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HERMITIAN SYMMETRIC SPACES, CYCLE SPACES, AND THE BARLET–KOZIARZ INTERSECTION METHOD FOR CONSTRUCTION OF HOLOMORPHIC FUNCTIONS

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Abstract. Under certain conditions, a recent method of Barlet and Koziarz [2] constructs enough holomorphic functions to give a direct proof of the Stein condition for a cycle space. Here we verify those conditions for open G_0–orbits on X, where G_0 is the group of a bounded symmetric domain and X is its compact dual viewed as a flag quotient manifold of the complexification G of G_0. This Stein result was known for a few years [9], and in fact a somewhat more precise result is known [12] for the flag domains to which we apply the Barlet–Koziarz method, but the proof here is much more direct and holds the possibility of greater generality. Also, some of the tools developed here apply directly to open orbits that need not be measurable, avoiding separate arguments of reduction to the measurable case.

1. Introduction

Let G_0 be a real semisimple Lie group, G its complexification, Q a parabolic subgroup of G and $X = G/Q$ the corresponding flag manifold, and D an open G_0–orbit on X. Let M_D denote the linear cycle space of D (see (2.6) and (2.7) below). The usual proof [9] that M_D is a Stein manifold, in the case where D is measurable, i.e. where D carries a G_0–invariant positive Radon measure, is rather indirect. One constructs a particular exhaustion function on D, uses it to verify that D is $(s+1)$–complete where s is the complex dimension of the cycles in M_D, and then in a rather technical way pushes the exhaustion function from D to M_D. This is done in such a way that (a slight modification of) the resulting function on M_D is a strictly plurisubharmonic exhaustion function there. If D is not measurable, one studies [10] a minimal flag covering of X and D.

Under certain conditions (2.11) the Barlet–Koziarz intersection method can show directly that M_D is Stein. That is the new element of this paper. One uses the intersection method to construct enough holomorphic functions to show that D is holomorphically convex, and at that point the Stein condition follows from some elementary conditions. The delicate points here are the G_0–orbit structure of the boundary of D in X, construction of a certain transversal X' to the elements of M_D, and an implicit application of some results from intersection theory on the Chow ring of X. See Proposition 2.13.

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In Section 2 we describe the Barlet–Koziarz method and specialize it to flag domains.

In the case where $G_0 = SU(p,q)$ and X is the complex Grassmann manifold $Gr(p,q)$, we verify the conditions (2.11) with classical computation. This is done in §3, and it is done in a way that indicates the procedure in a more general symmetric space setting. There are really two parts here: elucidation of the G_0–orbit structure of X and construction of a certain sort of transversal to the cycles. The result is Proposition 3.12.

The general hermitian symmetric space considerations are in Sections 4 and 5. The G_0–orbit structure is described in §4 using partial Cayley transforms, which leads more or less directly to construction of the transversals in §5. The result is Proposition 5.17.

The conditions (2.11) were first used by Barlet and Huckleberry where $G_0 = SL(n+1;\mathbb{R})$ and X is the complex projective space $P^n(\mathbb{C})$, so there is just one open orbit $D = P^n(\mathbb{C}) \setminus P^n(\mathbb{R})$. The cycles have codimension 1 in D so one can use complex projective lines for transversals. The technique described here started when I noticed that their considerations, codimension 1 cycles and transversal projective lines, held as well for the case $G_0 = SU(p,q)$ and $X = P^{p+q-1}(\mathbb{C})$. Later, Huckleberry and Simon [3] carried out a complete analysis of the case where $G_0 = SL(n+1;\mathbb{R})$ and X is an arbitrary flag manifold of $G = SL(n+1;\mathbb{C})$.

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2. The method

Here is the method used for our examples, extracted from the more general results of [2] and reformulated for consistency with the usual notation of complex flag manifolds and hermitian symmetric spaces.

Let D be an open submanifold of a complex projective variety X. Fix an integer $s \geqq 0$. Fix irreducible components $\mathcal{C}_s^0(D)$ and $\mathcal{C}_s^0(X)$ in the respective Barlet cycle spaces of (complex) dimension s in D and X, such that $\mathcal{C}_s^0(D) \subset \mathcal{C}_s^0(X)$. Choose

$$X' \subset X : \text{ closed nonsingular subvariety of } X \text{ such that}$$

$$\begin{aligned} &\text{(i) } \operatorname{codim}(X' \subset X) = s, \\ &\text{(ii) } X' \text{ meets every element } Y \in \mathcal{C}_s^0(D), \text{ and} \\ &\text{(iii) } X'' = X' \cap D \text{ is a Stein manifold.} \end{aligned} \tag{2.1}$$

Let

(2.2)
$$f : X'' \to C \text{ be a holomorphic function , and define}$$
$$F : \mathcal{C}^0_s(D) \to \mathbb{C} \text{ by } F(Y) = \sum\nolimits_{y \in X'(b) \cap Y} f(y).$$

Lemma 2.3. *Suppose that* (2.1) *holds. Let* $Y \in \mathcal{C}^0_s(D)$ *. Then* $X' \cap Y$ *is finite. In particular* F *is well defined in* (2.2)*.*

Proof. Hypothesis (2.1) says that X' meets Y. $X' \cap Y \subset D$ because $Y \subset D$. $X' \cap Y$ is the intersection of the compact variety Y with the Stein manifold $X'' = X' \cap D$, so it is a compact subvariety of X'', and thus finite. □

Now we can state a special case of [2, Proposition 1]), the basic step in the Barlet–Koziarz intersection method for construction of holomorphic functions on cycle spaces:

Proposition 2.4. *If* (2.1) *and* (2.2) *hold, then* F *is holomorphic on* $\mathcal{C}^0_s(D)$*.*

Now that we have a construction of holomorphic functions, we look at holomorphic convexity. Here is a special case of [2, Proposition 3].

Proposition 2.5. *For every* Y *in the boundary* $bd(\mathcal{C}^0_s(D))$ *of* $\mathcal{C}^0_s(D)$ *in* $\mathcal{C}^0_s(X)$*, suppose that there is a subvariety* $X' \subset X$ *such that* (i) X' *satisfies* (2.1) *and* (ii) $X' \cap Y$ *meets the boundary* $\mathrm{bd}(D)$ *of* D *in* X*. Then* $\mathcal{C}^0_s(D)$ *is holomorphically convex.*

Suppose in addition that if $Y_1 \neq Y_2$ *in* $\mathcal{C}^0_s(D)$ *then there is a closed subvariety* $X' \subset X$ *such that* (i) X' *satisfies* (2.1) *and* (ii) $X' \cap Y_1 \neq X' \cap Y_2$ *. Then* $\mathcal{C}^0_s(D)$ *is Stein.*

We reformulate Proposition 2.5 for the special case where G_0 is a real semisimple Lie group, G is its complexification, Q is a parabolic subgroup of G, X is the complex flag manifold G/Q, and D is an open G_0–orbit in X. We refer to this case as the "flag domain case."

In the flag domain case, we have a particular cycle

(2.6)
$$Y_0 = K_0(x_0) = K(x_0)$$

where $D = G_0(x_0)$, K_0 is an appropriately chosen maximal compact subgroup of G_0 , and K is the complexification of K_0 . See [7] for the fact that Y_0 is a complex flag manifold sitting as a maximal compact subvariety of D. Here $s = \dim_{\mathbb{C}} Y_0$ and we make use of

(2.7)
$$M_D := \text{ topological component of } Y_0 \text{ in } \{gY_0 \mid g \in G \text{ and } gY_0 \subset D\}.$$

rather than $\mathcal{C}^0_s(D)$. Observe that

(2.8)
$$E := \{g \in G \mid gY_0 = Y_0\} \text{ is a closed complex subgroup of } G.$$

Then M_D has topology and complex structure as an open submanifold of

(2.9)
$$M_X = \{gY_0 \mid g \in G\} \cong G/E.$$

Here (2.9) is the definition of M_X . See [9] or [12] for details.

We may assume that X is irreducible, i.e. that G is simple. For everything breaks up as a product according to the decomposition of G as a product of simple closed normal subgroups. If G_0/K_0 is not a bounded symmetric domain then the Lie algebra $\mathfrak{e}$ of E is a maximal subalgebra of the Lie algebra $\mathfrak{g}$ of G, so either $E = G$ with G_0 transitive on X, or K is the identity component E^0 of E. In these cases M_X is an affine variety. If G_0/K_0 is a bounded symmetric domain then there is another possibiity, $E = KM^{\pm}$, parabolic subgroup of G such that $G/KM^{\pm}$ is the hermitian symmetric flag manifold dual to G_0/K_0 . In this case M_X is a projective variety.

$\mathcal{C}^0_s(D)$ denotes the irreducible component of the Barlet cycle space of D that contains M_D , and $\mathcal{C}^0_s(X)$ is the irreducible component of the Barlet cycle space of X that contains M_X .

Lemma 2.10. *The inclusions $M_D \hookrightarrow \mathcal{C}^0_s(D)$ and $M_X \hookrightarrow \mathcal{C}^0_s(X)$ are holomorphic. If* (2.1) *and* (2.2) *hold, then $F : M_D \to \mathbb{C}$ is holomorphic.*

Proof. The complex structure on $\mathcal{C}^0_s(X)$ is G–invariant, so each G–orbit on $\mathcal{C}^0_s(X)$ is a complex submanifold, and in particular M_X is a complex submanifold of $\mathcal{C}^0_s(X)$. Now the open submanifold M_D of M_X is a complex submanifold of $\mathcal{C}^0_s(X)$ contained in $\mathcal{C}^0_s(D)$, thus is a complex submanifold of $\mathcal{C}^0_s(D)$. Thus the inclusions are holomorphic and the last statement is an immediate consequence of Proposition 2.4. $\square$

It is not quite as easy to carry Proposition 2.5 over from $\mathcal{C}^0_s(D)$ to M_D , but one can suitably adjust the statement and proof for the flag domain case. There (2.1) plus the hypothesis of the first part of Proposition 2.5 will be replaced by the condition:

(2.11) There is a closed complex submanifold $X' \subset X$ such that
(i) $\operatorname{codim}(X' \subset X) = s$,
(ii) $x_0 \in X' \cap Y_0$ and X' is transversal to Y_0 at x_0 ,
(iii) $X'' = X' \cap D$ is a Stein manifold, and
(iv) X' meets every G_0–orbit on $\mathrm{bd}(D)$.

The hypothesis of the second part of Proposition 2.5 will not be an issue.

A standard and straightforward intersection number argument gives

Lemma 2.12. *In the flag domain case, let X' be a closed complex submanifold of X such that* (i) $\operatorname{codim}(X' \subset X) = s$*,* (ii) *$X'$ meets the base cycle Y_0, and X' is transversal to Y_0 in at least one intersection point, and* (iii) *if $Y \in M_D$ then $X' \cap Y$ is finite. Then $X' \cap Y$ is non–empty for every $Y \in M_D$.*

Proposition 2.13. *In the flag domain case, if* (2.11) *holds, then M_D is Stein.*

Proof. Write $\mathrm{cl}(\cdot)$ and $\mathrm{bd}(\cdot)$ for Zariski closure and for boundary in the compact variety $\mathcal{C}_s^0(X)$. Then M_X is Zariski–open in $\mathrm{cl}(M_X)$ because the action of G on $\mathcal{C}_s^0(X)$ is algebraic. Now $\mathrm{cl}(M_X)$ is a compact subvariety of $\mathcal{C}_s^0(X)$, and $\mathrm{cl}(M_X)$ is a disjoint union $M_X \cup F$ where $F \subset \mathrm{bd}(M_X)$ is a union of lower–dimensional subvarieties. Now $\mathrm{bd}(M_D)$ is a disjoint union $(\mathrm{bd}(M_D) \cap F) \cup B$. Here $B \subset \mathrm{bd}(\mathcal{C}_s^0(D))$ because $B \subset M_X$.

Let $\{Y_n\} \subset M_D$ with $\{Y_n\} \to Y \in \mathrm{bd}(M_D)$. Now either $Y \in B$ or $Y \in \mathrm{bd}(M_X)$.

Suppose $Y \in B$. $\mathcal{C}_s^0(D)$ is holomorphically convex by Proposition 2.5 and Lemma 2.12. Thus we have $F : \mathcal{C}_s^0(D) \to \mathbb{C}$ holomorphic with $\lim |F(Y_n)| = \infty$.

Suppose $Y \in \mathrm{bd}(M_X)$. Then $\mathrm{bd}(M_X)$ is not empty, so M_X cannot be projective. Now M_X is affine, thus Stein, so we have $F : M_X \to \mathbb{C}$ holomorphic with $\lim |F(Y_n)| = \infty$.

Now M_D is holomorphically convex.

If $Y_1 \neq Y_2$ in $\mathcal{C}_s^0(D)$ and $x \in Y_1 \cap Y_2 \cap X'$ then we have $g \in G_0$ with $g(x) \in Y_1 \setminus Y_2$, and we replace X' by $g(X')$. So, as in the second part of Proposition 2.5, M_D is Stein. □

3. Grassmann manifold example

In this Section we work out the case where X is the complex Grassmann manifold $Gr(p,q)$ of q–dimensional linear subspaces of $\mathbb{C}^{p+q}$, and G_0 is an indefinite special unitary group $SU(p,q)$. This illustrates the situation where X is an hermitian symmetric flag manifold, compact dual to a bounded symmetric domain G_0/K_0, treated in Sections 4 and 5 below. Of course one can skip this Section and go directly to §4 and §5.

Notation. If $\{u_1, \ldots, u_\ell\}$ are linearly independent vectors in $\mathbb{C}^{p+q}$ then $[u_1 \wedge \cdots \wedge u_\ell]$ denotes their span. Fix a basis $\{e_1, \ldots, e_{p+q}\}$ of $\mathbb{C}^{p+q}$ in which the hermitian form defining $G_0 = SU(p,q)$ is given by $h(u,v) = \left(\sum_{1 \leqq i \leqq p} u_i \overline{v_i}\right) - \left(\sum_{1 \leqq i \leqq q} u_{p+i} \overline{v_{p+i}}\right)$. Here $u = \sum u_i e_i$ and $v = \sum v_i e_i$. If W is a subspace of $\mathbb{C}^{p+q}$ and $0 \leqq s \leqq \dim W$ then $Gr(s,W)$ denotes the Grassmann manifold of s–dimensional subspaces of W. So $Gr(s,W) \cong Gr(r,s) \cong Gr(s, \mathbb{C}^{r+s})$ where $\dim W = r+s$.

The $SU(p,q)$–orbits on $Gr(p,q)$ are given by the restriction of h to the elements of the orbit. So the orbits are the

$$
\begin{aligned}
D_{a,b,c} &= \{x \in Gr(p,q) \mid x \text{ has sign and rank } (+,-,0) = (a,b,c)\} \\
&= SU(p,q)([e_1 \wedge \cdots \wedge e_a \wedge e_{p+1} \wedge \cdots \wedge e_{p+b} \wedge f_1 \wedge \cdots \wedge f_c]) \qquad (3.1) \\
&\quad \text{where } f_j = e_{p+1-j} + e_{p+q+1-j}
\end{aligned}
$$

Here of course $a \leqq p$, $b \leqq q$, $a+b+c = q$, and $c \leqq \min(p,q)$. The boundary is given by degeneration of h on the elements of the orbit. Thus an orbit

$$D_{a',b',c'} \text{ is in the closure of } D_{a,b,c} \text{ if and only if } a' \leqq a \text{ and } b' \leqq b. \qquad (3.2)$$

In particular, the open orbits are the

$$D_a = D_{a,q-a,0} = SU(p,q)([e_1 \wedge \cdots \wedge e_a \wedge e_{p+1} \wedge \cdots \wedge e_{p+q-a}]) \tag{3.3}$$

and their boundaries are the union of the orbits given by

$$\mathrm{bd}\,(D_a) = \bigcup_{\substack{0 \leqq i \leqq a, 0 \leqq j \leqq q-a, \\ 0 < i+j \leqq \min(p,q)}} D_{a-i,q-a-j,i+j}. \tag{3.4}$$

Note that the bounded symmetric domain G_0/K_0 here is D_0, and its boundary orbits are the $D_{0,q-j,j}$ for $1 \leqq q \leqq \min(p,q)$. The boundary orbit

$$D_{0,q-\min(p,q),\min(p,q)}$$

is the Bergman–Shilov boundary of D_0.

Fix $D = D_a$ as in (3.3). It will be convenient to denote

$$\mathbb{C}^{p+q} = S^+ \oplus S^- \oplus T^+ \oplus T^-, \tag{3.5}$$

where $S^+ = [e_1 \wedge \cdots \wedge e_a]$ and $T^+ = [e_{a+1} \wedge \cdots \wedge e_p]$ are positive definite, and $S^- = [e_{p+1} \wedge \cdots \wedge e_{p+q-a}]$ and $T^- = [e_{p+q-a+1} \wedge \cdots \wedge e_{p+q}]$ are negative definite. Then $D_a = SU(p,q)(S^+ \oplus S^-)$. The "base cycle" in D_a is

$$\begin{aligned} Y_0 &= S(U(p) \times U(q))([e_1 \wedge \cdots \wedge e_a \wedge e_{p+1} \wedge \cdots \wedge e_{p+q-a}]) \\ &= Gr(a, [e_1 \wedge \cdots \wedge e_p]) \times Gr(q-a, [e_{p+1} \wedge \cdots \wedge e_{p+q}]). \end{aligned} \tag{3.6}$$

In particular $s = \dim_{\mathbb{C}} Y_0 = a(p+q-2a)$. If $g \in SL(p+q;\mathbb{C})$ and $Y = gY_0 \subset D_a$, then

$$\begin{aligned} U =& g(S^+ \oplus T^+) \gg 0,\ V = g(S^- \oplus T^-) \ll 0,\ \mathbb{C}^{p+q} = U \oplus V, \text{ and} \\ Y =& Y_{U,V} = \{U' \oplus V' \mid U' \subset U, \dim U' = a, V' \subset V, \dim V' = q-a\}. \end{aligned} \tag{3.7}$$

Let $G_1 \cong GL(2a;\mathbb{C})$ denote the general linear group on $S^+ \oplus T^-$, let $G_2 \cong GL((p-a)+(q-a);\mathbb{C})$ denote the general linear group on $T^+ \oplus S^-$, and write $S(G_1 \times G_2)$ for the determinant 1 elements of the product as a subgroup of $SL(p+q;\mathbb{C})$. Our transverse manifold will be

$$\begin{aligned} X' = X'_a &= S(G_1 \times G_2)(S^+ \oplus S^-) \\ &= G_1(S^+) \times G_2(S^-) = Gr(a, S^+ \oplus T^-) \times Gr(q-a, T^+ \oplus S^-). \end{aligned} \tag{3.8}$$

Then $X'_a \cap D_a$ consists of all $W^+ \oplus W^-$ where W^+ is an (a–dimensional) maximal positive definite subspace of $S^+ \oplus T^-$ and W^- is a (($q-a$)–dimensional) maximal negative definite subspace of $T^+ \oplus S^-$. In particular

Lemma 3.9. *$X'_a \cap D_a$ is a bounded symmetric domain, and thus is a Stein manifold.*

The next lemma illustrates Lemma 2.12 in our Grassmann manifold setting.

Lemma 3.10. *The manifold X'_a meets every cycle $Y \in M_D$.*

Proof. Express $Y = Y_{U,V}$ as in (3.7). If $v \in \mathbb{C}^{p+q}$ decompose $v = v_{s,+} + v_{t,+} + v_{s,-} + v_{t,-}$ with $v_{s,\pm} \in S^\pm$ and $v_{t,\pm} \in T^\pm$. Now $v \mapsto v_{s,+} + v_{t,+}$ has negative definite kernel, hence is one to one on U, so U has a basis $\{u_i = e_i + u_{i,s,-} + u_{i,t,-} \mid 1 \leqq i \leqq p\}$. Similarly V has a basis $\{v_j = e_{p+j} + v_{j,s,+} + v_{j,t,+} \mid 1 \leqq j \leqq q\}$.

Some $g_1 \in G_1$, with block form matrix $\left(\begin{smallmatrix} I & 0 \\ * & I \end{smallmatrix}\right)$ relative to $S^+ \oplus T^-$, sends u_i to $u'_i = e_i + u_{i,s,-}$ for $1 \leqq i \leqq a$. Similarly some $g_2 \in G_2$, with block form matrix $\left(\begin{smallmatrix} I & 0 \\ * & I \end{smallmatrix}\right)$ relative to $S^- \oplus T^+$, sends v_j to $v'_j = e_{p+j} + v_{j,s,+}$ for $1 \leqq j \leqq q-a$. Now we have $g = (g_1, g_2) \in G_1 \times G_2$ such that $g^{-1}[u_1 \wedge \cdots \wedge u_a]$ has basis $\{u'_i = e_i + u_{i,s,-} \mid 1 \leqq i \leqq a\}$ and $g^{-1}[v_1 \wedge \cdots \wedge v_{q-a}\}$ has basis $\{v'_j = e_{p+j} + v_{j,s,+} \mid 1 \leqq j \leqq q-a\}$.

If we add an appropriate linear combination of these v'_j to u'_i we kill off their $u_{i,s,-}$ summands, changing u'_i to $u''_i = e_i + u''_{i,s,+}$ for $1 \leqq i \leqq a$. However each $u''_{i,s,+}$ is in the span S^+ of $\{e_1, \ldots, e_a\}$. Now $g^{-1}[u_1 \wedge \cdots \wedge u_a \wedge v_1 \wedge \cdots \wedge v_{q-a}]$ has basis $\{e_i \mid 1 \leqq i \leqq a\} \cup \{v'_j \mid 1 \leqq j \leqq q-a\}$. If we add an appropriate linear combination of these e_i to v'_j we kill off their $v'_{j,s,+}$ summands, changing v'_j to e_{p+j}. Now $g^{-1}[u_1 \wedge \cdots \wedge u_a \wedge v_1 \wedge \cdots \wedge v_{q-a}]\}$ has basis $\{e_i \mid 1 \leqq i \leqq a\} \cup \{e_{p+j} \mid 1 \leqq j \leqq q-a\}$.

We have proved that $g(S^+ \oplus S^-) = [u_1 \wedge \cdots \wedge u_a \wedge v_1 \wedge \cdots \wedge v_{q-a}]$. Thus $g(S^+ \oplus S^-) \in Y = Y_{U,V}$. But $g(S^+ \oplus S^-) \in X'_a$ because g was constructed as an element of $G_1 \times G_2$. So X'_a meets Y. □

Lemma 3.11. *X'_a satisfies* (2.11)(i), (2.11)(iii) *and* (2.11)(iv) *for the open orbit D_a.*

Proof. It follows from the definition (3.8) that the codimension of X' in $Gr(p,q)$ is $pq - (a^2 + (p-a)(q-a)) = a(p+q-2a) = s$.

Lemma 3.9 says that $X'_a \cap D_a$ is Stein.

The linear space $S^+ \otimes T^-$ on which G_1 acts, has a subspace R_1 of rank and sign $(a-i, 0, i)$ given by $R_1 = [e_1 \wedge \ldots e_{a-i} \wedge f''_1 \wedge \cdots \wedge f''_i]$ where $f''_k = e_{a+1-k} + e_{p+q+1-k}$. The linear space $T^+ \otimes S^-$ on which G_2 acts, has a subspace R_2 of rank and sign $(0, q-a-j, j)$, given by $R_2 = [e_{p+1} \wedge \ldots e_{p+q-a-j} \wedge f'_1 \wedge \cdots \wedge f'_j]$ where $f'_k = e_{p+1-k} + e_{p+q-a+1-k}$. Thus X'_a contains the space $R_1 \oplus R_2$, which has rank and sign $(a-i, q-a-j, i+j)$. In other words, X'_a meets the orbit $D_{a-i,q-a-j,i+j}$. In view of (3.4) now X'_a meets every boundary orbit of D_a. That completes the proof. □

Proposition 2.13 combines with Lemmas 3.10 and 3.11 to give us

Proposition 3.12. *Let D be an open orbit of $SU(p,q)$ on the Grassmann manifold $Gr(p,q)$. Then the Barlet–Koziarz intersection method shows that M_D is Stein.*

Proof. We have all of (2.11) except (2.11)(ii). But (2.11)(ii) was only used to show that X'_a meets every cycle $Y \in M_D$, and we proved that directly as Lemma 3.10. Thus, essentially as in Proposition 2.13, we conclude that M_D is Stein. □

4. Cayley transforms and boundary structure

In this Section we recall the explicit real group orbit structure of flag manifolds that are the compact duals to bounded symmetric domains. That uses the Cayley transform methods of [5], [6] and [7], as described in [8], and extends (3.1), (3.2), (3.3) and (3.4) to the more general setting of hermitian symmetric spaces.

Let $B = G_0/K_0$ be an irreducible hermitian symmetric space of noncompact type, in other words an irreducible bounded symmetric domain. The real Lie algebra decomposes as usual into $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{m}_0$ where $\mathfrak{m}_0$ represents the real tangent space, $[\mathfrak{k}_0, \mathfrak{m}_0] = \mathfrak{m}_0$, $[\mathfrak{m}_0, \mathfrak{m}_0] = \mathfrak{k}_0$, and $\mathfrak{k}_0$ acts irreducibly (but not absolutely irreducibly) on $\mathfrak{m}_0$. The complexified Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{m}^+ + \mathfrak{k} + \mathfrak{m}^- \text{ where } [\mathfrak{k}, \mathfrak{m}^\pm] = \mathfrak{m}^\pm, [\mathfrak{m}^\pm, \mathfrak{m}^\pm] = 0, \text{ and } [\mathfrak{m}^-, \mathfrak{m}^+] = \mathfrak{k}. \tag{4.1}$$

The holomorphic tangent space of B is represented by $\mathfrak{m}^+$, and $\mathfrak{m}^- = \overline{\mathfrak{m}^+}$ represents the antiholomorphic tangent space. The algebra $\mathfrak{k}$ acts irreducibly on each of $\mathfrak{m}^\pm$. From (4.1) we have

$$\mathfrak{q} = \mathfrak{k} + \mathfrak{m}^- : \text{ parabolic subalgebra of } \mathfrak{g} \quad \text{with nilradical } \mathfrak{m}^-, \text{ reductive part } \mathfrak{k}. \tag{4.2}$$

We may assume that G_0 is contained as a real form in the connected simply connected complex simple Lie group G with Lie algebra $\mathfrak{g}$. Let $Q \subset G$ denote the parabolic subgroup with Lie algebra $\mathfrak{q}$ as in (4.2). Denote

$$X = G/Q \text{ complex flag manifold and } x_0 = 1Q \in X \text{ base point.} \tag{4.3}$$

The *Borel embedding* of B is

$$B \cong G_0(x_0), \text{ open } G_0\text{–orbit on } X. \tag{4.4}$$

Finally, we note that one can identify X with the compact dual symmetric space G_u/K_0 of $B = G_0/K_0$. For that, G_u is the compact real form of G with Lie algebra $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\, \mathfrak{m}_0$, and G_u acts transitively on X with isotropy K_0 at x_0.

Choose a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$; it also is a Cartan subalgebra of $\mathfrak{g}_0$. Fix any positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Extend it to a positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ by requiring that $\mathfrak{m}^+$ be a sum of positive root spaces, thus that $\mathfrak{m}^-$ be a sum of negative root spaces. Roots α, β are *strongly orthogonal*, written $\alpha \perp\!\!\!\perp \beta$, if neither of $\alpha \pm \beta$ is a root. Consider the "cascade construction"

$$\begin{aligned} &\Psi = \{\psi_1, \dots, \psi_\ell\}, \text{ maximal set constructed by:} \\ &\quad \psi_1 \text{ is the maximal root in } \Delta^+ \text{ (always noncompact),} \\ &\quad \psi_{i+1} \text{ is a maximal root in } \Delta(\mathfrak{m}^+, \mathfrak{t}) \text{ with } \psi_{i+1} \perp \{\psi_1, \dots, \psi_i\}. \end{aligned} \tag{4.5}$$

Any two different roots of Ψ are strongly orthogonal, so Ψ is a maximal set of strongly orthogonal noncompact positive roots. Any maximal set of strongly orthogonal noncompact positive roots is. K_0–conjugate to the one constructed in (4.5).

If $\alpha \in \Delta$ then $h_\alpha \in \sqrt{-1}\ \mathfrak{t}_0$ is defined by $\frac{2\alpha(h)}{\langle\alpha,\alpha\rangle} = \langle h_\alpha, h\rangle$ for all $h \in \mathfrak{t}$. One can choose root vectors $e_\alpha \in \mathfrak{g}^\alpha$, normalized by $[e_\alpha, e_{-\alpha}] = h_\alpha$, such that the

$$x_{\alpha,0} = e_\alpha + e_{-\alpha} \text{ and } y_{\alpha,0} = \sqrt{-1}\ (e_\alpha - e_{-\alpha}) \text{ for } \alpha \in \Delta^+(\mathfrak{m},\mathfrak{t}) \tag{4.6}$$

form a real basis of $\mathfrak{m}_0$. Here $[x_{\alpha,0}, y_{\alpha,0}] = -2\sqrt{-1}\ h_\alpha$. The almost complex structure J is given by a central element $z \in \mathfrak{k}_0$; in the basis (4.6) it is

$$Jx_{\alpha,0} = [z, x_{\alpha,0}] = y_{\alpha,0} \text{ and } Jy_{\alpha,0} = [z, y_{\alpha,0}] = -x_{\alpha,0}. \tag{4.7}$$

Similarly (or consequently) $\mathfrak{m}_u = \sqrt{-1}\ \mathfrak{m}_0$ has a real basis consisting of the

$$x_\alpha = \sqrt{-1}\ x_{\alpha,0} = \sqrt{-1}\ (e_\alpha + e_{-\alpha}) \text{ and } y_\alpha = \sqrt{-1}\ y_{\alpha,0} = -(e_\alpha - e_{-\alpha}) \tag{4.8}$$

for $\alpha \in \Delta^+(\mathfrak{m},\mathfrak{t})$. From maximal strong orthogonality of Ψ we have maximal commutative subspaces

$$\mathfrak{a}_0 = \textstyle\sum_{\psi\in\Psi} x_{\psi,0}\mathbb{R} \subset \mathfrak{m}_0 \text{ and } \mathfrak{a}_u = \textstyle\sum_{\psi\in\Psi} x_\psi\mathbb{R} \subset \mathfrak{m}_u\ . \tag{4.9}$$

Given $\psi \in \Psi$ we have the 3–dimensional simple algebra

$$\mathfrak{g}[\psi] = \mathfrak{g}^\psi + \mathfrak{g}^{-\psi} + h_\psi\mathbb{C} \cong \mathfrak{sl}(2;\mathbb{C}) \text{ in } \mathfrak{g}, \tag{4.10}$$

its noncompact real form

$$\mathfrak{g}_0[\psi] = \mathfrak{g}[\psi] \cap \mathfrak{g}_0 = \text{ Span}\,\{x_{\psi,0}, y_{\psi,0}, \sqrt{-1}\ h_\psi\} \cong \mathfrak{sl}(2;\mathbb{R}), \tag{4.11}$$

and its compact real form

$$\mathfrak{g}_u[\psi] = \mathfrak{g}[\psi] \cap \mathfrak{g}_u = \text{ Span}\,\{x_\psi, y_\psi, \sqrt{-1}\ h_\psi\} \cong \mathfrak{su}(2), \tag{4.12}$$

They define 3–dimensional simple subgroups

$$G[\psi] \subset G \text{ for } \mathfrak{g}[\psi], \quad G_0[\psi] \subset G_0 \text{ for } \mathfrak{g}_0[\psi], \quad G_u[\psi] \subset G_u \text{ for } \mathfrak{g}_u[\psi]. \tag{4.13}$$

Strong orthogonality of Ψ says that $[\mathfrak{g}[\psi], \mathfrak{g}[\psi']] = 0$ for $\psi, \psi' \in \Psi$ with $\psi \neq \psi'$. If $\Gamma \subset \Psi$ it follows that the

$$\mathfrak{g}[\Gamma] = \sum_{\psi\in\Gamma} \mathfrak{g}[\psi] \subset \mathfrak{g}, \quad \mathfrak{g}_0[\Gamma] = \sum_{\psi\in\Gamma} \mathfrak{g}_0[\psi] \subset \mathfrak{g}_0, \quad \mathfrak{g}_u[\Gamma] = \sum_{\psi\in\Gamma} \mathfrak{g}_u[\psi] \subset \mathfrak{g}_u \tag{4.14}$$

are well defined subalgebras that are Lie algebra direct sums. The corresponding groups

$$\begin{aligned} G[\Gamma] = \prod_{\psi\in\Gamma} G[\psi] \subset G, \quad G_0[\Gamma] = \prod_{\psi\in\Gamma} G_0[\psi] \subset G_0, \\ G_u[\Gamma] = \prod_{\psi\in\Gamma} G_u[\psi] \subset G_u \end{aligned} \tag{4.15}$$

are locally direct products. Their orbits at the base point are

$$\text{polydisk}: B[\Gamma] = G_0[\Gamma](x_0), \text{ product of } |\Gamma| \text{ unit disks, and} \tag{4.16}$$
$$\text{polysphere}: X[\Gamma] = G[\Gamma](x_0) = G_u[\Gamma](x_0), \text{ product of } |\Gamma| \text{ Riemann spheres.}$$

The (partial) Cayley transforms follow the classical model for the Riemann spheres of (4.16). If $\Gamma \subset \Psi$ we define the *partial Cayley transform*

$$c_\Gamma = \prod_{\psi \in \Gamma} c_\psi \in G_u[\Psi] \subset G_u \quad \text{where} \quad c_\psi = \exp(\tfrac{\pi}{4} y_\psi) \in G_u[\psi]. \tag{4.17}$$

Now define points and orbits in X by

$$\text{if } \Gamma, \Sigma \subset \Psi \text{ are disjoint, then } x_{\Gamma,\Sigma} = c_\Gamma c_\Sigma^2 x_0 \text{ and } D_{\Gamma,\Sigma} = G_0(x_{\Gamma,\Sigma}). \tag{4.18}$$

We reformulate the Orbit Structure Theorem ([7, Theorem 10.6], or see [8, §7]):

Theorem 4.19. *Assume that $B = G_0/K_0$ is irreducible, so G is simple. The G_0–orbits on X are just the $D_{\Gamma,\Sigma}$ where Γ and Σ are disjoint subsets of Ψ. An orbit $D_{\Gamma',\Sigma'}$ is in the closure of $D_{\Gamma,\Sigma}$ (where $\Gamma \cap \Sigma = \emptyset = \Gamma' \cap \Sigma'$) if and only if the cardinalities $|\Sigma'| \leqq |\Sigma|$ and $|\Sigma \cup \Gamma| \leqq |\Sigma' \cup \Gamma'|$. In particular*

(i) $D_{\Gamma',\Sigma'} = D_{\Gamma,\Sigma}$ *if and only if both* $|\Gamma| = |\Gamma'|$ *and* $|\Sigma| = |\Sigma'|$*;*
(ii) *the number of G_0–orbits on X is $(\ell+1)(\ell+2)$ where $\ell = |\Psi|$;*
(iii) *$D_{\Gamma,\Sigma}$ is open in X if and only if Γ is empty, so the open G_0–orbits are the $\ell+1$ orbits $\{D_0, \dots, D_\ell\}$ where $D_{|\Sigma|} = D_{\emptyset,\Sigma}$ for $0 \leqq |\Sigma| \leqq \ell$;*
(iv) $\mathrm{bd}\,(D_i)$ *is the union of the $D_{\Gamma',\Sigma'} \neq D_i$ with $|\Sigma'| \leqq i \leqq |\Sigma' \cup \Gamma'|$; and*
(v) *$D_{\Psi,\emptyset}$ is the unique closed G_0–orbit on X; it is in the closure of every orbit and is the Bergman–Shilov boundary of $B = D_0$.*

Theorem 4.19 gives us the boundary information that we will need in order to extend Proposition 3.12 from Grassmann manifolds with $G_0 = SU(p,q)$, to all hermitian symmetric spaces. The boundary orbit information (3.1), (3.2), (3.3) and (3.4), for the case where $G_0 = SU(p,q)$, can of course, be extracted from Theorem 4.19.

5. Construction and analysis of the transverse variety

Retain the setup and notation of Section 4. In this section we will construct subvarieties $X_i' \subset X$ for $0 \leqq i \leqq \ell$ such that X_i' satisfies (2.11) for the open orbit D_i. For this we have to define certain subspaces of $\mathfrak{g}_0$ using the partial Cayley transforms (4.17).

Fix a subset $\Sigma \subset \Psi$. It is a fact [6] that $\mathrm{Ad}(c_\Sigma^4)$ has square 1 as an automorphism of $\mathfrak{g}$, and that it preserves both $\mathfrak{m}$ and $\mathfrak{g}_0$. Define

$$\begin{aligned}
&\mathfrak{g}^\Sigma = \{\xi \in \mathfrak{g} \mid \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} \quad \text{and} \quad \mathfrak{g}_0^\Sigma = \mathfrak{g}^\Sigma \cap \mathfrak{g}_0 \ ,\\
&\mathfrak{k}^\Sigma = \{\xi \in \mathfrak{k} \mid \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} \quad \text{and} \quad \mathfrak{k}_0^\Sigma = \mathfrak{k}^\Sigma \cap \mathfrak{g}_0 \ ,\\
&\mathfrak{m}^\Sigma = \{\xi \in \mathfrak{m} \mid \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} \quad \text{and} \quad \mathfrak{m}_0^\Sigma = \mathfrak{m}^\Sigma \cap \mathfrak{g}_0 \ ,\\
&\mathfrak{r}^\Sigma = \{\xi \in \mathfrak{m} \mid \mathrm{Ad}\,(c_\Sigma^4)\xi = -\xi\} \quad \text{and} \quad \mathfrak{r}_0^\Sigma = \mathfrak{r}^\Sigma \cap \mathfrak{g}_0 \ .
\end{aligned} \tag{5.1}$$

So $\mathfrak{g}^\Sigma = \mathfrak{k}^\Sigma + \mathfrak{m}^\Sigma$ and $\mathfrak{g}_0^\Sigma = \mathfrak{k}_0^\Sigma + \mathfrak{m}_0^\Sigma$.

The case of [7, Lemma 11.6] (or see [8, Lemma 9.10]), where Γ is empty, says

Lemma 5.2. *Define $f : K_0 \times \mathfrak{m}_0^\Sigma \times \mathfrak{r}_0^\Sigma \to G_0$ by $f(k,\xi,\eta) = k\exp(\xi)\exp(\eta)$. Then f is a real analytic diffeomorphism of $K_0 \times \mathfrak{m}_0^\Sigma \times \mathfrak{r}_0^\Sigma$ onto G_0.*

Our use of Lemma 5.2 will require a refinement of the notation (5.1). If $\Gamma \subset \Phi$ we note that the centralizer of $\mathfrak{g}[\Psi \setminus \Gamma]$ has form (subspace of $\mathfrak{t}$) + $\sum_{\alpha \perp \Psi\setminus\Gamma} \mathfrak{g}^\alpha$. It is a reductive algebra with semisimple part

$$\mathfrak{g}_\Gamma = \mathfrak{k}_\Gamma + \mathfrak{m}_\Gamma : \text{ derived algebra of } \mathfrak{t} + \sum_{\alpha\perp\Psi\setminus\Gamma} \mathfrak{g}^\alpha. \tag{5.3}$$

which in turn has real forms

$$\mathfrak{g}_{\Gamma,0} = \mathfrak{g}_0 \cap \mathfrak{g}_\Gamma = \mathfrak{k}_{\Gamma,0} + \mathfrak{m}_{\Gamma,0} \quad \text{and} \quad \mathfrak{g}_{\Gamma,u} = \mathfrak{g}_u \cap \mathfrak{g}_\Gamma = \mathfrak{k}_{\Gamma,u} + \mathfrak{m}_{\Gamma,u} \ . \tag{5.4}$$

Of course we have the corresponding analytic groups

$$G_\Gamma \subset G, \quad G_{\Gamma,0} \subset G_0 \quad \text{and} \quad G_{\Gamma,u} \subset G_u \tag{5.5}$$

and their orbits

$$X_\Gamma = G_\Gamma(x_0) = G_{\Gamma,u}(x_0) \subset X \quad \text{and} \quad B_\Gamma = G_{\Gamma,0}(x_0) = B \cap X_\Gamma \ . \tag{5.6}$$

Note that $\mathfrak{g}_\Gamma$ has Γ as its maximal set of strongly orthogonal noncompact roots. In effect, these groups and spaces repeat the situation of G_0 and X with Ψ reduced to Γ. Passage from G_0 to $G_{\Gamma,0}$ was exemplified in §3 by passage from $SU(p,q)$ to $SU(p-a,q-a)$ with $a = |\Psi \setminus \Gamma|$.

We combine the idea of (5.1) with that of (5.3) and (5.4). If $\Sigma \subset \Phi \subset \Psi$, define

$$\begin{aligned}
&\mathfrak{g}_\Phi^\Sigma = \{\xi \in \mathfrak{g}_\Phi \mid \ \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} = \mathfrak{g}_\Phi \cap \mathfrak{g}^\Sigma \text{ and } \mathfrak{g}_{\Phi,0}^\Sigma = \mathfrak{g}_\Phi^\Sigma \cap \mathfrak{g}_0 \ ,\\
&\mathfrak{k}_\Phi^\Sigma = \{\xi \in \mathfrak{k}_\Phi \mid \ \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} = \mathfrak{k}_\Phi \cap \mathfrak{k}^\Sigma \text{ and } \mathfrak{k}_{\Phi,0}^\Sigma = \mathfrak{k}_\Phi^\Sigma \cap \mathfrak{k}_0 \ ,\\
&\mathfrak{m}_\Phi^\Sigma = \{\xi \in \mathfrak{m}_\Phi \mid \ \mathrm{Ad}\,(c_\Sigma^4)\xi = \xi\} = \mathfrak{m}_\Phi \cap \mathfrak{m}^\Sigma \text{ and } \mathfrak{m}_{\Phi,0}^\Sigma = \mathfrak{m}_\Phi^\Sigma \cap \mathfrak{m}_0 \ .
\end{aligned} \tag{5.7}$$

Then of course $\mathfrak{g}_\Phi^\Sigma = \mathfrak{k}_\Phi^\Sigma + \mathfrak{m}_\Phi^\Sigma$. We have real forms

$$\mathfrak{g}_{\Phi,0}^\Sigma = \mathfrak{g}_0 \cap \mathfrak{g}_\Phi^\Sigma = \mathfrak{k}_{\Phi,0}^\Sigma + \mathfrak{m}_{\Phi,0}^\Sigma \quad \text{and} \quad \mathfrak{g}_{\Phi,u}^\Sigma = \mathfrak{g}_u \cap \mathfrak{g}_\Phi^\Sigma = \mathfrak{k}_{\Phi,u}^\Sigma + \mathfrak{m}_{\Phi,u}^\Sigma \tag{5.8}$$

and analytic groups

$$G_\Phi^\Sigma \subset G, \quad G_{\Phi,0}^\Sigma \subset G_0 \quad \text{and} \quad G_{\Phi,u}^\Sigma \subset G_u \tag{5.9}$$

and their orbits

$$X^\Sigma_\Phi = G^\Sigma_\Phi(x_0) = G^\Sigma_{\Phi,u}(x_0) \subset X \quad \text{and} \quad B^\Sigma_\Phi = G^\Sigma_{\Phi,0}(x_0) = B \cap X^\Sigma_\Phi \ . \tag{5.10}$$

Here notice $G^\emptyset_\Phi = G_\Phi$, $G^\emptyset_{\Phi,0} = G_{\Phi,0}$, $G^\emptyset_{\Phi,u} = G_{\Phi,u}$, $X^\emptyset_\Phi = X_\Phi$, and $B^\emptyset_\Phi = B_\Phi$.

Lemma 5.11. *If* $\Sigma \subset \Psi$ *then*

$$G^\Sigma(x_0) = X^\Sigma \cong \left(X_{\Psi\setminus\Sigma} \times X^\Sigma_\Sigma\right) = \left(G_{\Psi\setminus\Sigma}(x_0) \times G^\Sigma_\Sigma(x_0)\right) \ \textit{and}$$
$$G^\Sigma_0(x_0) = B^\Sigma \cong \left(B_{\Psi\setminus\Sigma} \times B^\Sigma_\Sigma\right) = \left(G_{\Psi\setminus\Sigma,0}(x_0) \times G^\Sigma_{\Sigma,0}(x_0)\right).$$

Proof. This is essentially [7, Theorem 11.8 (1d)] with $\Gamma = \emptyset$. It is based on the argument in the proof (see [7, p. 1215]) that $\mathfrak{g}^\Sigma = \mathfrak{g}_{\Psi\setminus\Sigma} \oplus \mathfrak{g}^\Sigma_\Sigma \oplus \mathfrak{l}_{\emptyset,\Sigma}$ where $\mathfrak{l}_{\emptyset,\Sigma} \subset \mathfrak{k}$. □

Lemma 5.12. *If* $\Sigma \subset \Psi$ *then*

$$G^\Sigma(x_{\emptyset,\Sigma}) = c^2_\Sigma G^\Sigma(x_0) = c^2_\Sigma X^\Sigma \ \textit{and}\ G^\Sigma_0(x_{\emptyset,\Sigma}) = c^2_\Sigma G^\Sigma_0(x_0) = c^2_\Sigma B^\Sigma \ .$$

Proof. This also is implicit in [7, Theorem 11.8 (1d)] with $\Gamma = \emptyset$. By construction, $\mathrm{Ad}(c^2_\Sigma)$ normalizes $\mathfrak{g}^\Sigma$. Thus $G^\Sigma(x_{\emptyset,\Sigma}) = G^\Sigma(c^2_\Sigma x_0) = c^2_\Sigma G^\Sigma(x_0) = c^2_\Sigma X^\Sigma$.

$\mathrm{Ad}(c^2_\Sigma)_{\mathfrak{g}^\Sigma}$ has square 1 and commutes with both the Cartan involution θ and the complex conjugation $\xi \mapsto \overline{\xi}$ of $\mathfrak{g}$ over $\mathfrak{g}_0$. So $\mathrm{Ad}(c^2_\Sigma)$ preserves both $\mathfrak{g}^\Sigma_0$ and its decomposition $\mathfrak{g}^\Sigma_0 = \mathfrak{k}^\Sigma_0 + \mathfrak{m}^\Sigma_0$. Now $\mathrm{Ad}(c^2_\Sigma)$ normalizes G^Σ_0 and its maximal compact subgroup K^Σ_0. Thus $G^\Sigma_0(x_{\emptyset,\Sigma}) = G^\Sigma_0(c^2_\Sigma x_0) = c^2_\Sigma G^\Sigma_0(x_0) = c^2_\Sigma B^\Sigma$. □

Fix the open orbit $D = D_{|\Sigma|}$ and let Y_0 denote its base cycle, the maximal compact subvariety $K(x_{\emptyset,\Sigma}) = K_0(x_{\emptyset,\Sigma})$. Now use Lemma 5.2 to define a map

$$\pi : D \to Y_0 \text{ by } \pi(k\exp(\xi)\exp(\eta))(x_{\emptyset,\Sigma}) = k(x_{\emptyset,\Sigma}). \tag{5.13}$$

The maps β_k of the Orbit Fibration Theorem ([7, Theorem 11.8], or see [8, §9]) are well defined C^ω fibrations[1]. When $\Gamma = \emptyset$ the Orbit Fibration Theorem implies

Lemma 5.14. *The map* π *of* (5.13) *is a well defined* K_0*–equivariant real analytic fibration of* D *over* Y_0*. The fiber over* $x_{\emptyset,\Sigma}$ *is* $c^2_\Sigma B^\Sigma = c^2_\Sigma(B_{\Psi\setminus\Sigma} \times B^\Sigma_\Sigma)$.

Proof. The fact that π is a real analytic fibration, is contained in [7] and [8] for $\Gamma = \emptyset$. It is also shown there that the fiber $\pi^{-1}(x_{\emptyset,\Sigma}) = G^\Sigma_0(x_{\emptyset,\Sigma})$. The precise description of the fiber, asserted here, now follows from Lemmas 5.11 and 5.12. □

Lemma 5.15. *If* $\Sigma \subset \Psi$ *then* $X^\Sigma = c^2_\Sigma X^\Sigma = c^2_\Sigma(X_{\Psi\setminus\Sigma} \times X^\Sigma_\Sigma)$*, and it satisfies* (2.11) *for the open* G_0*–orbit* $D = D_{|\Sigma|}$ *on* X*.*

[1] There may be a problem with holomorphicity of the β_k but that is not of concern to us here.

Proof. Use Lemma 5.14 to see that $\dim c_\Sigma^2 X^\Sigma = \dim c_\Sigma^2 B^\Sigma = \dim \pi^{-1}(x_{\emptyset,\Sigma}) = \dim D - \dim Y_0$, so the codimension of $\dim c_\Sigma^2 X^\Sigma$ in X is $\dim Y_0$, as required for (2.11)(i).

The base point $x_{\emptyset,\Sigma}$ of $D = D_{|\Sigma|}$ is contained in both X^Σ and in Y_0, and Lemma 5.14 ensures that the intersection at that point is transversal. Thus we have (2.11)(ii).

Now we must show that $c_\Sigma^2 X^\Sigma \cap D = c_\Sigma^2 B^\Sigma$, which is a bounded symmetric domain, thus Stein. This is a consequence of [11, Theorem 3.8] as follows. Our X is both the hermitian symmetric space X of [11] and the complex flag manifold Z of [11]. Our x_0 is the base point for both. Now the G_0–orbits on X are the $D_{\Gamma,\Sigma}$ as described in Theorem 4.19. Our $\mathrm{Ad}(c_\Sigma^4)$ is the involutive automorphism σ of [11]. The Cartan involution θ satisfies $\theta(y_\psi) = -y_\psi$, so $\theta(c_\psi) = c_\psi^{-1}$ and this $\theta(c_\Sigma) = c_\Sigma^{-1}$, so $\theta \mathrm{Ad}(c_\Sigma^4) = \mathrm{Ad}(c_\Sigma^{-4})\theta = \mathrm{Ad}(c_\Sigma^4)\theta$; so $\mathrm{Ad}(c_\Sigma^4)$ commutes with θ as required. Now our G^Σ and its real form G_0^Σ, fixed point groups in G and G_0 of $\mathrm{Ad}(c_\Sigma^4)$, are the fixed point groups M and M_0 of σ in [11]. Our $X^\Sigma = G^\Sigma(x_0)$ is the space $F = M(z)$ in [11]. Note that $\mathfrak{g}[\Psi] \subset \mathfrak{g}^\Sigma$, so our Ψ is the $\Psi^{\mathfrak{m}}$ of [11]. Thus [11, Theorem 3.8], especially the last sentence of the theorem, which is hidden at the top of page 400 there, gives the following. As Φ and Γ range over disjoint pairs of subsets of Ψ,

$$\begin{aligned} &G_0^\Sigma(x_{\Gamma,\Phi}) \mapsto G_0(x_{\Gamma,\Phi}) = D_{\Gamma,\Phi} \text{ is a one–one map} \\ &\text{from the set of } G_0^\Sigma\text{–orbits on } X^\Sigma \text{ onto the set of } G_0\text{–orbits on } X. \end{aligned} \tag{5.16}$$

Conclusion: $G_0^\Sigma(x_{\emptyset,\Sigma}) = c_\Sigma^2 B^\Sigma$ is the only G_0^Σ–orbit in $c_\Sigma^2 X^\Sigma \cap D$. So $c_\Sigma^2 X^\Sigma \cap D = c_\Sigma^2 B^\Sigma$ is a bounded symmetric domain, thus Stein, as required for (2.11)(iii).

According to Theorem 4.19 the G_0–orbits in the boundary of $D = D_\Sigma$ are those $D_{\Gamma',\Sigma'} \neq D$ with $\Sigma' \subset \Sigma \subset (\Sigma' \cup \Gamma')$. But as noted above, $\mathfrak{g}[\Psi] \subset \mathfrak{g}^\Sigma$, so every $x_{\Gamma',\Sigma'} \in X^\Sigma = c_\Sigma^2 X^\Sigma$. Thus X^Σ meets every G_0–orbit on the boundary of D. Now we have (2.11)(iv). This completes the verification of (2.11), and thus the proof of Lemma 5.15. □

Proposition 2.13 combines with Lemma 5.15 to give us

Proposition 5.17. *Let D be an open G_0–orbit the hermitian symmetric flag manifold X. Then the Barlet–Koziarz intersection method shows that M_D is Stein.*

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