equiv 1\ (d)$
	$n \not\equiv 0\ (9),$ $\Gamma \neq \langle A, B^3\rangle \times \mathbb{O}^*$	$\mathrm{A}_{a,b} : \gamma_{k,\ell} \mapsto \gamma_{ak,b\ell}$	$(a,m) = (b,n/3) = 1,$ $b \equiv 1\ (d)$
	$n \equiv 0\ (9),\ d \not\equiv 0\ (3),$ $\Gamma = \langle A, B^{3^v}\rangle \times \mathbb{O}_v^*$	$\mathrm{A}_{a,b,c} : \xi_{k,\ell,j} \mapsto \xi_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1,$ $b \equiv 1\ (d)$
	$n \equiv 0\ (9),\ d \not\equiv 0\ (3),$ $\Gamma \neq \langle A, B^{3^v}\rangle \times \mathbb{O}_v^*$	$\mathrm{A}_{a,b,c} : \gamma_{k,\ell,j} \mapsto \gamma_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1,$ $b \equiv 1\ (d)$
	$d \equiv 0\ (3)$	$\mathrm{A}_{a,b} : \eta_{k,\ell} \mapsto \eta_{ak,b\ell}$	$(a,m) = (b,n) = 1, b \equiv 1\ (d)$
V	all	$\mathrm{A}_{a,b,c} : \iota_{k,\ell,j} \mapsto \iota_{ak,b\ell,cj}$	$(a,m) = (b,n) = 1,\ c = \pm 1,$ $b \equiv 1\ (d)$
VI	all	$\mathrm{A}_{a,b} : \kappa_{k,\ell} \mapsto \kappa_{ak,b\ell}$	$(a,m) = (b,n) = 1,\ b \equiv 1\ (d)$

The conditions in Proposition 5.3 are crucial for this paper, especially the condition $b \equiv 1\ (d)$. If $\Gamma = \langle A, B\rangle$ is of Type I, then $\mathrm{A}_{a,b}$ comes from any $\psi \in \mathrm{Aut}(\Gamma)$ with $\psi(A) = A^a$ and $\psi(B) = B^b A^w$. Here $(a,m) = (b,n) = 1$ because ψ preserves the order of a group element. Since $BAB^{-1} = A^r$ we must have $\psi(B)\psi(A)\psi(B)^{-1} = \psi(A^r)$, in other words $A^{ar^b} = A^{ar}$, so $ar^b \equiv ar\ (m)$. Since $(a,m) = 1$ this says $r^{b-1} \equiv 1\ (m)$. As d is the order of r in K_m now d divides $b-1$, which is why $b \equiv 1\ (d)$.

It is important also to remember that the representations listed in in column 5 of the table of Proposition 4.19 are not necessarily distinct: see the equivalence conditions described just after their defining equations as listed in column 6 of that table. The part of the equivalence

conditions that involve the parameter ℓ is as follows.

(5.4)

Type	Representations	Conditions
I	$\pi_{k,\ell}$	$\ell d \equiv \ell' d\ (n)$
II	$\alpha_{k,\ell}$ or $\beta_{k,\ell}$	$\ell d \equiv \epsilon\ell' d\ (n_1)$ and $\ell d \equiv \epsilon\ell' d\ (n_2)$
III	$\nu_{k,\ell}$ $(v=1)$, $\nu_{k,\ell,j}$ $(v>1)$ or $\mu_{k,\ell}$ $(v=0)$	$3^v\ell d \equiv 3^v\ell' d\ (n)$
IV	$\psi_{k,\ell,j}$ $(v=1)$ or $\xi_{k,\ell,j}$ $(v>1)$	$3^v\ell d \equiv 3^v\ell' d\ (n)$
	$\gamma_{k,\ell}$ $(v=1)$, $\gamma_{k,\ell,j}$ $(v>1)$ or $\eta_{k,\ell}$ $(v=0)$	$3^v\ell d \equiv \epsilon 3^v\ell' d\ (n_1)$ and $3^v\ell d \equiv 3^v\ell' d\ (n_2)$
V	$\iota_{k,\ell,j}$	$\ell d \equiv \ell' d\ (n)$
VI	$\kappa_{k,\ell}$	$\ell d \equiv \epsilon\ell' d\ (n_1)$ and $\ell d \equiv \epsilon\ell' d\ (n_2)$

We will need this later.

6 The Isospectral Groups $\mathcal{A}(\Gamma)$ and $\mathcal{A}(q,\Gamma)$

We define $\mathcal{A}(\Gamma)$ to be the group of transformations of $\mathcal{F}_{\mathbf{C}}(\Gamma)$ obtained by omitting the condition $b \equiv 1\ (d)$ in Proposition 5.3. In other words,

Definition 6.1 Let Γ be a finite group that admits a fixed point free representation. Let $\mathcal{F}_{\mathbf{C}}(\Gamma)$ denote the set of all equivalence classes of irreducible complex fixed point free representations of Γ. Then the *isospectral group* $\mathcal{A}(\Gamma)$ is the group of transformations of $\mathcal{F}_{\mathbf{C}}(\Gamma)$, given as follows, where we retain the notation of Proposition 4.19.

(6.2)

Type	Case	Transformations on $\mathcal{F}_{\mathbf{C}}(\Gamma)$	Conditions
I	all	$\mathrm{A}_{a,b} : \pi_{k,\ell} \mapsto \pi_{ak,b\ell}$	$(a,m) = (b,n) = 1$
II	all	$\mathrm{A}_{a,b} : \alpha_{k,\ell} \mapsto \alpha_{ak,b\ell}$ or $\mathrm{A}_{a,b} : \beta_{k,\ell} \mapsto \beta_{ak,b\ell}$	$(a,m) = (b,n) = 1$
III	$n \not\equiv 0\ (9)$	$\mathrm{A}_{a,b} : \nu_{k,\ell} \mapsto \nu_{ak,b\ell}$	$(a,m) = (b,n/3) = 1$
	$n \equiv 0\ (9)$, $d \not\equiv 0\ (3)$	$\mathrm{A}_{a,b,c} : \nu_{k,\ell,j} \mapsto \nu_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1$
	$d \equiv 0\ (3)$	$\mathrm{A}_{a,b} : \mu_{k,\ell} \mapsto \mu_{ak,b\ell}$	$(a,m) = (b,n) = 1$
IV	$n \not\equiv 0\ (9)$, $\Gamma = \langle A, B^3\rangle \times \mathbb{O}^*$	$\mathrm{A}_{a,b,c} : \psi_{k,\ell,j} \mapsto \psi_{ak,b\ell,cj}$	$(a,m) = (b,n/3) = 1$, $c = \pm 1$
	$n \not\equiv 0\ (9)$, $\Gamma \neq \langle A, B^3\rangle \times \mathbb{O}^*$	$\mathrm{A}_{a,b} : \gamma_{k,\ell} \mapsto \gamma_{ak,b\ell}$	$(a,m) = (b,n/3) = 1$,
	$n \equiv 0\ (9)$, $d \not\equiv 0\ (3)$, $\Gamma = \langle A, B^{3^v}\rangle \times \mathbb{O}_v^*$	$\mathrm{A}_{a,b,c} : \xi_{k,\ell,j} \mapsto \xi_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1$
	$n \equiv 0\ (9)$, $d \not\equiv 0\ (3)$, $\Gamma \neq \langle A, B^{3^v}\rangle \times \mathbb{O}_v^*$	$\mathrm{A}_{a,b,c} : \gamma_{k,\ell,j} \mapsto \gamma_{ak,b\ell,cj}$	$(a,m) = (b,n/3^v) = (c,3) = 1$
	$d \equiv 0\ (3)$	$\mathrm{A}_{a,b} : \eta_{k,\ell} \mapsto \eta_{ak,b\ell}$	$(a,m) = (b,n) = 1$
V	all	$\mathrm{A}_{a,b,c} : \iota_{k,\ell,j} \mapsto \iota_{ak,b\ell,cj}$	$(a,m) = (b,n) = 1$, $c = \pm 1$
VI	all	$\mathrm{A}_{a,b} : \kappa_{k,\ell} \mapsto \kappa_{ak,b\ell}$	$(a,m) = (b,n) = 1$

A close look at Table 6.2 yields

Proposition 6.3 *The isospectral group $\mathcal{A}(\Gamma)$ is transitive on $\mathcal{F}_{\mathbf{C}}(\Gamma)$.*

$\mathcal{A}(q,\Gamma)$ is defined to be the group of transformations of $\mathcal{F}_{\mathbf{R}}(q,\Gamma)$ defined by the action of $\mathcal{A}(\Gamma)$ on the irreducible summands. Despite Proposition 6.3, one cannot expect $\mathcal{A}(q,\Gamma)$ to be transitive on $\mathcal{F}_{\mathbf{R}}(q,\Gamma)$. The whole point of looking at $\mathcal{A}(\Gamma)$ and $\mathcal{A}(q,\Gamma)$ is

Theorem 6.4 *Let $\rho_1, \rho_2 \in \mathcal{F}_{\mathbf{R}}(q,\Gamma)$ belong to the same $\mathcal{A}(q,\Gamma)$–orbit. Then the spherical space forms $S^q/\rho_1(\Gamma)$ and $S^q/\rho_2(\Gamma)$ are strongly isospectral.*

CAUTION. We do not assert the converse. We do not know exactly when isospectrality of $S^q/\rho_1(\Gamma)$ and $S^q/\rho_2(\Gamma)$ will imply that ρ_1 and ρ_2 belong to the same $\mathcal{A}(q,\Gamma)$–orbit.

Proof. Let $\mathrm{B} \in \mathcal{A}(q,\Gamma)$ such that $\rho_2 = \rho_1 \circ \mathrm{B}$. Let $\mathrm{A} \in \mathcal{A}(\Gamma)$ be an element whose action on the irreducible summands of elements of $\mathcal{F}_{\mathbf{R}}(q,\Gamma)$ defines B. In view of Corollary 2.13 it suffices to find a bijective map

$$p = p_{\mathrm{A}} : \Gamma \to \Gamma \text{ such that } \phi(p(\gamma)) = \mathrm{A}(\phi)(\gamma)) \text{ for all } \gamma \in \Gamma \text{ and all } \phi \in \mathcal{F}_{\mathbf{C}}(\Gamma). \tag{6.5}$$

For then, of course, $\rho_2(p(\gamma))$ is $SO(q+1)$–conjugate to $\mathrm{B}(\rho_1)(\gamma)$ for every $\gamma \in \Gamma$.

In the notation of (6.2), one such choice of maps p is given by

(6.6)

Type	Case	Map $\mathrm{A}_{a,b}$ or $\mathrm{A}_{a,b,c}$	Map p Implementing $\mathrm{A}_{a,b}$ or $\mathrm{A}_{a,b,c}$
I	all	$\pi_{k,\ell} \mapsto \pi_{ak,b\ell}$	$A^iB^j \mapsto A^{ai}B^{bj}$
II	all	$\alpha_{k,\ell} \mapsto \alpha_{ak,b\ell}$ or $\beta_{k,\ell} \mapsto \beta_{ak,b\ell}$	$R^hA^iB^j \mapsto R^hA^{ai}B^{bj}$
III	$n \not\equiv 0\ (9)$	$\nu_{k,\ell} \mapsto \nu_{ak,b\ell}$	$A^iB^{3j}X \mapsto A^{ai}B^{3bj}X,\ X \in \langle B^{n/3}, P, Q\rangle$
	$n \equiv 0\ (9)$, $d \not\equiv 0\ (3)$	$\nu_{k,\ell,j} \mapsto \nu_{ak,b\ell,cj}$	$A^iB^{3^\nu j}B^{(n/3^\nu)h}X \mapsto A^{ai}B^{3^\nu bj}B^{(n/3^\nu)ch}X,$ $X \in \langle P, Q\rangle$
	$d \equiv 0\ (3)$	$\mu_{k,\ell} \mapsto \mu_{ak,b\ell}$	$A^iB^jX \mapsto A^{ai}B^{bj}X,\ X \in \langle P, Q\rangle$
IV	$n \not\equiv 0\ (9)$, $\Gamma \cong \langle A, B^3\rangle \times \mathbb{O}^*$	$\psi_{k,\ell,j} \mapsto \psi_{ak,b\ell,cj}$	$A^iB^{3j}B^{(n/3)h}X \mapsto A^{ai}B^{3bj}B^{(n/3)ch}X,$ $X \in \langle P, Q, R\rangle$
	$n \not\equiv 0\ (9)$, $\Gamma \not\cong \langle A, B^3\rangle \times \mathbb{O}^*$	$\gamma_{k,\ell} \mapsto \gamma_{ak,b\ell}$	$A^iB^{3j}X \mapsto A^{ai}B^{3bj}X,\ X \in \langle B^{n/3}, P, Q, R\rangle$
	$n \equiv 0\ (9)$, $d \not\equiv 0\ (3)$	$\xi_{k,\ell,j} \mapsto \xi_{ak,b\ell,cj}$ or $\gamma_{k,\ell,j} \mapsto \gamma_{ak,b\ell,cj}$	$A^iB^{3^\nu j}B^{(n/3^\nu)h}X \mapsto A^{ai}B^{3^\nu bj}B^{(n/3^\nu)ch}X,$ $X \in \langle P, Q, R\rangle$
	$d \equiv 0\ (3)$	$\eta_{k,\ell} \mapsto \eta_{ak,b\ell}$	$A^iB^jX \mapsto A^{ai}B^{bj}X,\ X \in \langle P, Q, R\rangle$
V	all	$\iota_{k,\ell,j} \mapsto \iota_{ak,b\ell,cj}$	$A^iB^jX \mapsto A^{ai}B^{bj}X,\ X \in \mathbb{I}^*$
VI	all	$\kappa_{k,\ell} \mapsto \kappa_{ak,b\ell}$	$A^iB^jX \mapsto A^{ai}B^{bj}X,\ X \in \langle S, \mathbb{I}^*\rangle$

Now (6.5) is proved for every $\mathrm{A} \in \mathcal{A}(\Gamma)$, and Theorem 6.4 follows. □

7 Isospectrality with dimension conditions

Let $\mathcal{F}_{\mathbf{R}}(\Gamma)$ denote the set of all $\mathbb{R}$–irreducible fixed point free real representations of Γ. If $|\Gamma| \leqq 2$ then $\mathcal{F}_{\mathbf{R}}(\Gamma) = \mathcal{F}_{\mathbf{C}}(\Gamma)$; otherwise

$$\mathcal{F}_{\mathbf{R}}(\Gamma) = \{\zeta \oplus \overline{\zeta} \mid \zeta \in \mathcal{F}_{\mathbf{C}}(\Gamma)\}. \tag{7.1}$$

All $\zeta \in \mathcal{F}_{\mathbf{C}}(\Gamma)$ have the same degree $\delta(\Gamma)$, so now all $\rho \in \mathcal{F}_{\mathbf{R}}(\Gamma)$ have the same degree, 1 if $|\Gamma| \leqq 2$ and $2\delta(\Gamma)$ if $|\Gamma| > 2$. We will refer to this degree as $\delta_{\mathbf{R}}(\Gamma)$. If $q = \delta_{\mathbf{R}}(\Gamma) - 1$ and $\rho \in \mathcal{F}(q,\Gamma)$ we will say that the spherical space form $S^q/\rho(\Gamma)$ is *irreducible*. This means irreducible as spherical space form, *not* as riemannian manifold, because it is automatic that $S^q/\rho(\Gamma)$ is irreducible as riemannian manifold whenever $q \geqq 2$. Combining Proposition 6.3 and Theorem 6.4 we have

Corollary 7.2 *Let Γ be a finite group that has a fixed point free representations. Let M_1 and M_2 be irreducible spherical space forms with fundamental groups isomorphic to Γ. Then M_1 and M_2 are strongly isospectral.*

Corollary 7.2 is trivial when Γ is cyclic. It was obtained for noncyclic groups of Type I by Ikeda [15, Theorem 1] for 0–isospectrality, then by Gilkey [11, Theorem 0.1(a)], and then by another method essentially also by Ikeda [17, Theorem 2].

Corollary 7.3 *Let $q = 2p - 1$ where p is prime. Consider spherical space forms $M = S^q/G$ and $M' = S^q/G'$ where G is not cyclic. Then the following are equivalent:* (i) *M and M' are isospectral,* (ii) *M and M' are strongly isospectral, and* (iii) *$G \cong G'$.*

Proof. If M and M' are isospectral then $G \cong G'$ by [13, Theorem 3.1]. This implication does not require G to be noncyclic. Now suppose $G \cong G'$. So $G = \rho_1(\Gamma)$ and $G' = \rho_2(\Gamma)$ where Γ is a fixed point free group and the ρ_i are fixed point free real representations of degree $q + 1 = 2p$. As p is prime and Γ is not cyclic now each $\rho_i \in \mathcal{F}_{\mathbb{R}}(\Gamma)$. Thus the spherical space forms M and M' are irreducible, and strong isospectrality follows from Corollary 7.2. We have shown that (i) implies (iii) and that (iii) implies (ii). Since (ii) implies (i), the proof is complete. □

Corollary 7.4 *Two 3-dimensional spherical space forms are isospectral if and only if they are isometric.*

This is immediate from Proposition 4.19 and Corollary 7.3 if the fundamental groups are noncyclic. See [12, Theorem I] for the cyclic case and another approach the other cases. A result of Berger ([4], or see [5, Proposition E.IV.18]) says that a compact riemannian 3–manifold isospectral to a manifold of constant positive sectional curvature K must also have constant positive curvature K. Thus [12, Theorem II]

> if a compact riemannian 3–manifold M is isospectral to a spherical space form S^3/G then M is isometric to S^3/G.

Corollary 7.5 *Two spherical space forms S^5/G and S^5/G', G not cyclic, are isospectral if and only if they are isometric.*.

This is immediate from Proposition 4.19 and Corollary 7.3. It was first proved [13, Theorem 3.8] by a careful analysis of the meromorphic function F_G based on the fact that $G = \rho(\Gamma)$ with Γ of Type I with $d = \delta(\Gamma) = 3$. A result of Tanno [26, Theorem A] says that a compact riemannian 5–manifold, isospectral to a manifold of constant positive sectional curvature K, must also have constant positive curvature K. Thus [13, Theorem 3.9]

> if a compact riemannian 5–manifold M is isospectral to a spherical space form S^5/G, where G is not cyclic, then M is isometric to S^5/G.

8 Isospectrality and homogeneity

We look at the special case of homogeneous spherical space forms. Our argument is a bit more direct than that of [12, Theorem III].

Lemma 8.1 *Let M be a spherical space form S^{2r-1}/G with $|G| > 2$. Recall the meromorphic function $F_G(z) = \frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(zI-g)}$. Then M is homogeneous as a riemannian manifold if, and only if, every pole $\lambda \neq \pm 1$ of F_G is a pole of order exactly r.*

Proof. According to [28, Theorem 4], M is homogeneous if and only if every $g \neq \pm I$ in G has just two eigenvalues $\lambda \neq \overline{\lambda}$, each of multiplicity r. In other words, M is homogeneous if and if, whenever $g \neq \pm I$ in G, the corresponding term $\frac{1-z^2}{\det(zI-g)}$ of F_G is $\frac{1-z^2}{(z-\lambda)^r(z-\overline{\lambda})^r}$. Since the poles $\neq \pm 1$ of F_G are just the eigenvalues of the elements $\neq \pm I$ of G, the Lemma follows. □

Theorem 8.2 *Let $M = S^q/G$ and $M' = S^q/G'$ be isospectral spherical space forms. Suppose that M is homogeneous. Then M and M' are isometric. In particular M' is also homogeneous.*

Proof. M' is homogeneous by Lemma 8.1, so G and G' each is cyclic, binary dihedral or binary polyhedral by [27]. In fact Lemma 8.1 shows that, for each divisor u of $|G| = |G'|$, the groups G and G' have the same number of elements of order exactly u, those with eigenvalues that are primitive u^{th} roots of 1.

G is cyclic if and only if it has $\varphi(|G|)$ elements of order exactly $|G|$, so if G is cyclic then $G \cong G'$.

Let $G \cong \mathbb{D}_m^*$. If G' is not binary dihedral then it must be binary polyhedral. If $G' \cong \mathbb{T}^*$ then $G \cong \mathbb{D}_6^*$, which has 14 elements of order exactly 4, while G' only has 6. Thus $G' \ncong \mathbb{T}^*$. If $G' \cong \mathbb{O}^*$ then $G \cong \mathbb{D}_{12}^*$, which has 26 elements of order exactly 4, while G' only has 9. Thus $G' \ncong \mathbb{O}^*$. If $G' \cong \mathbb{I}^*$ then $G \cong \mathbb{D}_{30}^*$, which has 62 elements of order exactly 4, while G' has only 45. Thus $G' \ncong \mathbb{I}^*$. Now G' is binary dihedral and $G \cong G'$.

Let G be binary polyhedral. The consideration of elements of order exactly 4, just above, forces $G \cong G'$.

Now in any case $G \cong G'$, so M is isometric to M' by [27]. □

9 Extension to Grassmann manifolds

Consider the real Grassmann manifold $G_{s,h} = G_{s,h}(\mathbb{R})$ of oriented s–planes in an oriented $\mathbb{R}^h$. Then $G_{s,h} = SO(h)/SO(s) \times SO(h-s)$, the (real) dimension $\dim G_{s,h} = s(h-s)$, and $SO(h)$ is the identity component $I^0(G_{s,h})$ of the group $I(G_{s,h})$ of all isometries of $G_{s,h}$. In this Section we extend our results on spherical space forms to results on riemannian quotient manifolds of $G_{s,h}$. Spherical space forms are the case $s = 1$ with dimension $q = h - 1$.

The even–dimensional case is discussed in [30]; or see [31, Theorem 9.3.1], as follows. Define isometries

β: oriented orthogonal complementation, isometry of $G_{s,2s}(\mathbb{R})$,

ω: change of orientation, isometry of $G_{s,h}(\mathbb{R})$,

g_v: isometry $\begin{pmatrix} I_{h-v} & 0 \\ 0 & -I_v \end{pmatrix}$ of $G_{s,h}(\mathbb{R})$, and

j: isometry $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ of $G_{s,2m}(\mathbb{R})$.

Here $\beta(\mathrm{Span}\{v_1, \ldots, v_s\}) = \mathrm{Span}\{v_{s+1}, \ldots, v_{2s}\}$ whenever $\{v_1, \ldots, v_{2s}\}$ is an oriented orthonormal basis of the oriented $\mathbb{R}^{2s}$, where Span refers to the oriented span of an ordered set of linearly independent vectors. If s is odd then $\omega = -I \in O(h)$. If s is even and $h = 2s$ then $\beta^2 = 1$. If s is odd and $h = 2s$ then $\beta^2 = \omega$.

Proposition 9.1 *Let $G \neq \{1\}$ be a group of isometries acting freely on $G_{s,h}$ where $s(h-s)$ is even. Then G is conjugate in $I(G_{s,h})$ to one of the groups*

1. $\{1, \omega\} \cong Z_2$
2. $\{1, \omega j\} \cong Z_2$ with h even
3. $\{1, \beta g_{2v}\} \cong Z_2$ with $h = 2s$ and $0 \leqq 2v < s$
4. $\{1, \beta g_{2v}, \omega, \omega\beta g_{2v}\} \cong Z_2 \times Z_2$ with $h = 2s$ and $0 \leqq 2v < s$
5. $\{1, \beta g_{2v-1}, \omega, \omega\beta g_{2v-1}\} \cong Z_4$ with $h = 2s$ and $0 \leqq 2v - 1 < s$

Conversely each of these groups acts freely on $G_{s,h}$, and two distinct ones must be non–conjugate in $I(G_{s,h})$.

The odd dimensional case is somewhat more interesting. From now on, $s(h-s)$ is odd, that is, s is odd and h is even. The isometry group is given by [31, (9.4.2)]:

$$I(G_{s,h}) = O(h) \text{ if } h \neq 2s, \quad I(G_{s,h}) = O(h) \cup \beta O(h) \text{ if } h = 2s. \tag{9.2}$$

An element $g \in O(h)$ has a fixed point on $G_{s,h}$ if and only if it is conjugate to an element of the isotropy subgroup $S(O(s) \times O(h-s))$, in other words just when it has $+1$ for an eigenvalue, that is, precisely when it has a fixed point on S^{h-1} [31, Lemma 9.4.3]. Thus the riemannian quotient manifolds $G_{s,h}/G$, for which $G \subset O(h)$, correspond one–one with the spherical space forms S^{h-1}/G [31, Theorem 9.4.4].

Note that $-I \in SO(h)$ changes the orientation of every $x \in G_{s,h}$, because s is odd. When $h = 2s$, we have $\beta^2 = -I$, $\beta g = g\beta$ for every $g \in SO(h)$, and $\beta g = -Ig\beta$ for $g \in O(h) \setminus SO(h)$. This is useful for the study of riemannian quotient manifolds $G_{s,2s}/G$, for which $G \not\subset O(2s)$, especially since $G \cap O(h) \subset SO(h)$. The result [31, Theorem 9.4.11] is:

Proposition 9.3 *Let G be a finite group of isometries acting freely on the oriented real Grassmann manifold $G_{s,h}$, s odd and h even. Then*

(i) $G = \rho(\Gamma)$ where Γ is a fixed point free group and $\rho \in \mathcal{F}_{\mathbf{R}}(h-1,\Gamma)$; or

(ii) Γ is a fixed point free group of Type I with $n \equiv 0$ (8), $h = 2s$, $\rho \in \mathcal{F}_{\mathbf{R}}(h-1,\Gamma)$, ρ_β is defined by $\rho_\beta(A) = \rho(A)$ and $\rho_\beta(B) = \beta\rho(B)$, and $G = \rho_\beta(\Gamma)$; or

(iii) Γ is a fixed point free group of Type I with $n \equiv 4$ (8), $h = 2s$, $\rho \in \mathcal{F}_{\mathbf{R}}(h-1,\Gamma)$, and $G = \rho(\Gamma) \cup \beta\rho(\Gamma)$.

Conversely, the groups G described just above under (i), (ii) and (iii), act freely on $G_{s,h}$. The three cases are distinguished by: (i) $G \subset O(h)$, (ii) $G \not\subset O(h)$ and $\beta \notin G$, and (iii) $\beta \in G$.

Theorem 9.4 *Suppose that $\dim G_{s,h} = s(h-s)$ is odd. Consider two riemannian quotient manifolds*

$M_1 = G_{s,h}/\rho_1(\Gamma)$ and $M_2 = G_{s,h}/\rho_2(\Gamma)$ as in Proposition 9.3*(i); or*

$M_1 = G_{s,h}/\rho_{1,\beta}(\Gamma)$ and $M_2 = G_{s,h}/\rho_{2,\beta}(\Gamma)$ as in Proposition 9.3*(ii); or*

$M_1 = G_{s,h}/(\rho_1(\Gamma) \cup \beta\rho_1(\Gamma))$ and $M_2 = G_{s,h}/(\rho_2(\Gamma) \cup \beta\rho_2(\Gamma))$ as in Proposition 9.3*(iii). Then*

1. M_1 is isometric to M_2 if and only if $S^{h-1}/\rho_1(\Gamma)$ is isometric to $S^{h-1}/\rho_2(\Gamma)$.

2. If $\rho_1, \rho_2 \in \mathcal{F}_{\mathbf{R}}(h-1,\Gamma)$ belong to the same $\mathcal{A}(h-1,\Gamma)$-orbit then M_1 and M_2 are strongly isospectral.

Proof. The $S^{h-1}/\rho_i(\Gamma)$ are isometric just when $\rho_1(\Gamma) = g_0\rho_2(\Gamma)g_0^{-1}$ for some $g_0 \in I(S^{h-1}) = O(h)$. In Case (*i*) of Proposition 9.3, since β centralizes $SO(h)$ and $\rho_i(\Gamma) \subset SO(h)$, this is equivalent to the condition that $\rho_1(\Gamma) = g_0\rho_2(\Gamma)g_0^{-1}$ for some $g_0 \in I(G_{s,h})$. In Cases (*ii*) and (*iii*) of Proposition 9.3 we have $n \equiv 0$ (4), so $-I \in \rho_i(\Gamma) \cap SO(h)$. Then if $g_2 \in \rho_2(\Gamma)$ we have $g_0\ \beta g_2\ g_0^{-1} = g_0\beta g_0^{-1}\ g_0 g_2 g_0^{-1} = ((\det g_0)I)\ \beta\ g_0 g_2 g_0^{-1}$, so $g_0\ \beta g_2\ g_0^{-1} \in \rho_1(\Gamma)$ if and only if $g_0 g_2 g_0^{-1} \in \rho_1(\Gamma)$. Thus $S^{h-1}/\rho_i(\Gamma)$ are isometric just when (Case (*ii*)) $\rho_{1,\beta}(\Gamma) = g_0\ \rho_{2,\beta}(\Gamma)\ g_0^{-1}$ for some $g_0 \in I(G_{s,h})$ or (Case (*iii*)) $(\rho_1(\Gamma) \cup \beta\rho_1(\Gamma)) = g_0\ (\rho_2(\Gamma) \cup \beta\rho_2(\Gamma))\ g_0^{-1}$ for some $g \in I(G_{s,h})$. The first assertion is proved.

Suppose that $\rho_1, \rho_2 \in \mathcal{F}_{\mathbf{R}}(h-1,\Gamma)$ belong to the same $\mathcal{A}(h-1,\Gamma)$–orbit. Recall that they have image in the special (determinant 1) orthogonal group $SO(h)$. Then the proof of Theorem 6.4, based on Table 6.6, shows that each conjugacy class in $SO(h)$ meets $\rho_1(\Gamma)$ and $\rho_2(\Gamma)$ in the same number of elements. Express $M_i = G_{s,h}/G_i$. Recall that β centralizes $SO(h)$ and $-I \in \rho_i(\Gamma)$. Now each conjugacy class in $I'(G_{s,h}) = SO(h) \cup \beta SO(h)$ meets G_1 and $\mathbb{G}_2$ in the same number of elements. Corollary 2.13 says that M_1 and M_2 are strongly isospectral. □

Sunada's Theorem [25, Theorem 1] in place of Corollary 2.13 shows 0–isospectrality directly and can be used for strong isospectrality. Ikeda's variation [17, Theorem 1] on Sunada's Theorem in place of Corollary 2.13 shows p–isospectrality for every p when the $\rho_i(\Gamma)$ are given by Case (iii) of Proposition 9.3.

Now most of the results of Section 7 extend immediately from spherical space forms to riemannian quotients of odd dimensional Grassmann manifolds, as follows. In the notation of Proposition 9.3, we will say that a riemannian quotient (Case (*i*)) $G_{s,h}/\rho(\Gamma)$, (Case (*ii*)) $G_{s,h}/\rho_\beta(\Gamma)$, or (Case (*iii*)) $G_{s,h}/(\rho(\Gamma) \cup \beta\rho(\Gamma))$, is *irreducible* if $\rho \in \mathcal{F}_{\mathbf{R}}(\Gamma)$.

Corollary 9.5 *Let Γ be a finite group that has fixed point free representations. Let M_1 and M_2 be irreducible riemannian quotient manifolds of the same odd dimensional Grassmann manifold with fundamental groups isomorphic to Γ and in the same case* (i), (ii) *or* (iii) *of* Proposition 9.3. *Then M_1 and M_2 are strongly isospectral.*

Compare the case of Corollary 9.5 for Proposition 9.3(i) with [17, Theorem 2].

Corollary 9.6 *Let $h = 2p$ where p is prime. Consider riemannian quotient manifolds $M = G_{s,h}/G$ and $M' = G_{s,h}/G'$, where G is not cyclic, both in the same case of* Proposition 9.3. *Then the following are equivalent:* (i) *M and M' are isospectral,* (ii) *M and M' are strongly isospectral, and* (iii) *$G \cong G'$.*

The analog of Corollary 7.4 carries no new information, for odd dimensionality of $G_{s,4}$ requires $s = 1$ or $s = 3$, and we are looking at $G_{1,4} \cong G_{3,4} \cong S^3$. The analog of Corollary 7.5 applies to $G_{1,6} \cong G_{5,6} \cong S^5$ and to $G_{3,6}$, and the latter says something new.

Corollary 9.7 *Two riemannian quotient manifolds of $G_{3,6}$, in the same case of* Proposition 9.3, *with noncyclic fundamental groups, are isospectral if and only if they are isometric.*

At this point it is clear that further progress will require an understanding of the action of ω on $L^2(G_{s,h})$, and also of the action of β on $L^2(G_{s,2s})$. We will study those actions in [32].

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