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Linear cycle spaces in flag domains

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Abstract. Let $Z = G/Q$, a complex flag manifold, where G is a complex semisimple Lie group and Q is a parabolic subgroup. Fix a real form $G_0 \subset G$ and consider the linear cycle spaces M_D, spaces of maximal compact linear subvarieties of open orbits $D = G_0(z) \subset Z$. In general M_D is a Stein manifold. Here the exact structure of M_D is worked out when G_0 is a classical group that corresponds to a bounded symmetric domain B. In that case M_D is biholomorphic to B if a certain double fibration is holomorphic, is biholomorphic to $B \times \overline{B}$ otherwise. There are also a number of structural results that do not require G_0 to be classical.

1. Introduction

Fix a connected simply connected complex simple Lie group G and a parabolic subgroup Q. That defines a connected irreducible complex flag manifold $Z = G/Q$. Let $G_0 \subset G$ be a real form and K_0 a maximal compact subgroup with complexification K.

If $D = G_0(z)$ is an open G_0–orbit on Z, then for an appropriate choice of base point $z \in D$, $Y = K_0(z) = K(z)$ is a maximal [1] compact complex submanifold of D [11]. The *linear cycle space* is

(1.1) $$M_D : \text{component of } Y \text{ in } \{gY \mid g \in G \text{ and } gY \subset D\}.$$

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[1] See [11] for a geometric proof and [8] for an analytic proof.

M_D is an open submanifold of the complex flag manifold $M_Z = \{gY \mid g \in G\} \cong G/J$ where[2] $J = \{g \in G \mid gY = Y\}$, thus also is a complex manifold. It is also known ([13], [14]) that M_D is a Stein manifold. We are going to sharpen that result when G_0 is of hermitian type.

There are two structurally distinct types of open orbits D, as follows.

1.2. Definition. *Consider the double fibration*

$$\begin{array}{ccc} & G_0/(L_0 \cap K_0) & \\ {}^{\pi_D}\swarrow & & \searrow^{\pi_B} \\ D = G_0/L_0 & & G_0/K_0 \end{array}$$

The open orbit $D \subset Z$ is said to be of **holomorphic type** *if there are G_0–invariant complex structures on $G_0/(L_0 \cap K_0)$ and G_0/K_0 such that π_D and π_B are simultaneously holomorphic, of* **nonholomorphic type** *if there is no such choice.*

Now we can state our main result. It is an immediate consequence of Proposition 3.9 and Theorems 3.8 and 5.1 below. For G_0 of hermitian type we write B and $\overline{B}$ for G_0/K_0 with the two G_0–invariant complex structures.

1.3. Theorem. *Let G_0 be a classical simple Lie group of hermitian type. Let $D = G_0(z) \subset Z = G/Q$ be an open G_0–orbit. If D is of holomorphic type then the linear cycle space M_D is biholomorphic either to B or to $\overline{B}$. If D is not of holomorphic type then M_D is biholomorphic to $B \times \overline{B}$.*

Theorem 1.3 extends a number of earlier results. In his work on periods of integrals on algebraic manifolds ([3], [4]), Griffiths set up moduli spaces M_D for certain classes of compact Kaehler manifolds. Wells [9] worked out an explicit parameterization of the M_D when $D \cong SO(2r, s)/U(r) \times SO(s)$. He used that parameterization to verify that the corresponding M_D are Stein, but he drew no connections between the structure of G_0 and the structure of M_D. Then Wells and Wolf [10] proved that M_D is a Stein manifold whenever the open orbit $D = G_0(z) \subset Z$ is of the form G_0/L_0 with L_0 compact. This was done in order to prove Fréchet convergence of certain Poincaré series for construction of automorphic cohomology related to Griffiths' period domains, and here some tentative connections were drawn between the structure of G_0 and M_D. Patton and Rossi [7] looked at the case $G_0 = SU(p, q)$ where Z is the Grassmannian of $(r + s)$–planes in $\mathbb{C}^{p+q}$ and D is the open orbit consisting of the $(r + s)$–planes of a fixed indefinite signature (r, s). Thus G_0 is of hermitian type and D is not of holomorphic type. This is the first instance in which close connections are indicated between the structure of G_0 and the structure of M_D. Recently

[2] In earlier work on this topic ([13], [14]) we used L to denote the G–stabilizer of Y. Here we use J for that stabilizer, reserving L for the reductive part of Q.

Wolf proved that M_D is Stein whenever D is an open G_0–orbit on Z; see [13] for the measurable case and [14] for the general case. Also recently, Dunne and Zierau [2] worked out the cases $G_0 = SO(2n, 1)$ with D indefinite hermitian symmetric, and also the cases $G_0 = SU(p, q)$ with D arbitrary. In the $SU(p, q)$ case they found that $M_D \cong B \times \overline{B}$. And very recently Novak ([5], [6]) studied the cases where $G_0 = Sp(n; \mathbb{R})$ and $D \cong Sp(n; \mathbb{R})/U(r, s)$ with $n = r + s$ and $rs \neq 0$. (Here $rs \neq 0$ is the condition that D is not of holomorphic type.) She proved $M_D \cong B \times \overline{B}$ in those cases. In the case where G_0 is classical and of hermitian type, Theorem 1.3 confirms a conjecture of Akhieser and Gindikin [1] that a certain extension of G_0/K_0 is a Stein manifold. See [16] for a discussion of applications of Theorem 1.3 to representation theory.

The remainder of the introduction is devoted to some preliminary notation and facts. The Lie algebra of G is denoted by $\mathfrak{g}$ and we let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the real form of $\mathfrak{g}$ corresponding to G_0. We consider the Cartan involution θ of G_0 corresponding to K_0. We extend θ to a holomorphic automorphism of G and a complex linear automorphism of $\mathfrak{g}$, thus decomposing

(1.4) $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, decomposition into ± 1 eigenspaces of θ.

The Lie algebra of K_0 is $\mathfrak{k}_0$, and $K = G^\theta$ is the complexification of K_0. K_0 is connected and is the G_0–normalizer of $\mathfrak{k}_0$, and K is connected because G is connected and simply connected.

From this point on we assume that G_0 is of hermitian symmetric type, that is,

(1.5) $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_-$ where K_0 acts irreducibly on each of $\mathfrak{s}_\pm$ and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$

where $\xi \mapsto \overline{\xi}$ denotes complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$. Set $S_\pm = \exp(\mathfrak{s}_\pm)$. So $S_- = \overline{S_+}$ where $g \mapsto \overline{g}$ also denotes complex conjugation of G over G_0. Then

(1.6)
the $\mathfrak{p}_\pm = \mathfrak{k} + \mathfrak{s}_\pm$ are parabolic subalgebras of $\mathfrak{g}$ with $\mathfrak{p}_- = \overline{\mathfrak{p}_+}$,
the $P_\pm = KS_\pm$ are parabolic subgroups of G with $P_- = \overline{P_+}$, and
the $X_\pm = G/P_\pm$ are hermitian symmetric flag manifolds.

Note that $X_- = \overline{X_+}$ in the sense of conjugate complex structure, for $\mathfrak{s}_+$ represents the holomorphic tangent space of X_- and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ represents the holomorphic tangent space of X_+. Let $x_\pm = 1 \cdot P_\pm \in X_\pm$, so $G_0/K_0 \cong G_0(x_\pm) \subset X_\pm$. We denote

(1.7)
$B = G_0/K_0$: symmetric space G_0/K_0 with the complex structure of $G_0(x_-)$,
$\overline{B} = \overline{G_0/K_0}$: space G_0/K_0 with the (conjugate) complex structure of $G_0(x_+)$.

The distinction between $\mathfrak{s}_-$ and $\mathfrak{s}_+$ in (1.5) is made by a choice of positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h} = \overline{\mathfrak{h}} \subset \mathfrak{k}$ of $\mathfrak{g}$. The choice is made so that $\mathfrak{s}_+$ is spanned by positive root spaces and consequently $\mathfrak{s}_-$ is spanned by negative root spaces.

We can view the complex flag manifold $Z = G/Q$ as the set of G–conjugates of $\mathfrak{q}$. Then $gQ = z \in Z = G/Q$ corresponds to $Q_z = \mathrm{Ad}(g)Q = \{g \in G \mid g(z) = z\}$ as well as its Lie algebra $\mathfrak{q}_z = \mathrm{Ad}(g)\mathfrak{q}$. Since $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \subset \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, complex conjugation acts on the root spaces by $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}$. Thus a G_0–orbit in $Z = G/Q$ is open if and only if it is of the form $G_0(z)$ in such a way that $\mathfrak{h} \subset \mathfrak{q}_z$. This choice of z in the open orbit amounts to a choice of G_0–conjugate of $\mathfrak{q}_z$, and some such conjugate must contain $\mathfrak{h}_0$ because all compact Cartan subalgebras of $\mathfrak{g}_0$ are G_0–conjugate. In other words, our standing assumption (1.5) that G_0 be of hermitian type, implies that all open G_0–orbits on Z are measurable. It also follows that we may choose a base point $z \in D$ so that $K_0(z) = K(z)$, a maximal compact complex submanifold of D. We fix such a base point and set $Y = K(z)$. See [11].

Fix an open orbit $D = G_0(z) \subset Z$ as above. We may suppose $\mathfrak{h} \subset \mathfrak{q} = \mathfrak{q}_z$ and $Q = Q_z$. Since D is measurable we decompose $\mathfrak{q} = \mathfrak{l} + \mathfrak{r}_-$ where $\mathfrak{h} \subset \mathfrak{l}$, where $\mathfrak{r}_-$ is the nilradical of $\mathfrak{q}$, and where $\mathfrak{l} = \mathfrak{q} \cap \overline{\mathfrak{q}}$ is a reductive complement (Levi component). Here $L_0 = G_0 \cap Q$ is connected and is a real form of the analytic subgroup $L \subset G$ with Lie algebra $\mathfrak{l}$. Its Lie algebra is the real form $\mathfrak{l}_0 = \mathfrak{g}_0 \cap \mathfrak{l}$ of $\mathfrak{l}$.

The orbits of holomorphic type are further characterized by [15, Prop. 1.11], which is an extension of [13, Prop. 1.3]. It says

1.8. Proposition. *The following conditions are equivalent:* (a) *the open orbit D is of holomorphic type,* (b) *either $\mathfrak{s} \cap \mathfrak{r}_+ = \mathfrak{s}_+ \cap \mathfrak{r}_+$ or $\mathfrak{s} \cap \mathfrak{r}_+ = \mathfrak{s}_- \cap \mathfrak{r}_+$,* (c) *either $\mathfrak{s}_- \cap \mathfrak{r}_+ = 0$ or $\mathfrak{s}_- \cap \mathfrak{r}_- = 0$,* (d) *one of $\mathfrak{q} \cap \mathfrak{p}$ and $\mathfrak{q} \cap \overline{\mathfrak{p}}$ is a parabolic subalgebra of $\mathfrak{g}$,* (e) *there is a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ such that both $\mathfrak{r}_+$ and $\mathfrak{s}_+$, or both $\mathfrak{r}_+$ and $\mathfrak{s}_-$, are sums of positive root spaces,* (f) *there is a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{q}$ is defined by a subset of the corresponding simple root system Ψ, and Ψ contains just one $\mathfrak{g}_0$–noncompact root.*

2. An embedding for the linear cycle space

The linear cycle space M_D is the component of $Y = K_0(z) = K(z)$ in $\{gY \mid g \in G \text{ and } gY \subset D\}$ as in (1.1). Here Y is a maximal compact subvariety of the open orbit $D = G_0(z)$. As before, $J = \{gY \mid g \in G\}$ so M_D is an open submanifold of the complex homogeneous space $M_Z \cong G/J$. By [13], Prop. 1.3 we know that if D is of holomorphic type then J is one of $KP_\pm$ and if D is of nonholomorphic type then J is a finite extension of K.

Recall the notation (1.6); $X_- \times X_+$ is a complex flag manifold $(G \times G)/(P_- \times P_+)$. Both the diagonal subgroup $\delta G \subset G \times G$ and the product $G_0 \times G_0$ are real forms of $G \times G$, so each of them acts on the complex flag manifold $X_- \times X_+$ with only finitely many orbits [11]. Let $(x_-, x_+) \in X_- \times X_+$ denote the base point $(1P_-, 1P_+)$. Thus $B \times \overline{B} = (G_0 \times G_0)(x_-, x_+)$. Our goal is to identify this with M_D in the nonholomorphic case. We start with

2.1. Lemma. *$(G_0 \times G_0)(x_-, x_+) \subset \delta G(x_-, x_+) \subset X_- \times X_+$, and both of these orbits are open in $X_- \times X_+$.*

Remark. Novak [6] was the first to see the key role of this sort of embedding.

Proof. Let $g_1, g_2 \in G_0$. Use $G_0 \subset S_+ K S_-$ to write $g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-)$ with $k \in K$ and $\xi_\pm \in \mathfrak{s}_\pm$. Then

$$\begin{aligned}(g_1 x_-, g_2 x_+) &= \delta g_2 (g_2^{-1} g_1 x_-, x_+) = \delta g_2 (\exp(\xi_+) x_-, x_+)\\ &= \delta g_2 (\exp(\xi_+) x_-, \exp(\xi_+) x_+)\\ &= \delta g_2 \, \delta \exp(\xi_+)(x_-, x_+) \in \delta G(x_-, x_+)\end{aligned}$$

shows that $(G_0 \times G_0)(x_-, x_+) \subset \delta G(x_-, x_+) \subset X_- \times X_+$. They are open because $G_0(x_-) = B$ is open in X_- and $G_0(x_+) = \overline{B}$ is open in X_-, so they all have full dimension. □

The isotropy subgroup of δG at (x_-, x_+) is $\{(g, g) \in G \times G \mid gx_- = x_- \text{ and } gx_+ = x_+\}$, in other words $\{(g, g) \in G \times G \mid g \in P_- \cap P_+ = K\}$. Thus

(2.2) δG has isotropy subgroup δK at (x_-, x_+), i.e. $\delta G(x_-, x_+) \cong G/K$.

We combine (2.2) with Lemma 2.1. That gives us the first part of

2.3. Proposition. *There is a natural holomorphic embedding of $B \times \overline{B}$ into G/K. Let $\pi : G/K \to G/J = M_Z$ be the natural projection. If the open G_0–orbit $D \subset Z$ is not of holomorphic type, then π is injective on $B \times \overline{B}$.*

Proof. Suppose that D is not of holomorphic type. Let $g_1, g_1', g_2, g_2' \in G_0$ and suppose $\pi(g_1 x_-, g_2 x_+) = \pi(g_1' x_-, g_2' x_+)$. As in the argument of Lemma 2.1, write

$$g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-) \text{ so } (g_1 x_-, g_2 x_+) = \delta g_2 \, \delta \exp(\xi_+)(x_-, x_+).$$

Similarly, this time reversing roles of the two factors,

$$g_1'^{-1} g_2' = \exp(\xi_-') k' \exp(\xi_+') \text{ so } (g_1' x_-, g_2' x_+) = \delta g_1' \, \delta \exp(\xi_-')(x_-, x_+).$$

The hypothesis $\pi(g_1 x_-, g_2 x_+) = \pi(g_1' x_-, g_2' x_+)$ now provides $j \in J$ such that $g_2 \exp(\xi_+) = g_1' \exp(\xi_-) j$. In other words, $(g_1')^{-1} g_2 \in S_- j S_+$.

Let $\{w_i\}$ be a set of representatives of the double coset space $W_K \backslash W_G / W_K$ for the Weyl groups of G and K. The Bruhat decomposition of G for X_+ is $G = \bigcup_i P_- w_i P_+$, the real group G_0 is contained in the cell $P_- P_+$ for $w_i = 1$, and G_0 does not meet any other cell $P_- w_i P_+$.

Since D is of nonholomorphic type $J \subset N_{G_u}(K_0)K$, so we may write $j = nk$ with $n \in N_{G_u}(K_0)$ and $k \in K$. Express $n = wk_0$ with $w \in \{w_i\}$ and $k_0 \in K_0$. Then $j = k'' w k''' \in KwK$ with $k'', k''' \in K$, so $(g_1')^{-1} g_2 = \exp(\xi'_-) k'' w k''' \exp(-\xi_+) \in P_- w P_+$. In particular G_0 meets $P_- w P_+$, so $w = 1 \in W_K$ and $j \in K$. This shows $g_2 \exp(\xi_+)K = g_1' \exp(\xi'_-)K$. Now

$$\begin{aligned}(g_1 x_-, g_2 x_+) &= \delta g_2 \, \delta \exp(\xi_+)(x_-, x_+) \\ &= \delta g_1' \, \delta \exp(\xi'_-)(x_-, x_+) = (g_1' x_-, g_2' x_+)\end{aligned}$$

as asserted. That completes the proof. □

3. $B \times \overline{B} \supset M_D$

In this Section we prove: (a) $M_D \subset B \times \overline{B}$ when the open orbit $D = G_0(z) \subset Z$ is not of holomorphic type and (b) $M_D = B$ or $\overline{B}$ when D is of holomorphic type. Here G_0 is of hermitian symmetric type. That is the standing hypothesis in this paper.

3.1. Lemma. *One or both of $\Delta(\mathfrak{r}_+ \cap \mathfrak{s}_\pm, \mathfrak{h})$ contains a long root of $\mathfrak{g}$.*

Proof. If all the roots of $\mathfrak{g}$ are of the same length there is nothing to prove. Now assume that there are two root lengths. The only cases are (i) $G_0 = Sp(n; \mathbb{R})$ up to a covering and (ii) $G_0 = SO(2, 2k+1)$ up to a covering.

Consider case (i). $D = G_0(z) \subset Z$ is open and $\mathfrak{q} = \mathfrak{q}_z$. The positive root system is adapted to $\mathfrak{q} = \mathfrak{l} + \mathfrak{r}_-$, so $\mathfrak{r}_-$ is spanned by negative root spaces. Let γ be the maximal root. Then $\gamma \in \Delta(\mathfrak{r}_+, \mathfrak{h})$ and γ is long. Every compact root of $\mathfrak{g}_0 = \mathfrak{sp}(n; \mathbb{R})$ is short. So γ is noncompact, hence contained in one of $\mathfrak{s}_\pm$. Now Lemma 3.1 is proved in case (i).

Consider case (ii). Then $\mathfrak{g}$ has a simple root system of the form $\{\alpha_1, \dots, \alpha_{k+1}\}$ with α_1 noncompact and the other α_i compact. Here $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqq i \leqq k$ and $\alpha_{k+1} = \epsilon_{k+1}$ with the ϵ_i mutually orthogonal and of the same length. The noncompact positive roots are the $\alpha_1 + \cdots + \alpha_m$ with $1 \leqq m \leqq k+1$ and the $(\alpha_1 + \cdots + \alpha_m) + 2(\alpha_{m+1} + \cdots + \alpha_{k+1})$. All are long except for $\alpha_1 + \cdots + \alpha_{k+1} = \epsilon_1$, which is short. Now at least one of $\Delta(\mathfrak{r}_+ \cap \mathfrak{s}_\pm, \mathfrak{h})$ contains a long root unless both $\mathfrak{r}_+ \cap \mathfrak{s}_+ = \mathfrak{g}_{\epsilon_1}$ and $\mathfrak{r}_+ \cap \mathfrak{s}_- = \mathfrak{g}_{-\epsilon_1}$. That is impossible because $\mathfrak{r}_+$ is nilpotent. Now Lemma 3.1 is proved in case (ii), and that completes the proof. □

Interchange $\mathfrak{s}_+$ and $\mathfrak{s}_-$ if necessary so that $\Delta(\mathfrak{r}_+ \cap \mathfrak{s}_+, \mathfrak{h})$ contains at least one long root. The G_0–orbit structure of $X_\pm$ is given in [12]. This is summarized as follows. Construct

$$\Psi^{\mathfrak g} = \{\gamma_1, \dots, \gamma_t\} : \text{maximal set of strongly orthogonal noncompact positive roots of } \mathfrak g \tag{3.2}$$

as in [15, (3.2)]: γ_1 is the maximal root and, at each stage, the next γ_{i+1} a maximal root in $\Delta^+(\mathfrak s_+, \mathfrak h)$ that is orthogonal to $\{\gamma_1, \dots, \gamma_i\}$. Then $\Psi^{\mathfrak g}$ consists of long roots, and any maximal set of strongly orthogonal long roots in $\Delta^+(\mathfrak s_+, \mathfrak h)$ is $W(K_0, H_0)$–conjugate to $\Psi^{\mathfrak g}$. In fact, any two subsets of $\Psi^{\mathfrak g}$ with the same cardinality are $W(K_0, H_0)$–conjugate. In particular, by modifying the choice of z within $K(z)$ we may assume that

$$\Psi^{\mathfrak g} \text{ meets } \Delta(\mathfrak r_+, \mathfrak h). \tag{3.3}$$

Using the notation and normalizations of [15], Sect. 3 we have

$$\begin{aligned}
&e_{-\gamma} : \text{ root vector for } \gamma \in \Delta(\mathfrak h)\\
&x_\gamma, y_\gamma, h_\gamma : \text{ spanning } \mathfrak g[\gamma] \simeq \mathfrak{sl}_2\\
&c_\gamma, c_\Gamma = \prod_{\gamma \in \Gamma} c_\gamma : \text{ Cayley transforms}\\
&G[\Gamma] = \prod_{\gamma \in \Gamma} G[\gamma],\\
&G[\gamma] = \text{ three dimensional subgroup corresponding to } \mathfrak g[\gamma].
\end{aligned}$$

The G_0 orbits on X_- are all of the form

$$G_0 c_\Gamma c_\Sigma^2(x_-), \quad \Gamma, \Sigma \text{ disjoint subsets of } \Psi^{\mathfrak g}. \tag{3.4}$$

The boundary of $B = G_0(x_-) \subset X_-$ consists of the orbits in (3.4) with $\Sigma = \emptyset$. The boundary orbits are described further by

$$G_0(c_\Gamma x_-) = K_0 G_0[\Psi^{\mathfrak g} \setminus \Gamma](c_\Gamma x_-). \tag{3.5}$$

One may use the Cayley transforms to gain some information about the the G_0–orbit structure of $Z = G/Q$. In particular we use the following fact.

3.6. Lemma. *Suppose $\Gamma \subset \Psi^{\mathfrak g} \cap \Delta(\mathfrak r_+, \mathfrak h)$. If $\Gamma \cap \Delta(\mathfrak r_+, \mathfrak h)$ is non–empty, then $c_\Gamma(z)$ is not contained in any open G_0–orbit on Z.*

Proof. The isotropy subgroup of G_0 at $c_\Gamma(z)$ has Lie algebra $\mathfrak g_0 \cap \mathfrak q'$ where $\mathfrak q' = \mathrm{Ad}(c_\Gamma)\mathfrak q$. If $\gamma \in \Gamma \cap \Delta(\mathfrak r_+, \mathfrak h)$ then, by [15, (3.5)], $\mathrm{Ad}(c_\Gamma)(e_{-\gamma}) = \mathrm{Ad}(c_\Gamma)(\frac{1}{2}(x_\gamma + \sqrt{-1}y_\gamma)) = \frac{1}{2}(x_\gamma - \sqrt{-1}h_\gamma)$. But $x_\gamma, \sqrt{-1}h_\gamma \in \mathfrak g_0$, so now $\mathrm{Ad}(c_\Gamma)(e_{-\gamma}) \in \mathfrak g_0 \cap \mathfrak q'$. Evidently $\mathrm{Ad}(c_\Gamma)(e_\gamma) \notin \mathfrak g_0 \cap \mathfrak q'$. Conclusion: $\mathfrak g_0 \cap \mathfrak q'$ is not reductive. As the G_0–orbits on Z are measurable, now $G_0(c_\Gamma(z))$ cannot be open in Z ([11], Theorem 6.3). □

We'll also need a topological lemma:

3.7. Lemma. *Let X_1 and X_2 be topological spaces, let $B_i \subset X_i$ be open subsets, and let $M \subset (X_1 \times X_2)$ be a connected open subset such that* (i) *M meets $B_1 \times B_2$ and* (ii) $M \cap (\mathrm{bd}\,(B_1) \times B_2) = \emptyset = M \cap (B_1 \times \mathrm{bd}\,(B_2))$. *Then $M \subset (B_1 \times B_2)$.*

Proof. $(X_1 \times X_2) \setminus M$ is closed in $(X_1 \times X_2)$ because M is open, contains $(\mathrm{bd}\,(B_1) \times B_2) \cup (B_1 \times \mathrm{bd}\,(B_2))$ by (ii), and thus contains the closure of $(\mathrm{bd}\,(B_1) \times B_2) \cup (B_1 \times \mathrm{bd}\,(B_2))$. That closure contains the boundary of the open set $B_1 \times B_2$. Thus

$$M = \Big(M \cap (B_1 \times B_2)\Big) \cup \Big(M \cap \big((X_1 \times X_2) \setminus \mathrm{closure}\,(B_1 \times B_2)\big)\Big).$$

As M is connected and meets $B_1 \times B_2$, now $M \subset (B_1 \times B_2)$. □

Now we come to the main result of this Section:

3.8. Theorem. *Let G_0 be of hermitian type, let $Z = G/Q$ be a complex flag manifold, and let $D = G_0(z) \subset Z = G/Q$ be an open G_0–orbit that is not of holomorphic type. View $B \times \overline{B} \subset M_Z$ as in* Proposition 2.3 *and $M_D \subset M_Z$ as usual. Then $M_D \subset B \times \overline{B}$.*

Proof. Retain the notation of Sect. 2. Suppose that $(g_1 x_-, g_2 x_+)$ belongs to the boundary of $B \times \overline{B}$ in $X_- \times X_+$. The closure of $G_0 K S_-$ in G is contained in $S_+ K S_-$, and similarly the closure of $G_0 K S_+$ in G is contained in $S_- K S_+$. So the boundary of $B \times \overline{B}$ in $X_- \times X_+$ is contained in G/K. That allows us to write $g_2^{-1} g_1 = \exp(\xi_+) k \exp(\xi_-)$ with $\xi_\pm \in \mathfrak{s}_\pm$ and $k \in K$, as before. We will prove that $g_2 \exp(\xi_+) Y \not\subset D$, that is, $g_2 \exp(\xi_+) Y \notin M_D$. The Theorem will follow. The proof breaks into three cases, according to the way $(g_1 x_-, g_2 x_+)$ sits in the boundary of $B \times \overline{B}$.

<u>Case 1</u>. Here $g_1 x_- \in \mathrm{bd}\,(B)$ and $g_2 x_+ \in \overline{B}$ with $g_1, g_2 \in G$. We may suppose $g_2 \in G_0$. Then $g_2^{-1} g_1 x_-$ also belongs to the boundary of B in X_-, so $g_2^{-1} g_1 x_- \in k_0 G_0[\Psi^{\mathfrak{g}} \setminus \Gamma](c_\Gamma(x_-))$ for some $k_0 \in K_0$ and $\Gamma \subset \Psi^{\mathfrak{g}}$ by (3.5). Thus $g_2^{-1} g_1(x_+) = k_0 g_0 c_\Gamma(x_-)$, $g_0 \in G_0[\Psi^{\mathfrak{g}} \setminus \Gamma]$. Using [15, (3.4)], [15, (3.5)], and strong orthogonality of $\Psi^{\mathfrak{g}}$, decompose

$$g_0 = \prod_{\Psi^{\mathfrak{g}} \setminus \Gamma} \Big(\exp(\xi_{+,\psi}) k_\psi \exp(\xi_{-,\psi})\Big) \text{ and}$$
$$c_\Gamma = \prod_{\Gamma} \Big(\exp(\sqrt{-1} e_\gamma) \exp(\sqrt{2} h_\gamma) \exp(\sqrt{-1} e_{-\gamma})\Big)$$

with $\xi_{\pm,\psi} \in \mathfrak{g}_{\pm\psi}$. Set $\xi_{\pm,\gamma} = \sqrt{-1} e_{\pm\gamma}$ for $\gamma \in \Gamma$. Now

$$(g_1 x_-, g_2 x_+) = \delta g_2\, \delta \exp(\mathrm{Ad}\,(k_0) \xi'_+)(x_-, x_+) \text{ where } \xi'_+ = \sum\nolimits_{\psi \in \Psi^{\mathfrak{g}}} \xi_{+,\psi}\,.$$

At the cost of changing k_0 within K_0, and in view of (3.3), we may assume $\Gamma \cap \Delta(\mathfrak{r}_+, \mathfrak{h}) \neq \emptyset$. Then $c_\Gamma(z) = c_{\Gamma\cap\Delta(\mathfrak{r}_+,\mathfrak{h})}(z)$ is not contained in any open G_0–orbit on Z, by Lemma 3.6. In particular $c_\Gamma(z) \notin D$. Now $\exp(\xi_+)(k_0 z) = \exp(\mathrm{Ad}(k_0)(\xi'_+))(k_0 z) = k_0 \exp(\xi'_+)(z) = k_0 g_0 c_\Gamma(z) \notin D$, thus $g_2 \exp(\xi_+) Y \not\subset D$.

Case 2. Here $g_1 x_- \in B$ and $g_2 x_+ \in \mathrm{bd}\,(\overline{B})$. The argument is exactly as in Case 1, but with the rôles of B and $\overline{B}$ reversed. Here note that this reversal of rôles replaces $\Psi^{\mathfrak{g}}$ by $-\Psi^{\mathfrak{g}}$ and c_Γ by $c_{-\Gamma}$.

Case 3. Here $g_1 x_- \in \mathrm{bd}\,(B)$ and $g_2 x_+ \in \mathrm{bd}\,(\overline{B})$. Then M_D is connected, M_D meets $B \times \overline{B}$ because $Y \in M_D \cap (B \times \overline{B})$, and $M_D \cap \left(\mathrm{bd}\,(B) \times \overline{B}\,\right) = \emptyset = M_D \cap \left(B \times \mathrm{bd}\,(\overline{B})\right)$ by Cases 1 and 2. Case 3 now follows from Lemma 3.7. □

The same type of argument applies to prove the following.

Proposition 3.9. *Suppose D is of holomorphic type. Then M_D is biholomorphic to either B or $\overline{B}$*

Proof. We may assume that $M_Z = X_- = G/KS_-$ by switching $\mathfrak{s}_\pm$ if necessary. It is clear that $gY \subset D$ for $g \in G_0$, so $B \subset M_D$. Now suppose that gx_- (for some $g \in G$) is in the boundary of $B \subset X_-$. Then $gx_- = g_0 c_\Gamma(x_-)$ for some $g_0 \in G_0$ and some $\Gamma \neq \emptyset$. Conjugating by an element of K_0 we may assume $\Gamma \cap \Delta(\mathfrak{r}_+, \mathfrak{h}) \neq \emptyset$. Now, for $\Gamma' = \Gamma \cap \Delta(\mathfrak{r}_+, \mathfrak{h})$, gY contains $g_0 c_\Gamma(z) = g_0 c_{\Gamma'}(z)$. By Lemma 3.6 that is not in an open orbit. □

4. A reduction for the inclusion $B \times \overline{B} \subset M_D$

We reduce to the case where Q is a Borel subgroup of G:

4.1. Proposition. *Suppose that, if Q is a Borel subgroup of G, then $B \times \overline{B} \subset M_D$ whenever D is an open G_0–orbit on G/Q that is not of holomorphic type. Then the same is true when Q is any parabolic subgroup of G.*

Proof. The maximal compact subvariety in the open orbit $D = G(z) \subset Z$ is $Y = K(z) = K_0(z)$. We may, and do, take Q to be the G–stabilizer of z; in other words we assume that $\mathfrak{q} = \mathfrak{q}_z$. Let $Q' \subset Q$ be any parabolic subgroup of G contained in Q such that $G_0 \cap Q'$ contains a compact Cartan subgroup $H_0 \subset K_0$ of G_0, let $Z' = G/Q'$ be the corresponding flag manifold, and let $\pi : Z' \to Z$ denote the associated G–equivariant projection $gQ' \mapsto gQ$. Write $z' \in Z'$ for the base point $1Q'$. Then $D' = G_0(z')$ is open in Z' because $\mathfrak{g}_0 \cap \mathfrak{q}'$ contains a compact Cartan subalgebra of $\mathfrak{g}_0$. We have set things up so that $Y' = K(z') = K_0(z')$ is a maximal compact subvariety of D'.

Since D is not of holomorphic type, both intersections $\mathfrak{r}_- \cap \mathfrak{s}_\pm$ are nonzero. But $\mathfrak{r}_-$ is contained in the nilradical $\mathfrak{r}'_-$ of $\mathfrak{q}'$. Now both intersections $\mathfrak{r}'_- \cap \mathfrak{s}_\pm$ are nonzero, so D' is not of holomorphic type.

If $g \in G$ with $gY' \subset D'$ then $gK_0 \subset G_0Q'$, so $gK_0 \subset G_0Q$ and thus $gY \subset D$. In other words, π maps $M_{D'}$ to M_D. This map is an injection equivariant for the correspondence of Proposition 2.3. If the inclusion holds for Z' then $B \times \overline{B} \subset M_{D'} \subset M_D$, so it holds for Z. The assertion of the Proposition is the case where Q' is a Borel subgroup. □

5. $B \times \overline{B} \subset M_D$ when G is classical

In this section we prove a partial counterpart of Theorem 3.8:

5.1. Theorem. *Suppose that G is a classical group and that its real form G_0 is of hermitian type. Let $Z = G/Q$ be a complex flag manifold, and let $D = G_0(z) \subset Z = G/Q$ be an open G_0–orbit that is not of holomorphic type. View $B \times \overline{B} \subset M_Z$ as in* Proposition 2.3 *and $M_D \subset M_Z$ as usual. Then $B \times \overline{B} \subset M_D$.*

We run through the classical cases. By Proposition 4.1 we may assume that Q is a Borel subgroup so that Z is the full flag. In each case, the standard basis of $\mathbb{C}^m$ will be denoted $\{e_1, \dots, e_m\}$. Without further comment we will decompose vectors as $v = \sum v_j e_j$. We will have symmetric bilinear forms $(\cdot,\cdot)$ or antisymmetric bilinear forms $\omega(\cdot,\cdot)$ on $\mathbb{C}^m$ and the term *isotropic* will refer only to those bilinear forms. We will also have hermitian forms $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^m$, and the term *signature* will refer only to those hermitian forms. In each case the flag manifold Z is described as a flag of subspaces $z = (z_1 \subset \cdots \subset z_m)$ in some $\mathbb{C}^m$ with $\dim z_j = j$, usually with $m = 2n$ or $m = n$. As we run through the cases, B and $\overline{B}$ are described in terms of such flags, as in [12]. Then we give explicit descriptions of (i) the embeddings of Sect. 2, (ii) the full flag and its open G_0–orbits, and (iii) we describe the G-action on M_Z, in such a way that the result of Theorem 5.1 is easily visible.

If $\{f_1, \dots, f_\ell\}$ is a linearly independent subset in a vector space V then $[f_1 \wedge \cdots \wedge f_\ell]$ denotes its span.

Type I: $B = \{Z \in \mathbb{C}^{p\times q} \mid I - Z^*Z >> 0\}$. Here $G = SL(n;\mathbb{C})$ and $G_0 = SU(p,q)$, indefinite unitary group defined by the hermitian form $\langle u, v\rangle = \sum_{j=1}^{p} v_j\overline{w_j} - \sum_{j=1}^{q} v_{p+j}\overline{w_{p+j}}$ with $p+q=n$.

The hermitian symmetric flag $X_- = G/KS_-$ is identified with the Grassmannian of q–planes in $\mathbb{C}^n$, the base point $x_- = [e_{p+1} \wedge \cdots \wedge e_{p+q}]$, and $B = G_0(x_-)$ consists of the negative definite q–planes. Similarly, $X_+ = G/KS_+$ is identified with the Grassmannian of p–planes in $\mathbb{C}^n$, $x_+ = [e_1 \wedge \cdots \wedge e_p]$, and $\overline{B} = G_0(x_+)$ consists of the positive definite p–planes. The embedding

$$B \times \overline{B} \subset G/K = G(x_-, x_+) \subset X_- \times X_+$$

of Sect. 2 is given by

$$(5.2)\quad \begin{aligned} B \times \overline{B} &= \{(V, W) \subset (X_- \times X_+) \mid V \text{ negative definite and} \\ &\qquad W \text{ positive definite}\} \\ &\text{and} \\ G/K &= G(x_-, x_+) \\ &= \{(V, W) \in (X_- \times X_+) \mid V \text{ and } W \text{ transverse in } \mathbb{C}^n\}. \end{aligned}$$

The full flag manifold is $Z = \{z = (z_1 \subset \cdots \subset z_{n-1}) \mid \dim z_j = j\}$. By Witt's Theorem, if two subspaces $U, U' \subset \mathbb{C}^n$ have the same signature and nullity (relative to the hermitian form $\langle\cdot,\cdot\rangle$) then there exists $g \in U(p, q)$ with $gU = U'$, and of course we can scale and choose $g \in G_0 = SU(p, q)$. It follows that the G_0–orbits on the full flag $Z = G/Q$ are determined by the rank and signature sequences of the subspaces in the flag. Let $r = (r_1, \ldots, r_{n-1})$ and $s = (s_1, \ldots, s_{n-1})$ consist of integers such that $0 \leqq r_1 \leqq \cdots \leqq r_{n-1} \leqq p$, $0 \leqq s_1 \leqq \cdots \leqq s_{n-1} \leqq q$, and $r_j + s_j = j$ for all j. Then r and s define a point $z_{r,s} \in Z$ and an open G_0–orbit $D_{r,s} \subset Z$ by

$$(5.3)\quad \begin{aligned} z_{r,s} &= (z_{r,s,1}, \ldots, z_{r,s,n-1}) \text{ where} \\ z_{r,s,j} &= [e_1 \wedge \cdots \wedge e_{r_j} \wedge e_{p+1} \wedge \cdots \wedge e_{s_j}] \text{ and} \\ D_{r,s} &= G_0(z_{r,s}) = \{z = (z_1, \ldots, z_{n-1}) \mid z_j \\ &\qquad \text{has signature } (r_j, s_j) \text{ for all } j\}. \end{aligned}$$

Each pair r, s is obtained by choosing p of the numbers from 1 to $p + q$, the indices at which $r_j > r_{j-1}$, so the number of pairs r, s is $\binom{n}{p} = \frac{n!}{p!q!}$, which is the quotient $|W_G|/|W_K|$ of the orders of the Weyl groups. As these $D_{r,s}$ are distinct open orbits, it follows from [11, Corollary 4.7] that they are exactly the open G_0–orbits on Z.

Fix r and s. Let $(V, W) \in G/K \subset (X_- \times X_+)$. Define

$$(5.4)\quad Y_{V,W} = \{z \in Z \mid \dim z_j \cap V = s_j \text{ and } \dim z_j \cap W = r_j \text{ for all } j\}.$$

We set $D = D_{r,s}$ so $Y = K(z_{r,s}) = Y_{x_-,x_+}$. If $g \in G$ then $gY = Y_{gx_-,gx_+}$. If $(V, W) \in B \times \overline{B}$ then $Y_{V,W} \subset D_{r,s}$, so $Y_{V,W} \in M_{D_{r,s}}$. Thus $(V, W) \mapsto Y_{V,W}$ defines a map $\eta : B \times \overline{B} \to M_{D_{r,s}}$. If $r_1 = \cdots = r_q = 0$ then $r_{q+j} = j$ for $1 \leqq j \leqq p$ and $\eta(V, W)$ depends only on V; if $s_1 = \cdots = s_p = 0$ then $s_{p+j} = j$ for $1 \leqq j \leqq q$ and $\eta(V, W)$ depends only on W; those are the cases where $D_{r,s}$ is of holomorphic type. In the nonholomorphic cases, η injects $B \times \overline{B}$ into $M_{D_{r,s}}$ and we have $B \times \overline{B} \subset M_{D_{r,s}}$. Theorem 5.1 is verified when B is of type I.

Type II: $B = \{Z \in \mathbb{C}^{n\times n} \mid Z = {}^tZ \text{ and } I - Z\cdot Z^* >> 0\}$. Here $G = Sp(n;\mathbb{C})$ and $G_0 = Sp(n;\mathbb{R})$. These are the complex and real symplectic groups, defined by the antisymmetric bilinear form $\omega(v,w) = \sum_{j=1}^n (v_j w_{n+j} - v_{n+j}w_j)$ on $\mathbb{C}^{2n}$ and $\mathbb{R}^{2n}$, respectively. Here it is more convenient to realize G_0 as $G \cap U(n,n)$ where $U(n,n)$ is the unitary group of the hermitian form $\langle v, w\rangle = \sum_{j=1}^n v_j\overline{w_j} - \sum_{j=1}^n v_{n+j}\overline{w_{n+j}}$, and we do that.

The hermitian symmetric flag $X_- = G/KS_-$ is identified with the Grassmannian of ω–isotropic n–planes in $\mathbb{C}^{2n}$, the base point $x_- = [e_{n+1}\wedge\cdots\wedge e_{2n}]$, and $B = G_0(x_-)$ consists of the negative definite ω–isotropic n–planes. Similarly, $X_+ = G/KS_+$ is identified with the Grassmannian of ω–isotropic n–planes in $\mathbb{C}^{2n}$, $x_+ = [e_1\wedge\cdots\wedge e_n]$, and $\overline{B} = G_0(x_+)$ consists of the positive definite ω–isotropic n–planes. The embedding

$$B \times \overline{B} \subset G/K = G(x_-, x_+) \subset X_- \times X_+$$

of Sect. 2 is given by

$$\begin{aligned} B\times\overline{B} = \{(V,W) \subset (X_-\times X_+) \mid\ & V \text{ negative definite and} \\ & W \text{ positive definite}\} \\ \text{and}\quad & \\ G/K = G(x_-,x_+)\ & \\ = \{(V,W) \in (X_-\times X_+) \mid\ & V \text{ and } W \text{ transverse in } \mathbb{C}^{2n}\}. \end{aligned} \tag{5.5}$$

The full flag is $Z = \{z = (z_1 \subset \cdots \subset z_{n-1}) \mid \text{each } z_j \text{ is isotropic with } \dim z_j = j\}$. One extends Witt's Theorem from $(\mathbb{C}^{2n}, \langle\cdot,\cdot\rangle)$ to prove

5.6. Lemma. *Let $U_1, U_2 \subset \mathbb{C}^{2n}$ be ω–isotropic subspaces of the same nondegenerate signature for $\langle\cdot,\cdot\rangle$. Then there exists $g \in G_0$ with $gU_1 = U_2$.*

Somewhat as in the Type I case it will follow that the *open* G_0–orbits on the full flag $Z = G/Q$ are determined by the signature sequences of the subspaces in the flag. Let $r = (r_1,\ldots,r_n)$ and $s = (s_1,\ldots,s_n)$ consist of integers such that $0 \leqq r_1 \leqq \cdots \leqq r_n \leqq n$, $0 \leqq s_1 \leqq \cdots \leqq s_n \leqq n$, and $r_j + s_j = j$ for all j. Then r and s define a point $z_{r,s} \in Z$ and a G_0–orbit $D_{r,s} \subset Z$ by

$$\begin{aligned} z_{r,s} &= (z_{r,s,1} \subset \cdots \subset z_{r,s,n}) \text{ where} \\ z_{r,s,j} &= [e_1\wedge\cdots\wedge e_{r_j}\wedge e_{2n-s_j+1}\wedge\cdots\wedge e_{2n}] \text{ and} \\ D_{r,s} &= G_0(z_{r,s}) \\ &= \{z = (z_1\subset\cdots\subset z_n) \mid z_j \text{ has signature } (r_j,s_j) \text{ for all } j\}. \end{aligned} \tag{5.7}$$

The last equality uses Lemma 5.6.

5.8. Proposition. *The $D_{r,s}$ are exactly the open G_0–orbits on Z, and they are distinct.*

Proof. The G_0–stabilizer of $z_{r,s}$ is the maximal torus consisting of diagonal unitary matrices, so $D_{r,s}$ is open in Z by dimension. If $D_{r,s} = D_{r',s'}$ then (5.7) forces $r = r'$ and $s = s'$. Now the open orbits $D_{r,s}$ are distinct. Each pair r, s is obtained by choosing a set of numbers from 1 to n, the indices at which $r_j > r_{j-1}$, so the number of pairs r, s is 2^n, which is the quotient $|W_G|/|W_K|$ of the orders of the Weyl groups. As these $D_{r,s}$ are distinct open orbits, it follows from [11, Corollary 4.7] that they are exactly the open G_0–orbits on Z. □

Fix r and s. Let $(V, W) \in G/K \subset (X_- \times X_+)$. Define

(5.9) $$Y_{V,W} = \{z \in Z \mid \dim z_j \cap V = s_j \text{ and } \dim z_j \cap W = r_j \text{ for all } j\}.$$

We set $D = D_{r,s}$ so $Y = K(z_{r,s}) = Y_{x_-,x_+}$. If $g \in G$ then $gY = Y_{gx_-,gx_+}$. If $(V, W) \in B \times \overline{B}$ then $Y_{V,W} \subset D_{r,s}$, so $Y_{V,W} \in M_{D_{r,s}}$. Thus $(V, W) \mapsto Y_{V,W}$ defines a map $\eta : B \times \overline{B} \to M_{D_{r,s}}$. If $r_1 = \cdots = r_n = 0$ then $\eta(V, W)$ depends only on V; if $s_1 = \cdots = s_n = 0$ then $\eta(V, W)$ depends only on W; those are the cases where $D_{r,s}$ is of holomorphic type. In the nonholomorphic cases, η injects $B \times \overline{B}$ into $M_{D_{r,s}}$ and we have $B \times \overline{B} \subset M_{D_{r,s}}$. Theorem 5.1 is verified when B is of type II.

Type III: $B = \{Z \in \mathbb{C}^{n\times n} \mid Z = -{}^tZ \text{ and } I - Z \cdot Z^* >> 0\}$. Here $G = SO(2n; \mathbb{C})$, special orthogonal group defined by the symmetric bilinear form $(v, w) = \sum_{j=1}^{n}(v_j w_{n+j} + v_{n+j} w_j)$ on $\mathbb{C}^{2n}$, and $G_0 = SO^*(2n)$, the real form with maximal compact subgroup $U(n)$. We realize G_0 as $G \cap U(n, n)$ where $U(n, n)$ is the unitary group of the hermitian form $\langle v, w\rangle = \sum_{j=1}^{n} v_j \overline{w_j} - \sum_{j=1}^{n} v_{n+j} \overline{w_{n+j}}$.

The hermitian symmetric flags $X_\pm = G/KS_\pm$ are identified with the two choices of connected component in the Grassmannian of isotropic (relative to $(\cdot,\cdot)$) n–planes in $\mathbb{C}^{2n}$. The components in question are specified by orientation. X_- has base point $x_- = [e_{n+1} \wedge \cdots \wedge e_{2n}]$, $X_- = G(x_-)$, and $B = G_0(x_-)$ consists of the negative definite isotropic n–planes in X_-. Similarly, X_+ has base point $x_+ = [e_1 \wedge \cdots \wedge e_n]$, and $X_+ = G(x_+)$, and $\overline{B} = G_0(x_+)$ consists of the positive definite isotropic n–planes in X_+. The embedding

$$B \times \overline{B} \subset G/K = G(x_-, x_+) \subset X_- \times X_+$$

of Sect. 2 is given by

(5.10) $$\begin{aligned} B \times \overline{B} &= \{(V, W) \subset (X_- \times X_+) \mid V \text{ negative definite and } \\ &\qquad W \text{ positive definite}\} \\ &\text{and} \\ G/K &= G(x_-, x_+) \\ &= \{(V, W) \in (X_- \times X_+) \mid V \text{ and } W \text{ transverse in } \mathbb{C}^{2n}\}. \end{aligned}$$

$Z = \{z = (z_1 \subset \cdots \subset z_n) \mid \text{each } z_j \text{ is isotropic with } z_n \in X_- \text{ and } \dim z_j = j\}$ is the full flag. Here of course the z_j are linear subspaces of $\mathbb{C}^{2n}$. One could require $z_n \in X_+$ instead, with the same results, but it is necessary to make a choice. Witt's Theorem extends from $(\mathbb{C}^{2n}, \langle\cdot,\cdot\rangle)$ as follows.

5.11. Lemma. *Let $U_1, U_2 \subset \mathbb{C}^{2n}$ be $(\cdot,\cdot)$–isotropic subspaces of the same nondegenerate signature for $\langle\cdot,\cdot\rangle$. If $\dim U_i = n$ then assume also that the U_i are contained in the same $X_\pm$. Then there exists $g \in G_0$ with $gU_1 = U_2$.*

As in the Type II case it will follow that the *open* G_0–orbits on the full flag $Z = G/Q$ are determined by the signature sequences of the subspaces in the flag. Let $r = (r_1, \dots, r_{n-1})$ and $s = (s_1, \dots, s_{n-1})$ consist of integers such that $0 \leqq r_1 \leqq \cdots \leqq r_{n-1} \leqq n-1$, $0 \leqq s_1 \leqq \cdots \leqq s_{n-1} \leqq n-1$, and $r_j + s_j = j$ for all j. Then r and s specify integers r_n and s_n such that (i) $r_{n-1} \leqq r_n \leqq n$, (ii) $s_{n-1} \leqq s_n \leqq n$, (iii) $r_n + s_n = n$, and (iv) $[e_1 \wedge \cdots \wedge e_{r_n} \wedge e_{2n-s_n+1} \wedge \cdots \wedge e_{2n}] \in X_-$. In effect, (iv) is a parity condition on r_n. Now r and s define a point $z_{r,s} \in Z$ and a G_0–orbit $D_{r,s} \subset Z$ by

$$\begin{aligned} z_{r,s} &= (z_{r,s,1} \subset \cdots \subset z_{r,s,n}) \text{ where} \\ z_{r,s,j} &= [e_1 \wedge \cdots \wedge e_{r_j} \wedge e_{2n-s_j+1} \wedge \cdots \wedge e_{2n}] (j < n), \\ z_{r,s,n} &\in X_-, \text{ and} \\ D_{r,s} &= G_0(z_{r,s}) \\ &= \{z = (z_1 \subset \cdots \subset z_n) \mid z_j \text{ has signature } (r_j, s_j) \text{ for all } j\}. \end{aligned} \tag{5.12}$$

The last equality uses Lemma 5.11.

5.13. Proposition. *The $D_{r,s}$ are exactly the open G_0–orbits on Z, and they are distinct.*

Proof. The G_0–stabilizer of $z_{r,s}$ is the maximal torus consisting of diagonal unitary matrices, so $D_{r,s}$ is open in Z by dimension. If $D_{r,s} = D_{r',s'}$ then (5.12) forces $r = r'$ and $s = s'$. Now the open orbits $D_{r,s}$ are distinct. Each pair r, s is obtained by choosing a set of numbers from 1 to $n-1$, the indices at which $r_j > r_{j-1}$, so the number of pairs r, s is 2^{n-1}, which is the quotient $|W_G|/|W_K|$ of the orders of the Weyl groups. As these $D_{r,s}$ are distinct open orbits, it follows from [11, Corollary 4.7] that they are exactly the open G_0–orbits on Z. □

Fix r and s. Let $(V, W) \in G/K \subset (X_- \times X_+)$. Define

$$Y_{V,W} = \{z \in Z \mid \dim z_j \cap V = s_j \text{ and } \dim z_j \cap W = r_j \text{ for all } j\}. \tag{5.14}$$

We set $D = D_{r,s}$ so $Y = K(z_{r,s}) = Y_{x_-,x_+}$. If $g \in G$ then $gY = Y_{gx_-,gx_+}$. If $(V, W) \in B \times \overline{B}$ then $Y_{V,W} \subset D_{r,s}$, so $Y_{V,W} \in M_{D_{r,s}}$. Thus $(V, W) \mapsto Y_{V,W}$ defines a map $\eta : B \times \overline{B} \to M_{D_{r,s}}$. If $r_1 = \cdots = r_n = 0$ then $\eta(V, W)$ depends only on V; if $s_1 = \cdots = s_n = 0$ then $\eta(V, W)$ depends only on W; those are the cases where $D_{r,s}$ is of holomorphic type. In the nonholomorphic cases, η injects $B \times \overline{B}$ into $M_{D_{r,s}}$ and we have $B \times \overline{B} \subset M_{D_{r,s}}$. Theorem 5.1 is verified when B is of type III.

Type IV: $B = \{Z \in \mathbb{C}^n \mid 1 + |{}^t Z \cdot Z|^2 - 2Z^* \cdot Z > 0$ *and* $I - Z^* \cdot Z > 0\}$. Here $G = SO(2+n;\mathbb{C})$, special orthogonal group defined by the symmetric bilinear form $(v, w) = \sum_{j=1}^{2} v_j w_j - \sum_{j=3}^{2+n} v_j w_j$ on $\mathbb{C}^{2+n}$, and G_0 is the identity component of $SO(2, n)$. We view G_0 as the identity component of $G \cap U(2, n)$ where $U(2, n)$ is defined by the hermitian form $\langle v, w\rangle = (v, \overline{w})$.

The hermitian symmetric flags $X_\pm = G/KS_\pm$ are each identified with the space of $(\cdot,\cdot)$ isotropic lines in $\mathbb{C}^{2+n}$. $X_\pm$ has base point $x_\pm = [e_1 \pm ie_2]$. $B = G_0(x_-)$ and $\overline{B} = G_0(x_+)$, and each consists of the $\langle\cdot,\cdot\rangle$ positive definite $(\cdot,\cdot)$ isotropic lines. The embedding

$$B \times \overline{B} \subset G/K = G(x_-, x_+) \subset X_- \times X_+$$

of Sect. 2 is given by

$$\begin{aligned} &B \times \overline{B} = \{(V, W) \in (X_- \times X_+) \mid V \text{ and } W \text{ positive definite}\}\\ &\text{and } G/K = G(x_-, x_+) = \{(V, W) \in (X_- \times X_+) \mid V \not\perp W\}. \end{aligned} \tag{5.15}$$

Here "positive definite" refers to the hermitian form $\langle\cdot,\cdot\rangle$ and "$\perp$" refers to the symmetric bilinear form $(\cdot,\cdot)$.

The full flag manifold Z is a connected component of $\widetilde{Z} = \{z = (z_1 \subset \cdots \subset z_m) \mid z_j$ isotropic subspace of $\mathbb{C}^{2+n}$ and $\dim z_j = j\}$. Here $m = [\frac{n}{2}] + 1$. If n is odd then $Z = \widetilde{Z}$, in other words $\widetilde{Z}$ is connected. If n is even then $\widetilde{Z}$ has two topological components. In any case

$$Z_+ = G([(e_1 + ie_2) \wedge (e_3 + ie_4) \wedge \cdots \wedge (e_{2m-1} + ie_{2m})])$$

is a connected component in the variety of all maximal isotropic subspaces of $\mathbb{C}^{2+n}$, and

$$\begin{aligned} Z = \{z = (z_1 \subset \cdots \subset z_m) \mid z_j \text{ isotropic in } \mathbb{C}^{2+n}, \dim z_j = j,\\ \text{and } z_m \in Z_+\}. \end{aligned} \tag{5.16}$$

Witt's Theorem extends from $(\mathbb{C}^{2+n}, \langle\cdot,\cdot\rangle)$ as follows.

5.17. Lemma. *Let* $U_1, U_2 \subset \mathbb{C}^{2+n}$ *be* $(\cdot,\cdot)$*–isotropic subspaces of the same nondegenerate signature for* $\langle\cdot,\cdot\rangle$. *Then there exists* $g \in O(2+n;\mathbb{C}) \cap U(2, n)$ *with* $gU_1 = U_2$.

As in the earlier cases it will follow that the *open* G_0–orbits on the full flag $Z = G/Q$ are determined by the signature sequences of the subspaces in the flag. Let $1 \leqq k \leqq m$, and define points $z_k^\pm \in Z$ and G_0–orbits $D_k^\pm \subset Z$, by

$$
\begin{aligned}
z_k^\pm &= (z_{k,1}^\pm \subset \cdots \subset z_{k,m}^\pm) \text{ where}\\
&\qquad z_{k,j}^\pm = [(e_3 + ie_4) \wedge \cdots \wedge (e_{2j+1} + ie_{2j+2}]\\
&\qquad\qquad \text{for } j < k,\\
&\qquad z_{k,j}^\pm = [(e_1 \pm ie_2) \wedge (e_3 + ie_4) \wedge \cdots \wedge (e_{2j-1} + ie_{2j}]\\
&\qquad\qquad \text{for } j \geqq k\\
\text{and } D_k^\pm &= G_0(z_k^\pm)\\
&= \{z \in Z \mid z_j \text{ has signature } (0, j)\\
&\qquad \text{for } j < k, (1, j-1) \text{ for } j \geqq k,\\
&\qquad\qquad \text{and } z_j \text{ meets } G_0(x_\pm) \text{ for } j \geqq k\}.
\end{aligned}
\tag{5.18}
$$

The last equality uses Lemma 5.17.

5.19. Proposition. *The $D_k^\pm$ are exactly the open G_0–orbits on Z, and they are distinct. The $D_k^+ \cup D_k^-$ are the open $(O(2+n;\mathbb{C}) \cap U(2,n))$–orbits on Z.*

Proof. The G_0–stabilizer of $z_k^\pm$ is the maximal torus consisting of independent rotations of the planes $[e_1 \wedge e_2]$ through $[e_{2m-1} \wedge e_{2m}]$, so D_k is open in Z by dimension. If $D_k^\epsilon = D_{k'}^{\epsilon'}$ $(\epsilon, \epsilon' = \pm)$ then (5.18) shows that $(k, \epsilon) = (k', \epsilon')$. Now the open G_0–orbits $D_k^\pm$ are distinct, and the $D_k^+ \cup D_k^-$ are open $(O(2+n;\mathbb{C}) \cap U(2,n))$–orbits on Z.

There are $2m$ pairs k, ϵ. Whether n is even or odd, the quotient $|W_G|/|W_K|$ of the orders of the Weyl groups is $2m$. As the $D_k^\pm$ are distinct open orbits, it follows from [11, Corollary 4.7] that they are exactly the open G_0–orbits on Z. □

Fix k and ϵ. Let $(V, W) \in G/K = (X_- \times X_+)$. So $V = [v]$ and $W = [w]$ where $v, w \in \mathbb{C}^{2+n}$ are isotropic vectors with $(v, w) \neq 0$. Define

$$
\begin{aligned}
Y_{V,W} =&\{z \in Z \mid \dim z_j \cap [v \wedge w] = 0 \text{ and } \dim z_j \cap [v \wedge w]^\perp = j\\
&\quad \text{for } j < k,\\
&\quad \dim z_j \cap [v \wedge w] = 1 \text{ and } \dim z_j \cap [v \wedge w]^\perp = j-1\\
&\quad \text{for } j \geqq k,\\
&\quad v \in z_j \text{ if } \epsilon = + \text{ and } j \geqq k;\quad w \in z_j \text{ if } \epsilon = - \text{ and } j \geqq k\}.
\end{aligned}
\tag{5.20}
$$

Here $\perp$ refers to the symmetric bilinear form. Also, note that the only isotropic vectors in $[v \wedge w]$ are the multiples of v and the multiples of w.

We set $D = D_k^\pm$ so $Y = K(z_k^\pm) = Y_{x_-,x_+}$. If $g \in G$ then $gY = Y_{gx_-,gx_+}$.

5.21. Lemma. *If $(V, W) \in B \times \overline{B}$ then $Y_{V,W} \subset D_k^\pm$, so $Y_{V,W} \in M_{D_k^\pm}$.*

Proof. First consider D_k^+. Let $z' \in Y_{V,W}$. For $j \geqq k$ we have $v \in z'_j$. As $V \in B$ it is positive definite for $\langle\cdot,\cdot\rangle$, so we need only check that $z'_j \cap [v \wedge w]^\perp$ is negative definite for $\langle\cdot,\cdot\rangle$.

Let $u \in z_j' \cap [v \wedge w]^\perp$. Here $\perp$ refers to the symmetric bilinear form $(\cdot, \cdot)$. If $\langle u, u\rangle \geqq 0$ then $U = [u]$ is in the closure of B or in the closure of $\overline{B}$. In the first case the pair (U, W) sits in G/K by the remarks at the beginning of the proof of Theorem 3.8. Then $(u, w) \neq 0$, contradicting $u \in [v, w]^\perp$. Similarly, in the second case the pair $(V, U) \in G/K$, so $(v, u) \neq 0$, contradicting $u \in [v \wedge w]^\perp$. We have verified that $z_j' \cap [v \wedge w]^\perp$ is negative definite for $\langle \cdot, \cdot \rangle$. □

Now $(V, W) \mapsto Y_{V,W}$ defines a map $\eta : B \times \overline{B} \to M_{D_k^\pm}$. If $k = 1$ and $\epsilon = +$ then $\eta(V, W)$ depends only on V; if $k = 1$ and $\epsilon = -$ then $\eta(V, W)$ depends only on W; those are the cases where $D_k^\pm$ is of holomorphic type. In the nonholomorphic cases, η injects $B \times \overline{B}$ into $M_{D_k^\pm}$ and we have $B \times \overline{B} \subset M_{D_k^\pm}$. Theorem 5.1 is verified when B is of type IV, and that completes its proof. □

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