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# REAL GROUPS TRANSITIVE ON COMPLEX FLAG MANIFOLDS

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Abstract. Let $Z = G/Q$ be a complex flag manifold. The compact real form $G_u$ of $G$ is transitive on $Z$. If $G_0$ is a noncompact real form, such transitivity is rare but occasionally happens. Here we work out a complete list of Lie subgroups of $G$ transitive on $Z$ and pick out the cases that are noncompact real forms of $G$.

## 0. The problem

Let $Z = G/Q$ be a complex flag manifold where $G$ is a complex connected semisimple Lie group and $Q$ is a parabolic subgroup. Let $G_0$ be a real form of $G$. If $G_0$ is the compact real form, then it is transitive on $Z$. On a number of occasions the question has come up as to whether any noncompact real form of $G$ can be transitive on $Z$. Here I'll record the answer. The rough answer is "yes, but just a few." The precise answer, Corollaries 1.7 and 2.3 below, follows from a more general classification, Theorems 1.6 and 2.2. This more general classification uses a technique of D. Montgomery [M], together with some results of [W1] that depend in an essential way on a classification [O1] of A. L. Onishchik.

After this paper was written I learned of Onishchik's book [O2]. There is some overlap for compact groups, but there are no inclusions.

## 1. The solution for irreducible flags

We formulate the problem in terms of transitive subgroups. Let $G_u$ be the compact real form of $G$, so $Z = G_u/(G_u \cap Q)$ and $G_u \cap Q$ is the compact real form of the reductive part of $Q$. Let $A \subset G$ be a closed subgroup that is transitive on $Z$. The identity component $A^0$ of $A$ is transitive on $Z$, because $Z$ is connected, so a maximal compact subgroup $B^0 \subset A^0$ already is transitive on $Z$, according to Montgomery [M]. We may replace $A$ by a conjugate and assume $B = A \cap G_u$. So

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now we have several expressions:

$$\begin{aligned} Z = G/Q &= G_u/(G_u \cap Q) = A/(A \cap Q) = B/(B \cap Q) \\ &= A^0/(A^0 \cap Q) = B^0/(B^0 \cap Q). \end{aligned} \tag{1.1}$$

According to [W1, Prop. 3.1] there are just a few possibilities for a homogeneous almost–hermitian manifold $Z$ to have distinct expressions such as $G_u/L_u$ and $B^0/(B^0 \cap L_u)$, where $G_u$ is the identity component of the group of all almost–hermitian isometries, $G_u$ is simple, $L_u$ is the centralizer of a torus subgroup of $G_u$, and $B^0 \subsetneqq G_u$ with $B^0$ connected. They are :

(1.2) $Z = P^{2n-1}(\mathbb{C}) = SU(2n)/U(2n-1) = Sp(n)/(Sp(n-1)\cdot U(1))$, complex projective space,

(1.3) $Z = SO(2r+2)/U(r+1) = SO(2r+1)/U(r)$, unitary structures on $\mathbb{R}^{2r+2}$,

(1.4) $Z = SO(7)/(SO(5)\cdot SO(2)) = G_2/U(2)$, 5–dimensional complex quadric, and

(1.5) $Z = SO(8)/(SO(6)\cdot SO(2)) = \{Spin(7)/Z_2\}/U(3)$, 6–dimensional complex quadric.

This applies in our situation because $L_u = G_u \cap Q$ is the centralizer of a torus subgroup of $G_u$, and $Z$ has a $G_u$–invariant hermitian metric.

Now return to the expression $Z = G/Q$. $G$ (and thus $G_u$) is simple. Let $A \subsetneqq G$ be a closed subgroup that is transitive on $Z$ and let $B$ be its maximal compact subgroup. We may assume $B = A \cap G_u$. Then $B \subsetneqq G_u$, $B^0$ is transitive on $Z$, and the expression $Z = G_u/L_u = B^0/(B^0 \cap L_u)$ is given above. In each case the group $B^0$ is simple, so $A^0$ has Levi decomposition $A^0 = A^0_{ss}A^0_{rad}$ into semisimple part and solvable radical, where $B^0$ is a maximal compact subgroup of $A^0_{ss}$. We run through the 4 possibilities listed above.

For (1.2), $G = SL(2n;\mathbb{C})$ and $B^0 = Sp(n)$. The semisimple Lie groups with maximal compact subgroup $Sp(n)$ are $Sp(n), Sp(n;\mathbb{C})$, the quaternionic linear group $SL(n;\mathbb{H})$, and, for $n = 4$, the real group $F_{4,C_4}$. But $F_4$ does not have a representation of degree 8, in other words $F_4 \not\subset G$, so now $A^0_{ss}$ is one of $Sp(n), Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$. Each of them is irreducible on $\mathbb{C}^{2n}$, so the unipotent radical of the algebraic hull of $A^0$ acts trivially on $\mathbb{C}^{2n}$ and the center of the reductive part of $A^0$ acts by scalars. As $G$ acts effectively and by transformations of determinant 1 on $\mathbb{C}^{2n}$ now $A^0_{ss} = A^0$, so $A^0$ is one of $Sp(n), Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$. If $g \in G$ normalizes $A^0$, then some element $g' \in gA^0$ centralizes $A^0$, because $A^0$ has no rational outer automorphism. As $A^0$ is irreducible on $\mathbb{C}^{2n}$ now $g'$ is scalar (and thus acts trivially on $Z$). Thus $A = A^0F$ where $F$ can be any subgroup of the center $\{e^{2\pi ik/2n}I \mid 0 \leqq k < 2n\}$ of $G$.

For (1.3), $G = SO(2r+2;\mathbb{C})$ and $B^0 = SO(2r+1)$. The semisimple Lie groups with maximal compact subgroup $SO(2r+1)$ are $SO(2r+1), SO(2r+1;\mathbb{C}), SO(1,2r+1)$, and $SL(2r+1;\mathbb{R})$. But $A^0_{ss} = SL(2r+1;\mathbb{R})$ would give $SL(2r+1;\mathbb{C}) \subset SO(2r+2;\mathbb{C})$, so the respective dimensions would satisfy $4r^2+4r \leqq 2r^2+3r+1$, forcing $r = 0$ and $Z =$ (point). Thus[1] $A^0_{ss} \neq SL(2r+1;\mathbb{R})$. Now $A^0_{ss}$ is one of $SO(2r+1), SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$. The last one acts irreducibly on $\mathbb{C}^{2r+2}$, and there $A^0_{ss} = A^0$ as above. For the first two, recall that $SO(2r+1)$ is absolutely irreducible on the tangent space $\mathfrak{so}(2r+2)/\mathfrak{so}(2r+1)$ of the sphere $S^{2r+1}$, so $A^0_{rad}$ has Lie algebra reduced to 0, and again $A^0_{ss} = A^0$. Now $A^0$ is one of $SO(2r+1), SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$. If $g \in G$ normalizes $A^0$, then some

[1] The author thanks the referee for a comment that improved and clarified his treatment of this $SL(2r+1;\mathbb{R})$ case.

element $g' \in gA^0$ centralizes $A^0$, because $A^0$ has no rational outer automorphism. Thus either $A = A^0$ or $A/A^0$ has order 2 where $A$ is one of $O(2r+1), O(2r+1;\mathbb{C})$, and $SO(1,2r+1)\cdot\{\pm I\}$.

For (1.4), $G = SO(7;\mathbb{C})$ and $B^0 = G_2$. The semisimple Lie groups with maximal compact subgroup $G_2$ are $G_2$ and its complexification $G_{2,\mathbb{C}}$. They are irreducible on $\mathbb{C}^7$ and have no rational outer automorphisms, so, as before, $A^0$ is either $G_2$ or $G_{2,\mathbb{C}}$, and if $g \in G$ normalizes $A^0$, then some element $g' \in gA^0$ centralizes $A^0$. This forces $g'$ to be central in $SO(7;\mathbb{C})$, so $g' = 1$ and $A = A^0$. Thus $A$ is either $G_2$ or $G_{2,\mathbb{C}}$.

Finally, (1.5) is obtained from the case $r = 3$ of (1.3) by applying the triality automorphism, so it does not give us anything more.

In summary,

**Theorem 1.6.** *Consider a complex flag manifold $Z = G/Q$. Suppose that $Z$ is irreducible, i.e., that $G$ is simple. Then the closed subgroups $A \subset G$ transitive on $Z$, $G_u \neq A \neq G$, are precisely those given as follows:*

1. *$Z = SU(2n)/U(2n-1) = P^{2n-1}(\mathbb{C})$ complex projective $(2n-1)$–space; $G = SL(2n;\mathbb{C})$ and $A = A^0F$ where $A^0$ is one of $Sp(n)$, $Sp(n;\mathbb{C})$ and $SL(n;\mathbb{H})$, and $F$ is any subgroup of the center $\{e^{2\pi ik/2n}I \mid 0 \leqq k < 2n\}$ of $G$. Here $F$ acts trivially on $Z$, so $A$ and $A^0$ have the same action on $Z$.*
2. *$Z = SO(2r+2)/U(r+1)$, unitary structures on $\mathbb{R}^{2r+2}$; $G = SO(2r+2;\mathbb{C})$ and $A = A^0F$ where $A^0$ is one of $SO(2r+1)$, $SO(2r+1;\mathbb{C})$, and $SO(1,2r+1)$, and where $F$ is any subgroup of the center $\{\pm I\}$ of $G$. Here $F$ acts trivially on $Z$, so $A$ and $A^0$ have the same action on $Z$.*
3. *$Z = SO(7)/(SO(5)\cdot SO(2))$, 5–dimensional complex quadric; $G = SO(7;\mathbb{C})$ and $A$ is either the compact connected group $G_2$ or its complexification $G_{2,\mathbb{C}}$.*

Picking out the cases where $A$ is a real form of $G$ we have

**Corollary 1.7.** *Consider a complex flag manifold $Z = G/Q$. Suppose that $Z$ is irreducible, i.e., that $G$ is simple. Then the (connected) noncompact real forms $G_0 \subset G$ transitive on $Z$ are precisely those given as follows:*

1. *$Z = SU(2n)/U(2n-1) = P^{2n-1}(\mathbb{C})$ complex projective $(2n-1)$–space; $G = SL(2n;\mathbb{C})$ and $G_0$ is the quaternion linear group $SL(n;\mathbb{H})$, which has maximal compact subgroup $Sp(n)$.*
2. *$Z = SO(2r+2)/U(r+1)$, unitary structures on $\mathbb{R}^{2r+2}$; $G = SO(2r+2;\mathbb{C})$ and $G_0$ is the Lorentz group $SO(1,2r+1)$, which has maximal compact subgroup $SO(2r+1)$.*

## 2. The solution for flag manifolds in general

We complete the solution of the problem by reducing it to the case where $Z$ is irreducible.

**Proposition 2.1.** *Decompose $G = \prod G_i$, the local direct product of complex connected simple Lie groups. Thus $Z = \prod Z_i$, the product of irreducible flag manifolds $Z_i = G_i/Q_i$ where $Q_i = Q \cap G_i$. Then $A^0 = \prod A_i^0$ with $A_i^0 = A^0 \cap G_i$ and $B^0 = \prod B_i^0$ with $B_i^0 = B^0 \cap G_i$. The groups $A_i^0$ and $B_i^0$ are connected, simple, and transitive on $Z_i$.*

*Proof.* The solvable radical of $A^0$ is contained in a Borel subgroup of $G$, and thus has a fixed point on $Z$. It is normal in the transitive group $A^0$ so it fixes every point. Thus $A^0$ is semisimple. Similarly $B^0$ is semisimple.

Let $\pi_i : G \to G_i$ denote the projection. The compact connected group $\pi_i(B^0)$ is transitive on $Z_i$. So it must be the compact real form $G_{u,i} = G_i \cap G_u$ of $G_i$ or one of the compact connected transitive groups described in (1.2), (1.3) or (1.4). (Recall that (1.5) is in fact a special case of (1.3).) In all cases, $\pi_i(B^0)$ is nontrivial and simple. Now $\pi_i$ annihilates all but one of the simple factors of $B^0$. Obviously no simple factor of $B^0$ is annihilated by every $\pi_i$. So now $B^0 = \prod B^0_\alpha$ where the $B^0_\alpha$ are simple and where the index set $I$ for $G = \prod_I G_i$ is a disjoint union of subsets $I_\alpha$ with $B^0_\alpha \subset \prod_{i \in I_\alpha} G_i$. The proof of Proposition 2.1 is reduced to the case where $B^0$ (and thus also $A^0$) is simple, and there it is reduced to the proof that $G_u$ is simple.

We may now assume $B^0$ simple. Suppose that $G_u$ is not simple. Projecting to $G_1 \times G_2$ we may assume $G = G_1 \times G_2$. View the isomorphisms $\pi_i : B^0 \cong \pi_i(B^0)$ as identifications. Denote $E_i = \pi_i(B^0_{\mathbb{C}})$, the complexification of the image of $B^0$ in $G_i$. Denote $E_{u,1} = \pi_i(B^0)$, the compact real form of $E_i$. Denote $P_i = E_i \cap Q_i$, the parabolic subgroup of $E_i$ that is its isotropy subgroup in $Z_i$, so $Z_i = E_i/P_i$. Now $B^0_{\mathbb{C}} = \{(e,e) \mid e \in E_1\}$, $B^0_{\mathbb{C}} \cap Q = \{(p,p) \mid p \in (P_1 \cap P_2)\}$, and $Z = B^0_{\mathbb{C}}/(B^0_{\mathbb{C}} \cap Q) \cong E_1/(P_1 \cap P_2)$. In particular $P_1 \cap P_2$ is a parabolic subgroup of $E_1$. Compute complex dimensions: $\dim E_1 - \dim(P_1 \cap P_2) = \dim B^0 - \dim(B^0 \cap Q) = \dim Z = \dim Z_1 + \dim Z_2 = (\dim E_1 - \dim P_1) + (\dim E_1 - \dim P_2)$. On the Lie algebra level this says $\dim \mathfrak{e}_1 = \dim \mathfrak{p}_1 + \dim \mathfrak{p}_2 - \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2)$, in other words $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra of $\mathfrak{e}_1$ we have a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{s}$ with $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. In the root order such that $\mathfrak{s}$ is the sum of $\mathfrak{h}$ and the negative root spaces, no parabolic containing $\mathfrak{s}$ can contain the root space for the maximal root. This contradicts $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. The contradiction proves $G_u$ simple and completes the proof. □

Combining Proposition 2.1 with Theorem 1.6 we have

**Theorem 2.2.** *Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the closed subgroups $A \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = A^0 F$ where $A^0 = \prod A_i$ with $A_i = (A \cap G_i)^0$, and $A_i$ is equal to $G_i$, or to its compact real form $G_{u,i}$, or to one of the three types listed in* Theorem 1.6*, and $F$ is any subgroup of the center of $G$. Here $F$ acts trivially on $Z$, so $A$ and $A^0$ have the same action on $Z$.*

Picking out the cases where $A$ is a real form of $G$ we have, as in Corollary 1.7,

**Corollary 2.3.** *Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the (connected) real forms $G_0 \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = \prod A_i$ where $A_i = A \cap G_i$ either is the compact real form $G_{u,i}$ of $G_i$ or is one of the two types listed in* Corollary 1.7*.*

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