http://dx.doi.org/10.1090/conm/262/04181

Contemporary Mathematics
Volume **262**, 2000

Isoclinic Spheres and Flat Homogeneous Pseudo–Riemannian Manifolds

Joseph A. Wolf

ABSTRACT. The structure theory ([**3**], [**8**]) for complete flat homogeneous pseudo–riemannian manifolds reduces the classification to the solution of some systems of quadratic equations. There is no general theory for that, but new solutions are found here by essentially the same construction as that used for isoclinic spheres in Grassmann manifolds [**4**]. It is interesting to speculate on a possible direct geometric relation between those constant positive curvature riemannian spheres and the "corresponding" flat homogeneous pseudo–riemannian manifolds.

0.Introduction.

The structure of flat homogeneous riemannian manifolds is rather trivial [**2**]: they are the quotients $M = \mathbb{E}^n/\Gamma$ of an euclidean n–space by a discrete group of pure translations. Thus M is isometric to the product of an euclidean space with a flat torus. In the pseudo–riemannian case, however, there is the possibility of nontrivial holonomy [**3**]. A rather basic structure theory is worked out in [**3**] for complete flat homogeneous pseudo–riemannian manifolds, and a more refined structure theory is worked out in [**8**]. In Section 1 we recall that structure theory and indicate how solutions to the systems of quadratic equations for the holonomy lead to examples — and eventual classification. In Section 2 we describe an easy method that produces some solutions to the systems of quadratic equations. We use it in Section 3 to produce a family of solutions for each of the four real division algebras. We use it again in Section 4 to produce families for solutions which depend on "translational representations" of real Clifford algebras. Those translational representations were developed in [**4**] for the study of "isoclinic spheres" in Grassmann manifolds. We recall the relevant part of that theory ([**9**], [**4**]) in Section 5 and note the connection between isoclinic spheres and complete flat homogeneous pseudo–riemannian manifolds. Finally, in Section 6, we give some examples of parallel geometric structures whose construction takes advantage of our methods.

2000 *Mathematics Subject Classification.* Primary 22E40, 53C50; Secondary 22E15, 57S30.
Research supported the Alexander von Humboldt Foundation.

1.Structure Theory.

We recall the structure theory of [**3**] and [**8**]. Let $\mathbb{R}^{p,q}$ denote the real vector space $\mathbb{R}^{p+q}$ with the nondegenerate bilinear form $b(x,y) = \sum_{1 \leqq i \leqq p} x_i y_i - \sum_{1 \leqq i \leqq q} x_{p+i} y_{p+i}$, let $\mathbb{E}^{p,q}$ denote the corresponding pseudo–euclidean space, and let $O(p,q)$ denote the orthogonal group of $\mathbb{R}^{p,q}$.

Let M be a complete connected flat homogeneous pseudo–riemannian manifold. Then (the appropriate) $\mathbb{E}^{p,q}$ is the universal pseudo–riemannian covering manifold of M, so $M = \mathbb{E}^{p,q}/\Gamma$ where Γ is the group of deck transformations of the universal cover $\pi : \mathbb{E}^{p,q} \to M$. The full isometry group $\mathbf{I}(\mathbb{E}^{p,q})$ is the semidirect product $\mathbb{R}^{p+q} \cdot O(p,q)$, which acts by $(t, A) : x \mapsto t + Ax$. Homogeneity of M is equivalent to the condition that the centralizer of Γ in $\mathbf{I}(\mathbb{E}^{p,q})$ is transitive on $\mathbb{E}^{p,q}$, just as in the riemannian case [**2**]. Also, in the presence of homogeneity one need only check proper discontinuity of a discrete subgroup $\Gamma \subset \mathbf{I}(\mathbb{E}^{p,q})$ at a single point, typically the origin, where one need only verify that the translation components of the elements of Γ form a discrete subset of $\mathbb{R}^{p,q}$.

Fix a properly discontinuous discrete subgroup $\Gamma \subset \mathbf{I}(\mathbb{E}^{p,q})$ as above, such that $M = \mathbb{E}^{p,q}/\Gamma$ is homogeneous. In [**3**] it is shown that $\mathbb{R}^{p,q} = U \oplus W \oplus U^*$ where U is a totally isotropic subspace of some dimension s, where $W \cong \mathbb{R}^{p-s,q-s}$ with $U^\perp = U \oplus W$, and with U^* totally isotropic and paired to U. This is done as follows. As $\gamma = (t, A) : x \mapsto t + Ax$ ranges over Γ, the subspace U is the sum of the ranges of the $A - I$, so A has matrix block form $\begin{pmatrix} I & 0 & \alpha \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ in a corresponding skew basis of $\mathbb{R}^{p,q}$. Here "skew basis" means a basis $\{u_1, \dots, u_s; v_1, \dots, v_{p+q-2s}; w_1, \dots, w_s\}$ where $b(u_i, u_j) = b(u_i, v_j) = b(v_j, w_k) = b(w_j, w_k) = 0$, $b(u_i, w_j) = \delta_{i,j}$, and $b(v_i, v_j) = \delta_{i,j}$ if $1 \leqq i \leqq p - s$, $b(v_i, v_j) = -\delta_{i,j}$ if $p - s + 1 \leqq i \leqq p + q - 2s$. Note that s is even, $s \leqq \min(p,q)$, each $t \in U^\perp$, and Γ is free abelian on some number $m \leq p + q - s$ of generators $\gamma_i = (t_i, A_i)$. The t_i are linearly independent and the α_i are antisymmetric.

In [**8**] this was refined as follows. In general let p, q, s and m be integers $\geqq 0$ with s even, $s \leqq \min(p,q)$ and $m \leqq p + q - s$. Let $\mathbf{t} = \{t_1, \dots . t_m\} \subset \mathbb{R}^{p+q}$ be a linearly independent set of column vectors with last s entries 0. Let t_i' denote the $s \times 1$ column vector consisting of the first s entries of t_i. Let $\mathbf{a} = \{\alpha_1, \dots, \alpha_m\}$ be a set of antisymmetric real $s \times s$ matrices such that (i) some real linear combination is nonsingular and (ii) t_i' is not in the range of α_i. Then the

(1.1) **basic datum:** $\delta = \delta(p, q, s, m, \mathbf{t}, \mathbf{a})$

defines a group of rigid motions of $\mathbb{E}^{p,q}$ as follows. Let $U \subset \mathbb{R}^{p,q}$ be an s–dimensional totally isotropic subspace, $\mathbb{R}^{p,q} = U \oplus W \oplus U^*$ the corresponding decomposition as above, with each $t_i \in U^\perp = U \oplus W$. In an associated skew basis, $A_i \in O(p,q)$ corresponds to α_i. Now we have the

(1.2) **associated group:** $\Gamma_\delta = \langle \gamma_1, \dots, \gamma_m \rangle$ where $\gamma_i = (t_i, A_i)$ as above.

PROPOSITION 1.3.. *Let $\delta = \delta(p, q, s, m, \mathbf{t}, \mathbf{a})$ be a basic datum. Then the associated group Γ_δ acts freely and properly discontinuously on $\mathbb{E}^{p,q}$, and $M_\delta = \mathbb{E}^{p,q}/\Gamma_\delta$ is a complete connected flat pseudo–riemannian manifold.*

THEOREM 1.4.. *Fix a basic datum $\delta = \delta(p, q, s, m, \mathbf{t}, \mathbf{a})$, a decomposition $\mathbb{R}^{p,q} = U \oplus W \oplus U^*$, and the associated group Γ_δ, as above. If $v \in \mathbb{R}^{p,q}$ express $v =$*

$v' + v'' + v'''$ with $v' \in U, v'' \in W$ and $v''' \in U^*$. Let $T = \mathrm{Span}\{t_i\}$. Then $t_i \mapsto \alpha_i$ extends uniquely to a linear map $t \mapsto \alpha_t$ from T to $\mathrm{Span}\{\alpha_i\}$. Define $S_v : T \to U$ by $S_v(t) = t' + \alpha_t v'''$. Then $M_\delta = \mathbb{E}^{p,q}/\Gamma_\delta$ is homogeneous if and only if, for every $v \in \mathbb{R}^{p,q}$, there is a linear map $\tilde{S}_v : U^\perp \to U$ such that (i) $\tilde{S}_v|_T = S_v$ and (ii) $S'_v = \tilde{S}_v|_U$ preserves each of the antisymmetric bilinear forms α_i on U.

There is also a result in [**8**] on just when two quotients M_δ and $M_{\delta'}$ are equivalent, when we have $\delta = \delta(p,q,s,m,\mathbf{t},\mathbf{a})$ and $\delta' = \delta(p,q,s,m,\mathbf{t}',\mathbf{a}')$. However we will not need that result here.

The condition of Theorem 1.4 is the system

$$S'_v \cdot \alpha_i \cdot {}^tS'_v = \alpha_i \text{ for } v \in \mathbb{R}^{p,q} \text{ and } 1 \leqq i \leqq m \tag{1.5}$$

of quadratic equations. Here, by a change of $\mathbb{Z}$–basis in the integral span of the $\{t_i\}$ we may suppose that every nonzero α_i is either zero or nonsingular. Thus the homogeneity condition becomes the requirement that certain linear transformations belong to certain intersections of real symplectic groups. In general little is known about the intersection of real symplectic groups, but obviously it contains the identity element. Our examples will be based on constructions that lead to $S'_v = I$, where (1.5) is automatic.

2.A Sufficient Condition.

One does not yet have an effective general method for finding solutions to the quadratic system (1.5), but in this Section we exhibit a special method that produces some solutions, and in the following sections we see several interesting classes of complete flat homogeneous pseudo–riemannian manifolds based on that method. It seems unlikely that this special method leads to all relevant solutions to (1.5).

THEOREM 2.1.. *Let $\mathcal{A}$ be a real vector subspace of the space of real antisymmetric $s \times s$ matrices such that*

$$\text{if } \alpha \in \mathcal{A} \text{ then either } \alpha = 0 \text{ or } \alpha \text{ is nonsingular.} \tag{2.2}$$

Let $\delta = \delta(p,q,s,m,\mathbf{t},\mathbf{a})$, be a basic datum such that $\mathbf{a}$ is a sequence of elements of $\mathcal{A}$. Then the complete flat pseudo–riemannian manifold $\mathbf{E}^{p,q}/\Gamma_\delta$ is homogeneous.

Proof. Let $T = \mathrm{Span}\{t_i\}$. We can assume that $T \cap U$ has basis $\{u_1, \dots, u_r\}$. For $1 \leqq i \leqq r$ now $\gamma_i = (u_i, A_i) \in \Gamma_\delta$ with $A_i = \begin{pmatrix} 1 & 0 & \alpha_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\alpha_i \in \mathcal{A}$. If $\alpha_i \neq 0$ then α_i is nonsingular, so $u_i \in U$ is in the image of $A_i - I$, and γ_i has a fixed point. Contradiction. Thus $\alpha_i = 0$ for $1 \leqq i \leq r$. So $S_v(u_i) = u_i + \alpha_i v''' = u_i$ for $1 \leqq i \leqq r$.

We can now extend $S_v : T \cap U \to U$ to $\widetilde{S}_v : U^\perp \to U$ with $\widetilde{S}_v|_U = S'_v = I$ and $\widetilde{S}_v|_W$ arbitrary. Thus each $S'_v \cdot \alpha_{e_i} \cdot {}^tS'_v = \alpha_{e_i}$. Homogeneity follows. ☐

CONSTRUCTION 2.3.. One applies Theorem 2.1 as follows. Fix a real vector subspace $\mathcal{A}$ of the space of real antisymmetric $s \times s$ matrices such that: if $\alpha \in \mathcal{A}$ then either $\alpha = 0$ or α is nonsingular, as in 2.2. Choose an integer $m \geqq 0$. Choose elements α_1 , ... , $\alpha_m \in \mathcal{A}$, not necessarily distinct and not necessarily nonzero. Let $\mathbf{a} = \{\alpha_1, \dots, \alpha_m\}$ and $\mathcal{A}_\mathbf{a} = \mathrm{Span}\{\alpha_1, \dots, \alpha_m\}$. Choose integers $p, q \geqq 0$ such that $s \leqq \min\{p,q\}$, $m \leqq p + q - s$, and $\dim \mathcal{A}_\mathbf{a} \leqq p + q - 2s$. Choose linearly independent column vectors $t_1, \ldots, t_m \in \mathbb{R}^{p,q}$ such that, for $1 \leqq i \leqq m$, (i) the last s entries of t_i are zero and (ii) if $\alpha_i \neq 0$ then also the first s entries of t_i are zero. Set $\mathbf{t} = \{t_1, \ldots, t_m\}$. Then $\delta = \delta(p, q, s, m, \mathbf{t}, \mathbf{a})$, is a basic datum to which Theorem 2.1 applies, so $\mathbf{E}^{p,q}/\Gamma_\delta$ is homogeneous.

Theorem 2.1 is complemented by the following result, which leads to constructions of relevant spaces $\mathcal{A}$.

PROPOSITION 2.4.. *Let $\mathcal{A} = \mathrm{Span}\{e_1, \ldots, e_\ell\}$ where each e_i is an antisymmetric nonsingular $s \times s$ real matrix. Suppose that $i \neq j$ implies $e_ie_j + e_je_i = 0$. Then e is nonsingular whenever $0 \neq e \in \mathcal{A}$.*

Proof. For $\left(\sum_i a_ie_i\right)^2 = \sum_i a_i^2e_i^2 + \sum_{i<j}(a_ia_je_ie_j + a_ja_ie_je_i) = \sum_i a_i^2e_i^2$, which is negative definite unless every $a_i = 0$. □

3.Examples Based On Division Algebras.

Each of the real division algebras $\mathbb{F}$ gives us an example of the space $\mathcal{A}$ of Proposition 2.4 through multiplication by its pure imaginary elements. The case $\mathbb{F} = \mathbb{R}$ is not interesting, so we omit it.

EXAMPLE 3.1.. $\mathbb{F} = \mathbb{C} = \mathrm{Span}\{1, \mathbf{i}\}$, complex number field viewed as a division algebra over $\mathbb{R}$. Denote $r(t) = \left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix}\right)$. Consider a vector space $U = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$, $s/2$ summands. Let e be the linear transformation $r(t_1) \oplus \cdots \oplus r(t_{s/2})$ of U where each $t_i \neq 0$. Here e is multiplication by $t_m\mathbf{i}$ on m^{th} summand $\mathbb{R}^2 \cong \mathbb{C}$, and $\mathcal{A} = \mathrm{Span}\{e\}$.

EXAMPLE 3.2.. $\mathbb{F} = \mathbb{H} = \mathrm{Span}\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, real quaternion division algebra. Consider a vector space $U = \mathbb{R}^4 \oplus \cdots \oplus \mathbb{R}^4$, $s/4$ summands. Define
e_1: multiplication by $t_{1,m}\mathbf{i}$ on the m^{th} summand $\mathbb{R}^4 \cong \mathbb{H}$,
e_2: multiplication by $t_{2,m}\mathbf{j}$ on the m^{th} summand $\mathbb{R}^4 \cong \mathbb{H}$,
e_3: multiplication by $t_{3,m}\mathbf{k}$ on the m^{th} summand $\mathbb{R}^4 \cong \mathbb{H}$,
where each $t_{a,m} \in \mathbb{R}$ with $t_{a,m} \neq 0$. Here $\mathcal{A} = \mathrm{Span}\{e_1, e_2, e_3\}$.

EXAMPLE 3.3.. $\mathbb{F} = \mathbb{O} = \mathrm{Span}\{1, \mathbf{f}_1, \ldots, \mathbf{f}_7\}$, real Cayley division algebra. Consider a vector space $U = \mathbb{R}^8 \oplus \cdots \oplus \mathbb{R}^8$, $s/8$ summands. Define
e_a: multiplication by $t_{a,m}\mathbf{f}_a$ on the m^{th} summand $\mathbb{R}^8 \cong \mathbb{O}$, for $1 \leqq a \leqq 7$,
where each $t_{a,m} \in \mathbb{R}$ with $t_{a,m} \neq 0$. Here $\mathcal{A} = \mathrm{Span}\{e_1, \ldots, e_7\}$.

REMARK 3.4.. Note that in each of the examples in Section 3, the eigenspaces of the various e_i are well aligned with each other. In general, of course, things will be more complicated.

4.Examples Based On Clifford Algebras.

An appropriate class [**4**] of representations of real Clifford algebras provides some more interesting examples of the phenomenon of Theorem 2.1 and Proposition 2.4. We recall the relevant definitions and properties of those algebras and representations.

DEFINITION 4.1.. The **Clifford algebra** on $\mathbb{R}^r$ is the associative algebra $\mathcal{C}_r$ with multiplicative unit 1, generators $\{f_1, \ldots, f_r\}$, and defining identities $f_if_j + f_jf_i = -2\delta_{i,j}$. So $f_if_j + f_jf_i = 0$ for $i \neq j$ and $f_i^2 = -1$.

DEFINITION 4.2.. Let $\phi : \mathcal{C}_r \to \mathbb{R}^{s\times s}$ be an associative algebra homomorphism (to the algebra of real $s \times s$ matrices) with $\phi(1) = I$. Fix an orthogonal direct sum decomposition $\mathbb{R}^s = V \oplus V^\perp$ where $\dim V = s/2$. If

(i) each $\phi(f_i)$ is in the orthogonal group $O(s)$ and

(ii) each $\phi(f_i)(V) = V^\perp$

then ϕ is a **translational representation** of $\mathcal{C}_r$ on $\mathbb{R}^s$ with **basepoint** V.

The connection with Theorem 2.1 and Proposition 2.4 is

LEMMA 4.3.. *Let ϕ be a translational representation of the real Clifford algebra $\mathcal{C}_r$. Then each $\phi(f_i)$ is antisymmetric and nonsingular. Thus, in view of* Proposition 2.4, *every nonzero element of $\mathcal{A} = \phi(\mathrm{Span}\{f_1, \dots, f_r\})$ is antisymmetric and nonsingular.*

Proof. Let V be the basepoint of ϕ. Note that $\phi(f_i)V = V^\perp$ and $\phi(f_i) \in O(n)$ imply $\phi(f_i)V^\perp = V$.

We have $s = 2t$, even, where $t = \dim V$. Now there is an orthonormal basis $\mathbf{v} = \{v_1, \dots, v_{2t}\}$ of $\mathbb{R}^s$ such that $\{v_1, \dots, v_t\}$ is a basis of V and $\{v_{t+1}, \dots, v_{2t}\}$ is a basis of $V^\perp$. In this basis, $\phi(f_i)$ has matrix of the form $\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)$ in the basis $\mathbf{v}$. Compute $-I = \phi(f_i)^2 = \left(\begin{smallmatrix} xy & 0 \\ 0 & yx \end{smallmatrix}\right)$. So $xy = -I$. But x is orthogonal so $y = -x^{-1} = -{}^t x$. Now $\phi(f_i)$ has matrix of the form $\left(\begin{smallmatrix} 0 & x \\ -{}^t x & 0 \end{smallmatrix}\right)$ in the basis $\mathbf{v}$. So $\phi(f_i)$ has matrix that is antisymmetric in the basis $\mathbf{v}$, thus antisymmetric in every orthonormal basis of $\mathbb{R}^s$. Also, $\phi(f_i)$ is nonsingular because it has nonsingular square $-I$. $\square$

Here are the basic facts on existence and uniqueness of translational representations.

PROPOSITION 4.4.. [**4**] *Any two translational representations of $\mathcal{C}_r$ on $\mathbb{R}^s$ are orthogonally equivalent.*

PROPOSITION 4.5.. [**4**] *Let $s = 2^{4a+b+1}u$, u odd, $0 \leqq b \leqq 3$. Then the following are equivalent.*

(1.) $r \leqq 8s + 2^b$.

(2.) *$\mathcal{C}_{r-1}$ has an associative algebra representation on $\mathbb{R}^{s/2}$.*

(3.) *$\mathcal{C}_r$ has a translational representation on $\mathbb{R}^s$.*

EXAMPLE 4.6.. Let $s = 2^{4a+b+1}u$, u odd, $0 \leq b \leq 3$, and $r \leq 8s + 2^b$. Let ϕ be a translational representation of $\mathcal{C}_r$ on $\mathbb{R}^s$. Define $e_i = \phi(f_i)$. Here $\mathcal{A} = \mathrm{Span}\{e_1, \dots, e_r\}$. One can also produce somewhat more general examples of this type by scaling on various summands, as in the division algebra examples of Section 3.

5.Isoclinic Spheres.

Let $\mathbb{F}$ be a real division algebra. Let $\mathbb{F}^s$ be the space of $s \times 1$ (column) vectors over $\mathbb{F}$, viewed as a right vector space so that linear transformations act on the left by matrix multiplication. Put the standard positive definite inner product $\langle x, y\rangle = \sum x_i \bar{y}_i$ on $\mathbb{F}^s$ and let $U(s;\mathbb{F})$ be its unitary group. Two linear subspaces $U, V \subset \mathbb{F}^s$ of the same dimension k are called **isoclinic** if the orthogonal projection of $\mathbb{F}^s$ onto U multiplies the length of vectors in V by some constant $|\cos(\theta)|$. Then of course $k \leqq s/2$. Sets of mutually isoclinic k–planes in $\mathbb{F}^s$ were studied, as subsets of the Grassmann manifold $G_{k,s}(\mathbb{F})$ of all k–planes in $\mathbb{F}^s$, in [**4**], [**5**] and [**1**]. An isoclinic closure operation was defined, and the isoclinically closed sets S of mutually isoclinic k–planes in $\mathbb{F}^s$ were shown to be closed totally geodesic submanifolds of $G_{k,s}(\mathbb{F})$. Further, it was shown that S is a riemannian symmetric space of rank 1, hence a sphere or a real, complex, quaternionic projective space or the Cayley projective plane. In the first of these cases we refer to S as an **isoclinic sphere**.

The starting point in the classification of isoclinic spheres is the following construction of isoclinic spheres in $G_{s/2,s}(\mathbb{F})$ where s is even and $\mathbb{F}$ is associative. Let $\mathcal{C}_r(\mathbb{F})$ denote the Clifford algebra on $\mathbb{F}^r$, defined as in Definition 4.1, except that more generally it is an algebra over $\mathbb{F}$. Define translational representation of $\mathcal{C}_r(\mathbb{F})$ on $\mathbb{F}^s$ as in Definition 4.2 with $U(s;\mathbb{F})$ in place of $O(s)$. Now let ϕ be a translational representation of $\mathcal{C}_r(\mathbb{F})$ on $\mathbb{F}^s$, with base point $V \subset \mathbb{F}^s$. Then

$$S = \phi(\mathrm{Span}\{1, f_1, f_2, \dots, f_r\})(V) \tag{5.1}$$

is an isoclinic sphere in $G(\frac{s}{2}, s; \mathbb{F})$. Compare this to

$$\mathcal{A} = \phi(\mathrm{Span}\{f_1, f_2, \dots, f_r\}). \tag{5.2}$$

Now rephrase Example 4.6 as

EXAMPLE 5.3.. Let S be an isoclinic sphere in $G_{s/2,s}(\mathbb{R})$ and realize S in the form $\phi(\mathrm{Span}\{1, f_1, f_2, \dots, f_r\})(V)$ where ϕ is a translational representation of $\mathcal{C}_r$ on $\mathbb{R}^s$ with basepoint V. Then $\mathcal{A} = \mathrm{Span}\{e_i, \dots, e_r\}$ satisfies the conditions of Theorem 2.1.

Now there is a formal correspondence from the isoclinic spheres $S \subset G_{s/2,s}(\mathbb{R})$ of (5.1) to the large family of complete flat homogeneous pseudo–riemannian manifolds, determined by the space $\mathcal{A}$ of (5.2), of nonsingular antisymmetric real $s \times s$ matrices. It would be very interesting to find the geometric basis for this algebraic correspondence.

6.Parallel Structures.

Finally, we look at parallel tangent structures on $M_\delta = \mathbb{E}^{p,q}/\Gamma_\delta$. In the riemannian setting, and in the context of homogeneous spaces of real semisimple Lie groups, the pseudo–kaehler and pseudo–hyperkähler structures have been (and continue to be) studied extensively. The Heisenberg structures and their generalizations have been used extensively to study hypoellipticity of various systems of PDE. Here we give examples which show that those structures also occur in the setting of complete flat homogeneous pseudo–riemannian manifolds.

Parallel tangent structures on M_δ are sets of parallel fields of linear transformations on the tangent spaces of M_δ, classically called parallel vector 1–forms. They are the structures obtained from sets $\mathbf{S}$ of linear transformations of $\mathbb{R}^{p,q}$ (viewed as its own tangent space at 0), parallel–translated to every point of $\mathbb{R}^{p,q}$ and pushed down from $\mathbb{E}^{p,q}$ to M_δ, for which the push–down is well defined. That push–down is well defined just when each element of $\mathbf{S}$ is centralized by the holonomy group of M_δ. The holonomy group in question is

$$H_\delta = \{A \in \mathbb{R}^{n \times n} \mid \text{ there is an element of the form } (t, A) \text{ in } \Gamma_\delta\}\ . \tag{6.1}$$

Let $g = (g_{i,j})$ in block form relative to a skew basis of $\mathbb{R}^{p,q}$. Then $Ag = gA$ for every $A = \begin{pmatrix} I & 0 & \alpha \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in H_\delta$, if and only if,

$$(6.2) \qquad g \text{ has form } \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \text{ with } a\alpha = \alpha f \text{ for every } A \in H_\delta \ .$$

Now we look at a few examples of interesting structure.

EXAMPLE 6.3.. **Pseudo–Kähler Structure.** Fix a translational representation ϕ of $\mathcal{C}_{r+2}$ on $\mathbb{R}^s$. Let $\delta = \delta(p,q,s,m,\mathbf{t},\mathbf{a})$ be any basic datum constructed as in Construction 2.3 with $p+q$ even, from $\mathcal{A} = \text{Span}\{\phi(f_1),\ldots,\phi(f_r)\}$. Let $\mathbf{j} = \phi(f_{r+1})\phi(f_{r+2})$. Note that $\phi(f_\ell)$ commutes with $\mathbf{j}$ for $1 \leqq \ell \leqq r$. Let $\mathbf{J}$ be the block–diagonal matrix $\text{diag}\{\mathbf{j},\mathbf{j}',\mathbf{j}\}$ where $\mathbf{j}'$ is any orthogonal $(p+q-2s)\times(p+q-2s)$ matrix with square $-I$. Then $\mathbf{J} \in O(p,q)$, $\mathbf{J}^2 = -I$ and, by (6.2), $\mathbf{J}$ commutes with every element of H_δ. Thus $\mathbf{J}$ defines an (obviously integrable) parallel almost–complex structure on M_δ , and $\mathbf{J}$ together with the pseudo–riemannian metric forms a pseudo–kähler structure on M_δ .

EXAMPLE 6.4.. **Pseudo–Hyperkähler Structure.** Fix a translational representation ϕ of $\mathcal{C}_{r+3}$ on $\mathbb{R}^s$. Let $\delta = \delta(p,q,s,m,\mathbf{t},\mathbf{a})$ be any basic datum constructed as in Construction 2.3 with $p+q$ divisible by 4, from $\mathcal{A} = \text{Span}\{\phi(f_1),\ldots,\phi(f_r)\}$. Define

$$\mathbf{i} = \phi(f_{r+1})\phi(f_{r+2}),\ \mathbf{j} = \phi(f_{r+2})\phi(f_{r+3}), \text{ and } \mathbf{k} = \phi(f_{r+3})\phi(f_{r+1}).$$

Compute that each has square $-I$, that $\mathbf{ij} = \mathbf{k}$, and that $\phi(f_\ell)$ commutes with each of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, for $1 \leqq \ell \leqq r$. Now define block–diagonal matrices

$$\mathbf{I} = \text{diag}\{\mathbf{i},\mathbf{i}',\mathbf{i}\},\ \mathbf{J} = \text{diag}\{\mathbf{j},\mathbf{j}',\mathbf{j}\}, \text{ and } \mathbf{K} = \text{diag}\{\mathbf{k},\mathbf{k}',\mathbf{k}\},$$

where $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ generate a quaternion algebra on $\mathbb{R}^{p+q-2s}$. (Here $p+q-2s$ is divisible by 4 since s is even and we assume that 4 divides $p+q$.) Now as above, each of $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ belongs to $O(p,q)$ and defines an integrable parallel almost–complex structure on M_δ . Those structures anticommute and thus form a pseudo–hyperkähler structure.

EXAMPLE 6.5.. **Heisenberg Structure.** The criterion (6.2) is trivially satisfied when a, d, and f are identity matrices. Thus each element of the "slightly generalized Heisenberg group"

$$(6.6) \qquad H_{s,p+q-2s}(\mathbb{R}) : \text{all real } \begin{pmatrix} I & x & z \\ 0 & I & y \\ 0 & 0 & I \end{pmatrix} \text{ where}$$
$$x \text{ is } s \times (p+q-2s),\ y \text{ is } (p+q-2s) \times s,\ z \text{ is } s \times s$$

satisfies (6.2), and we have an elementwise–parallel Heisenberg group structure on M_δ . Note that its Lie group product rule can be written $(x,y,z)(x',y',z') = (x+x',y+y',z+z'+xy')$.

The corresponding "slightly generalized Heisenberg algebra"

$$(6.7) \qquad \mathfrak{h}_{s,p+q-2s}(\mathbb{R}) : \text{all real } \begin{pmatrix} 0 & \xi & \zeta \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix} \text{ where}$$
$$\xi \text{ is } s \times (p+q-2s),\ \eta \text{ is } (p+q-2s) \times s,\ \zeta \text{ is } s \times s$$

of course also satisfies (6.2), and gives an elementwise–parallel Heisenberg algebra structure on M_δ. Its Lie algebra product rule can be written $[(\xi, \eta, \zeta), (\xi', \eta', \zeta')] = (0, 0, \xi\eta' - \eta\xi')$.

Examples 6.3 and 6.4 can be combined with the Heisenberg–type constructions (6.6) and (6.7), giving complex and quaternionic parallel Heisenberg structures on M_δ. Compare with the constructions of semidirect product groups in [**6**] and [**7**].

References

[1] D. B. Shapiro & J. A. Wolf, On isoclinic closure, Illinois J. Math **19** (1975), 316–317.

[2] J. A. Wolf, Sur la classification des variétés riemanniènnes homogenènes à courbure constante, C. R. Acad. Sci. Paris **250** (1960), 3443–3445.

[3] ———, Homogeneous manifolds of zero curvature, Trans. Amer. Math. Soc. **104** (1962), 462–469.

[4] ———, Geodesic spheres in Grassmann manifolds, Illinois J. Math **7** (1963), 425–446.

[5] ———, Elliptic spaces in Grassmann manifolds, Illinois J. Math **7** (1963), 447–462.

[6] ———, Representations of certain semidirect product groups, J. Funct. Analysis **19** (1975), 339–372.

[7] ———, Classification and Fourier inversion for parabolic subgroups with square integrable nilradical, Memoirs Amer. Math. Soc., Number 225, 1979.

[8] ———, Flat homogeneous pseudo–riemannian manifolds, Geometriae Dedicata **57** (1995), 111–120.

[9] Y.-C. Wong, Isoclinic n–planes in euclidean $2n$–space, Clifford parallels in elliptic $(2n-1)$–space, and the Hurwitz matrix equations, Mem. Amer. Math. Soc. **41**, 1961.

Institut für Mathematik, Ruhr–Universität Bochum, D-44780 Bochum, Germany

Department of Mathematics, University of California, Berkeley, California 94720–3840, U.S.A.

E-mail address: `jawolf@math.berkeley.edu`