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Flag Duality

Dedicated to the memory of Alfred Gray

ALAN T. HUCKLEBERRY[1⋆] and JOSEPH A. WOLF[2⋆⋆]
[1]*Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany. e-mail: ahuck@cplx.ruhr-uni-bochum.de*
[2]*Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany. e-mail: jawolf@math.berkeley.edu*

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Abstract. We introduce a duality on complex flag manifolds that extends the usual point-hyperplane duality of complex projective spaces. This has consequences for the structure of the linear cycle spaces of flag domains, especially when those flag domains are not measurable.

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1. Introduction

If G is a complex semi-simple group, Q a parabolic subgroup and $Z = G/Q$ the associated flag manifold, then any real form G_0 has only finitely many orbits in Z (see [3] for this and other general results). In particular G_0 has open orbits on Z. Let D be one of those open orbits.

A maximal compact subgroup K_0 of G_0 has an essentially unique complex orbit in D. Given the standard root theoretic set-up at the neutral point $z \in Z$, there is a canonical way of choosing K_0 so that $K_0(z) := Y_0$ is a complex submanifold of Z. The moduli space of linear cycles is then defined to be

$$M_D := \{g(Y_0) : g \in G \text{ and } g(Y_0) \subset D\}.$$

In the measurable case, i.e., that where D possesses a G_0-invariant pseudo-Kählerian metric, a great deal is known about the complex geometry of M_D and associated G_0-representations on spaces of holomorphic functions on M_D (see, e.g., [6]).

The goal of the present note is to set up a duality which should facilitate a study of nonmeasurable flag domains $D = G_0(z)$. This can be used at least in a conceptual way to compare the cycle space M_D to that of its measurable model $\widetilde{D}$.

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⋆⋆ Research partially supported by the Alexander von Humboldt-Stiftung. Permanent address: Department of Mathematics, University of California, Berkeley, CA 94720–3840, U.S.A.

Operating from this point of view we show in the case of the unique open orbit D of $G_0 = Sl_{n+1}(\mathbb{R})$ acting in the usual way on $Z = \mathbb{P}_n(\mathbb{C})$ that the connected component of the moduli space M_D which contains Y_0 can be identified with that of $\widetilde{D}$.

2. Background and Notation

Fix a complex semisimple Lie group G, a parabolic subgroup $Q = Q_\Phi$ where Φ is a subset of the simple root system relative to a root order and a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Express $\mathfrak{q} = \mathfrak{q}_\Phi = \mathfrak{q}^r + \mathfrak{q}^{-n}$ where the nilradical $\mathfrak{q}^{-n} = \sum_{\Phi^n} \mathfrak{g}_{-\alpha}$. Then the *opposite* parabolic is $\mathfrak{q}^- = \mathfrak{q}^r + \mathfrak{q}^{+n}$ where the nilradical $\mathfrak{q}^{+n} = \sum_{\Phi^n} \mathfrak{g}_{+\alpha}$. The *dual parabolic* is the complex conjugate (of $\mathfrak{g}$ over $\mathfrak{g}_0$) $\mathfrak{q}^* = \overline{\mathfrak{q}^-}$ of the opposite.

Similarly the parabolic subgroups $Q^- \subset G$ opposite to Q and $Q^* \subset G$ dual to Q are the parabolic subgroups with respective Lie algebras $\mathfrak{q}^-$ and $\mathfrak{q}^*$. Our flag duality will be between

$$Z = G/Q \quad \text{and} \quad Z^* = G/Q^*, \tag{2.1}$$

where Q^* is the parabolic subgroup of G dual to Q.

Fix a real form $G_0 \subset G$ and suppose that $Q = Q_z$ where $D = G_0(z)$ is an open G_0-orbit in Z. In other words, $\mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}$. Sending each root to its negative, and then applying complex conjugation, it follows that $\mathfrak{q}^* + \overline{\mathfrak{q}^*} = \mathfrak{g}$. Thus $\mathfrak{q}^* = \mathfrak{q}^*_{z^*}$, the isotropy subalgebra of $\mathfrak{g}$ at a point $z^* \in Z^*$ such that $D^* = G_0(z^*)$ is an open orbit in Z^*. D^* is the *dual* of D.

Fix a Cartan involution θ of G_0 that preserves the Cartan subalgebra $\mathfrak{h} = \overline{\mathfrak{h}}$ used here. Then the corresponding maximal compact subgroup $K_0 = G_0^\theta$ has complex orbit $Y = K_0(z)$ in D, and similarly has complex orbit $Y^* = K_0(z^*)$ in D^*.

One also knows [5, lemma 2.2] that $\mathfrak{p} := \mathfrak{q} \cap \mathfrak{q}^*$ is a parabolic subalgebra of $\mathfrak{g}$. The reader should be careful here: our $\mathfrak{q}$ is the $\mathfrak{r}$ of [5], and our $\mathfrak{p}$ is the $\mathfrak{q}$ of [5]. Recall that $\mathfrak{q}$ and $\mathfrak{q}^*$ are G-conjugate if and only if the open G_0-orbits on Z are measurable.

P denotes the parabolic subgroup of G corresponding to $\mathfrak{p}$. Now we have three complex flag manifolds, $Z = G/Q$, $Z^* = G/Q^*$ and $X = G/P$. Note $P = Q \cap Q^*$. We thus have a G-equivariant holomorphic double fibration

$$\begin{tikzcd} & X = G/P \arrow[dl, "\mu"'] \arrow[dr, "\mu^*"] & \\ Z = G/Q & & Z^* = G/Q^* \end{tikzcd} \tag{2.2}$$

The G-equivariance of course implies G_0-equivariance. Let $Q = Q_z$ for the open orbit $D = G_0(z) \subset Z$. Then $Q^* = Q^*_{z^*}$ for the open orbit $D^* = G_0(z^*) \subset Z^*$, so we expect a correspondence of open G_0-orbits on Z and Z^*. Before making this precise let us fix the notation clearly.

(2.3a) $\mathfrak{q}_z = \mathfrak{q} = \mathfrak{q}^r + \mathfrak{q}^{-n}$ is the 'starting' parabolic subalgebra of $\mathfrak{g}$, $Q_z = Q = Q^r Q^{-n}$ is the corresponding parabolic subgroup of G, and $Z = G/Q$ is the associated flag manifold.

(2.3b) $D = G_0(z)$ is an open G_0-orbit on Z.

(2.3c) $\mathfrak{q}^*_{z^*} = \mathfrak{q}^* = \overline{\mathfrak{q}^r + \mathfrak{q}^{+n}}$ is the 'starting' parabolic subalgebra of $\mathfrak{g}$, $Q^*_{z^*} = Q^* = \overline{Q^r Q^{+n}}$ is the corresponding parabolic subgroup of G, and $Z^* = G/Q^*$ is the associated flag manifold.

(2.3d) $D^* = G_0(z^*)$ is the open G_0-orbit on Z^* given by the point $z^* \in Z^*$ whose complex isotropy algebra is $\mathfrak{q}^*$.

(2.3e) $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{q}^*$ is the parabolic subalgebra, $P = Q \cap Q^*$ parabolic subgroup, $X = G/P$ flag manifold, and μ and μ^* are the projections of the equivariant holomorphic double fibration above. $E = \mu^{-1}(D)$ and $E^* = (\mu^*)^{-1}(D^*)$.

(2.3f) $x \in X$ is the base point, defined by $\mathfrak{p} = \mathfrak{p}_x$, and $\widetilde{D}$ is the open (see Lemma 2.4 below) orbit $G_0(x)$.

LEMMA 2.4. *$\widetilde{D} = G_0(x)$ is open in X, $D = \mu(\widetilde{D})$, and $D^* = \mu^*(\widetilde{D})$.*

Proof. The first statement is [5, lemma 2.3]. The second is $\mu(\widetilde{D}) = \mu(G_0(x)) = G_0(\mu(x)) = G_0(z) = D$. The third is $\mu^*(\widetilde{D}) = \mu^*(G_0(x)) = G_0(\mu^*(x)) = G_0(z^*) = D^*$. $\square$

LEMMA 2.5. *$Y := K_0(z) \in M_D$, $Y^* := K_0(z^*) \in M_{D^*}$, and $\widetilde{Y} := K_0(x) \in M_{\widetilde{D}}$. Furthermore $\mu(\widetilde{Y}) = Y$ and $\mu^*(\widetilde{Y}) = Y^*$.*

Proof. The alignments that the various $K_0(\cdot)$ be complex submanifolds are conditions on the Cartan involution θ defining $K_0 = G_0^\theta$: that it preserve $\mathfrak{h}_0$. That is the same for $\mathfrak{q}$, for $\mathfrak{q}^*$, and for $\mathfrak{p}$. The further statements follow as in Lemma 2.3 with K_0 in place of G_0. $\square$

LEMMA 2.6. *$\mu : \widetilde{Y} \to Y$ and $\mu^* : \widetilde{Y} \to Y^*$ are biholomorphic diffeomorphisms.*

Proof. This is [5, lemma 2.5] for Y, and the situation is symmetric for Y^*. $\square$

3. The Duality Automorphism

As before $\mathfrak{g}_u$ is the compact real form $\mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$ of $\mathfrak{g}$ where $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ under its Cartan involution θ, and of course we extend θ to $\mathfrak{g}$ and restrict it to $\mathfrak{g}_u$. The fundamental Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ of $\mathfrak{g}_0$, the sum of its intersections with $\mathfrak{k}_0$ and $\mathfrak{s}_0$, defines the Cartan subalgebra $\mathfrak{h}_u = \mathfrak{t}_0 + \sqrt{-1}\,\mathfrak{a}_0$ of $\mathfrak{g}_u$.

PROPOSITION 3.1. *There is an involutive automorphism ν of $\mathfrak{g}_u$ that preserves $\mathfrak{h}_u$, sends every root to its negative, and commutes with θ.*

Proof. We may assume that the symmetric space G_u/K_0 is irreducible, for otherwise everything decomposes as a product with that irreducibility. Now we run through some cases.

Case 1: θ is inner on G_u. By general symmetric space theory, $\theta = \mathrm{Ad}(t)$ for some $t \in K_0$, so K_0 contains a maximal torus of G_u. Then t is contained in every maximal torus of K_0, in particular in the maximal torus $T_0 = \exp(\mathfrak{t}_0)$ of G_u. Note here that $\mathfrak{a}_u = 0$. Extend the map $\xi \mapsto -\xi$ from $\mathfrak{t}_0$ to an automorphism ν of order 2 of $\mathfrak{g}_u$. Then ν preserves $\mathfrak{h}_u = \mathfrak{t}_0$ and sends every root to its negative, but also $\nu\theta = \nu\,\mathrm{Ad}(t) = \mathrm{Ad}(t^{-1})\nu = \theta^{-1}\nu = \theta\nu$.

For the rest of the proof θ is an outer automorphism on $\mathfrak{g}_u$. As in the inner case we extend the map -1 from $\mathfrak{h}_u$ to an involutive automorphism ν of $\mathfrak{g}_u$, and ν preserves $\mathfrak{h}_u$ and sends every root to its negative. Now we must show that ν commutes with θ.

Case 2: θ is outer on G_u, and G_u is not simple. Then $\mathfrak{g}_u = \mathfrak{m}_u \oplus \mathfrak{m}_u$ for some compact simple Lie algebra $\mathfrak{m}_u$, $\mathfrak{h}_u = \mathfrak{r}_u \oplus \mathfrak{r}_u$ for a Cartan subalgebra $\mathfrak{r}_u$ of $\mathfrak{m}_u$, and θ is the interchange $(\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$. Also $\mathfrak{k}_0 = \mathfrak{g}_u^\theta$ is the diagonal, and ν has form $(\xi_1, \xi_2) \mapsto (\phi(\xi_1), \phi(\xi_2))$ for an involutive automorphism ϕ of $\mathfrak{m}_u$ that is -1 on $\mathfrak{r}_u$ and thus sends every root of $(\mathfrak{m}_u, \mathfrak{r}_u)$ to its negative. So $\nu\theta(\xi_1, \xi_2) = \nu(\xi_2, \xi_1) = (\phi(\xi_2), \phi(\xi_1)) = \theta(\phi(\xi_1), \phi(\xi_2)) = \theta\nu(\xi_1, \xi_2)$.

Case 3: $\mathfrak{g}_0 = \mathfrak{sl}(n; \mathbb{R})$. Then $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{C})$ and $\mathfrak{g}_u$ is the Lie algebra $\mathfrak{su}(n)$, of the special unitary group $U(n)$. We use the Cartan subalgebra

$$\mathfrak{h}_u = \{\, \mathrm{diag}\,\{ia_1, \ldots, ia_n\} \mid a_j \text{ real}, \ \sum a_j = 0\}.$$

Let $A = A^{-1}$ denote the antidiagonal, 1's from the upper right-hand corner to the lower left and 0's elsewhere, for example $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $n = 2$. Define $\nu(\xi) = A\xi A^{-1} = A\xi A$ and note that ν sends a root $\epsilon_i - \epsilon_j$ to $\epsilon_j - \epsilon_i$. Now compute $\theta\nu(\xi) = \theta(A\xi A) = -{}^t(A\xi A) = {}^tA(-{}^t\xi)\,{}^tA = A(-{}^t\xi)A = \nu\theta(\xi)$, that is, $\theta\nu = \nu\theta$.

Case 4: θ is outer on G_u, G_u is simple, and -1 is in the Weyl group of K_0. Here we have a Weyl group element $w_0 \in W(K_0, T_0)$ such that $w_0(\xi) = -\xi$ for all $\xi \in \mathfrak{t}_0$. Represent $w_0 = \mathrm{Ad}(s)|_{\mathfrak{t}_0}$ with, of course, $s \in K_0$. Then $\theta(s) = s$, we define $\nu = \mathrm{Ad}(s)\theta$, and we have $\nu(\xi) = -\xi$ for all $\xi \in \mathfrak{t}_0$. But we need

LEMMA 3.2. *$\nu(\xi) = -\xi$ for all $\xi \in \mathfrak{h}_u$.*

Proof. Let w_1 be the element of the Cartan group of $\mathfrak{g}_u$ – the Weyl group extended by adjoining outer automorphisms – that sends every $\xi \in \mathfrak{h}$ to its negative. Then $w_1 = \omega|_{\mathfrak{h}}$ where ω is an outer automorphism of $\mathfrak{g}_u$ that normalizes $\mathfrak{h}_u$ and whose square is inner. Now both θ and $\omega\,\mathrm{Ad}(s)$ are outer automorphisms of $\mathfrak{g}_u$ whose squares are inner. A result of de Siebenthal [1] says that their fixed point sets have the same rank. For θ that is of course rank $K_0 = \dim T_0$. But the fixed point set of $\omega\,\mathrm{Ad}(s)$ contains T_0, so it cannot be larger, and this forces $\mathrm{Ad}(s)(\xi) = \xi$ for every $\xi \in \mathfrak{a}$. Now $\nu(\xi) = -\xi$ for all $\xi \in \mathfrak{a}$, and this completes the argument that $\nu(\xi) = -\xi$ for all $\xi \in \mathfrak{h}_u$. □

Continuation of proof of Proposition 3.1. We have $\theta(s) = s$, so θ commutes with w_0 and thus commutes with ν. It remains only to check that $\nu^2 = 1$, in other words that $\mathrm{Ad}(s)^2 = 1$, i.e. that s^2 is central in G_u.

Note that $w_0^2 = 1$ in the Weyl group $W = W(K_0, T_0)$, so $s^2 \in T_0$ and $w(s^2) = s^2$ for all $w \in W$. We may assume G centerless; that only impinges on the center of G_u. This assumption made, K_0 is centerless because G_0/K_0 is irreducible and θ is outer.

Express $s^2 = \exp(\sigma)$ for some $\sigma \in \mathfrak{t}_0$. Here σ is unique up to a root lattice translation. So we minimize $||\sigma||$ by choosing it in a fundamental closed cell $\mathcal{C}$. In this regard, recall that every positive root system $\Delta^+ = \Delta^+(\mathfrak{k}_0, \mathfrak{t}_0)$ defines such a cell from its simple root system $\{\psi_j\}$ and its maximal root μ by

$$\mathcal{C} = \{\xi \in \mathfrak{t}_0 \mid \mu(-i\xi) \leqq 1 \text{ and each } \psi_j(-i\xi) \geqq 0\}.$$

The important property is that each element of K_0 is conjugate to some $\exp(\zeta)$, $\zeta \in \mathcal{C}$, and ζ is unique modulo the root lattice and the action of $W = W(K_0, T_0)$. Thus $w(\sigma) = \sigma$ for all $w \in W$. But the action of W on $\mathfrak{t}_0$ is a sum of nontrivial irreducible representations because $\mathfrak{k}_0$ is semisimple. It follows that $\sigma = 0$. Now $s^2 = 1$, so s^2 is central in G_u as required.

Completion of proof of Proposition 3.1. By classification, if θ is outer and $\mathfrak{g}_u$ is simple then $\mathfrak{g}_0$ is one of

(a) $\mathfrak{sl}(n; \mathbb{R})$, Lie algebra of the real special linear group, which has maximal compact subgroup $K_0 \cong SO(n)$,
(b) $\mathfrak{sl}(m; \mathbb{H})$, Lie algebra of the quaternion special linear group, which has maximal compact subgroup $K_0 \cong Sp(m)$,
(c) $\mathfrak{so}(2u+1, 2v+1)$, Lie algebra of the indefinite orthogonal group that has maximal compact subgroup $K_0 \cong SO(2u+1) \times SO(1+2v)$,
(d) $\mathfrak{e}_{6,\mathfrak{c}_4}$, Lie algebra of the real group G_0 of type E_6 with maximal compact subgroup K_0 of type C_4,
(e) $\mathfrak{e}_{6,\mathfrak{f}_4}$, Lie algebra of the real group G_0 of type E_6 with maximal compact subgroup K_0 of type F_4.

Here (a) is Case 3 above, and (b), (c), (d) and (e) are covered by Case 4 above. That completes the proof of Proposition 3.1. □

Now we use the map ν to start the duality theory.

PROPOSITION 3.3. *Let ν be as in Proposition* 3.1. *Then $\alpha\colon G \to G$ by $\alpha(g) = \overline{\nu(g)}$, complex conjugation of G over G_0, has the properties*

(1) *α^2 is the identity,*
(2) *$\alpha(Q) = Q^*$ and $\alpha(Q^*) = Q$,*
(3) *the induced map ϕ: $Z \to Z^*$ given by $gQ \mapsto \alpha(g)Q^*$ and the other induced map ϕ^{-1}: $Z^* \to Z$ given by $gQ^* \mapsto \alpha(g)Q$, are biholomorphic diffeomorphisms,*
(4) *$\alpha(G_0) = G_0$, and ϕ maps an arbitrary G_0-orbit $G_0(z) \subset Z$ onto a G_0-orbit $G_0(\phi(z)) \subset Z^*$, and*
(5) *ϕ: $D \to D^*$ defines a biholomorphic diffeomorphism M_ϕ: $M_D \to M_{D^*}$ of linear cycle spaces.*

Proof. The automorphism $\nu : G_u \to G_u$ of Proposition 3.1 extends to G and commutes with θ, and thus preserves G_0. Now evidently $\alpha(G_0) = G_0$. In fact, if $g \in G_0$ then $\alpha(g) = \nu(g)$ so $\alpha^2(g) = \nu^2(g) = g$. As α^2 is a holomorphic automorphism of G, $\alpha^2 = 1$.

Compute $d\alpha(\mathfrak{q}) = \overline{d\nu(\mathfrak{q})} = \overline{\mathfrak{q}^-} = \mathfrak{q}^*$, so $\alpha(Q) = Q^*$, and also $Q = \alpha^2(Q) = \alpha(Q^*)$. It follows immediately that α induces the maps ϕ: $Z \to Z^*$ and ϕ^{-1}: $Z^* \to Z$, as asserted, at the real analytic level. Since $\alpha(G_0) = G_0$ it follows as well that ϕ and ϕ^{-1} map G_0-orbits to G_0-orbits.

The holomorphic tangent space to Z at the base point z_0, the one that corresponds to $\mathfrak{q}$, is given by $\mathfrak{q}^n = \sum_{\beta\in\Phi^n} \mathfrak{g}_\beta$. The holomorphic tangent space to Z^* at the base point z_0^*, the one that corresponds to $\mathfrak{q}^*$, is given by $(\mathfrak{q}^*)^n = \overline{\mathfrak{q}^{-n}} = d\alpha(\mathfrak{q}^n)$. Thus ϕ: $Z \to Z^*$ is holomorphic, and the same argument shows that ϕ^{-1}: $Z^* \to Z$ is holomorphic.

Let $Y = K_0(z_0)$ be the base point in M_D. Similarly $Y^* = K_0(z_0^*)$ is the base point in M_{D^*}. Note $\alpha(K_0) = K_0$ so $\phi(Y) = Y^*$. Thus, if $g \in G$ then $\phi(gY) = \alpha(g)\phi(Y) = \alpha(g)Y^*$. But $\phi(D) = D^*$ so $gY \subset D$ exactly when $\alpha(g)Y^* \subset D^*$. In other words, $gY \in M_D$ if and only if $\phi(gY) \in M_{D^*}$. Thus ϕ defines a real analytic diffeomorphism M_ϕ: $M_D \to M_{D^*}$ of linear cycle spaces. It is holomorphic because ϕ is holomorphic. □

4. The Case of Complex Projective Space

Here we discuss the cycle space of the unique open orbit D of $G_0 = Sl_{n+1}(\mathbb{R})$ in the complex projective space $\mathbb{P}_n(\mathbb{C})$. Our goal is to prove $M_D = M_{\widetilde{D}}$. We begin with some notation.

The action of $G := Sl_{n+1}(\mathbb{C})$ on $Z = \mathbb{P}_n(\mathbb{C})$ is defined by its standard representation on $V := \mathbb{C}^{n+1}$. The dual representation on V^* defines its action on $Z^* = \mathbb{P}(V^*)$. A point in Z (resp. Z^*) is a complex line L in V (resp. a hyperplane H).

Let $\langle e_0, \dots, e_n\rangle$ be the standard basis for V and choose $L_0 = \mathbb{C}.(e_0 + ie_1)$ as a base point in $Z = \mathbb{P}(V)$. It follows that the orbit $D = G_0(L_0)$ is open. In fact, its complement is the set of real points $Z(\mathbb{R}) = \mathbb{P}(V(\mathbb{R}))$ which is also a G_0-orbit. If the maximal compact subgroup K_0 of G_0 is chosen to be $K_0 := SO_{n+1}(\mathbb{R})$, then $Y_0 := K_0(L_0)$ is the unique complex K_0-orbit in D. It is the quadric hypersurface $Y_0 = \{[z_0 : \ldots : z_n] : \sum z_j^2 = 0\}$. The complex group $K^{\mathbb{C}}$ has two orbits in Z, the above quadric and its complement.

Let H_0 be the projective tangent hyperplane to Y_0 at the point L_0 regarded as a hyperplane in V. Then $H_0 = \{(z_0, \ldots, z_n) : z_0 + iz_1 = 0\} = ((e_0 + ie_1, e_2, \ldots, e_n))$. It follows that $K_0(H_0) = K_0^{\mathbb{C}}(H_0) =: Y_0^* \subset \mathbb{P}(V^*)$ is the dual quadric of tangent hyperplanes to Y_0. It is likewise the unique complex K_0-orbit in the unique open G_0-orbit D^* in Z^*.

Define Q (resp. Q^*) to be the G_0-isotropy at L_0 (resp. H_0) and let $P = Q \cap Q^*$. Then $X = G/P$ is the flag manifold of lines L contained in hyperplanes H in V. We let $(L \subset H)$ denote a point in X. The projection $\pi\colon X \to Z$ (resp. $\pi^* : X \to Z^*$) is defined by $(L \subset H) \mapsto L$ (resp. $(L \subset H) \mapsto H$). Note that the π and π^*-fibers are $(n-1)$-dimensional projective spaces.

4.1. THE G_0-ORBIT STRUCTURE

For the sake of completeness we outline the proof of the following elementary

PROPOSITION 4.1. *The group G_0 has 5 orbits in X. In ascending order of codimension they are*

(1) *The unique open orbit*

$$\widetilde{D} = \{(L \subset H) : L \neq \bar{L},\ H \neq \bar{H}, \bar{L} \not\subset H\}.$$

(2) *The top-dimensional boundary orbit*

$$\Sigma := \{(L \subset H) : L \neq \bar{L},\ H \neq \bar{H},\ \bar{L} \subset H\}.$$

(3) *Two intermediate orbits which are exchanged by flag duality:*

$$M^* := \{(L \subset H) : L = \bar{L}, H \neq \bar{H}\}$$

and

$$M := \{(L \subset H) : L \neq \bar{L}, H = \bar{H}\}.$$

(4) *The minimal orbit*

$$X(\mathbb{R}) := \{(L \subset H) : L = \bar{L}, H = \bar{H}\}.$$

Remark. We do not consider the case of $Z := \mathbb{P}_1(\mathbb{C})$. For $Z = \mathbb{P}_2(\mathbb{C})$ the orbits Σ, M^* and M coincide.

Proof. For (1) note that given $(L_1 \subset H_1)$ and $(L_2 \subset H_2)$ in $\widetilde{D}$, since G_0 acts transitively on the complement of the real points in Z, we may assume that $L_1 = L_2 =: L$.

Let $E := L \oplus \bar{L}$ and $E_j := H_j \cap \bar{H}_j$, $j = 1, 2$. It follows that $V = E \oplus E_1 = E \oplus E_2$ and, since all of these spaces are defined over $\mathbb{R}$, there exists $T \in G_0$ such that $T|_E = id_E$ and $T(E_1) = E_2$. Since $H_j = L \oplus E_j$, $j = 1, 2$, it follows that T maps $(L_1 \subset H_1)$ to $(L_2 \subset H_2)$. Consequently $\widetilde{D}$ is a G_0-orbit.

For (2) let $E = L \oplus \bar{L}$ as above and let $\widetilde{E}$ be a complementary subspace in V which is defined over $\mathbb{R}$. To prove that Σ is a G_0-orbit it suffices to remark that there exits $T \in G_0$ which fixes E pointwise, stabilizes $\widetilde{E}$ and interchanges the hyperplanes $H_1 \cap \widetilde{E}$ and $H_2 \cap \widetilde{E}$ which are not defined over $\mathbb{R}$ in $\widetilde{E}$.

If $L_1 = L_2 = L$ and $L = \bar{L}$, then we let $E = L$ and argue as in (2) to show that M^* is a G_0-orbit. The dual argument handles M.

The transitivity of the G_0-action on $X(\mathbb{R})$ can be proved in a similar way. □

Remark. It is a simple matter to compute the dimensions of all orbits. For this we first note that, since the π-fibers are $(n-1)$-dimensional, it follows that $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} \widetilde{D} = 2n - 1$.

Now $\pi|_\Sigma$ and $\pi^*|_\Sigma$ map Σ onto the open G_0-orbits in Z and Z^* respectively. For example, the $\pi|_\Sigma$-fiber over L can be identified with the complement of the real points in the $(n-2)$-dimensional projective space of hyperplanes H which contain both L and $\bar{L}$. Thus Σ is 2-codimensional (over $\mathbb{R}$) in X.

Analogously, since $\pi|_{M^*}$ maps M^* surjectively onto the real points in Z and its fiber over a point L is the set of hyperplanes H containing L with $H \neq \bar{H}$, it follows that $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} M^* = n + 2(n-1) = 3n - 2$.

Finally, $\dim_{\mathbb{R}} X(\mathbb{R}) = \dim_{\mathbb{C}} X = 2n - 1$. □

4.2. TRANSVERSALITY OF CYCLE INTERSECTION WITH INTERMEDIATE ORBITS

Let $\widetilde{Y}_0$ be the base cycle in $\widetilde{D}$, $g \in G$ an arbitrary element of the complex group and $\widetilde{Y} := g(\widetilde{Y}_0)$. Now $\widetilde{Y}_0$ maps to Y_0 and Y_0^* respectively and, since Y_0^* is the dual quadric of tangent hyperplanes to Y_0, it follows that a point $(L \subset H) \in \widetilde{Y}_0$ consists of $L \in Y_0$ and the hyperplane H which corresponds to the projective tangent plane of Y_0 at L. Since G acts by linear transformations, this holds for $(L \subset H) \in \widetilde{Y}$ as well, i.e., $L \in Y$ and H corresponds to the tangent hyperplane of Y at L. We use this fact to prove the following transversality statement.

PROPOSITION 4.2. *At any point p of $\widetilde{Y} \cap M$ (resp. $\widetilde{Y} \cap M^*$) the tangent spaces $T_p\widetilde{Y}$ and T_pM (resp. $T_p\widetilde{Y}$ and T_pM^*) are transversal in T_pX.*

Proof. We give the proof for $p \in \widetilde{Y} \cap M$.

Let $Y := \pi(\widetilde{Y})$ be the associated cycle in Z and define $B := \pi^{-1}(Y)$. Since M is a G_0-orbit which is mapped surjectively to the open G_0-orbit in Z and B is π-saturated, it follows that B intersects M transversally at p. Thus $\dim_{\mathbb{R}} T_p(B \cap M) = 3n - 4$.

If p is the flag $(L \subset H)$, then, recalling that $\mathbb{P}(H)$ is the projective tangent space of Y at $\pi(p) = \mathbb{P}(L)$, it follows that the pre-image $(\pi^*|_B)^{-1}(H)$ can be identified with $\mathbb{P}(H) \cap Y$. This in an $(n-2)$-dimensional quadric cone with vertex at p. Its tangent space at p generates the full tangent space $T_p F^*$ of the fiber $(\pi^*)^{-1}(H)$. Since $F^* \subset M$, it follows that $T_p F^* \subset T_p(B \cap M)$.

Now suppose that $\widetilde{Y}$ is not transversal to M at p. In this case

$$\dim(T_p \widetilde{Y} \cap T_p M) > \dim_{\mathbb{R}} \widetilde{Y} + (3n - 4) - \dim_{\mathbb{R}} B = n - 2.$$

But, since the cycle $\widetilde{Y}$ intersects the π^*-fibers transversally, it follows that

$$T_p F^* \oplus (T_p \widetilde{Y} \cap T_p M) \subset T_p(B \cap M)$$

which, contrary to the transversality of the intersection $B \cap M$, implies that

$$\dim_{\mathbb{R}} T_p(B \cap M) > 2(n-1) + (n-2) = 3n - 4.$$

□

4.3. CYCLE INTERSECTION WITH THE TOP-DIMENSIONAL BOUNDARY ORBIT

Our goal here is to prove the following

PROPOSITION 4.3. *Let $\widetilde{Y}_t$, $0 \leq t \leq 1$, be a continuous curve of cycles in X with $\widetilde{Y}_0$ the base cycle in $\widetilde{D}$ and let $Y_t = \pi(\widetilde{Y}_t)$ be the associated curve of cycles in Z. If $\widetilde{Y}_1 \cap \Sigma \neq \emptyset$, then there exists $t \in (0, 1)$ with $Y_t \cap Z(\mathbb{R}) \neq \emptyset$.*

For the proof it is convenient to introduce some notation. Here we deal with projective lines E which are defined over $\mathbb{R}$, i.e., one-dimensional linear subspaces of $Z = \mathbb{P}_n(\mathbb{C})$ which are invariant with respect to the anti-holomorphic involution τ which is induced from complex conjugation on $\mathbb{C}^{n+1}$.

If E is such a line, then $E(\mathbb{R}) := \mathrm{Fix}(\tau|_E)$ divides E into two components which are interchanged by τ, i.e., $E \setminus E(\mathbb{R}) = E_1 \dot{\cup} E_2$ and $\tau(E_1) = E_2$.

The basic cycle Y_0 is also defined over $\mathbb{R}$, but $\mathrm{Fix}(\tau|_{Y_0}) = \emptyset$. Thus $Y_0 \cap E$ consists of two distinct points $z_j \in E_j$, $j = 1, 2$.

Proof of Proposition 4.3. An intersection point $x_1 \in \widetilde{Y}_1 \cap \Sigma$ is a flag $(L \subset H)$ with $L \neq \bar{L}$ and $\bar{L} \subset H$. Recall that $\pi(x_1) =: z_1$ is a point in the quadric Y_1 with projective tangent plane $\mathbb{P}(H)$. Thus the projective line $E := \mathbb{P}(L \oplus \bar{L})$, which is defined over $\mathbb{R}$, is tangent to Y_1 at z_1.

Since $E.Y_1 = 2$, it follows that $E \cap Y_1 = \{z_1\}$. Without loss of generality we may assume that $z_1 \in E_1$ and therefore $E_2 \cap Y_1 = \emptyset$.

On the other hand, for t sufficiently small, $Y_t \cap E_j \neq \emptyset$, $j = 1, 2$. By continuity it therefore follows that $Y_t \cap E(\mathbb{R}) \neq \emptyset$ for some intermediate $t \in (0, 1)$. □

4.4. THE EQUALITY OF CYCLES SPACES

As was indicated above, we may regard M_D and M_{D^*} as being identified with subspaces of the full space of linear cycles in X, e.g., $M_D \cong \{\widetilde{Y} = g(\widetilde{Y}_0) : \pi(\widetilde{Y}) \subset D\}$.

Since $\pi(\widetilde{D}) = D$, it is clear that in this sense $M_{\widetilde{D}} \subset M_D$. Of course both moduli spaces contain the orbit $G_0(Y_0)$ of the base cycle which is connected. Let M°_D, $M^\circ_{\widetilde{D}}$ and $M^\circ_{D^*}$ denote the connected components of the respective cycle spaces which contain this orbit.

THEOREM 4.4. $M^\circ_D = M^\circ_{\widetilde{D}} = M^\circ_{D^*}$.

Proof. It is sufficient to show that $\partial M^\circ_{\widetilde{D}} \cap M^\circ_D = \emptyset$. For this note first of all that the boundary $\partial M_{\widetilde{D}}$ in the full space of linear cycles in X is defined by the condition $\widetilde{Y} \cap \partial \widetilde{D} \neq \emptyset$. Since $\partial \widetilde{D}$ is semi-algebraic, it follows that $\partial M_{\widetilde{D}}$ is likewise semi-algebraic. Therefore, at least generically, for $\widetilde{Y}_1 \in \partial M^\circ_{\widetilde{D}}$ it is possible to find a curve $\widetilde{Y}_t$, $0 \leq t \leq 1$, beginning at the neutral cycle $\widetilde{Y}_0$ with $\widetilde{Y}_y \subset \widetilde{D}$ for $0 \leq t < 1$ and $\widetilde{Y}_1 \cap \partial \widetilde{D} \neq \emptyset$.

Now $\widetilde{Y}_1 \cap M^* = \widetilde{Y}_1 \cap M = \emptyset$, because it is only possible for $\widetilde{Y}_1$ to intersect these orbits transversally (Proposition 4.2). Furthermore, $\widetilde{Y}_1 \cap \Sigma = \emptyset$, because otherwise $Y_t \cap Z(\mathbb{R}) \neq \emptyset$ for some $t \in (0, 1)$

Thus the only possible nonempty intersection is $\widetilde{Y}_1 \cap X(\mathbb{R})$ which implies that Y_1 and Y_1^* are boundary points of M_D and M_{D^*} as well. Since this holds at generic boundary points, the result follows. □

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