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Holomorphic Double Fibration Transforms

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Dedicated to the memory of Harish-Chandra

ABSTRACT. Let D be a noncompact complex manifold which fits into a holomorphic double fibration $D \leftarrow W \rightarrow M$. We describe the construction of a transform from the Dolbeault cohomology space $H^s(D, \mathcal{O}(\mathbb{E}))$ into a space of holomorphic sections of a bundle on M, under somewhat mild conditions on the fibration (and the bundle $\mathbb{E} \rightarrow D$). When D is a flag domain for a semisimple Lie group the space M will be the linear cycle space M_D of D. This specifies a natural holomorphic double fibration. The corresponding Dolbeault cohomology spaces support many interesting irreducible representations. Our transform provides new realizations of these representations on spaces of holomorphic sections over M_D. This article surveys the background and presents new results on the detailed structure of M_D.

1. Introduction

Double fibration transforms have appeared in mathematics in a variety of contexts, mostly analytic or geometric, over many years. Many of the integral transforms in special function theory can be viewed as single or double fibration transforms. This is more explicit, however, in integral geometry, starting with the Crofton formulae and Chern's intersection theory of two homogeneous spaces G/H

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and $K\backslash G$ of the same group, through the classical Radon and X-ray transforms, the Gelfand-Graev horocycle transforms [**12**] which extend the Radon transform to Riemannian and Lorentzian spaces of constant negative curvature, Helgason's group-theoretic reformulation of the Gelfand-Graev horocycle transform, which applies in general to riemannian symmetric spaces of noncompact type, and the Fourier inversion formulae for various classes of non-unimodular Lie groups.

Here we study certain holomorphic double fibration transforms connected with real group orbits $D = G_0(z) \subset Z = G/Q$ on complex flag manifolds [**38**] and their linear cycle spaces [**33**], [**42**], [**43**], [**47**]. We apply those transforms to the representation theory of the real group G_0. A real analytic double fibration transform occurs in [**3**] and [**48**] as an important tool in the study of cohomology representations; a degenerate case occurs in [**25**]. It appears that holomorphic fibration transforms were first used in construction of automorphic cohomology [**33**]; see §7 for more details. The case $G_0 = \mathrm{SU}(2,2)$ is also used in physics [**10**]. In that context it is called the "Penrose Transform".

We discuss holomorphic double fibration transforms in general in §2. In §3 we recall some basic facts on real group orbits in complex flag manifolds, their linear cycle spaces, and cohomologies of homogeneous holomorphic vector bundles. Highest weight (holomorphic) representations are discussed in §4, where in fact the linear cycle space is not yet needed. We then discuss the linear cycle space in some detail in §§6 and 7, verifying the conditions needed for injectivity of the double fibration transform, and we show how this injectivity yields an identity theorem and a closed range theorem for cohomology. Then, in §7, we describe some closely related matters: convergence of Poincaré series, Gindikin-Akhiezer extensions of symmetric spaces, the Barlet Space, and a few closed range theorems. The Appendix extends the range of our structural result for the linear cycle space.

2. General transforms

Let D be a complex manifold. Later it will be an open orbit of a real reductive group G_0 on a complex flag manifold $Z = G/Q$ of its complexification. We suppose that D fits into a *holomorphic double fibration*, in other words that there are complex manifolds M and W with holomorphic fibrations as follows:

(2.1)
$$\begin{tikzcd} & W \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D & & M \end{tikzcd}$$

Given a coherent analytic sheaf $\mathcal{E} \to D$ we construct a coherent sheaf $\mathcal{E}' \to M$ and a transform

(2.2)
$$P : H^s(D;\mathcal{E}) \to H^0(M;\mathcal{E}')$$

under mild conditions on (2.1). In fact we give several variations on the construction. This construction is fairly standard (see, for example, [**6**], [**22**] and [**16**]), but we need some specific details.

Pull-back. The first step is to pull cohomology back from D to W. Let $\mu^{-1}(\mathcal{E}) \to W$ denote the inverse image sheaf. As μ is open, it is the sheaf defined by the presheaf whose value at an open set $\widetilde{U} \subset W$ is $\Gamma(U, \mathcal{E})$ where $U = \mu(\widetilde{U})$. Here, as usual, Γ denotes the space of sections. For every integer $r \geqq 0$ there is a natural map

$$\mu^{(r)} : H^r(D; \mathcal{E}) \to H^r(W; \mu^{-1}(\mathcal{E})) \tag{2.3}$$

given on the Čech cocycle level by $\mu^{(r)}(c)(\sigma) = c(\mu(\sigma))$ where $c \in Z^r(D; \mathcal{E})$ and where $\sigma = (w_0, \ldots, w_r)$ is a simplex.

PROPOSITION 2.4. (See [**7**].) *Suppose that the fiber F of $\mu : W \to D$ is connected and that $H^r(F; \mathbb{C}) = 0$ for $1 \leqq r \leqq p-1$. Then the map* (2.3) *is an isomorphism for $r \leqq p-1$ and is injective for $r = p$. In particular, if the fibers of μ are contractible then* (2.3) *is an isomorphism for all r.*

As usual, if X is a complex manifold then $\mathcal{O}_X \to X$ denotes its structure sheaf, the sheaf of germs of holomorphic $\mathbb{C}$-valued functions on X. If $\mathbb{E} \to X$ is a holomorphic vector bundle then $\mathcal{O}(\mathbb{E}) \to X$ is its sheaf of germs of holomorphic sections.

Denote $\mu^*(\mathcal{E}) = \mu^{-1}(\mathcal{E}) \otimes_{\mu^{-1}(\mathcal{O}_D)} \mathcal{O}_W$. It is a sheaf of $\mathcal{O}_W$-modules. If it happens that $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$, then $\mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E}))$, where $\mu^*(\mathbb{E})$ is the pull-back bundle. In any case, $[\sigma] \mapsto [\sigma] \otimes 1$ defines a map $i : \mu^{-1}(\mathcal{E}) \to \mu^*(\mathcal{E})$ which in turn specifies maps in cohomology, the coefficient morphisms

$$i_p : H^p(W; \mu^{-1}(\mathcal{E})) \to H^p(W; \mu^*(\mathcal{E})) \quad \text{for } p \geqq 0. \tag{2.5}$$

Our natural pull-back maps are the compositions

$$j^{(p)} : H^p(D; \mathcal{E}) \to H^p(W; \mu^*(\mathcal{E})) \quad \text{for } p \geqq 0 \tag{2.6}$$

of (2.3) and (2.5), that is, $j^{(p)} = i_p \cdot \mu^{(p)}$.

Consider the case $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$. Then $\mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E}))$, we realize these sheaf cohomologies as Dolbeault cohomologies, and the pull-back maps (2.6) are given by pulling back $[\omega] \mapsto [\mu^*(\omega)]$ on the level of differential forms.

Push-down. In order to push the $H^q(W; \mu^*(\mathcal{E}))$ down to cohomologies on M we assume that

$$\nu : W \to M \text{ is a proper map and } M \text{ is a Stein manifold.} \tag{2.7}$$

Consider the Leray direct image sheaves[1] $\mathcal{R}^p(\mu^*(\mathcal{E})) \to M$. The Grauert Direct Image Theorem[2] [**13**] ensures that each of the $\mathcal{R}^p(\mu^*(\mathcal{E})) \to M$ is a coherent sheaf of $\mathcal{O}_M$-modules. As M is Stein

$$H^q(M; \mathcal{R}^p(\mathcal{E})) = 0 \quad \text{for} \quad p \geqq 0 \text{ and } q > 0. \tag{2.8}$$

Thus the Leray spectral sequence collapses and gives

$$H^p(W; \mu^*(\mathcal{E})) \cong H^0(M; \mathcal{R}^p(\mu^*(\mathcal{E}))). \tag{2.9}$$

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[1]Whenever $\gamma : W \to X$ is a holomorphic map, the p-th Leray direct image sheaf $\mathcal{R}^p(\mathcal{F}) \to X$ is defined by the presheaf $U \mapsto H^p(\gamma^{-1}(U); \mathcal{F}|_{\gamma^{-1}(U)})$.

[2]Grauert's theorem says that if $\gamma : W \to X$ is a proper holomorphic map, if $p \geq 0$, and if $\mathcal{F} \to W$ is a coherent analytic sheaf, then $\mathcal{R}^p(\mathcal{F})$ is a coherent sheaf of $\mathcal{O}_X$-modules.

DEFINITION 2.10. The *holomorphic double fibration transform* for the holomorphic double fibration (2.1) is the composition

$$P : H^p(D;\mathcal{E}) \to H^0(M;\mathcal{R}^p(\mu^*(\mathcal{E}))) \tag{2.11}$$

of the maps (2.6) and (2.9).

One wants two things in (2.11): that P be injective, and that there be an explicit description of its image. Assuming (2.7), injectivity of P is equivalent to injectivity of $j^{(p)}$ in (2.6). There are several ways to ensure this. The most general is the vanishing condition in Theorem 2.18 below. Another, more specific to our situation, which we carry out in Section 4, uses the fact that the fibers of μ are Stein manifolds. Finally, in some cases one knows that $H^p(D;\mathcal{E})$ is an irreducible representation space for a group under which all our constructions are equivariant, so P is an intertwining operator, thus zero or injective.

The relative Dolbeault complex. In order to address the injectivity question just mentioned, we need some basic facts on the relative Dolbeault complex of a holomorphic fibration $\mu : W \to D$.

Let $T^{1,0}_{\mu,w} = T^{1,0}(\mu^{-1}(\mu(w)))$, the holomorphic tangent space at w of the fiber $\mu^{-1}(\mu(w))$ of μ containing w. Since μ is a holomorphic fibration,

$$\mathbb{T}^{1,0}_\mu(W) = \bigcup_{w\in W} T^{1,0}_{\mu,w}$$

is a holomorphic sub-bundle of the holomorphic tangent bundle $\mathbb{T}^{1,0}(W) \to W$. Define the sheaf of germs of *relative holomorphic p-forms* on W with respect to μ by

$$\Omega^p_\mu(W) = \mathcal{O}(\textstyle\bigwedge^p \mathbb{T}^{1,0}_\mu(W)^*) = \mathcal{O}(\textstyle\bigwedge^p(\mathbb{T}^*_\mu(W))^{1,0}) \text{ as sheaf over } W.$$

Let $\Omega^p(W) \to W$ denote the sheaf $\mathcal{O}(\bigwedge^p \mathbb{T}^{1,0}(W)^*) \to W$ of ordinary holomorphic p-forms, as usual. If $\eta \in \Gamma(U;\Omega^p_\mu(W))$ there exists $\omega \in \Gamma(U;\Omega^p)$ such that $\eta(w) = \omega(w)|_{\bigwedge^p \mathbb{T}_\mu(W)}$. Thus

$$p : \Omega^p(W) \to \Omega^p_\mu(W), \text{ by } p(\omega)(w) = \omega(w)|_{\bigwedge^p \mathbb{T}_{\mu,w}},$$

is surjective.

The relative exterior differential is

$$\partial_\mu(\eta) = p(\partial(\omega)), \text{ where } \eta = p(\omega).$$

It is clear that ∂_μ is well defined, $\partial_\mu^2 = 0$ and p is a map of complexes.

LEMMA 2.12. *$\mu^{-1}\mathcal{O}_D \to \Omega^\bullet_\mu(W)$ is a resolution of the sheaf $\mu^{-1}\mathcal{O}_D \to W$.*

PROOF. Note that $\Omega^0(W) = \Omega^0_\mu(W)$. The kernel of $\partial_\mu : \Omega^0_\mu(W) \to \Omega^1_\mu(W)$ is $\mu^{-1}\mathcal{O}_D$. For if f is a holomorphic function and $\partial_\mu f = 0$ then f is constant in the fiber directions, so $f \in \mu^{-1}(\mathcal{O}_D)$. The exactness of $\Omega^\bullet_\mu(W)$ follows from an argument analogous to that of the Dolbeault Lemma. □

Let $\mathcal{E} \to D$ be a coherent analytic sheaf. Tensor $\mu^{-1}(\mathcal{E})$ over $\mu^{-1}(\mathcal{O}_D)$ with the resolution of Lemma 2.12. That gives a resolution

$$0 \to \mu^{-1}\mathcal{E} \to \mu^*(\mathcal{E}) \to \Omega^1_\mu(\mathcal{E}) \to \cdots \to \Omega^m_\mu(\mathcal{E}) \to 0 \tag{2.13}$$

of $\mu^{-1}\mathcal{E}$. We will use it to obtain an injectivity condition for

$$H^s(W;\mu^{-1}\mathcal{E}) \to H^s(W;\mu^*(\mathcal{E})). \tag{2.14}$$

Recall that a complex of sheaves

$$(2.15) \qquad 0 \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \mathcal{C}^m \to 0$$

over a space W leads to spectral sequences $\{'E_r^{p,q},\ 'd_r\}$ and $\{''E_r^{p,q},\ ''d_r\}$ converging to the hypercohomology, with

$$(2.16) \qquad 'E_2^{p,q} = H^p(W;\mathcal{H}^q(W;\mathcal{C}^\bullet)) \text{ and } ''E_2^{p,q} = H_d^q(H^p(W;\mathcal{C}^\bullet)),$$

as follows. For the first, $\mathcal{H}^q(W;\mathcal{C}^\bullet)$ is the cohomology sheaf, associated to the presheaf

$$U \to \frac{\operatorname{Ker}\{\Gamma(U;\mathcal{C}^q) \to \Gamma(U;\mathcal{C}^{q+1})\}}{\operatorname{Im}\{\Gamma(U;\mathcal{C}^{q-1}) \to \Gamma(U;\mathcal{C}^q)\}}.$$

For the second, the differentials are the coefficient morphisms induced by the differentials of (2.15).

LEMMA 2.17. *Suppose that* (1) $0 \to \mathcal{S} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \mathcal{C}^m \to 0$ *is exact and that* (2) *there is an integer s such that $H^p(W;\mathcal{C}^q) = 0$ for all $p < s$ and for $q = 1, 2, \ldots, m$. Then $H^s(W;\mathcal{S}) \to H^s(W;\mathcal{C}^0)$ is injective.*

PROOF. Hypothesis (1) says $\mathcal{H}^q(W;\mathcal{C}^\bullet) = 0$ for $q > 0$. In the notation (2.16) now

$$'E_2^{p,q} = \begin{cases} H^p(W;\mathcal{S}) & \text{for } q = 0 \text{ and all } p, \\ 0 & \text{for } q > 0 \text{ and all } p. \end{cases}$$

Hypothesis (2) gives us

$$''E_2^{p,q} = \begin{cases} 0 & \text{for } p < s \text{ and } 1 \leqq q \leqq m, \\ \operatorname{Ker} d_q / \operatorname{Im} d_{q-1} & \text{for } p = s \text{ and all } q. \end{cases}$$

Therefore

$$H^s(W;\mathcal{S}) = \operatorname{Ker}\{d_0 : H^s(W;\mathcal{C}^0) \to H^s(W;\mathcal{C}^1)\}.$$

In particular, $H^s(W;\mathcal{S}) \to H^s(W;\mathcal{C}^0)$ is injective. □

Injectivity. We return to the situation of a double fibration (2.1), a holomorphic vector bundle $\mathbb{E} \to D$, and a coherent analytic sheaf $\mathcal{E} \to D$. Eventually we will take $\mathcal{E} = \mathcal{O}(\mathbb{E})$ where $\mathbb{E} \to D$ is a negative homogeneous holomorphic vector bundle over a flag domain.

THEOREM 2.18. *Suppose that the fiber F of $\mu : W \to D$ is connected and, for some fixed integer $s \geqq 0$, that $H^r(F;\mathbb{C}) = 0$ for $1 \leqq r < s$. Assume* (2.7) *that $\nu : W \to M$ is a proper map and that M is a Stein manifold. Suppose further that $H^p(\nu^{-1}(Y');\Omega_\mu^q(\mathbb{E})|_{\nu^{-1}(Y')}) = 0$ for all $Y' \in M$, all $p < s$, and $1 \leqq q \leqq m$. Then $P : H^s(D;\mathcal{E}) \to H^0(M;\mathcal{R}^s(\mu^*(\mathcal{E})))$ is injective.*

PROOF. The assumption on F ensures, as in Proposition 2.4, that

$$\mu^{(s)} : H^s(D;\mathcal{E}) \to H^s(W;\mu^{-1}(\mathcal{E}))$$

is injective. Now the Leray spectral sequence for $\mu : W \to M$ and $\Omega_\mu^q(\mathcal{E}) \to W$ and the Stein condition on M give $H^p(W;\Omega_\mu^q(\mathcal{E})) \cong H^0(M;\mathcal{R}^p(\Omega_\mu^q(\mathcal{E})))$. The vanishing assumption for certain $H^p(\nu^{-1}(Y');\Omega_\mu^q(\mathcal{E})|_{\nu^{-1}(Y')})$ says $\mathcal{R}^p(\Omega_\mu^q(\mathcal{E})) = 0$ for $p < s$ and $1 \leqq q \leqq m$. Now $H^p(W;\Omega_\mu^q(\mathcal{E})) = 0$ for $p < s$ and $1 \leqq q \leqq m$, and $H^s(W;\Omega_\mu^q(\mathcal{E})) \cong H^0(M;\mathcal{R}^s(\Omega_\mu^q(\mathcal{E})))$ for $1 \leqq q \leqq m$. Using the notation (2.5), Lemma 2.17 says that

$$i_s : H^s(W;\mu^{-1}\mathcal{E}) \to H^s(W;\mu^*\mathcal{E})$$

is injective. In view of Proposition 2.4 now the map

$$j^{(s)} : H^s(D;\mathcal{E}) \to H^s(W;\mu^*(\mathcal{E}))$$

of (2.6) is injective. In view of (2.9) we conclude that the double fibration transform $P : H^s(D;\mathcal{E}) \to H^0(M;\mathcal{R}^s(\mu^*(\mathcal{E})))$ is injective. □

REMARK 2.19. The image of $P : H^s(D;\mathcal{E}) \to H^0(M;\mathcal{R}^s(\mu^*(\mathcal{E})))$ may be identified with the kernel of the coefficient morphism $H^s(W;\mu^*(\mathcal{E})) \to H^s(W;\Omega^1_\mu(\mathcal{E}))$ specified by d_0. In effect, the Leray spectral sequence identifies this kernel with the kernel of the corresponding map $H^0(M;\mathcal{R}^s(\mu^*(\mathcal{E}))) \to H^0(M;\mathcal{R}^s(\Omega^1_\mu(\mathcal{E})))$.

REMARK 2.20. In the cases of interest to us, $\mathcal{E} = \mathcal{O}(\mathbb{E})$ for some holomorphic vector bundle $\mathbb{E} \to D$, and P has an explicit formula. Let ω be an $\mathbb{E}$-valued $(0,s)$-form on D representing a Dolbeault cohomology class $[\omega] \in H^s_{\overline{\partial}}(D,\mathbb{E})$. Note that $\mathcal{R}^s(\mu^*(\mathcal{E})) = \mathcal{O}(\mathbb{H}^s(\mu^*(\mathbb{E})|_{\mu^{-1}(Y)})$ where the latter bundle has fiber $H^s(Y';\mu^*(\mathbb{E})|_{\mu^{-1}(Y')})$ over $Y' \in M$. Thus $P([\omega])(Y')$ is the section of $\mathcal{R}^s(\mu^*(\mathcal{E})) \to M$ whose value at $Y' \in M$ is $\mu^*(\omega)|_{\nu^{-1}(Y')}$. In other words,

$$P([\omega])(Y') = [\mu^*(\omega)|_{\nu^{-1}(Y')}] \in H^0_{\overline{\partial}}(M;\mathbb{H}^s(\mu^*(\mathbb{E})|_{\nu^{-1}(Y)})). \tag{2.21}$$

This is most conveniently interpreted by viewing $P([\omega])(Y')$ as the Dolbeault class of $\omega|_{Y'}$, and by viewing $Y' \mapsto [\omega|_{Y'}]$ as a holomorphic section of the holomorphic vector bundle over M whose fiber at Y' is $H^s(Y';\mu^*(\mathbb{E})|_{\nu^{-1}(Y')})$.

3. Flag domains and cohomology

Basic structure. Let G be a complex semisimple Lie group, Q a parabolic subgroup of G, and $Z = G/Q$ the corresponding complex flag manifold. We write $\mathfrak{g}$ and $\mathfrak{q}$ for the respective Lie algebras of G and Q, and Q is the G-normalizer of $\mathfrak{q}$, so we can view Z as the set of G-conjugates of $\mathfrak{q}$. The correspondence is $z \leftrightarrow \mathfrak{q}_z$ where $\mathfrak{q}_z$ is the Lie algebra of the isotropy subgroup Q_z of G at z.

Fix a real form G_0 of G. So its Lie algebra is a real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Write $x \mapsto \overline{x}$ for complex conjugation of G over G_0 and of $\mathfrak{g}$ over $\mathfrak{g}_0$. We recall some of the basic facts about G_0-orbits on Z [**38**]; or see [**44**].

If $z \in Z$ then $\mathfrak{q}_z \cap \overline{\mathfrak{q}_z}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We may assume that $\mathfrak{h} = \overline{\mathfrak{h}}$, in other words that $\mathfrak{h}$ is the complexification of a Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ of $\mathfrak{g}_0$. There is a choice of positive root system $\Delta^+ = \Delta^+(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{q}_z$ is the standard parabolic subalgebra $\mathfrak{q}_\Phi$ defined by some subset $\Phi \subset \Psi$ where $\Psi = \Psi(\mathfrak{g},\mathfrak{h},\Delta^+)$ is the corresponding simple root system. In other words, $\mathfrak{q}_z = \mathfrak{q}_\Phi$ where

$$\begin{aligned} \Phi^r &= \{\alpha \in \Delta \mid \alpha \text{ is a linear combination of elements of } \Phi\}, \\ \Phi^n &= \{\alpha \in \Sigma^+ \mid \alpha \notin \Phi^r\}, \text{ and} \\ \mathfrak{q}_\Phi &= \mathfrak{q}^r_\Phi + \mathfrak{q}^{-n}_\Phi \text{ with } \mathfrak{q}^r_\Phi = \mathfrak{h} + \sum_{\alpha\in\Phi^r} \mathfrak{g}_\alpha \text{ and } \mathfrak{q}^{-n}_\Phi = \sum_{\alpha\in\Phi^n} \mathfrak{g}_{-\alpha}. \end{aligned} \tag{3.1}$$

It follows that G_0 acts on Z with only finitely many orbits; in particular there are open orbits. We refer to the open orbits as *flag domains*. As G_0-invariant open subsets of Z, the flag domains $D \subset Z$ are G_0-homogeneous complex manifolds.

A flag domain $D = G_0(z) \subset Z$ is called *measurable* if it carries a G_0-invariant volume element. This is the type of flag domain currently of most importance in representation theory. More precisely, the following conditions are equivalent:

(3.2a) The orbit $G_0(z)$ is measurable.
(3.2b) $G_0 \cap Q_z$ is the G_0-centralizer of a (compact) torus subgroup of G_0.
(3.2c) D has a G_0-invariant possibly-indefinite Kaehler metric, thus a G_0-invariant measure obtained from the volume form of that metric.
(3.2d) $\overline{\Phi^r} = \Phi^r$, and $\overline{\Phi^n} = -\Phi^n$ where $\mathfrak{q}_z = \mathfrak{q}_\Phi$.
(3.2e) $\mathfrak{q}_z \cap \overline{\mathfrak{q}_z}$ is reductive, i.e. $\mathfrak{q}_z \cap \overline{\mathfrak{q}_z} = \mathfrak{q}_z^r \cap \overline{\mathfrak{q}_z^r}$.
(3.2f) $\mathfrak{q}_z \cap \overline{\mathfrak{q}_z} = \mathfrak{q}_z^r$.
(3.2g) $\overline{\mathfrak{q}}$ is Ad(G)-conjugate to the parabolic subalgebra $\mathfrak{q}^- = \mathfrak{q}^r + \mathfrak{q}^n$ opposite to $\mathfrak{q}$.

In particular, since (3.2g) is independent of choice of z, if one open G_0-orbit on Z is measurable then all open G_0-orbits are measurable.

Condition (3.2d) holds whenever the Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ of $\mathfrak{g}_0$ corresponds to a compact Cartan subgroup $H_0 \subset G_0$. (Here $\mathfrak{h} = \overline{\mathfrak{h}}$ is the Cartan subalgebra relative to which $\mathfrak{q}_z = \mathfrak{q}_\Phi$.) For in that case $\overline{\alpha} = -\alpha$ for every $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. In particular, if G_0 has discrete series representations, so that by a result of Harish-Chandra it has a compact Cartan subgroup, then every open G_0-orbit on Z is measurable. Condition (3.2d) is also automatic if Q is a Borel subgroup of G, and more generally Condition (3.2g) provides a quick test for measurability.

Compact subvarieties. We now fix $z \in Z$ such that $D = G_0(z)$ is open in Z. For convenience we suppose that z is the base point in $Z = G/Q$, so $Q = Q_z$ and $\mathfrak{q} = \mathfrak{q}_z$. For notational consistency with many papers in this area, we write L for the Levy component Q^r of Q. For reasons that will appear at the end of Section 3, we write R_- for the unipotent radical Q^{-n}. So D is measurable if and only if $L \cap G_0$ is a real form L_0 of L, and in that case $D \cong G_0/L_0$.

Fix a Cartan involution θ of G_0 that stabilizes the Cartan subgroup $H_0 \subset G_0$, and denote its fixed point sets on G_0 and G by $K_0 = G_0^\theta$ and $K = G^\theta$. Then K_0 is a maximal compact subgroup of G_0 and K is its complexification. $L \cap K_0$ is a real form of $L \cap K$ and $K_0(z) \cong K_0/(L \cap K_0)$.

As D is open we may assume $\mathfrak{h}$ chosen so that $H_0 \cap K_0$ is a Cartan subgroup of K_0, in other words so that H_0 is a *fundamental* Cartan subgroup of G_0. Use $\mathfrak{h}$ for the standard Weyl basis construction of a θ-stable compact real form $\mathfrak{g}_u \subset \mathfrak{g}$. Then $G_0 \cap G_u = K_0$ and $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{l}) + (\mathfrak{k} \cap \mathfrak{r}_-) + (\mathfrak{k} \cap \mathfrak{r}_+)$. Thus $K(z) \cong K/(K \cap Q)$ is a complex flag submanifold of Z, and K_0 acts transitively on it. In summary,

LEMMA 3.3. $K(z) = K_0(z)$; *in particular it is a compact complex submanifold of* D.

We write Y for the compact complex submanifold $K_0(z) \subset D$.

EXAMPLE 3.4. Let Z be the complex projective space $P^n(\mathbb{C})$ and let $G_0 = SU(n,1)$. Let $\{e_1, \ldots, e_{n+1}\}$ denote the standard basis of $\mathbb{C}^{n+1}$, relative to which the hermitian form defining G_0 is $\langle u, v\rangle = \left(\sum_{1 \leq a \leq n} u_a \overline{v_a}\right) - u_{n+1}\overline{v_{n+1}}$. Then G_0 has three orbits on Z: the (open) unit ball B in $\mathbb{C}^n$ inside Z, consisting of the negative definite lines; the $(2n-1)$-sphere S which is the boundary of B, consisting of the null lines; and the complement D of $B \cup S$, consisting of the positive definite lines. D is the non-convex open G_0-orbit on Z. Here Y is the hyperplane at infinity, complement to $\mathbb{C}^n$ in Z. In homogeneous coordinates $[z^1, \ldots, z^{n+1}]$, B is given by $\sum_{1 \leqq a \leqq n} |z^a|^2 < |z^{n+1}|^2$, S is given by $\sum_{1 \leqq a \leqq n} |z^a|^2 = |z^{n+1}|^2$, D is given by $\sum_{1 \leqq a \leqq n} |z^a|^2 > |z^{n+1}|^2$, and Y is given by $|z^{n+1}|^2 = 0$.

Y is maximal among the *linear* subvarieties of Z contained in D. This phenomenon leads us to the term *linear cycle space*, which we will define in Section 5 below.

Holomorphic vector bundles. Assume now that D is measurable and fix a (finite dimensional) homogeneous holomorphic vector bundle $\mathbb{E}_\chi \to D$. Thus the typical fiber E_χ is an $(L, \mathfrak{q})$-module, χ denotes the representations both of L and of $\mathfrak{q}$ on E_χ, and the complex structure on $\mathbb{E}_\chi$ is the one for which [**30**]

$$(3.5)\quad \begin{aligned} &f : G_0 \to E_\chi \text{ is a } \textit{section} \text{ if and only if } f(gl) = \chi(l)^{-1} f(g)\ \forall\, g \in G_0, l \in L_0 \\ &f : G_0 \to E_\chi \text{ is a } \textit{holomorphic} \text{ section if and only if, in addition,} \\ &\qquad f(g;\xi) + \chi(\xi) f(g) = 0 \text{ for } g \in G_0 \text{ and } \xi \in \mathfrak{q}. \end{aligned}$$

Here $f(g;\xi)$ refers to the complexification of the differential of the right translation action of G_0 on itself. For ξ in the Levy component $\mathfrak{l}$ of $\mathfrak{q}$, the "holomorphic" half of (3.5) follows from the "section" half. The Cauchy-Riemann equations for $\mathbb{E}_\chi$ come down to the "holomorphic" half of (3.5) for ξ in the nilradical $\mathfrak{r}_-$ of $\mathfrak{q}$. In effect, the antiholomorphic tangent space of D at z is identified with $\mathfrak{r}_-$. The holomorphic tangent space is identified with $\mathfrak{r}_+ = \overline{\mathfrak{r}_-} = \sum_{\Delta^+ \setminus \Phi} \mathfrak{g}_\alpha$. Of course $\mathcal{O}(\mathbb{E}_\chi) \to D$ denotes the sheaf of germs of holomorphic sections of $\mathbb{E}_\chi \to D$.

For more details on the structure of flag domains see [**38**], [**39**] and [**44**]. For more details on their geometry see [**33**], [**42**] and [**44**].

Our goal is to study the cohomologies $H^p(D; \mathcal{O}(\mathbb{E}_\chi))$ using the holomorphic double fibration (2.1). In Sections 4 and 5 we carry this out for many situations, defining and analyzing a natural double fibration and addressing the problem of injectivity raised in Section 2. For the remainder of this section we look at the structure of the $H^p(D; \mathcal{O}(\mathbb{E}_\chi))$ and we examine a certain "holomorphic type" property for flag domains.

Cohomologies. The sheaf cohomology spaces $H^p(D; \mathcal{O}(\mathbb{E}_\chi))$ can be computed by means of the Dolbeault complex, and we identify $H^p(D; \mathcal{O}(\mathbb{E}_\chi))$ with the Dolbeault cohomology $H^p_{\overline{\partial}}(D; \mathbb{E}_\chi)$. As $\mathfrak{r}_-$ is identified with the antiholomorphic tangent space, the complex

$$\left(A^\bullet(D; \mathbb{E}_\chi), \overline{\partial}\right) \text{ computes Dolbeault cohomology } H^p_{\overline{\partial}}(D; \mathbb{E}_\chi)$$

where

$$(3.6)\quad \begin{aligned} A^p(D; \mathbb{E}_\chi) = \{\omega : G_0 \to {\textstyle\bigwedge}^p(\mathfrak{r}_-)^* \otimes E_\chi \mid \\ \omega(gl) = \mathrm{Ad}\,(l)^{-1} \chi(l)^{-1} \omega(g) \text{ for } g \in G_0,\ l \in L_0\} \end{aligned}$$

We will have occasion to restrict cohomology classes from D to Y, and some formalism will be convenient for this. Whenever $U \subset V$ is a submanifold, $\mathbb{E} \to V$ is a vector bundle, and ω is an $\mathbb{E}$-valued differential form on V, the inclusion $i : U \to V$ gives us the restrictions

$$(3.7)\qquad R_{U,V}(\mathbb{E}) = i^*(\mathbb{E}) \text{ and } R_{U,V}(\omega) = i^*(\omega).$$

Here $R_{U,V}(\mathbb{E}) \to U$ is the bundle with fiber E_u at $u \in U$. Its transition functions are the restrictions of the transition functions of $\mathbb{E} \to V$. Also, $R_{U,V}(\omega)(u)$ is just the restriction of $\omega(u)$ to the appropriate exterior power of the tangent space of

$T_u(U)$. The point of this formalism is that we want to consider

(3.8) $R_{Y,D} : A^p(D;\mathbb{E}) \to A^p(Y;\mathbb{E}|_Y)$, by $R_{Y,D}(\omega)(y) = \omega(y)\Big|_{\Lambda^p T_y^{(0,s)}(Y)}$ for $y \in Y$.

If we represent $[\omega] \in H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$ by a form $\omega \in A^p(D;\mathbb{E}_\chi)$ and view $\omega(g)$ as an E_χ-valued linear function on $\bigwedge^p(\mathfrak{r}_-)$, then the restriction becomes

(3.9) $$R_{Y,D} : A^p(D;\mathbb{E}_\chi) \to A^p(Y;\mathbb{E}_\chi|_Y) \text{ by } R_{Y,D}(\omega)(k) = \omega(k)|_{\Lambda^p(\mathfrak{r}_-\cap\mathfrak{k})}.$$

This intertwines the respective $\overline{\partial}$ operators for $\mathbb{E}_\chi \to D$ and $\mathbb{E}_\chi|_Y \to Y$ and thus induces a map $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi) \to H^p_{\overline{\partial}}(Y;\mathbb{E}_\chi|_Y)$. The latter corresponds to the map on sheaf cohomology induced by $Y \hookrightarrow D$.

G_0 acts naturally on all the ingredients in the recipe for $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$, so it acts naturally on $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$. The sticky point here is the topology on $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$. In principle it need not be Hausdorff, in fact *a priori* the trivial subspace $\{0\}$ might be dense, so one needs a closed range theorem for $\overline{\partial}$. That is an extremely delicate point, not at all understood in general, and only known recently for flag domains. The current state of the matter is

THEOREM 3.10. *Let D be a measurable flag domain and let $\mathbb{E}_\chi \to D$ be a finite dimensional homogeneous holomorphic vector bundle. Then the differential $\overline{\partial}$ of the complex $\left(A^\bullet(D;\mathbb{E}_\chi),\overline{\partial}\right)$ has closed range, the cohomology $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$ is a nuclear Fréchet space, and the action of G_0 on $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$ is a strongly continuous representation.*

Various cases were proved in [**25**], [**33**], [**24**] and [**28**]. The theorem stated above (except for nuclearity of $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$, noted by Michor) is due to Wong [**48**]. Schmid-Wolf [**28**] (for Q Borel), and later Wong [**48**] (in general), prove closed range for $\overline{\partial}$ by showing that $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$ is a maximal globalization [**26**] of the corresponding cohomologically induced $(\mathfrak{g}, K_0)$-module. Thus the functorial properties of cohomologically induced modules apply to the Dolbeault cohomologies $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi)$. In particular,

THEOREM 3.11. *Let $s = \dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} K_0/(L_0 \cap K_0)$. Suppose that the representation χ of L is irreducible with highest weight λ where $\langle \lambda + \rho, \gamma\rangle < 0$ for all $\gamma \in \Delta(\overline{\mathfrak{r}_+}, \mathfrak{h})$, that is, for all $\gamma \in \Phi^n$. Then $H^p_{\overline{\partial}}(D;\mathbb{E}_\chi) = 0$ for $p \neq s$, and the representation of G_0 on $H^s_{\overline{\partial}}(D;\mathbb{E}_\chi)$ is topologically irreducible.*

All the tempered unitary highest weight representations of G_0, and many non-tempered ones as well, occur on the cohomologies just described. The highest weight representation case of Theorems 3.10 and 3.11 was done in [**24**] with the help of a certain holomorphic double fibration as in (3.12) below.

The two types of double fibrations. Consider the following variation on the double fibration (2.1):

(3.12)
$$\begin{tikzcd} & G_0/(L_0\cap K_0) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D = G_0/L_0 & & B = G_0/K_0 \end{tikzcd}$$

Decompose G_0 into a local direct product of a compact group with some noncompact simple groups. Then everything—including both double fibrations—decomposes accordingly. Thus we may, and do, assume that G_0 is a noncompact simple Lie group. This assumption made, G_0 has nontrivial unitary highest weight representations if and only if it is of *hermitian type*, that is, locally isomorphic to the analytic automorphism group of a bounded symmetric domain. In that case the domain $B = G_0/K_0$ has one of two possible invariant complex structures, and $G_0/(L_0 \cap K_0)$ has several invariant complex structures.

DEFINITION 3.13. A flag domain $D \subset Z$ is of *holomorphic type* if G_0 is of hermitian type and the G_0-invariant complex structures on G_0/K_0 and $G_0/(L_0 \cap K_0)$ can be chosen such that both μ and ν are holomorphic in (3.12). D is of *nonholomorphic type* if it is not of holomorphic type[3].

Assume from now on that G_0 is of hermitian type. Then the standard Cartan decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ into ± 1 eigenspaces of θ decompose further to

$$\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_- \text{ where } K_0 \text{ acts irreducibly on each of } \mathfrak{s}_\pm \text{ and } \mathfrak{s}_- = \overline{\mathfrak{s}_+}.$$

Set $S_\pm = \exp(\mathfrak{s}_\pm)$, so $S_- = \overline{S_+}$. Then the $\mathfrak{p}_\pm = \mathfrak{k} + \mathfrak{s}_\pm$ are parabolic subalgebras of $\mathfrak{g}$ with $\mathfrak{p}_- = \overline{\mathfrak{p}_+}$, the $P_\pm = KS_\pm$ are parabolic subgroups of G with $P_- = \overline{P_+}$, and the $X_\pm = G/P_\pm$ are hermitian symmetric flag manifolds. Note that $X_- = \overline{X_+}$ in the sense of conjugate complex structure, for $\mathfrak{s}_+$ represents the holomorphic tangent space of X_- and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ represents the holomorphic tangent space of X_+. Let $x_\pm = 1 \cdot P_\pm \in X_\pm$, so $G_0/K_0 \cong G_0(x_\pm) \subset X_\pm$. We denote

$$B = G_0/K_0 : \text{ symmetric space } G_0/K_0 \text{ with the complex structure of } G_0(x_-),$$

$$\overline{B} = \overline{G_0/K_0} : \text{ space } G_0/K_0 \text{ with the (conjugate) complex structure of } G_0(x_+).$$

The distinction between $\mathfrak{s}_-$ and $\mathfrak{s}_+$ in (1.3) is made by a choice of positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h} = \overline{\mathfrak{h}} \subset \mathfrak{k}$ of $\mathfrak{g}$. The choice is made so that $\mathfrak{s}_+$ is spanned by positive root spaces *in that system* and consequently $\mathfrak{s}_-$ is spanned by negative root spaces. However, Σ^+ need not be the same as the positive system Δ^+ relative to which $\mathfrak{r}_+$ is spanned by positive root spaces and $\mathfrak{r}_-$ is spanned by negative root spaces. The precise situation here is

PROPOSITION 3.14. *The following conditions are equivalent.*

(1) *D is of holomorphic type.*
(2) *One can choose $\Sigma^+ = \Delta^+$ in the discussion just above. In other words, $\mathfrak{g}$ has a positive root system relative to $\mathfrak{h}$ that contains both $\Delta(\mathfrak{s}_+, \mathfrak{h})$ and $\Delta(\mathfrak{r}_+, \mathfrak{h})$.*
(3) *Either $\mathfrak{r}_+ \cap \mathfrak{s}_+ = 0$ or $\mathfrak{r}_+ \cap \mathfrak{s}_- = 0$.*
(4) *Either $\mathfrak{q} \cap \mathfrak{p}_+$ or $\mathfrak{q} \cap \mathfrak{p}_-$ is a parabolic subalgebra of $\mathfrak{g}$.*

EXAMPLE 3.15. Let $G_0 = \mathrm{SU}(p, q)$ and let Z be the flag manifold of m-planes in $\mathbb{C}^{p+q}$. The open G_0-orbits on Z are the

$$D_{r,s} = \{z \in Z \mid z \text{ has signature } (r, s)\}.$$

Here $r + s = m$, and "signature" refers to the hermitian form that defines G_0. Then $D_{r,s}$ is of holomorphic type if and only if either $rs = 0$ or $(p - r)(q - s) = 0$.

EXAMPLE 3.16. Let G_0 be a connected simple group of hermitian type and let Q be a Borel subgroup of G. Then $z \in Z = G/Q$ lies in an open orbit if and only if $Q_z \cap G_0$ contains a compact Cartan subgroup T_0 of G_0. We suppose that

[3]Note here: if G_0 is not of hermitian type then D is of nonholomorphic type.

$T_0 \subset K_0$ and that $\mathfrak{q} = \mathfrak{q}_z$ has Levy component $\mathfrak{t}$. The open orbits are of the form $G_0(wz)$ where w belongs to the Weyl group $W(G,T)$. They are parameterized by $W(G_0,T_0)\backslash W(G,T) = W(\mathfrak{k},\mathfrak{t})\backslash W(\mathfrak{g},\mathfrak{t})$. See [**38**]. There are just two open orbits of holomorphic type, corresponding to the two (modulo $W(\mathfrak{k},\mathfrak{t})$) positive root systems with just one noncompact simple root.

EXAMPLE 3.17. Fix a positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g},\mathfrak{t})$ that has a unique noncompact simple root. Let Ψ denote the simple root system. Then D is of holomorphic type if $\mathfrak{q}_z = \mathfrak{q}_\Phi$ for some $z \in D$ and some $\Phi \in \Psi$. Every open G_0-orbit of holomorphic type is obtained this way for one of the two (modulo $W(\mathfrak{k},\mathfrak{t})$) such systems Σ^+. Since few positive root systems have just one noncompact simple root, this shows that flag domains of holomorphic type are relatively rare.

4. Highest weight representations and domains of holomorphic type

In this section we suppose that D is an open orbit of holomorphic type in our complex flag manifold Z. Thus we fix G_0-invariant complex structures on $B = G_0/K_0$ and on $G_0/(L_0 \cap K_0)$ such that both projections in the double fibration

$$\begin{tikzcd}[column sep=small] & G_0/(L_0 \cap K_0) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D = G_0/L_0 & & B = G_0/K_0 \end{tikzcd} \tag{4.1}$$

are holomorphic. Let χ be an irreducible finite dimensional unitary representation of L_0, let λ denote its highest weight, and let $\mathbb{E}_\chi \to D$ be the corresponding homogeneous holomorphic vector bundle. As before, $\mathcal{E}_\chi = \mathcal{O}(\mathbb{E}_\chi)$. We will give an embedding of $H^s(D;\mathcal{E}_\chi)$ into $H^0(B;\mathcal{E}_{\chi'})$ for a certain finite dimensional unitary representation χ' of K_0 and the corresponding homogeneous holomorphic vector bundle $\mathbb{E}_{\chi'} \to B$.

The double fibration (4.1) satisfies our basic conditions for double fibration transforms:

(1) $\nu : G_0/(L_0 \cap K_0) \to B$ is proper: in fact $\nu^{-1}(gK_0) = g \cdot Y$,
(2) $B = G_0/K_0$ is Stein,
(3) $\mu^{-1}(gL_0) = gF \cong g \cdot L_0/L_0 \cap K_0 \cong L_0/L_0 \cap K_0$. It is a hermitian symmetric space, thus contractible.

Let $W = G_0/(L_0 \cap K_0)$. Then the sheaf of germs of holomorphic relative p-forms for the fibration μ is $\Omega^p_\mu = \Omega^p_\mu(W) = \mathcal{O}(\bigwedge^p \mathbb{T}^{1,0}_\mu(W)^*)$. We write $\mu^*\mathbb{E}_\chi$-valued forms as $\Omega^p_\mu(\mathbb{E}) = \mathcal{O}(\bigwedge^p \mathbb{T}^{1,0}_\mu(W)^* \otimes \mu^*\mathbb{E}_\chi)$ for the corresponding sheaf of $\mathbb{E}_\chi$-valued forms.

The vanishing condition on the fibers of ν in Theorem 2.18 is

$$H^p(Y';\Omega^q_\mu(\mathcal{E}_\chi)|_{Y'}) = 0 \text{ for all } Y' \in B, \text{ all } p < s, \text{ and } 1 \leqq q \leqq m.$$

The bundle $\mathbb{T}^{1,0}_\mu(W) \to W$ has the same typical fiber $\mathfrak{l} \cap \mathfrak{s}_+$ as $\mathbb{T}^{1,0}(F)$. In fact by construction $\mathbb{T}^{1,0}_\mu(W)|_F = \mathbb{T}^{1,0}(F)$. So we express the vanishing condition as

$$H^p(Y;\mathcal{O}(\textstyle\bigwedge^q(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes \mathbb{E}_\chi)) = 0 \text{ for } p < s \text{ and } 1 \leqq q \leqq m. \tag{4.2}$$

Fix a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ containing both $\Delta(\mathfrak{s}_+, \mathfrak{t})$ and $\Delta(\mathfrak{r}_+, \mathfrak{t})$; see Proposition 3.14. The Bott-Borel-Weil Theorem shows that (4.2) holds whenever

$$\langle \lambda, \gamma \rangle < -C \tag{4.3}$$

for all $\gamma \in \Delta(\mathfrak{k} \cap \mathfrak{r}_+)$, for some sufficiently large constant C. One can choose

$$C = \max_{F \subset \Delta((\mathfrak{l} \cap \mathfrak{s}_-), \mathfrak{t})} \left\{ \sum_{\beta \in F} \langle \beta, \gamma \rangle \middle| \gamma \in \Delta((\mathfrak{k} \cap \mathfrak{r}_+), \mathfrak{t}) \right\}. \tag{4.4}$$

(See [**24**] for a more refined bound.) The bundle $\mathbb{E}_{\chi'} \to B$ is given by the Bott-Borel-Weil Theorem as well. Let $w_0 \in W(\mathfrak{k}, \mathfrak{t})$ such that

$$w_0\big(\Delta^+(\mathfrak{k} \cap \mathfrak{l}, \mathfrak{t}) \cup \Delta(\mathfrak{k} \cap \mathfrak{r}_-, \mathfrak{t})\big) = \Delta^+(\mathfrak{k} \cap \mathfrak{l}, \mathfrak{t}) \cup \Delta(\mathfrak{k} \cap \mathfrak{r}_+, \mathfrak{t}).$$

Then the highest weight of $E_{\chi'}$ is $\lambda' = w_0(\lambda + \rho_{\mathfrak{k}}) - \rho_{\mathfrak{k}}$. We now have

THEOREM 4.5. *Let D be an orbit of holomorphic type in Z and suppose that the highest weight λ of χ satisfies* (4.3). *Let χ' be the representation of K_0 of highest weight $\lambda' = w_0(\lambda + \rho_{\mathfrak{k}}) - \rho_{\mathfrak{k}}$ with w_0 as above. Then the double fibration transform P for* (4.1) *maps $H^s(D; \mathcal{E}_\chi)$ to $H^0(G_0/K_0; \mathcal{E}_{\chi'})$, and $P : H^s(D; \mathcal{E}_\chi) \to H^0(G_0/K_0; \mathcal{E}_{\chi'})$ is an injection.*

There is a slightly different route to Theorem 4.5 using the fact that the fibers of μ are Stein. The de Rham cohomology of the fiber $F = \mu^{-1}(z) \cong L_0/(L_0 \cap K_0)$ can be calculated from the complex of holomorphic forms,

$$0 \to H^0(F; \mathcal{O}(\textstyle\bigwedge^0 \mathbb{T}^{1,0}(F)^*)) \to \cdots \to H^0(F; \mathcal{O}(\textstyle\bigwedge^m \mathbb{T}^{1,0}(F)^*)).$$

In order to make effective use of the structure of $\mathfrak{g}$ here, we adopt some slightly unusual notation: if $\mathfrak{v}$ is an $(L \cap K)$-module we denote

$$\begin{aligned} &\mathbb{E}(\mathfrak{v}) \to F : \text{associated homogeneous holomorphic vector bundle,} \\ &\mathcal{E}(\mathfrak{v}) \to F : \text{sheaf of germs of holomorphic sections of } \mathbb{E}(\mathfrak{v}) \to F. \end{aligned} \tag{4.6}$$

Since $\mathfrak{l} \cap \mathfrak{s}_+$ represents the holomorphic tangent space to F, the complex of holomorphic forms that computed de Rham cohomology of F can now be written as

$$0 \to H^0(F; \mathcal{E}(\textstyle\bigwedge^0(\mathfrak{l} \cap \mathfrak{s}_+)^*)) \to \cdots \to H^0(F; \mathcal{E}(\textstyle\bigwedge^m(\mathfrak{l} \cap \mathfrak{s}_+)^*)). \tag{4.7}$$

Here the differentials are the ordinary exterior derivative d, which sends $\mathcal{E}(\bigwedge^i(\mathfrak{l} \cap \mathfrak{s}_+)^*)$ to $\mathcal{E}(\bigwedge^{i+1}(\mathfrak{l} \cap \mathfrak{s}_+)^*)$ because $\overline{\partial}$ annihilates holomorphic forms. Since $F = L_0/(L_0 \cap K_0)$ is contractible we have an exact sequence

$$0 \to \mathbb{C} \to H^0(F; \mathcal{E}(\textstyle\bigwedge^0(\mathfrak{l} \cap \mathfrak{s}_+)^*)) \to \cdots \to H^0(F; \mathcal{E}(\textstyle\bigwedge^m(\mathfrak{l} \cap \mathfrak{s}_+)^*)).$$

Tensoring typical fibers of the underlying bundles by the L_0-module E_χ, and then taking associated vector bundles, we obtain exact sequences of G_0-homogeneous holomorphic vector bundles over D, and of their sheaves of germs of holomorphic sections,

$$\begin{aligned} &0 \to \mathbb{E}_\chi \to \mathbb{H}^0(F; \mathcal{E}(\textstyle\bigwedge^0(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes E_\chi)) \to \cdots \to \mathbb{H}^0(F; \mathcal{E}(\textstyle\bigwedge^m(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes E_\chi)), \\ &0 \to \mathcal{E}_\chi \to \mathcal{H}^0(F; \mathcal{E}(\textstyle\bigwedge^0(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes E_\chi)) \to \cdots \to \mathcal{H}^0(F; \mathcal{E}(\textstyle\bigwedge^m(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes E_\chi)). \end{aligned} \tag{4.8}$$

The double complex

$$A^q(D; \mathbb{H}^0(F; \mathcal{E}(\textstyle\bigwedge^p(\mathfrak{l} \cap \mathfrak{s}_+)^* \otimes E_\chi))) \tag{4.9}$$

gives rise to a spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$\begin{aligned} E_1^{p,q} &= H^q(D; \mathcal{H}^0(F; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi))) \\ &= H^q(W_D; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)) \\ &= H^0(B; (\mathcal{H}^q(Y; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)|_Y))). \end{aligned} \tag{4.10}$$

Here recall the coset space expressions $D = G_0/L_0$, $W_D = G_0/(K_0 \cap L_0)$, $B = G_0/K_0$, and $Y = K_0/(K_0 \cap L_0)$. If $\lambda+\rho$ is sufficiently negative then $H^q(Y; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)|_Y) = 0$ for $q \neq s$, so we have

$$E_1^{p,q} = 0 \text{ for } q \neq s \text{ and } E_1^{p,s} = H^0(B; (\mathcal{H}^s(Y; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)|_Y))). \tag{4.11}$$

As $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ computes $E_2^{p,q}$ we now have

$$E_2^{p,q} = 0 \text{ for } q \neq s \text{ and } E_1^{p,s} = H^s(D; \mathcal{H}^0(F; \mathcal{E}(\bigwedge^p(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)|_Y))). \tag{4.12}$$

In particular, the spectral sequence collapses at E_2 and we have

$$\begin{aligned} H^s(D; \mathcal{E}_\chi) &= E_2^{0,s} \\ &= \operatorname{Ker}\left\{d_1 : H^0(B; \mathcal{H}^s(Y; \mathcal{E}_\chi|_Y)) \to H^0(B; \mathcal{H}^s(Y; \mathcal{E}(\mathfrak{l}\cap\mathfrak{s}_+)^* \otimes E_\chi)|_Y))\right\}. \end{aligned} \tag{4.13}$$

In the notation of (3.8), if $[\omega] \in H^s_{\overline{\partial}}(D; \mathbb{E})$ then

$$P([\omega])(y') = R_{Y',D}(\mu^*(\omega))(y') \text{ for } y' \in Y' \in M_D. \tag{4.14}$$

One immediate consequence of Theorem 4.5 is the Schmid Identity Theorem [**25**], [**33**], [**48**] for flag domains of holomorphic type. That is Corollary 4.15 below, and of course its application, Corollary 4.16, is exactly what one would expect.

COROLLARY 4.15. *Assume* (4.2). *If* $[\omega] \in H^s(D; \mathcal{E}_\chi)$ *has the property that* $R_{Y',D}(\omega) = 0$ *for each* $Y' = g \cdot Y$ *then* $[\omega] = 0$.

Spaces of smooth forms on D carry the C^∞ topology. Similarly we use the C^∞ topology on the spaces of holomorphic sections of holomorphic vector bundles over G_0/K_0. The kernel of $\overline{\partial}$ is a closed subspace of $A^s(D; \mathcal{E}_\chi)$. In order for $H^s(D; \mathcal{E}_\chi)$ to be a Fréchet space, the image of $\overline{\partial}$ must be closed.

COROLLARY 4.16. *Assume* (4.2). *In the topology induced by the natural* C^∞ *topology,* $H^s(D; \mathcal{E}_\chi)$ *is a Fréchet space, and the resulting action of* G *is a continuous representation.*

PROOF. The restriction map $R_{Y,D}$ of (4.14) is continuous in the C^∞ topology, for any compact complex submanifold $Y \subset D$. Since Y is compact, its cohomologies are finite dimensional, so $\overline{\partial}_Y$ has closed range. Suppose $\omega = \lim \overline{\partial}\eta_k$. Then $R_{Y,D}(\omega) = \lim R_{Y,D}(\overline{\partial}\eta_k) = \lim \overline{\partial} R_{Y,D}(\eta_k)$ is exact. Now ω is exact by Corollary 4.15. □

The proof given here, that $H^s(D; \mathcal{E}_\chi)$ is Fréchet, is close to the proofs given in [**25**], [**33**] and [**48**]. But our proof is somewhat different when D is not of holomorphic type; see Section 6.

COROLLARY 4.17. *When* D *is of holomorphic type the representations* $H^s(D; \mathcal{E}_\chi)$ *are highest weight representations.*

PROOF. Theorem 4.5 realizes $H^s(D; \mathcal{E}_\chi)$ as a subrepresentation of the highest weight representation $H^0(G_0/K_0; \mathcal{E}_{\chi'})$. □

Some of the representations $H^s(D;\mathcal{E}_\chi)$ are unitarized in [**24**], and our spectral sequence argument above is taken from [**24**]. In [**6**] the Bernstein-Gelfand-Gelfand resolution is used to explicitly calculate the differential operators defining the image of P.

5. The linear cycle space: basic setup

We now modify the double fibration (3.12), replacing B by the linear cycle space M_D and $G_0/(L_0 \cap K_0)$ by the appropriate incidence space W_D. If D is of holomorphic type, so we can use (3.12) for a double fibration transform as described in Section 2, then nothing changes. But in the important case where D is not of holomorphic type, this replaces (3.12) by a holomorphic double fibration which we can in fact use to construct a double fibration transform of the sort described in Section 2.

Lemma 3.3 gives us a particular maximal compact complex submanifold $Y = K(z) = K_0(z)$ in our flag domain $D = G_0(z) \subset Z$.

The G-stabilizer J of Y is a closed (because Y is compact) complex (because Y is a complex submanifold) subgroup of G, so

$$M_Z = \{gY \mid g \in G\} \cong G/J$$

has a natural structure of complex manifold. Since Y is compact and D is open in Z, the subset $\{gY \mid g \in G\}$ is open in M_Z. Thus $\{gY \mid g \in G\}$ has a natural structure of complex manifold. See [**33**]. Its incidence space

$$W_Z = \{(z', Y') \mid z' \in Y' \in M_Z\}$$

thus also has a natural structure of complex manifold, and we have a G-equivariant holomorphic double fibration

$$\begin{tikzcd} & W_Z \arrow[dl, "\widetilde{\mu}"'] \arrow[dr, "\widetilde{\nu}"] & \\ Z & & M_Z \end{tikzcd} \tag{5.1}$$

where $\widetilde{\mu}(z', Y') = z'$ and $\widetilde{\nu}(z', Y') = Y'$.

DEFINITION 5.2. The *linear cycle space* M_D of D is the topological component of Y in $\{gY \mid g \in G \text{ and } gY \subset D\}$. Its *incidence space* is $W_D = \{(z', Y') \mid z' \in Y' \subset D\}$.

M_D inherits a complex manifold structure as an open subset of M_Z, as follows. First, $\{gY \mid g \in G \text{ and } gY \subset D\}$ is open in M_Z because D is open in Z and Y is compact. Second, M_D is open in $\{gY \mid g \in G \text{ and } gY \subset D\}$. Similarly, W_D inherits a complex manifold structure as an open subset of W_Z. Thus, inside (5.1) we have a smaller G_0-equivariant holomorphic double fibration

$$\begin{tikzcd} & W_D \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ D & & M_D \end{tikzcd} \tag{5.3}$$

where $\mu(z', Y') = z'$ and $\nu(z', Y') = Y'$.

We compare the double fibrations (3.12) and (5.3). Assuming G_0 simple, there are two sharply different possibilities for the complex isotropy subgroup $J = \{g \in G \mid gY = Y\}$. Specifically, the following is proved in [**33**] and [**42**].

PROPOSITION 5.4. *Let G_0 be simple. Then either* (1) *D is of holomorphic type, $J = KS_\pm$, $M_Z = G/J$ is a projective algebraic variety, and* $\dim_{\mathbb{C}} M_D = \frac{1}{2}\dim_{\mathbb{R}} G_0/K_0$, *or* (2) *$D$ is of nonholomorphic type, J is a finite extension of K, $M_Z = G/J$ is an affine algebraic variety, and* $\dim_{\mathbb{C}} M_D = \dim_{\mathbb{R}} G_0/K_0$.

REMARK 5.5. The incidence space $W_Z \cong G/(Q \cap J)$ varies with the two types as follows. If D is of holomorphic type then $Q \cap J$ is (for proper choice of $\mathfrak{s}_\pm$) a parabolic subgroup of G, so W_Z also is a projective algebraic variety. If D is of nonholomorphic type then $Q \cap J$ is not parabolic.

REMARK 5.6. Suppose that D is of nonholomorphic type. In most cases $J = K$, More precisely, one always has $K \subset J \subset N_{G_u}(K) \cdot K$, and in most but not all cases $K = N_{G_u}(K) \cdot K$. Here $G_u \subset G$ is the compact real form such that $G_u \cap G_0 = K_0$. For example, if $G_0 = \mathrm{SU}(p,q)$ with $p \neq q$ then $K = N_{G_u}(K) \cdot K$, that is, $N_{G_u}(K) = K_0$. But if $p = q$ then $N_{G_u}(K) = K_0 \cdot \{1, \sigma\}$ where[4] $\sigma = \left(\begin{smallmatrix} 0 & I_p \\ -I_p & 0 \end{smallmatrix}\right)$. Now let $D = D_{r,s}$ as in Example 3.15 above. Decompose $\mathbb{C}^{p+q}$ as the orthogonal direct sum $U \oplus V$ where U is the positive definite C^p spanned by the first p standard basis vectors and V is the negative definite C^q spanned by the last q standard basis vectors. Then

$$Y = \{z \in Z \mid \dim(z \cap U) = r \text{ and } \dim(z \cap V) = s\}.$$

In particular $\sigma(Y) = Y$ if and only if $r = s$. Thus we have $J = K$ except when $p = q$ and $D = D_{r,s}$ with $r = s$; and in that exceptional case we have $J = K \cdot \{1, \sigma\}$.

REMARK 5.7. In both cases, M_D is a Stein manifold. See [**42**] and [**43**].

The holomorphic case. If D is of holomorphic type, then the G_0-equivariant holomorphic double fibrations (3.12) and (5.3) agree, by

PROPOSITION 5.8. (See [**42**] for the original proof, [**47**] for a more direct proof.) *If D is of holomorphic type, then M_D is biholomorphic to B or to $\overline{B}$, and* $W_D \cong G_0/(K_0 \cap L_0)$.

REMARK 5.9. If $D \cong G_0/L_0$ is of holomorphic type and L_0 is compact, so $L_0 \subset K_0$, then the natural projection $\pi : G_0/L_0 \to G_0/K_0$ is holomorphic. In that case, if $gY \in M_D$ then[5] $\pi(gY) = g\pi(Y) = gx_\pm = g_0 x_\pm$ for some $g_0 \in G_0$. Thus $g^{-1}g_0 \in KS_\pm$. As D is of holomorphic type now $gY = g_0 Y$. So $M_D = G_0/K_0$. This is the special case of Proposition 5.8 considered in [**33**]. Here (5.3) collapses: $W_D = D$, μ is the identity map, and ν is just the projection $\pi : G_0/L_0 \to G_0/K_0$.

6. The linear cycle space: domains of nonholomorphic type

Structure of M_D. We now suppose that D is of nonholomorphic type. Then the detailed structure of M_D remains elusive, except in some special cases, many of which are included in the following result from [**47**].

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[4]See [**37**] for a number of calculations of this sort.

[5]Here, as in Section 3, x_- is the base point of $B = G_0(x_-)$ and x_+ is the base point of $\overline{B} = G_0(x_+)$.

THEOREM 6.1. *If G_0 is a classical group of hermitian type, and D is a flag domain of nonholomorphic type, then M_D is biholomorphic to $B \times \overline{B}$.*

Some special cases of Theorem 6.1 are worked out in [**23**], in [**9**], and in [**18**] and [**19**]. Here is a straightforward example.

EXAMPLE 6.2. $G_0 = \mathrm{SU}(p,q)$, Z is the Grassmann manifold of m-planes in $\mathbb{C}^{p+q}$, and $D = D_{r,s}$ as in Example 3.15 (so $r+s=m$) . We assume that D is of nonholomorphic type, in other words, in this case, that both $0 < r < p$ and $0 < s < q$. Let $\{e_1, \ldots, e_{p+q}\}$ denote the standard basis of $\mathbb{C}^{p+q}$, relative to which the hermitian form defining G_0 is $\langle u, v\rangle = \sum_{1 \le a \le p} u_a \overline{v_a} - \sum_{1 \le b \le q} u_{p+b} \overline{v_{p+b}}$. Set

$$\begin{aligned} z_+ = &\ \mathrm{span}\,\{e_1, \ldots, e_p\} \text{ and } z_- = \ \mathrm{span}\,\{e_{p+1}, \ldots, e_{p+q}\}, \\ z_{r,s} = &\ \mathrm{span}\,\{e_1, \ldots, e_r; e_{p+1}, \ldots, e_{p+s}\}, \text{ so } Y_{r,s} = K_0(z_{r,s}) \in M_{D_{r,s}}. \end{aligned}$$

Whenever $\mathbb{C}^{p+q} = U \oplus V$ direct sum of a p-plane U and a q-plane V we set

$$Y_{U,V} = \{z' \in Z \mid \dim(z' \cap U) = r \text{ and } \dim(z' \cap V) = s\}, \text{ so } Y_{r,s} = Y_{z_+, z_-}.$$

Then $gY_{U,V} = Y_{gU,gV}$ for all $g \in G$. It follows that

$$\begin{aligned} M_Z &= \{Y_{U,V} \mid \mathbb{C}^{p+q} = U \oplus V \text{ with } \dim U = p \text{ and } \dim V = q\} \\ &= \{Y_{gz_+, gz_-} \mid g \in G\}, \end{aligned}$$

consisting of transverse pairs U, V where U is a p-plane and V is a q-plane. When $p \ne q$ or $r \ne s$ or both, $M_Z \cong GL(p+q;\mathbb{C})/(GL(p;\mathbb{C}) \times GL(q;\mathbb{C})) \cong G/K$. In any case,

$$\begin{aligned} M_{D_{r,s}} &= \{Y_{U,V} \mid U \text{ is positive definite and } V \text{ is negative definite}\} \\ &= \{Y_{gz_+, gz_-} \mid g \in G,\ gz_+ \text{ is positive definite and } gz_- \text{ is negative definite}\}. \end{aligned}$$

As g runs over G_0, gz_- runs through the bounded symmetric domain B of maximal negative definite subspaces of $(\mathbb{C}^{p+q}, \langle\cdot,\cdot\rangle)$, and gz_+ runs through the bounded symmetric domain $\overline{B}$ of maximal positive definite subspaces of $(\mathbb{C}^{p+q}, \langle\cdot,\cdot\rangle)$. That shows exactly how $M_{D_{r,s}} \cong B \times \overline{B}$.

The crucial first step in the proof of Theorem 6.1 is our placement of $B \times \overline{B}$ inside G/K. Use notation of Section 3 and the standard $G_0 \subset S_+ K S_-$. Then K is the intersection of the parabolic subgroups $P_\pm = KS_\pm$ so $G/K = \delta G(x_-, x_+) \subset X_- \times X_+$ where δ denotes the diagonal action of G on the product $X_- \times X_+$ of hermitian symmetric flag manifolds. Of course we also have $B \times \overline{B} = (G_0 \times G_0)(x_-, x_+) \subset X_- \times X_+$.

LEMMA 6.3. $B \times \overline{B} = (G_0 \times G_0)(x_-, x_+) \subset \delta G(x_-, x_+) = G/K \subset X_- \times X_+$.

PROOF. [**19**], [**47**]. Let $g_i \in G_0$, so $g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-)$ with $k \in K$, $\xi_\pm \in \mathfrak{s}_\pm$. Then

$$\begin{aligned} (g_1 x_-, g_2 x_+) &= \delta g_2(g_2^{-1} g_1 x_-, x_+) = \delta g_2(\exp(\xi_+) x_-, x_+) \\ &= \delta g_2(\exp(\xi_+) x_-, \exp(\xi_+) x_+) = \delta g_2\ \delta \exp(\xi_+)(x_-, x_+) \in \delta G(x_-, x_+) \end{aligned}$$

shows that $(G_0 \times G_0)(x_0^-, x_0^+) \subset \delta G(x_0^-, x_0^+) \subset X_- \times X_+$. □

Now $B \times \overline{B} \subset G/K \subset X_- \times X_+$. But $M_D \subset M_Z = G/J$, and while the identity component $J^0 = K$ we will have $J \ne K$ in general. Nevertheless, it is shown in [**47**] that

LEMMA 6.4. *If D is of nonholomorphic type then the natural projection $G/K \to G/J$ is injective on $B \times \overline{B}$.*

Theorem 6.1 is proved in [**47**]. Lemmas 6.3 and 6.4 above, show that comparison is possible, for they place both M_D and $B \times \overline{B}$ inside the same space $M_Z = G/J$. In [**47**] is it shown that

PROPOSITION 6.5. *If D is of nonholomorphic type and G_0 is of hermitian type then $M_D \subset B \times \overline{B}$.*

However, $B \times \overline{B} \subset M_D$ requires a case by case argument, which is carried out for the classical groups, in order to complete the proof of Theorem 6.1.

There are several cases not covered by Theorem 6.1 where the structure of M_D is known, due to R. O. Wells [**32**], to Akhiezer and Gindikin [**1**], and to Dunne and Zierau [**9**]. There are also a number of cases, the cases where D is indefinite Kähler symmetric space, covered in the Appendix below.

When D is of holomorphic type, it is immediate from Proposition 5.8 that M_D is a Stein manifold. When D is of nonholomorphic type and G_0 is a classical group of hermitian type, then it is immediate from Theorem 6.1 that M_D is a Stein manifold. However one knows in general that M_D is a Stein manifold; see [**42**] and [**43**].

The fiber of $\mu : W_D \to D$. The next ingredient in our construction of the holomorphic double fibration transform associated to (5.3) is

THEOREM 6.6. *If G_0 is a classical group of hermitian type, and D is a flag domain of nonholomorphic type, then the fibers of $\mu : W_D \to D$ are contractible.*

A proof will appear in [**20**]. The contractibility can be obtained by realizing the fiber as an iterated fibration of bounded symmetric domains. Here is an example that indicates the general idea.

EXAMPLE 6.7. $G_0 = \mathrm{SU}(p,q)$, Z is the Grassmann manifold of m-planes in $\mathbb{C}^{p+q}$, and $D = D_{r,s} = G_0(z_{r,s})$ as in Examples 3.15 and 6.2 (so $r+s=m$). Again, D is of nonholomorphic type, that is, both $0 < r < p$ and $0 < s < q$. Let $\{e_1, \ldots, e_{p+q}\}$ denote the standard basis of $\mathbb{C}^{p+q}$, relative to which the hermitian form defining G_0 is $\langle u, v\rangle = \sum_{1 \le a \le p} u_a \overline{v_a} - \sum_{1 \le b \le q} u_{p+b}\overline{v_{p+b}}$. Then $z_{r,s} = \mathrm{Span}\,\{e_1, \ldots, e_r; e_{p+1}, \ldots, e_{p+s}\}$ and $D_{r,s} = \{z' \in Z \mid z' \text{ has signature } (r,s)\}$. As in Example 6.2, now

$$Y_{U,V} = \{z' \in Z \mid \dim z' \cap U = r \text{ and } \dim z' \cap V = s\} \in W_Z$$

and $Y_{U,V} \in M_D$ just when U is positive definite and V is negative definite. So

$$\mu^{-1}(z_{r,s}) = \{Y_{U,V} \mid U >> 0 >> V,\ \dim(z_{r,s} \cap U) = r, \text{ and } \dim(z_{r,s} \cap V) = s\}.$$

Thus the possibilities for U and V are as follows. Write $\perp$ for orthogonality relative to the hermitian form $\langle \cdot, \cdot \rangle$. Then $U = U' \oplus U''$ where

(1) U' is an element of the bounded symmetric domain B'_+ consisting of all maximal positive definite subspaces of $z_{r,s}$,
(2) U'' ranges over the bounded symmetric domain B''_+ of all maximal positive definite subspaces of $(U')^\perp$.

Thus the pairs (U', U'') form the total space of a real analytic fiber bundle with base B'_+ and typical fiber B''_+. If $p \ne 2r$, in other words if $\dim U' \ne \dim U''$, then

U determines (U', U''). If $p = 2r$ then the condition $Y_{U,V} \in M_D$ determines which is which between U' and U''. Now the possibilities for U form a contractible space, the total space of a C^ω fiber bundle with base B'_+ and typical fiber B''_+.

Similarly, $V = V' \oplus V''$ where

(1) V' is an element of the bounded symmetric domain B'_- consisting of all maximal negative definite subspaces of $z_{r,s}$,

(2) V'' ranges over the bounded symmetric domain B''_- of all maximal negative definite subspaces of $(V')^\perp$.

The possibilities for V form a contractible space, the total space of a C^ω fiber bundle with base B'_- and typical fiber B''_-.

Conclusion: $\mu^{-1}(z_{r,s})$ is contractible. The same holds for any fiber, for if $z' \in D$ we have $g \in G_0$ with $z' = gz_{r,s}$, and $\mu^{-1}(z') = g\mu^{-1}(z_{r,s})$.

The transform. All the ingredients are now in place for the holomorphic double fibration transform $P : H^s(D; \mathcal{E}_\chi) \to H^0(M; \mathcal{R}^s(\mu^*(\mathcal{E}_\chi)))$ when G_0 is a classical group of hermitian type. In particular, M_D is explicitly identified as the Stein manifold $B \times \overline{B}$ and the fibers of $\mu : W_D \to D$ are shown contractible. Now, by Theorem 2.18, we must determine a condition on the highest weight λ of χ that will ensure

$$(6.8) \qquad H^p(Y'; \Omega^q_\mu(\mathcal{E}_\chi)|_{Y'}) = 0 \text{ for all } Y' \in M_D, \text{ all } p < s, \text{ and } 1 \leqq q \leqq m.$$

Recall from (3.1) that $\mathfrak{q} = \mathfrak{q}^r + \mathfrak{q}^{-n}$ has nilradical $\mathfrak{q}^{-n} = \sum_{\alpha \in \Phi^n} \mathfrak{g}_{-\alpha}$. Its opposite is its complex conjugate $\overline{\mathfrak{q}} = \mathfrak{q}^r + \mathfrak{q}^n$, which has nilradical $\mathfrak{q}^n = \sum_{\alpha \in \Phi^n} \mathfrak{g}_\alpha$. The sheaf of relative holomorphic p-forms for $\mu : W_D \to D$ is $\Omega^p_\mu = \mathcal{O}(\bigwedge^p \mathbb{T}^{1,0}_\mu(W_D)^*)$ as in Section 4, but here $\overline{\mathfrak{q}}/(\overline{\mathfrak{q}} \cap \mathfrak{k})$ replaces $\mathfrak{l} \cap \mathfrak{s}_+$, and the relative complex is given by the $\mathcal{E}(\bigwedge^q(\overline{\mathfrak{q}}/(\overline{\mathfrak{q}} \cap \mathfrak{k}))^* \otimes E_\chi)$. In terms of homogeneous vector bundles, (6.8) becomes

$$(6.9)\ H^p(Y'; \mathcal{E}(\textstyle\bigwedge^q(\overline{\mathfrak{q}}/(\overline{\mathfrak{q}} \cap \mathfrak{k}))^* \otimes E_\chi)) = 0, \text{ all } Y' \in M_D, \text{ all } p < s, \text{ and } 1 \leqq q \leqq m.$$

Let $\rho_{\hat{\mathfrak{k}}}$ denote half the sum of the positive roots of $\mathfrak{k}$, as usual, and recall that $\Delta^+(\mathfrak{q}^n, \mathfrak{h})$ consists of positive roots. Following the Bott-Borel-Weil Theorem, the negativity condition

$$(6.10)\ \langle \lambda + \beta + \rho_{\hat{\mathfrak{k}}}, \gamma \rangle < 0 \text{ whenever } \beta \text{ is a sum from } \Delta(\overline{\mathfrak{q}} \cap \mathfrak{s}, \mathfrak{h}) \text{ and } \gamma \in \Delta(\mathfrak{q}^n \cap \mathfrak{k})$$

gives $H^p(Y; \Omega^q_\mu(\mathcal{E}_\chi)|_Y) = 0$ for all $p < s$ and all $q \geqq 0$. In view of G-homogeneity of M_Z this holds with Y replaced by any $Y' \in M_D$. Thus (6.10) implies (6.8) and (6.9). We conclude that the double fibration transform

$$(6.11) \qquad P : H^s(D; \mathcal{E}_\chi) \to H^0(M; \mathcal{R}^s(\mu^*(\mathcal{E}_\chi)))$$

is injective when (6.10) holds.

In our case the map (2.3) is specialized to $\mu^{(s)} : H^s(D; \mathcal{E}_\chi) \to H^s(W_D; \mu^{-1}\mathcal{E}_\chi)$. Contractibility of the fiber of $\mu : W_D \to D$ shows that this is an isomorphism and that the map

$$(6.12) \qquad d_0 : H^0(M_D; \mathcal{R}^s(\mu^*(\mathcal{E}_\chi))) \to H^0(M_D; \mathcal{R}^s(\Omega^1_\mu(\mathcal{E}_\chi)))$$

has kernel equal to the image of P.

The bundle $\mathcal{R}^s(\mu^*(\mathcal{E}_\chi))$ can be computed from $\mu^*(\mathcal{E}_\chi)$ using the Bott-Borel-Weil Theorem. It is the $G_0 \times G_0$-homogeneous holomorphic vector bundle $\mathbb{F}_{\chi'} \to$

$B \times \overline{B}$ defined by the complex structures and the representation $\chi' \otimes 1$ of $K_0 \times K_0$, where

(6.13) χ' has highest weight $\lambda' = w_0(\lambda + \rho_{\mathfrak{k}}) - \rho_{\mathfrak{k}}$ and $w_0 \in W(\mathfrak{k} \cap \mathfrak{q}^r, \mathfrak{t})$ such that $w_0(\Delta^+(\mathfrak{k} \cap \mathfrak{q}^r, \mathfrak{t}) \cup \Delta(\mathfrak{k} \cap \mathfrak{q}^{-n}, \mathfrak{t})) = \Delta^+(\mathfrak{k} \cap \mathfrak{q}^r, \mathfrak{t}) \cup \Delta(\mathfrak{k} \cap \mathfrak{q}^n, \mathfrak{t})$.

See [**19**, §4] for this computation along with an explanation of the apparent asymmetry in the two factors of $K_0 \times K_0$.

$F_{\chi'}$ is the representation space for the representation $\chi' \otimes 1$ of $K_0 \times K_0$, $\mathbb{F}_{\chi'} \to B \times \overline{B}$ is the associated homogeneous holomorphic vector bundle, and $\mathcal{F}_{\chi'} \to B \times \overline{B}$ is the sheaf of germs of holomorphic sections. We summarize the results of this section as follows.

THEOREM 6.14. *Let D be an orbit of nonholomorphic type in Z and suppose that the highest weight λ of χ satisfies* (6.10)*. Let χ' be the representation of K_0 of highest weight $\lambda' = w_0(\lambda + \rho_{\mathfrak{k}}) - \rho_{\mathfrak{k}}$ as in* (6.13)*. Then the linear cycle space $M_D = B \times \overline{B}$, the Leray derived sheaf $\mathcal{R}^s(\Omega^1_\mu(\mathcal{E}_\chi)) = \mathcal{F}_{\chi'}$, the double fibration transform P for* (5.3) *maps $H^s(D; \mathcal{E}_\chi)$ to $H^0(B \times \overline{B}; \mathcal{F}_{\chi'})$, $P : H^s(D; \mathcal{E}_\chi) \to H^0(B \times \overline{B}; \mathcal{F}_{\chi'})$ is an injection, and the image of P is the kernel of the differential operator d_0 of* (6.12).

Now, somewhat as in (4.15), we have a version of the Identity Theorem [**25**], [**33**], [**48**] for flag domains of nonholomorphic type:

COROLLARY 6.15. *Assume* (6.10)*. If $[\omega] \in H^s(D; \mathcal{E}_\chi)$ such that $0 = [\omega|_{Y'}] \in H^s(Y'; \mathcal{E}_\chi|_{Y'})$ for all $Y' \in M_D$, then $[\omega] = 0$.*

Again, because of cohomology vanishing in degrees $\neq s$ we have

COROLLARY 6.16. *Assume* (6.10)*. In the topology induced by the natural C^∞ topology, $H^s(D; \mathcal{E}_\chi)$ is a Fréchet*[6] *space, and the resulting action of G is a continuous representation.*

7. Remarks

Fréchet convergence of ϑ-series. Currently ϑ-series and automorphic cohomology have only been studied for flag domains $D = G_0(z)$ such that the isotropy subgroup $G_0 \cap Q_z$ is compact. In that case, Wells and one of us [**33**] used the double fibration transform and the Stein nature of M_D to prove: if $\mathbb{E} \to D$ is a sufficiently negative homogeneous holomorphic vector bundle, if $[c] \in H^s(D; \mathcal{O}(\mathbb{E}))$ is a K_0-finite Dolbeault class, if c is chosen reasonably within its cohomology class, and if Γ is any discrete subgroup of G_0, then the Poincaré series $\sum_{\gamma \in \Gamma} \gamma^* c$ converges in the C^∞ topology to a representative $\vartheta[c] \in H^s(D; \mathcal{O}(\mathbb{E}))^\Gamma$ of a Γ-invariant cohomology class. Later work provided information on the image of the Poincaré series operator $\vartheta : H^s(D; \mathcal{O}(\mathbb{E})) \to H^s(D; \mathcal{O}(\mathbb{E}))^\Gamma$. For arbitrary discrete Γ this consists of completeness results—every invariant class realized as such a Poincaré series—for the various Lebesgue classes [**40**], [**31**]. For a class of discrete subgroups Γ that includes the arithmetic groups, Williams gave several arguments that $\dim H^s(D; \mathcal{O}(\mathbb{E}))^\Gamma < \infty$ [**34**], [**35**], [**36**]. The earlier one [**34**] uses an L_2-index theorem of Moscovici [**17**], but the most interesting one [**36**] uses the double fibration transform to map the invariant L_2 cohomology $H^s_2(D; \mathcal{O}(\mathbb{E}))^\Gamma$ into one of

[6] In fact this Fréchet space is nuclear. Compare Theorem 3.10 above.

Harish-Chandra's spaces $A_2(\nu, \Gamma)$ of automorphic forms [**14**], [**15**], which are known to be finite dimensional.

Gindikin-Akhiezer tubular extension. The linear cycle space is a G_0-invariant Stein extension of G_0/K_0. By a *complex extension* of a real analytic manifold M_0 we mean a complex manifold M containing M_0 as a totally real submanifold. If M is Stein and has an action of G_0 by holomorphic automorphisms extending a given action on M_0, then we call M a G_0-invariant *Stein extension.* Complex analysis on M is closely related to analysis on M_0. In the case $M_0 = G_0/K_0$, with the proper choice of G_0-invariant Stein extension, representations known to occur in a real analytic setting on M_0 may be given realizations in a holomorphic setting on M.

Akhiezer and Gindikin [**1**] published a conjecture, motivated by results in [**33**], on the structure of a certain G_0-invariant extension of G_0/K_0. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{s}_0$ in a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$. Then $\mathfrak{g}$ has compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\, \mathfrak{s}_0$ and the corresponding compact group has decomposition $G_u = K_0 \exp(\sqrt{-1}\, \mathfrak{a}_0) K_0$. Define

$$\text{(7.1)} \qquad \Omega : \text{component of } 1K \text{ in}$$
$$G_0\{\exp(iH) \mid H \in \mathfrak{a}_0 \text{ and } |\alpha(H)| < \pi/2, \ \ \forall \alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)\}K.$$

The particular G_0-invariant extension of G_0/K_0 studied in [**1**] is $M = \Omega$. It is conjectured that Ω is a Stein extension.

PROPOSITION 7.2. *If G_0 is a classical group of hermitian type, and D is a flag domain of nonholomorphic type, then $\Omega = M_D$.*

Proposition 7.2 has the same hypotheses as Theorem 6.1. The statement follows from the structure of Hermitian symmetric spaces given in [**38**] (or see [**39**]).

Several examples given in [**1**], with G_0 not of holomorphic type also have $\Omega = M_D$.

It would be interesting to determine all G_0-invariant Stein extensions of G_0/K_0 inside G/K. Of course G/K, being affine, is Stein. An example of a G_0-invariant Stein extension which is smaller that Ω is given in [**1**].

There is some evidence that M_D is the "correct" extension. First, the restriction of holomorphic sections from M_D to G_0/K_0 is one to one. Therefore, by Theorem 6.14, one may study the representations in cohomology either as real analytic sections on G_0/K_0 or as holomorphic sections on M_D. It appears that M_D is the maximal extension, or at least the maximal Stein extension, for which this is possible. Second, there is a strong "correctness" indication in [**2**], which studies the discrete series representations of $U(p,q)$ by realizing them as images of a Szegö maps into real analytic sections of certain bundles over G_0/K_0. The largest extension in G/K of G_0/K_0 to which the Szegö kernel extends holomorphically, is computed. For orbits of nonholomorphic type it is shown that the connected component of this extension is precisely M_D. So, again the representation is realized in a holomorphic setting.

A closely related problem is to determine the G_0-invariant Stein extensions of a semisimple symmetric space. See [**11**], [**21**] and [**29**] for results on the semisimple group manifold case.

The Barlet Space. Let D be a complex analytic space and fix an integer $m \geqq 0$. Then $\mathcal{C}_m(D)$ denotes the space of m-cycles on D, consisting of all finite integral linear combinations $Y = \sum n_i Y_i$ of m-dimensional, compact, pairwise distinct, irreducible analytic subsets of D. Our linear cycle space M_D is of course contained in $\mathcal{C}_s(D)$. The *Barlet space* of D is the union $\mathcal{C}(D) = \bigcup \mathcal{C}_m(D)$. Barlet [**4**] introduced a natural analytic structure on $\mathcal{C}(D)$. Barlet proved [**5**] that when D is a reduced $(q+1)$-complete complex space, then $\mathcal{C}_q(D)$ is Stein. If we could show that our linear cycle space M_D is a closed subvariety in $\mathcal{C}_s(D)$, then this would apply directly to M_D. It would not, however, give us any new information on the precise structure of M_D nor on our double fibration transforms.

See [**8**] for an exposition of the theory of cycle spaces, especially the Barlet space.

Closed range theorems. In realizations of topological representations on cohomologies one must always deal with the question of closed range for the coboundary. Typically that coboundary is the Dolbeault operator $\overline{\partial}$ or something close, and the cocycle spaces are either Hilbert or Fréchet. In the Fréchet setting at least, where the coboundaries are continuous, cohomology vanishing theorems lead to closed range theorems [**25**], [**33**], [**26**], [**27**], [**41**], [**48**], as in Corollary 6.16 above. Double fibrations give an identity theorem setting for proof of cohomology vanishing theorems as [**25**], [**33**] and Corollary 6.15 above, as well as the corresponding closed range theorems. In fact, given a general holomorphic double fibration (2.1) such that (i) μ has contractible fibers, (ii) ν is proper and (iii) M is Stein, one expects closed range theorems for an appropriate class of bundles $\mathbb{E} \to D$.

Appendix: The symmetric case

In this section we show that Theorem 6.1 holds when D is a pseudo-riemannian symmetric space. As before, our standing assumption is that G_0 is a simple Lie group of hermitian type, but we do not assume that G_0 is classical.

DEFINITION A.1. The open orbit $D = G_0(z) \subset Z$ is *symmetric* if it has the structure of a pseudo-riemannian symmetric space for G_0, that is, if L_0 is open in the fixed point set G_0^σ of an involutive automorphism[7] σ of G_0.

LEMMA A.2. *If one open G_0-orbit on Z is symmetric, then every open G_0-orbit on Z is symmetric.*

PROOF. Let $D = G_0(z) \subset Z$ be a symmetric open orbit. We may assume $Q = Q_z$ and $L_0 = G_0 \cap Q$ open in G_0^σ where σ is an involutive automorphism of G_0. Thus the reductive part $\mathfrak{l} = \mathfrak{q} \cap \overline{\mathfrak{q}}$ of $\mathfrak{q}$ is of the form $\mathfrak{g}^\sigma$ where σ is an involutive automorphism of G that preserves $\mathfrak{g}_0$. Now let $D' = G_0(z') \subset Z$ be another open orbit, $Q' = Q_{z'}$, $L' = Q' \cap \overline{Q'}$, and $L_0' = G_0 \cap L'$. Write $z' = g(z)$ where $g \in G_u$. Now $\sigma' = \mathrm{Ad}(g)\sigma\mathrm{Ad}(g^{-1})$ is an involutive automorphism of G such that $\mathfrak{l}' = \mathfrak{g}^{\sigma'}$.

We have $\overline{\mathfrak{l}'} = \mathfrak{l}'$ by construction because $L' = Q' \cap \overline{Q'}$. It follows that σ' and complex conjugation $\xi \mapsto \overline{\xi}$ commute. Now σ' preserves $\mathfrak{g}_0$, so the real form $\mathfrak{l}_0' = \mathfrak{g}_0 \cap \mathfrak{l}'$ of $\mathfrak{l}'$ satisfies $\mathfrak{l}' = \mathfrak{g}_0^{\sigma'}$. Thus D' is symmetric. □

In order to describe the G_0-orbit structure of Z it is best to start from the viewpoint of an open orbit of holomorphic type. For that we need

[7] In this case σ is the identity on $\mathfrak{l}_0$, thus on every compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{l}_0$, in particular on the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{k}_0$ of $\mathfrak{g}_0$, relative to whose complexification $\mathfrak{h}$ we have $\mathfrak{q}$ of the form $\mathfrak{q}_\Phi$. Now σ is the identity on, and thus preserves, $\mathfrak{h} \cap \mathfrak{k}_0$. It follows that $\sigma(\mathfrak{k}_0) = \mathfrak{k}_0$. Also, in this case L_0 is the identity component of G_0^σ because L_0 is connected.

LEMMA A.3. *There is an open G_0-orbit on Z that is of holomorphic type.*

PROOF. Let $D = G_0(z)$ be an open orbit in Z. Choose any positive root systems $\Delta^+(\mathfrak{l},\mathfrak{h})$ and $\Delta^+(\mathfrak{k},\mathfrak{h})$. Then we have positive systems

$$\Delta_0^+ = \Delta_0^+(\mathfrak{g},\mathfrak{h}) = \Delta^+(\mathfrak{l},\mathfrak{h}) \cup \Delta(\mathfrak{r}_+,\mathfrak{h}) \text{ and } \Delta_{\pm 1}^+ = \Delta_{\pm 1}^+(\mathfrak{g},\mathfrak{h}) = \Delta^+(\mathfrak{k},\mathfrak{h}) \cup \Delta(\mathfrak{s}_\pm,\mathfrak{h})$$

for $\Delta(\mathfrak{g},\mathfrak{h})$. Let $w_\pm \in W(\mathfrak{g},\mathfrak{h})$ be the Weyl group element that sends Δ_0^+ to $\Delta_{\pm 1}^+$ and let $g_\pm \in G$ represent $w_\pm$. Then [**38**] $G_0(g_\pm(z))$ is open in Z because $\mathrm{Ad}(g_\pm)\mathfrak{h} = \mathfrak{h}$. Further, $\mathfrak{s}_\pm \subset \mathrm{Ad}(g_\pm)(\mathfrak{l} + \mathfrak{r}_+)$, so $\mathrm{Ad}(g_\pm)(\mathfrak{r}_-) \cap \mathfrak{s}_\pm = 0$, and thus $G_0(g_\pm(z))$ is of holomorphic type by equivalence of (1) and (3) in Proposition 3.14. □

We need one more piece of background information: the structure of the hermitian symmetric submanifold of Z that serves [**46**] to pick out the G_0-orbits on Z. As θ and σ commute, the (± 1)-eigenspace decompositions

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0 \text{ under } \theta \text{ and } \mathfrak{g}_0 = \mathfrak{l}_0 + \mathfrak{r}_0 \text{ under } \sigma$$

are compatible. We define

$$\mathfrak{m} = \mathfrak{g}^{\theta\sigma} = (\mathfrak{l}\cap\mathfrak{k}) + (\mathfrak{r}\cap\mathfrak{s}) \text{ and } \mathfrak{m}_0 = \mathfrak{g}_0^{\theta\sigma} = (\mathfrak{l}_0\cap\mathfrak{k}_0) + (\mathfrak{r}_0\cap\mathfrak{s}_0). \tag{A.4}$$

Let M and M_0 denote the corresponding analytic subgroups of G and G_0. Since D is symmetric it is measurable (even without the assumption that G_0 be of hermitian type), so (3.2b) there exists $\xi \in \sqrt{-1}(\mathfrak{h}_0 \cap k_0)$ such that $\mathfrak{q} = \mathfrak{h} + \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h}),\alpha(\xi)\geqq 0} \mathfrak{g}_\alpha$. Then $\mathfrak{m}\cap\mathfrak{q} = (\mathfrak{l}\cap\mathfrak{k}) + (\mathfrak{r}_-\cap\mathfrak{s}) = \sum_{\alpha\in\Delta(\mathfrak{m},\mathfrak{h}),\alpha(\xi)\geqq 0} \mathfrak{g}_\alpha$, so $\mathfrak{m}\cap\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{m}$. Now $M(z)$ is a flag manifold and $M_0(z)$ is an open M_0-orbit. The isotropy subgroup of M_0 at z is $M_0 \cap Q = M_0 \cap L_0 = M_0 \cap K_0$, which is the maximal compact subgroup M_0^θ of M_0. Now $M_0(z)$ is a riemannian symmetric space of noncompact type with invariant complex structure. We have proved

LEMMA A.5. *If the open orbit $D = G_0(z) \subset Z$ is symmetric, then $F = M(z)$ is a complex flag manifold of M, the open orbit $M_0(z)$ is a hermitian symmetric space of noncompact type, and $M_0(z) \subset F$ is the Borel embedding.*

The proof of Lemma A.5 did not require that G_0 be of hermitian type.

THEOREM A.6. *Let G_0 be of hermitian type and let $D = G_0(z) \subset Z = G/Q$ be a symmetric flag domain that is not of holomorphic type. View $B\times\overline{B} \subset G/J \cong M_Z$ (using* Lemma 6.4*) and $M_D \subset M_Z$ as usual. Then $B \times \overline{B} = M_D$.*

PROOF. We have $B \times \overline{B} \supset M_D$ from Proposition 6.5, so we need only prove $B\times\overline{B} \subset M_D$. Let $g_1, g_2 \in G_0$ so $(g_1 x_-, g_2 x_+) \in B \times \overline{B}$. Express the element $g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-)$ with $\xi_\pm \in \mathfrak{s}_\pm$ and $k \in K$, as in Lemma 6.3. We must show that $g_2 \exp(\xi_+) Y \subset D$.

Let $||\xi||_{\mathfrak{g}}$ denote the K_0-invariant norm on $\mathfrak{s}_+$ as in Hermann's Convexity Theorem [**39**, p. 286]: $||\xi||_{\mathfrak{g}}$ is the operator norm of $\mathrm{ad}(\frac{1}{2}(\xi + \overline{\xi}))$ on $\mathfrak{g}$ relative to the positive definite hermitian form $(u,v) = -\langle u, \overline{v}\rangle$ where $\langle\cdot,\cdot\rangle$ is the Killing form. Let $\xi \in \mathfrak{s}_+$; then there exist $g_1, g_2 \in G_0$ such that $g_2^{-1} g_1 \in \exp(\xi) K S_-$ if and only if $||\xi||_{\mathfrak{g}} < 1$. For Theorem A.6 it now suffices to prove

$$\text{if } \xi \in \mathfrak{s}_+ \text{ with } ||\xi||_{\mathfrak{g}} < 1 \text{ then } \exp(\xi_+)(z) \in D. \tag{A.7}$$

For if $g_1, g_2 \in G_0$ with $g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-)$ then (A.5) will give us

$$g_2 \exp(\xi_+) Y = g_2 \exp(\xi_+) K_0(z) \subset g_2\{\exp(\mathrm{Ad}\,(k_0)\xi)(z) \mid k_0 \in K_0\} \subset g_2 D = D,$$

as required.

Write $\xi = \xi_+ + \xi_- + \xi_{\mathfrak{l}}$ with $\xi_+ \in \mathfrak{r}_+ \cap \mathfrak{s}_+$, $\xi_- \in \mathfrak{r}_- \cap \mathfrak{s}_+$, and $\xi_{\mathfrak{l}} \in \mathfrak{l} \cap \mathfrak{s}_+$. We assert

$$(A.8) \qquad ||\xi_+||_{\mathfrak{g}} \leqq ||\xi||_{\mathfrak{g}}.$$

To see this, recall [39] that if $\eta \in \mathfrak{s}_+$ is $\mathrm{Ad}(K_0)$-conjugate to $\sum_{\psi \in \Psi^{\mathfrak{g}}} z_\psi e_\psi$ then, for appropriate normalization of e_ϕ, one has $||\eta||_{\mathfrak{g}} = \sup_{\psi \in \Psi^{\mathfrak{g}}} |z_\psi|$. Decompose the simple root system $\Psi^{\mathfrak{g}} = \Psi(\mathfrak{g}, \mathfrak{h})$ as $\Psi^{\mathfrak{g}}_+ \cup \Psi^{\mathfrak{g}}_-$ where $\sigma(e_\psi) = \pm e_\psi$ for $\psi \in \Psi^{\mathfrak{g}}_\pm$. From this note $||\xi||_{\mathfrak{g}} = ||\sigma\xi||_{\mathfrak{g}}$. Now $||\xi_+ + \xi_-||_{\mathfrak{g}} = ||\frac{1}{2}(\xi - \sigma\xi)||_{\mathfrak{g}} \leqq \frac{1}{2}(||\xi||_{\mathfrak{g}} + ||\sigma\xi||_{\mathfrak{g}}) = ||\xi||_{\mathfrak{g}}$. Recall $\mathfrak{m} = \mathfrak{g}^{\theta\sigma}$, so $\mathfrak{m} = (\mathfrak{l} \cap \mathfrak{k}) + (\mathfrak{r} \cap \mathfrak{s})$. Decompose $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_- \oplus \mathfrak{z}$ where

(i) $\mathfrak{m}_+$ is generated by $(\mathfrak{r}_+ \cap \mathfrak{s}_+) + (\mathfrak{r}_- \cap \mathfrak{s}_-)$,
(ii) $\mathfrak{m}_-$ is generated by $(\mathfrak{r}_+ \cap \mathfrak{s}_-) + (\mathfrak{r}_- \cap \mathfrak{s}_+)$, and
(iii) $\mathfrak{z}$ is the center of the reductive Lie algebra $\mathfrak{m}$.

Note that $[\mathfrak{m}_+, \mathfrak{m}_-] = 0$ so the $\mathfrak{m}_\pm$ are ideals in $\mathfrak{m}$. Let $M_\pm$ denote the (closed normal) analytic subgroups of M for $\mathfrak{m}_\pm$. Then we have $k_\pm \in M_\pm \cap K_0$ such that $\mathrm{Ad}(k_+)\xi_+$ is of the form $\sum_{\psi \in \Psi^{\mathfrak{g}}_+} z_\psi e_\psi$ and $\mathrm{Ad}(k_-)\xi_-$ is of the form $\sum_{\psi \in \Psi^{\mathfrak{g}}_-} z_\psi e_\psi$. Here $[\mathfrak{m}_+, \mathfrak{m}_-] = 0$ implies $k_+k_- = k_-k_+$, $\mathrm{Ad}(k_-)\xi_+ = \xi_+$ and $\mathrm{Ad}(k_+)\xi_- = \xi_-$. Thus $||\xi_+ + \xi_-||_{\mathfrak{g}} = \max\{||\xi_+||_{\mathfrak{g}}, ||\xi_-||_{\mathfrak{g}}\}$. We already proved $||\xi_+ + \xi_-||_{\mathfrak{g}} \leqq ||\xi||_{\mathfrak{g}}$. Now $||\xi_+||_{\mathfrak{g}} \leqq ||\xi||_{\mathfrak{g}}$, and (A.8) is proved.

Let $\eta \in \mathfrak{m}_+ \cap \mathfrak{s}_+$, using the decomposition of $\mathfrak{m}$ described just above. Recall [39] that $||\eta||_{\mathfrak{m}_+}$ and $||\eta||_{\mathfrak{g}}$ are operator norms. Decompose $\mathfrak{m}_+ = \sum \mathfrak{m}_i$ into simple ideals. Each $\mathfrak{m}_i$ is θ-stable and $\mathrm{ad}(\eta)$-invariant so $||\eta||_{\mathfrak{m}_+} = \max_i ||\eta||_{\mathfrak{m}_i}$. The Killing form of $\mathfrak{m}_i$ is some positive multiple of the restriction of the Killing form of $\mathfrak{g}$. So on each $\mathfrak{m}_i$ the restriction of $\mathrm{ad}(\frac{1}{2}(\eta + \overline{\eta}))$ has the same operator norm whether computed relative to the Killing form of $\mathfrak{g}$ or the Killing form of $\mathfrak{m}_i$. We compute relative to the Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ using $(u, v) = -\langle u, \overline{v} \rangle$:

$$\begin{aligned} ||\eta||_{\mathfrak{m}_+} &= \max_i \sup_{0 \neq \zeta \in \mathfrak{m}_i} \left(\frac{\sqrt{([\frac{1}{2}(\eta + \overline{\eta}), \zeta], [(\frac{1}{2}(\eta + \overline{\eta})), \zeta])}}{\sqrt{(\zeta, \zeta)}} \right) \\ &= \sup_{0 \neq \zeta \in \mathfrak{m}} \left(\frac{\sqrt{([\frac{1}{2}(\eta + \overline{\eta}), \zeta], [(\frac{1}{2}(\eta + \overline{\eta})), \zeta])}}{\sqrt{(\zeta, \zeta)}} \right) \\ &\leqq \sup_{0 \neq \zeta \in \mathfrak{g}} \left(\frac{\sqrt{([\frac{1}{2}(\eta + \overline{\eta}), \zeta], [(\frac{1}{2}(\eta + \overline{\eta})), \zeta])}}{\sqrt{(\zeta, \zeta)}} \right) = ||\eta||_{\mathfrak{g}}. \end{aligned}$$

That proves

$$(A.9) \qquad \text{if } \eta \in \mathfrak{m}_+ \cap \mathfrak{s}_+ \text{ then } ||\eta||_{\mathfrak{m}_+} \leqq ||\eta||_{\mathfrak{g}}.$$

Lemma A.5 says that $F = M(z)$ is a complex flag manifold and $M_0(z) \subset F$ is a hermitian symmetric space of noncompact type Borel-embedded in its compact dual. In the notation just above, $F = F_+ \times F_-$ where each $F_\pm = M_\pm(z)$ is a complex flag and $M_0(z) = M_+(z) \times M_-(z)$ where each $M_\pm(z) \subset F_\pm$ is a hermitian symmetric space of noncompact type Borel-embedded in its compact dual. Thus,

$$(A.10) \qquad \text{if } \eta \in \mathfrak{m}_+ \cap \mathfrak{s}_+ \text{ with } ||\eta||_{\mathfrak{m}_+} < 1 \text{ then } \exp(\eta)z \in M_{+,0}(z).$$

Combine (A.8) and (A.9) to see $||\xi_+||_{\mathfrak{m}_+} \leqq ||\xi_+||_{\mathfrak{g}} \leqq ||\xi||_{\mathfrak{g}} < 1$. As $\xi_+ \in \mathfrak{m}_+ \cap s_+$, (A.10) forces $\exp(\xi_+)z \in M_{+,0}(z) \subset D$. That proves (A.7), completing the proof of Theorem A.6. □

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