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CAYLEY TRANSFORMS AND ORBIT STRUCTURE IN COMPLEX FLAG MANIFOLDS

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Abstract. Let $Z = G/Q$ be a complex flag manifold and G_0 a real form of G. Suppose that G_0 is the analytic automorphism group of an irreducible bounded symmetric domain and that some open G_0-orbit on Z is a semisimple symmetric space. Then the G_0-orbit structure of Z is described explicitly by the partial Cayley transforms of a certain hermitian symmetric sub–flag $F \subset Z$. This extends the results and simplifies the proof for the classical orbit structure description of [10] and [11], which applies when $F = Z$.

Introduction

Let $Z = G/Q$ be a complex flag manifold where G is a complex simple Lie group and Q is a parabolic subgroup. Let $G_0 \subset G$ be a real form of hermitian type. In other words G_0 corresponds to a bounded symmetric domain $B = G_0/K_0$. Suppose that at least one open G_0-orbit on Z is a semisimple symmetric space.

The main result, Theorem 3.8, gives a detailed description of the G_0-orbits in Z in terms of partial Cayley transforms. The orbits are given by $D_{\Gamma,\Sigma} = G_0(c_\Gamma c_\Sigma^2 z)$ where Γ and Σ are disjoint subsets of a certain set of strongly orthogonal roots, c_Γ and c_Σ are the corresponding Cayley transforms, and z is a certain base point in Z. We give precise conditions for two orbits $D_{\Gamma,\Sigma}$ to be equal, for one to be contained in the closure of another, for one to be open, etc..

The case where Z is the compact hermitian symmetric space dual to G_0/K_0 was worked out directly by Wolf in [10] and [11], worked out from earlier results of Korányi and Wolf [13] by Takeuchi [9] and again, later, by Lassalle [3]. The proof in our more general setting is independent of [10],

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[11] and [13]. It is self contained except for a few general results from [10] on open orbits in a flag manifold and a decomposition in [14]. Some points of the proof simplify the corresponding considerations in [10] and [11], as noted in Remarks 3.10 and 3.11.

See Wolf [10] for the more general setting of G_0-orbits on general complex flag manifolds G/Q with no symmetry requirements. See Matsuki [5] for the more general setting of H-orbits on G/Q where H is any subgroup that is, up to finite index, the fixed point set of an involutive automorphism of G. See Makarevič [4] for analogous considerations on symmetric R–spaces from the Jordan algebra viewpoint, where (although not formulated this way in [4]) primitive idempotents replace strongly orthogonal positive complementary roots as in Korányi–Wolf [2] and Drucker [1]. And see Richardson and Springer [7], [8] for the the dual setting of K-orbits on G/B where B is a Borel subgroup of G. The results are much more complicated in these more general settings. The virtue of the present paper is that it picks out a few somewhat more general settings in which the results are extremely simple.

Notation and some basic definitions are established in Section 1. Then in Section 2 we work out the geometric basis for the G_0-orbit structure of Z. We first prove every open G_0-orbit on Z is symmetric, and at least one such orbit $D = G_0/L_0$ takes part in a holomorphic double fibration $D \leftarrow G_0/(L_0 \cap K_0) \rightarrow B$. Fix that open orbit $D \subset Z$. We may assume that the base point $z \in Z$ is chosen so that the isotropy subgroup $L_0 \subset G_0$ is the identity component of the fixed point set G_0^σ of an involutive automorphism σ that commutes with the Cartan involution θ. Then $\theta\sigma$ is an involutive automorphism. Let M and M_0 denote the identity components of its respective fixed point sets on G and G_0. We then prove that $M_0(z)$ is an irreducible bounded symmetric domain, that $F = M(z)$ is an hermitian symmetric space and a complex sub–flag of Z, and that $M_0(z) \subset F$ is the Borel embedding.

In Section 3 we recall the partial Cayley transform theory [2], [13] for $M_0(z) \subset F$, and use it to specify the orbits $D_{\Gamma,\Sigma} = G_0(c_\Gamma c_\Sigma^2 z) \subset Z$. Then we state the main result as Theorem 3.8. The proof is carried out in Section 4 for open orbits and in Section 5 for orbits in general. It follows that the G_0-orbit structure of Z is exactly the same as the M_0-orbit structure of F.

The appendix contains a distance formula that simplifies our arguments under certain circumstances and gives an interpretation of the results in Section 4 in terms of Riemannian distance in Z.

1. The double fibration

Fix a connected simply connected complex simple Lie group G and a parabolic subgroup Q. This defines a connected irreducible complex flag manifold $Z = G/Q$. Let $G_0 \subset G$ be a real form, let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the corresponding real form of the Lie algebra of G, and fix a Cartan involution θ of G_0 and $\mathfrak{g}_0$. We extend θ to a holomorphic automorphism of G and a complex linear automorphism of $\mathfrak{g}$, thus decomposing

(1.1) $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, decomposition into ± 1 eigenspaces of θ.

Then the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0, K_0 has Lie algebra $\mathfrak{k}_0$, and $K = G^\theta$ is the complexification of K_0. K_0 is connected and is the G_0–normalizer of $\mathfrak{k}_0$, and K is connected because G is connected and simply connected.

The subspace $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\ \mathfrak{s}_0 \subset \mathfrak{g}$ is a compact real form of $\mathfrak{g}$. The corresponding real analytic subgroup $G_u \subset G$ is a compact real form of G.

We can view Z as the space of G–conjugates of $\mathfrak{q}$. Then $gQ = z \in Z = G/Q$ corresponds to $Q_z = \mathrm{Ad}(g)Q = \{g \in G \mid g(z) = z\}$ as well as its Lie algebra $\mathfrak{q}_z = \mathrm{Ad}(g)\mathfrak{q}$.

From this point on we assume that G_0 is of hermitian symmetric type, that is,

(1.2) $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_-$ where K_0 acts irreducibly on each of $\mathfrak{s}_\pm$ and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$

where $\xi \mapsto \overline{\xi}$ denotes complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$. Set $S_\pm = \exp(\mathfrak{s}_\pm)$. So $S_- = \overline{S_+}$ where $g \mapsto \overline{g}$ also denotes complex conjugation of G over G_0. Then

$$\begin{aligned} &\mathfrak{p} = \mathfrak{k} + \mathfrak{s}_- \text{ is a parabolic subalgebra of } \mathfrak{g}\ , \\ (1.3)\qquad &P = KS_- \text{ is a parabolic subgroup of } G \text{ and} \\ &X = G/P \text{ is an hermitian symmetric flag manifold.} \end{aligned}$$

Note that $\mathfrak{s}_+$ represents the holomorphic tangent space of X and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ represents the antiholomorphic tangent space. As above we can view X as the space of G–conjugates of $\mathfrak{p}$. Then $G_0/(G_0 \cap P) = G_0/K_0 = \mathrm{Ad}(G_0)\mathfrak{p}$ is an open subset of X and thus inherits an invariant complex structure. Define

$$\begin{aligned} &B = G_0/K_0: \\ (1.4)\qquad &\text{symmetric space } G_0/K_0 \text{ with} \\ &\text{the complex structure from } X. \end{aligned}$$

The distinction between $\mathfrak{s}_-$ and $\mathfrak{s}_+$ in (1.2) is made by a choice of positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h} = \overline{\mathfrak{h}} \subset \mathfrak{k}$ of $\mathfrak{g}$. The choice is made so that $\mathfrak{s}_+$ is spanned by positive root spaces and consequently $\mathfrak{s}_-$ is spanned by negative root spaces.

Under our standing assumption (1.2), Theorems 2.12 and 4.5 of [10] say that $G_0(z)$ is an open G_0-orbit if and only if $\mathfrak{q}_z$ contains a compact Cartan subalgebra. Since all compact Cartan subalgebras are G_0–conjugate we have

(1.5) a G_0-orbit in Z is open iff it contains some z such that $\mathfrak{h} \subset \mathfrak{q}_z$.

Now fix an open orbit $D = G_0(z)$ with z chosen as in (1.5). Write $\mathfrak{q}$ for $\mathfrak{q}_z$ and decompose $\mathfrak{q}$ in its Levi decomposition; $\mathfrak{q} = \mathfrak{l} + \mathfrak{r}_-$ with $\mathfrak{h} \subset \mathfrak{l}$ and $\mathfrak{r}_-$ the nilradical of $\mathfrak{q}$. Since $\mathfrak{h}_0 \subset \mathfrak{k}_0$, complex conjugation acts on root spaces by $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}$. Decomposing $\mathfrak{q}$ into $\mathfrak{h}$–root spaces we see that $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{l}$. Thus $G_0(z) = G_0/L_0$ with $L_0 = G_0 \cap Q$ and L_0 is a connected real reductive group having Lie algebra $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. In the terminology of [10] we have shown that all open orbits for G_0 (satisfying (1.2)) are measurable. Similarly $L_u = G_u \cap Q$ is connected and is the compact real form of L, and it has Lie algebra $\mathfrak{l}_u = \mathfrak{g}_u \cap \mathfrak{l}$.

1.6. **Definition.** The open orbit $D = G_0(z) \subset Z$ is *symmetric* if it has the structure of a pseudoriemannian symmetric space for G_0, that is, if L_0 is open in the fixed point set G_0^σ of an involutive automorphism[1] σ of G_0.

1.7. *Remark.* Suppose D is a symmetric orbit in Z. As mentioned above, one may choose the base point z, and the Levi factor $\mathfrak{l}$ of $\mathfrak{q}_z$, so that $\mathfrak{h} \subset \mathfrak{l}$. It follows that the involutive automorphism σ is conjugation by some element of H. Since the Cartan involution is also conjugation by an element of H, σ and θ commute.

In a root order for which $\mathfrak{r}_-$ is a sum of negative root spaces, its complex conjugate $\mathfrak{r}_+ = \overline{\mathfrak{r}_-}$ is a sum of positive root spaces, and $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}_- + \mathfrak{r}_+$. We will write $\mathfrak{r}$ for $\mathfrak{r}_- + \mathfrak{r}_+$ and $\mathfrak{r}_0$ for its real form $\mathfrak{g}_0 \cap \mathfrak{r}$, and $\mathfrak{r}_u$ for its "compact real form" $\mathfrak{g}_u \cap \mathfrak{r}$. Now Theorems 2.7 and 4.3 in [14] specialize to

1.8. **Proposition.** *$G_0 = K_0 \cdot \exp_{G_0}(\mathfrak{s}_0 \cap \mathfrak{r}_0) \cdot L_0$ and $G_u = K_0 \cdot \exp_{G_u}(\mathfrak{s}_u \cap \mathfrak{r}_u) \cdot L_u$. If D is symmetric, $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{r}_0 \cap \mathfrak{s}_0$, $\mathfrak{a}_u = \sqrt{-1}\,\mathfrak{a}_0$, $A_0 = \exp_{G_0}(\mathfrak{a}_0)$, and $A_u = \exp_{G_u}(\mathfrak{a}_u)$, then $G_0 = K_0 A_0 L_0$ and $G_u = K_0 A_u L_u$.*

1.9. **Definition.** Consider the C^∞ double fibration

$$D = G_0/L_0 \xleftarrow{\pi_D} G_0/(L_0 \cap K_0) \xrightarrow{\pi_B} G_0/K_0 = B. \tag{1.10}$$

The open orbit $D \subset Z$ is said to be of *holomorphic type* if there is a G_0–invariant complex structure on $G_0/(L_0 \cap K_0)$ and a choice of $\mathfrak{s}_\pm$ such that

$$\pi_D : G_0/(L_0 \cap K_0) \to D \text{ and } \pi_B : G_0/(L_0 \cap K_0) \to B$$

are simultaneously holomorphic, of *nonholomorphic type* if there is no such choice.

The orbits of holomorphic type are characterized by this extension of [12, Prop. 1.3]:

[1] In this case σ must commute with the Cartan involution θ such that $K_0 = G_0^\theta$, for $\theta(\mathfrak{h}) = \mathfrak{h}$ and that forces $\theta(\mathfrak{l}) = \mathfrak{l}$. Also, in this case, L_0 is the identity component of G_0^σ.

1.11. **Proposition.** *For the proper choice of* $\mathfrak{s}_+$ *in* (1.2), *the following conditions are equivalent:*

(a) *the open orbit* D *is of holomorphic type,*
(b) *either* $\mathfrak{s} \cap \mathfrak{r}_+ = \mathfrak{s}_+ \cap \mathfrak{r}_+$ *or* $\mathfrak{s} \cap \mathfrak{r}_+ = \mathfrak{s}_- \cap \mathfrak{r}_+$,
(c) *either* $\mathfrak{s}_- \cap \mathfrak{r}_+ = 0$ *or* $\mathfrak{s}_- \cap \mathfrak{r}_- = 0$,
(d) *one of* $\mathfrak{q} \cap \mathfrak{p}$ *and* $\mathfrak{q} \cap \overline{\mathfrak{p}}$ *is a parabolic subalgebra of* $\mathfrak{g}$,
(e) *there is a positive root system* $\Delta^+(\mathfrak{g}, \mathfrak{h})$ *such that both* $\mathfrak{r}_+$ *and* $\mathfrak{s}_+$ *are sums of positive root spaces,*
(f) *there is a positive root system* $\Delta^+(\mathfrak{g}, \mathfrak{h})$ *such that* $\mathfrak{q}$ *is defined by a subset of the corresponding simple root system* Ψ, *and* Ψ *contains just one* $\mathfrak{g}_0$*-noncompact root.*

Proof. Fix a G_0–invariant complex structure on $G_0/(L_0 \cap K_0)$. Then $\mathfrak{g} = (\mathfrak{l} \cap \mathfrak{k}) + \mathfrak{v}_+ + \mathfrak{v}_-$, vector space direct sum, where $\overline{\mathfrak{v}_+} = \mathfrak{v}_-$, where $\mathfrak{v}_+$ represents the holomorphic tangent space, and where $\mathfrak{v}_-$ represents the antiholomorphic tangent space. This defines a G_0-invariant almost–complex structure, and the integrability condition is that one (thus both) of $(\mathfrak{l} \cap \mathfrak{k}) + \mathfrak{v}_\pm$ be subalgebras of $\mathfrak{g}$.

Note that $d\pi_D$ is surjective. Thus the condition that π_D be holomorphic in (1.10) is that it maps the holomorphic tangent space of $G_0/(L_0 \cap K_0)$ onto the holomorphic tangent space of D, in other words that $\mathfrak{r}_+ \subset \mathfrak{v}_+ \subset \mathfrak{l} + \mathfrak{r}_+$ where $\mathfrak{r}_+$ represents the holomorphic tangent space of D. Similarly the condition that π_B be holomorphic is that, for the correct choice of $\mathfrak{s}_\pm$, say $\mathfrak{s}_+$, we have $\mathfrak{s}_+ \subset \mathfrak{v}_+ \subset \mathfrak{k} + \mathfrak{s}_+$. These two hold simultaneously if and only if

$$\mathfrak{r}_+ + \mathfrak{s}_+ \subset \mathfrak{v}_+ \subset (\mathfrak{l} + \mathfrak{r}_+) \cap (\mathfrak{k} + \mathfrak{s}_+) \subset (\mathfrak{l} \cap \mathfrak{k}) + \mathfrak{r}_+ + \mathfrak{s}_+ \ , \ \text{i.e.,}\ \mathfrak{r}_+ + \mathfrak{s}_+ = \mathfrak{v}_+ \ .$$

We have shown that D is of holomorphic type, which is condition (a), if and only if $(\mathfrak{r}_+ + \mathfrak{s}_+) \cap \overline{(\mathfrak{r}_+ + \mathfrak{s}_+)} = 0$, which is equivalent to condition (e). Now it is straightforward to verify equivalence of conditions (a) through (f). □

2. Some preliminary results

In this section we obtain some preliminary results for the open G_0-orbits on Z and construct the hermitian symmetric sub–flag whose Cayley transforms will give the G_0-orbit structure of Z.

2.1. **Lemma.** *If one open* G_0*-orbit on* Z *is symmetric, then every open* G_0*-orbit on* Z *is symmetric.*

Proof. Let $D = G_0(z) \subset Z$ be a symmetric open orbit. We may assume $Q = Q_z$ and $L_0 = G_0 \cap Q$ open in G_0^σ where σ is an involutive automorphism of G_0. Thus the reductive part $\mathfrak{l} = \mathfrak{q} \cap \overline{\mathfrak{q}}$ of $\mathfrak{q}$ is of the form $\mathfrak{g}^\sigma$ where σ is an involutive automorphism of G that preserves $\mathfrak{g}_0$. Now let $D' = G_0(z') \subset Z$ be another open orbit, $Q' = Q_{z'}$, $L' = Q' \cap \overline{Q'}$, and $L_0' = G_0 \cap L'$. Write $z' = g(z)$ where $g \in G_u$. Now $\sigma' = \mathrm{Ad}(g)\sigma\mathrm{Ad}(g^{-1})$ is an involutive automorphism of G such that $\mathfrak{l}' = \mathfrak{g}^{\sigma'}$.

Complex conjugation $\xi \mapsto \bar{\xi}$ of $\mathfrak{g}$ over $\mathfrak{g}_0$ preserves $\mathfrak{l}' = \mathfrak{q}_{z'} \cap \tau\mathfrak{q}_{z'}$. It thus commutes with σ'. Now σ' preserves $\mathfrak{g}_0$, so the real form $\mathfrak{l}'_0 = \mathfrak{g}_0 \cap \mathfrak{l}'$ of $\mathfrak{l}'$ satisfies $\mathfrak{l}'_0 = \mathfrak{g}_0^{\sigma'}$. Thus D' is symmetric. $\square$

Remark. Lemma 2.1 also follows from Remark 1.7.

In order to describe the G_0-orbit structure of Z it is best to start from the viewpoint of an open orbit of holomorphic type. For that we need

2.2. **Lemma.** *There is an open G_0-orbit on Z that is of holomorphic type.* (The argument will exhibit two open G_0-orbits of holomorphic type, but they can coincide.)

Proof. Let $D = G_0(z)$ be an open orbit in Z. Choose any positive root systems $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and $\Delta^+(\mathfrak{k}, \mathfrak{h})$. Then we have positive subsystems

$$\Delta_0^+ = \Delta_0^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{l}, \mathfrak{h}) \cup \Delta(\mathfrak{r}_+, \mathfrak{h}) \text{ and}$$
$$\Delta_{\pm 1}^+ = \Delta_{\pm 1}^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{k}, \mathfrak{h}) \cup \Delta(\mathfrak{s}_\pm, \mathfrak{h})$$

for $\Delta(\mathfrak{g}, \mathfrak{h})$. Let $w_\pm \in W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group element that sends Δ_0^+ to $\Delta_{\pm 1}^+$ and let $g_\pm \in G$ represent $w_\pm$. Then by (1.5) $G_0(g_\pm(z))$ is open in Z because $\mathrm{Ad}(g_\pm)\mathfrak{h} = \mathfrak{h}$. Further, $\mathfrak{s}_\pm \subset \mathrm{Ad}(g_\pm)(\mathfrak{l} + \mathfrak{r}_+)$, so $\mathrm{Ad}(g_\pm)(\mathfrak{r}_-) \cap \mathfrak{s}_\pm = 0$, and thus $G_0(g_\pm(z))$ is of holomorphic type by equivalence of (a) and (c) in Proposition 1.11. $\square$

Now we need one more piece of background: the structure of the hermitian symmetric submanifold of Z that later will pick out the G_0-orbits on Z. Let $D = G_0(z) \cong G_0/L_0$ as before, and suppose that D is symmetric. Then we have commuting involutive automorphisms θ and σ of G_0, $\mathfrak{g}_0$ and their complexifications such that $\mathfrak{l}_0 = \mathfrak{g}_0^\sigma$ and $\mathfrak{k}_0 = \mathfrak{g}_0^\theta$. The corresponding (± 1)–eigenspace decompositions are

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0 \text{ under } \theta \text{ and } \mathfrak{g}_0 = \mathfrak{l}_0 + \mathfrak{r}_0 \text{ under } \sigma. \tag{2.3}$$

Now define

$$\mathfrak{m} = \mathfrak{g}^{\theta\sigma} = (\mathfrak{l} \cap \mathfrak{k}) + (\mathfrak{r} \cap \mathfrak{s}) \text{ and } \mathfrak{m}_0 = \mathfrak{g}_0^{\theta\sigma} = (\mathfrak{l}_0 \cap \mathfrak{k}_0) + (\mathfrak{r}_0 \cap \mathfrak{s}_0). \tag{2.4}$$

Let M and M_0 denote the corresponding analytic subgroups of G and G_0.

Any parabolic subalgebra containing a Cartan subalgebra $\mathfrak{h}$ is specified by choosing a positive system of $\mathfrak{h}$-roots and specifying a subset of the simple roots (which will be in the nilradical). Therefore one can find some $\xi \in \mathfrak{h}^*$ so that $\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}), \alpha(\xi) \geq 0} \mathfrak{g}_\alpha$. In our situation $\mathfrak{h}_0 \subset \mathfrak{k}_0$ so we may choose $\xi \in \sqrt{-1}\mathfrak{h}_0^*$. It follows that $\mathfrak{m} \cap \mathfrak{q} = (\mathfrak{l} \cap \mathfrak{k}) + (\mathfrak{r}_- \cap \mathfrak{s}) = \sum_{\alpha \in \Delta(\mathfrak{m},\mathfrak{h}), \alpha(\xi) \geq 0} \mathfrak{g}_\alpha$, so $\mathfrak{m} \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{m}$. Now $M(z)$ is a flag manifold and $M_0(z)$ is an open M_0-orbit. The isotropy subgroup of M_0 at z is $M_0 \cap Q = M_0 \cap L_0 = M_0 \cap K_0$, which is the maximal compact subgroup M_0^θ of M_0. Now $M_0(z)$ is a riemannian symmetric space of noncompact type with invariant complex structure. We have proved Part (i) of

2.5. **Proposition.** *Let $D = G_0(z) \subset Z$ be a symmetric open orbit.*

(i) *$F = M(z)$ is a complex flag manifold of M, the open orbit $M_0(z)$ is an hermitian symmetric space of noncompact type, and $M_0(z) \subset F$ is the Borel embedding.*

(ii) *If D is of holomorphic type, then the bounded symmetric domain $M_0(z)$ is irreducible.*

Remark. Part (i) of Proposition 2.5 requires neither that G_0 be simple nor that it be of hermitian type. Part 2 requires both hermitian type and simplicity.

Proof. Part (i) is proved. To prove Part (ii) it suffices to show that there is a positive root system for $\mathfrak{g}$ such that (a) there is just one noncompact simple root (as in Proposition 1.11(f)) and (b) the Dynkin diagram of $\mathfrak{m}$ is obtained from the Dynkin diagram of $\mathfrak{g}$ by deleting a set of simple roots. Then either M_0 is compact or (which we now assume) it has just one noncompact local factor M_0'', corresponding to the component of the Dynkin diagram of $\mathfrak{m}$ that contains the noncompact simple root of $\mathfrak{g}$. Then $M_0(z) = M_0''(z)$, which is irreducible because M_0'' is simple.

We may, and do, assume $\mathfrak{r}_- \cap \mathfrak{s}_+ = 0 = \mathfrak{r}_+ \cap \mathfrak{s}_-$. By Lemma 1.11(c) and (f), we need only show that $\mathfrak{m}$ is the Levi component of a parabolic subalgebra $\mathfrak{w} = \mathfrak{m} + \mathfrak{u}_-$ of $\mathfrak{g}$ such that $\mathfrak{u}_- \cap \mathfrak{s}_+ = 0$. Then $G_0(e)$ is an open orbit of holomorphic type in $E = G/W$ where the base point $e = 1W \in E$, and Proposition 1.11 completes the argument.

First, $\mathfrak{u}_- = (\mathfrak{l} \cap \mathfrak{s}_-) + (\mathfrak{k} \cap \mathfrak{r}_+)$ is a commutative subalgebra of $\mathfrak{g}$ because

$$
\begin{aligned}
&[\mathfrak{l} \cap \mathfrak{s}_-, \mathfrak{l} \cap \mathfrak{s}_-] \subset [\mathfrak{s}_-, \mathfrak{s}_-] = 0,\\
&[\mathfrak{k} \cap \mathfrak{r}_+, \mathfrak{k} \cap \mathfrak{r}_+] \subset [\mathfrak{r}_+, \mathfrak{r}_+] = 0, \text{ and}\\
&[\mathfrak{l} \cap \mathfrak{s}_-, \mathfrak{k} \cap \mathfrak{r}_+] \subset [\mathfrak{l}, \mathfrak{r}_+] \cap [\mathfrak{k}, \mathfrak{s}_-] \subset \mathfrak{r}_+ \cap \mathfrak{s}_- = 0.
\end{aligned}
$$

Second, $\mathfrak{m} = (\mathfrak{k} \cap \mathfrak{l}) + (\mathfrak{r} \cap \mathfrak{s})$ is the fixed point set of an involution of $\mathfrak{g}$, in particular is a subalgebra of $\mathfrak{g}$. Third, $\mathfrak{m}$ normalizes $\mathfrak{u}_-$ because

$$
\begin{aligned}
&[\mathfrak{k} \cap \mathfrak{l}, \mathfrak{k} \cap \mathfrak{r}_+] \subset \mathfrak{k} \cap [\mathfrak{l}, \mathfrak{r}_+] \subset \mathfrak{k} \cap \mathfrak{r}_+ \ ,\\
&[\mathfrak{k} \cap \mathfrak{l}, \mathfrak{l} \cap \mathfrak{s}_-] \subset \mathfrak{l} \cap [\mathfrak{k}, \mathfrak{s}_-] \subset \mathfrak{l} \cap \mathfrak{s}_- \ ,\\
&[\mathfrak{r} \cap \mathfrak{s}, \mathfrak{k} \cap \mathfrak{r}_+] = [\mathfrak{r}_- \cap \mathfrak{s}_-, \mathfrak{k} \cap \mathfrak{r}_+] \subset [\mathfrak{r}_- \cap \mathfrak{r}_+] \cap [\mathfrak{s}_-, \mathfrak{k}] \subset \mathfrak{k} \cap \mathfrak{l} \ , \text{ and}\\
&[\mathfrak{r} \cap \mathfrak{s}, \mathfrak{l} \cap \mathfrak{s}_-] = [\mathfrak{r}_+ \cap \mathfrak{s}_+, \mathfrak{l} \cap \mathfrak{s}_-] \subset [\mathfrak{r}_+ \cap \mathfrak{l}] \cap [\mathfrak{s}_+, \mathfrak{s}_-] \subset \mathfrak{k} \cap \mathfrak{r}_+ \ .
\end{aligned}
$$

Now $\mathfrak{g}$ is simple, $\mathfrak{m} + \mathfrak{u}_-$ is a subalgebra with commutative nilradical u_-, and $\mathfrak{m}$ is the fixed point set of an involution of $\mathfrak{g}$. If $\mathfrak{u}_- = 0$ then $\mathfrak{k} = \mathfrak{l}$ and assertion (ii) is obvious. If $\mathfrak{u}_- \neq 0$ then $m + u_-$ is a parabolic subalgebra of $\mathfrak{g}$, and (ii) follows. □

Remark. For open symmetric orbits of holomorphic type the subgroups L and M may be described in terms of the Dynkin diagram as follows. Fix a positive system Δ^+ as in Proposition 1.11(f). Then the set of simple roots Π has exactly one noncompact root α and there is a root $\beta \in \Pi$, having coefficient one in the highest root, so that $\Pi \setminus \{\beta\}$ is a set of simple roots for $\mathfrak{l}$. Conversely, if $\beta \in \Pi$ is any simple root with coefficient one in the highest root then $\Pi \setminus \{\beta\}$ defines a subalgebra $\mathfrak{l}$ and there is a corresponding open symmetric orbit of holomorphic type. Given L, Δ^+ and β as above, the Lie algebra $\mathfrak{m}$ has a root system spanned by roots in Δ^+ for which the coefficients of α and β are equal. (This coefficient is 0 for roots in $\mathfrak{l} \cap \mathfrak{k}$ and is ± 1 for roots in $\mathfrak{r} \cap \mathfrak{s}$.) The minimal root in $\mathfrak{r}_+ \cap \mathfrak{s}_+$ along with $\Pi \setminus \{\alpha, \beta\}$ forms a system of simple roots for $\mathfrak{m}$. Thus $\mathfrak{m} \cap \mathfrak{q}$ is a maximal parabolic. This gives and alternative proof of the irreducibility of $M_0(z)$.

3. Cayley transforms and statement of theorem

Following Lemma 2.2 we fix an open G_0-orbit $D = G_0(z) \subset Z$ of holomorphic type. We may, and do, assume $\mathfrak{r}_- \cap \mathfrak{s}_+ = 0 = \mathfrak{r}_+ \cap \mathfrak{s}_-$.

As D is of holomorphic type, we can fix a positive root system

$$\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h}) \text{ such that } \Delta(\mathfrak{r}_+, \mathfrak{h}) \subset \Delta^+ \text{ and } \Delta(\mathfrak{s}_+, \mathfrak{h}) \subset \Delta^+ . \tag{3.1}$$

Then the maximal root γ_1 of $\Delta^+(\mathfrak{g}, \mathfrak{h})$ belongs to $\Delta(\mathfrak{r}_+ \cap \mathfrak{s}_+, \mathfrak{h})$ because it has positive coefficient along every simple root. Cascade down from γ_1 within the subset $\Delta^+(\mathfrak{m}, \mathfrak{h})$ of $\Delta^+(\mathfrak{g}, \mathfrak{h})$: at each stage, the next root γ_{i+1} is a maximal noncompact positive root orthogonal to $\{\gamma_1, \ldots, \gamma_i\}$, terminating the construction when no such root γ_{i+1} is available. This defines

$$\begin{aligned} &\Psi^{\mathfrak{m}} = \{\gamma_1, \ldots, \gamma_r\} : \\ &\text{maximal set of strongly orthogonal} \\ &\text{noncompact positive roots of } \mathfrak{m}. \end{aligned} \tag{3.2}$$

Here recall that roots $\alpha, \beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ are called *strongly orthogonal* if $\alpha + \beta \neq 0$ and neither of $\alpha \pm \beta$ is a root, so that $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = 0$.

Whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ we denote the corresponding 3–dimensional simple subalgebra $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ by $\mathfrak{g}[\alpha]$. If $\alpha, \beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ are strongly orthogonal then $[\mathfrak{g}[\alpha], \mathfrak{g}[\beta]] = 0$. If $\Sigma \subset \Delta(\mathfrak{g}, \mathfrak{h})$ is a set of strongly orthogonal roots then $\mathfrak{g}[\Sigma]$ denotes the Lie algebra direct sum $\sum_{\alpha \in \Sigma} \mathfrak{g}[\alpha]$. We write $G[\alpha]$ and $G[\Sigma]$ for the corresponding analytic subgroups of G. When the roots are noncompact we denote real forms of the Lie algebras by $\mathfrak{g}_0[\alpha] = \mathfrak{g}_0 \cap \mathfrak{g}[\alpha]$ and $\mathfrak{g}_0[\Sigma] = \mathfrak{g}_0 \cap \mathfrak{g}[\Sigma]$; then $G_0[\alpha]$ and $G_0[\Sigma]$ are the corresponding analytic subgroups of G_0. Finally, if $\gamma \in \Psi^{\mathfrak{m}}$ then $G[\gamma](z)$ is a Riemann sphere, and if $\Sigma \subset \Psi^{\mathfrak{m}}$, then $G[\Sigma](z)$ is the polysphere $\prod_{\gamma \in \Sigma} G[\gamma](z)$.

We make this a bit more explicit. For each $\alpha \in \Delta(\mathfrak{s}, \mathfrak{h})$ we choose $e_\alpha \in \mathfrak{g}_\alpha$ in such a way that (i) if $\alpha \in \Delta(\mathfrak{s}_+, \mathfrak{h})$ then the almost complex structure of $B = G_0/K_0$ sends $x_\alpha = e_\alpha + e_{-\alpha}$ to $y_\alpha = \sqrt{-1}\,(e_\alpha - e_{-\alpha})$ and sends y_α to

$-x_\alpha$ and (ii) $[e_\alpha, e_{-\alpha}] = h_\alpha \in \mathfrak{h}$. Thus, e_α, $e_{-\alpha}$ and h_α correspond to the standard basis of $\mathfrak{sl}(2;\mathbb{C})$:

$$e_\alpha \longleftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\alpha} \longleftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h_\alpha \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\mathfrak{a}_0 = \mathfrak{a}_0^{\mathrm{m}} : \text{ real span of the } y_\gamma = \sqrt{-1}(e_\gamma - e_{-\gamma}) \text{ for } \gamma \in \Psi^{\mathrm{m}} \tag{3.3}$$

is a maximal abelian subspace of $\mathfrak{r}_0 \cap \mathfrak{s}_0$, and we define Cayley transforms as in [13]:

$$c_\gamma = \exp\left(\tfrac{\pi\sqrt{-1}}{4}\, x_\gamma\right) \leftrightarrow \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{bmatrix} \tag{3.4}$$

so that

$$\mathrm{Ad}(c_\gamma) \text{ maps } x_\gamma \mapsto x_\gamma,\ y_\gamma \mapsto -h_\gamma,\ h_\gamma \mapsto y_\gamma; \quad \text{so } \mathrm{Ad}(c_\gamma^2)\mathfrak{h}_0 = \mathfrak{h}_0\ . \tag{3.5}$$

An easy $SL(2,\mathbb{C})$-calculation shows the $G_0[\gamma]$-orbits on $G[\gamma](z)$ are the lower hemisphere $G_0[\gamma](z)$, the equator $G_0[\gamma](c_\gamma z)$ and the upper hemisphere $G_0[\gamma](c_\gamma^2 z)$. Thus

$$\begin{gathered} \text{the } G_0[\Psi^{\mathrm{m}}]\text{-orbits on } G[\Psi^{\mathrm{m}}](z) \text{ are} \\ \text{the } G_0[\Psi^{\mathrm{m}}](c_\Gamma c_\Sigma^2 z)\ (\Gamma, \Sigma \subset \Psi^{\mathrm{m}},\ \text{disjoint}). \end{gathered} \tag{3.6}$$

Here $\Psi^{\mathrm{m}} \setminus (\Gamma \cup \Sigma)$ indexes the lower hemisphere factors, Γ indexes the equatorial factors, and Σ indexes the upper hemisphere factors. With this in mind we denote some G_0-orbits on Z by

$$D_{\Gamma,\Sigma} = G_0(c_\Gamma c_\Sigma^2 z) \text{ whenever } \Gamma \text{ and } \Sigma \text{ are disjoint subsets of } \Psi^{\mathrm{m}}\ . \tag{3.7}$$

Whenever we write $D_{\Gamma,\Sigma}$ it will be implicit that the subsets $\Gamma, \Sigma \subset \Psi^{\mathrm{m}}$ are disjoint. The main result of this paper is

3.8. **Theorem.** *Suppose that the open G_0-orbits on Z are symmetric. Let $D = G_0(z)$ be an open G_0-orbit of holomorphic type. In the notation above,*

(i) *Z is the union of the sets $D_{\Gamma,\Sigma}$,*
(ii) *$D_{\Gamma,\Sigma} = K_0 G_0[\Psi^{\mathrm{m}}](c_\Gamma c_\Sigma^2 z)$,*
(iii) *$D_{\Gamma,\Sigma}$ is open in Z if and only if $\Gamma = \varnothing$,*
(iv) *the closure of $D_{\Gamma,\Sigma}$ is the union of the $D_{\Gamma',\Sigma'}$ with $\Sigma' \subset \Sigma$ and $\Gamma \cup \Sigma \subset \Gamma' \cup \Sigma'$,*
(v) *$D_{\Gamma,\Sigma} = D_{\Gamma',\Sigma'}$ if and only if the cardinalities $|\Gamma| = |\Gamma'|$ and $|\Sigma| = |\Sigma'|$.*

In particular, the map $M_0(c_\Gamma c_\Sigma^2 z) \mapsto D_{\Gamma,\Sigma}$ *is a one to one map from the set of all* M_0*-orbits on* F *onto the set of all* G_0*-orbits on* Z*.*

3.9. *Remark.* Define $D_{a,b} = D_{\Gamma,\Sigma}$ with $a = |\Sigma|$, $b = |\Phi^{\mathfrak{m}}| - |\Gamma \cup \Sigma|$, and $\Gamma \cap \Sigma$ empty. This is well defined by Theorem 3.8(v). Now Theorem 3.8(iv) becomes: the closure of $D_{a,b}$ is the union of $D_{a',b'}$ with $0 \leqq a' \leqq a$ and $0 \leqq b' \leqq b$.

3.10. *Remark.* Theorem 3.8 gives a generalization of the orbit structure ([13], [10], [11]) for a noncompact real Lie group of hermitian symmetric type acting on the dual hermitian symmetric flag manifold. The proof here is independent of [11]. The orbit structure results of [13], [10] and [11] are recovered as the case $\sigma = \theta$. In particular Theorem 3.8(i) through Theorem 3.8(v) hold with Z replaced by $F = M(z)$ and $D_{\Gamma,\Sigma} = G_0(c_\Gamma c_\Sigma^2 z)$ replaced by $M_0(c_\Gamma c_\Sigma^2 z)$. The one to one correspondence between M_0-orbits on F and G_0-orbits on Z follows immediately.

3.11. *Remark.* The most delicate part of the proof of Theorem 3.8 is the proof that open orbits $D_{\varnothing,\Sigma}$ and $D_{\varnothing,\Sigma'}$ are distinct when $|\Sigma| \neq |\Sigma'|$. In the case $F = Z$ of [10] and [11], this follows immediately from a simple distance argument using geometric ideas from [14]. Those considerations prove that $\inf_{z' \in D_{\varnothing,\Sigma}} \mathrm{dist}_Z(z', K_0(z))^2 = |\Sigma|$ where dist_Z is the distance for a properly normalized G_u-invariant riemannian metric on Z. (This simplifies the rather elaborate argument in [10] and [11].) The same distance formula, properly interpreted for the pseudoriemannian context, holds in general as a corollary to Theorem 3.8. The proof of Theorem 3.8 would be simplified if one could prove the distance formula first, but we have not been able to do that. See the appendix for details.

4. The open orbits

We first check that Z is the union of the sets $D_{\Gamma,\Sigma}$. Note that $Z = G_u(z) = K_0 A_u(z)$ by Proposition 1.8, that $A_u(z) \subset G[\Psi^{\mathfrak{m}}](z)$, and that $G[\Psi^{\mathfrak{m}}](z)$ is the union of the $G_0[\Psi^{\mathfrak{m}}](c_\Gamma c_\Sigma^2 z)$ by (3.6). Thus Z is contained in the union of the $G_0(c_\Gamma c_\Sigma^2 z) = D_{\Gamma,\Sigma}$. Assertion (i) of Theorem 3.8 is proved.

The remainder of this section contains the proof of the following lemma.

4.1. **Lemma.** (a) *The* G_0*-orbits* $D_{\varnothing,\Sigma}$ *are open and* (b) $D_{\varnothing,\Sigma} = D_{\varnothing,\Sigma'}$ *just when* $|\Sigma| = |\Sigma'|$.

(We will see in Section 5 that every open orbit is of the form $D_{\varnothing,\Sigma}$.)

Proof. Fix a subset $\Sigma \subset \Psi^{\mathfrak{m}}$ and set $z' = c_\Sigma^2(z)$. Then the G_0–stabilizer of z' contains the compact Cartan subgroup H_0 of G_0, by (3.5) and (1.5) so

$$D_{\varnothing,\Sigma} = G_0(z') \text{ is open in } Z. \tag{4.2}$$

This proves (a).

Now the G_0–stabilizer of z' is a real form of the reductive part of $Q_{z'} = \mathrm{Ad}(c_\Sigma^2)Q$. It has Lie algebra $\mathfrak{l}_0' = \mathfrak{g}_0 \cap \mathfrak{l}'$ where $\mathfrak{l}' = \mathrm{Ad}(c_\Sigma^2)\mathfrak{q} \cap \overline{\mathrm{Ad}(c_\Sigma^2)\mathfrak{q}}$. If $\gamma \in \Psi^{\mathfrak{m}}$ then $\overline{c_\gamma} = \theta(c_\gamma) = \sigma(c_\gamma) = c_\gamma^{-1}$. Again from (3.5), $c_\Sigma^4 \in K_0$. If $\xi \in \mathfrak{g}_0$ then

$$\overline{\big(\mathrm{Ad}(c_\Sigma^2)\sigma\mathrm{Ad}(c_\Sigma^{-2})\big)(\xi)} = \big(\mathrm{Ad}(c_\Sigma^{-2})\sigma\mathrm{Ad}(c_\Sigma^2)\big)(\xi) = \big(\sigma\mathrm{Ad}(c_\Sigma^4)\big)(\xi) \in \mathfrak{g}_0 \ .$$

Thus $\mathfrak{g}_0$ is stable under $\sigma' = \mathrm{Ad}(c_\Sigma^2)\sigma\mathrm{Ad}(c_\Sigma^{-2})$. This shows $\mathfrak{l}_0' = \mathfrak{g}_0^{\sigma'}$. Now the G_0–stabilizer of z' is

$$L_0' : \text{identity component in the fixed point set of } \sigma' = \mathrm{Ad}(c_\Sigma^2)\sigma\mathrm{Ad}(c_\Sigma^{-2}). \tag{4.3}$$

In (4.3) notice that $\sigma'\theta = \mathrm{Ad}(c_\Sigma^2)\sigma\,\mathrm{Ad}(c_\Sigma^{-2})\theta = \mathrm{Ad}(c_\Sigma^2)\sigma\theta\,\mathrm{Ad}(c_\Sigma^2) = \mathrm{Ad}(c_\Sigma^2)\theta\sigma\,\mathrm{Ad}(c_\Sigma^2) = \theta\,\mathrm{Ad}(c_\Sigma^{-2})\sigma\,\mathrm{Ad}(c_\Sigma^2) = \theta\sigma'$. In particular $\theta\sigma'$ is another involutive automorphism of G that stabilizes G_0.

As in (2.3) and (2.4) we have the (±1)–eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{l}_0' + \mathfrak{r}_0'$ for σ', and we use the involutive automorphism $\theta\sigma'$ to define

$$\mathfrak{m}' = \mathfrak{g}^{\theta\sigma'} = (\mathfrak{l}' \cap \mathfrak{k}) + (\mathfrak{r}' \cap \mathfrak{s}) \text{ and } \mathfrak{m}_0' = \mathfrak{g}_0^{\theta\sigma'} = (\mathfrak{l}_0' \cap \mathfrak{k}_0) + (\mathfrak{r}_0' \cap \mathfrak{s}_0). \tag{4.4}$$

M' and M_0' denote the corresponding analytic subgroups of G and G_0.

As $\mathfrak{m}'$ is θ–stable, it decomposes as a direct sum $\mathfrak{m}' = \mathfrak{m}'_{\mathrm{nonc}} \oplus \mathfrak{m}'_{\mathrm{comp}}$ of ideals, where $\mathfrak{m}'_{\mathrm{nonc}}$ is generated by $\mathfrak{r}' \cap \mathfrak{s}$ and where $\mathfrak{m}'_{\mathrm{comp}} \subset \mathfrak{k}$. Note that $[(\mathfrak{r}'_- \cap \mathfrak{s}_+) + (\mathfrak{r}'_+ \cap \mathfrak{s}_-), (\mathfrak{r}'_+ \cap \mathfrak{s}_+) + (\mathfrak{r}'_- \cap \mathfrak{s}_-)] = 0$. Thus $\mathfrak{m}'$ decomposes further, as follows, into a finer direct sum of ideals:

$$\begin{gathered} \mathfrak{m}' = \mathfrak{m}_1' \oplus \mathfrak{m}_2' \oplus \mathfrak{e}' \text{ where} \\ \mathfrak{m}_1' \text{ is generated by } (\mathfrak{r}'_- \cap \mathfrak{s}_+) + (\mathfrak{r}'_+ \cap \mathfrak{s}_-), \\ \mathfrak{m}_2' \text{ is generated by } (\mathfrak{r}'_+ \cap \mathfrak{s}_+) + (\mathfrak{r}'_- \cap \mathfrak{s}_-), \\ \text{and } \mathfrak{e}' = \mathfrak{m}'_{\mathrm{comp}} \subset \mathfrak{k}. \end{gathered} \tag{4.6}$$

But (3.5) gives us $\mathrm{Ad}(c_\Sigma)^2 e_\gamma = e_{-\gamma}$ and $\mathrm{Ad}(c_\Sigma)^2 e_{-\gamma} = e_\gamma$ for $\gamma \in \Sigma$, $\mathrm{Ad}(c_\Sigma)^2 e_\gamma = e_\gamma$ and $\mathrm{Ad}(c_\Sigma)^2 e_{-\gamma} = e_{-\gamma}$ for $\gamma \in \Psi^{\mathfrak{m}} \setminus \Sigma$. It follows that $\mathrm{Ad}(c_\Sigma)^2(e_{\pm\gamma}) \in \mathfrak{r}'_{\mp}$ for $\gamma \in \Sigma$ and $\mathrm{Ad}(c_\Sigma)^2(e_{\pm\gamma}) \in \mathfrak{r}'_{\pm}$ for $\gamma \in \Psi^{\mathfrak{m}} \setminus \Sigma$. Thus, $\mathrm{Ad}(c_\Sigma)^2(e_\gamma - e_{-\gamma})$ is in $\mathfrak{m}_1'$ for $\gamma \in \Sigma$ and is in $\mathfrak{m}_2'$ for $\gamma \in \Psi^{\mathfrak{m}} \setminus \Sigma$. It follows from (3.3) that $\mathfrak{m}_1'$ has real rank $|\Sigma|$ and $\mathfrak{m}_2'$ has real rank $|\Phi^{\mathfrak{m}} \setminus \Sigma|$.

Suppose $D_{\varnothing,\Sigma'} = D_{\varnothing,\Sigma''}$. Let $\mathfrak{q}' = \mathrm{Ad}(c_{\Sigma'}^2)\mathfrak{q}$ as for Σ and set $\mathfrak{q}'' = \mathrm{Ad}(c_{\Sigma''}^2)\mathfrak{q}$. Let $z'' = c_{\Sigma''}^2 z$ and let L_0'' denote the isotropy subgroup of G_0 at z''. Express $\mathfrak{l}'' = \mathfrak{g}^{\sigma''}, \mathfrak{m}'' = \mathfrak{g}^{\theta\sigma''}$, etc. As $D_{\varnothing,\Sigma'} = D_{\varnothing,\Sigma''}$ we can choose $g_0 \in G_0$ with $\mathrm{Ad}(g_0)\mathfrak{q}'' = \mathfrak{q}'$. Then $\mathrm{Ad}(g_0)L_0'' = L_0'$, so $\mathrm{Ad}(g_0)H_0$ is just another compact Cartan subgroup of L_0'. Choose an element $\ell \in L_0'$ that conjugates $\mathrm{Ad}(g_0)H_0$ back to H_0. Now $\mathrm{Ad}(\ell g_0)$ preserves H_0 and sends $\mathfrak{q}''$ to

$\mathfrak{q}'$. It represents a Weyl group element $w \in W(G_0, H_0) = W(K_0, H_0)$. Here w sends the nilradical $\mathfrak{r}''_-$ of $\mathfrak{q}''$ to the nilradical $\mathfrak{r}'_-$ of $\mathfrak{q}'$. Taking complex conjugates, w sends $\mathfrak{r}''_+$ to $\mathfrak{r}'_+$. Evidently w preserves the $\mathrm{Ad}(K_0)$–invariant subspaces $\mathfrak{s}_+$ and $\mathfrak{s}_-$. Now w sends $\mathfrak{m}''_i$ to $\mathfrak{m}'_i$ for $i = 1, 2$, so $|\Sigma'| = |\Sigma''|$ by the real rank remark just after (4.6).

Conversely, let $\Sigma', \Sigma'' \subset \Psi^{\mathrm{m}}$. In view of Proposition 2.5, the subgroup of the Weyl group $W(M_0, H_0)$ that stabilizes Ψ^{m} acts as the group of all permutations of Ψ^{m}. See [6]. If $|\Sigma'| = |\Sigma''|$ we choose an element $w \in W(M_0, H_0)$ that sends Σ' to Σ'' and preserves Ψ^{m}. Let $k \in K_0$ represent w. Then $D_{\varnothing,\Sigma''} = G_0(c^2_{\Sigma''}z) = G_0(kc^2_{\Sigma'}z) = G_0(c^2_{\Sigma'}z) = D_{\varnothing,\Sigma'}$.

This completes the proof of the lemma. □

5. Proof for general orbits

We now consider orbits in general. In the notation of (4.5) the closure $\mathrm{cl}(D_{\varnothing,\Sigma})$ contains the closures $\mathrm{cl}(k\ G_0[\Psi^{\mathrm{m}}](z')) = k\ \mathrm{cl}(G_0[\Psi^{\mathrm{m}}](z'))$ for all $k \in K_0$. On the other hand, $\mathrm{cl}(G_0[\Psi^{\mathrm{m}}](z'))$ is closed in the polysphere $G[\Psi^{\mathrm{m}}](z')$, hence compact. Now $K_0{\cdot}\mathrm{cl}(G_0[\Psi^{\mathrm{m}}](z'))$ is compact, hence closed in Z. It follows from (4.5) that $\mathrm{cl}(D_{\varnothing,\Sigma}) = K_0{\cdot}\mathrm{cl}(G_0[\Psi^{\mathrm{m}}](z'))$. The $G_0[\Psi^{\mathrm{m}}]$-orbit structure of the polysphere $G[\Psi^{\mathrm{m}}](z')$ is given by (3.6). It follows that $\mathrm{cl}(G_0[\Psi^{\mathrm{m}}](z'))$ is the union of the $\mathrm{cl}(G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c^2_{\Sigma'}z))$, where Γ' and Σ' are disjoint subsets of Ψ^{m} and $\Sigma' \subset \Sigma \subset \Sigma' \cup \Gamma'$. As $\mathrm{cl}(D_{\varnothing,\Sigma})$ is G_0-invariant, this proves the closure of $D_{\varnothing,\Sigma}$ in Z is

$$(5.1)\qquad \begin{aligned} \mathrm{cl}(D_{\varnothing,\Sigma}) &= \bigcup\nolimits_{\Sigma'\subset\Sigma\subset\Sigma'\cup\Gamma',\Sigma'\cap\Gamma'=\varnothing} K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c^2_{\Sigma'}z) \\ &= \bigcup\nolimits_{\Sigma'\subset\Sigma\subset\Sigma'\cup\Gamma',\Sigma'\cap\Gamma'=\varnothing} D_{\Gamma',\Sigma'}\ . \end{aligned}$$

If $\Gamma', \Sigma' \subset \Psi^{\mathrm{m}}$ are disjoint and if D is an open G_0-orbit on Z then

$$(5.2)\qquad D_{\Gamma',\Sigma'} \subset \mathrm{cl}(D) \text{ iff } D \text{ is of the form } D_{\varnothing,\Sigma} \text{ with } \Sigma' \subset \Sigma \subset \Sigma' \cup \Gamma'.$$

In particular the $D_{\varnothing,\Sigma}$ are the only open orbits, and

$$(5.3)\qquad D_{\Gamma',\Sigma'} \text{ is in the closure of exactly } |\Gamma'| + 1 \text{ open orbits.}$$

Now fix pairs $\Gamma', \Sigma' \subset \Psi^{\mathrm{m}}$ and $\Gamma'', \Sigma'' \subset \Psi^{\mathrm{m}}$ of disjoint subsets of Ψ^{m}. If $|\Gamma'| = |\Gamma''|$ and $|\Sigma'| = |\Sigma''|$ then, as in the argument that $|\Sigma'| = |\Sigma''|$ implies $D_{\varnothing,\Sigma'} = D_{\varnothing,\Sigma''}$, [6] provides $w \in W(M_0, H_0)$ that sends Γ' to Γ'', sends Σ' to Σ'', and preserves Ψ^{m}. It follows that $K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma''}c^2_{\Sigma''}z) = K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c^2_{\Sigma'}z)$, in particular that $D_{\Gamma'',\Sigma''} = D_{\Gamma',\Sigma'}$.

Conversely suppose $D_{\Gamma',\Sigma'} = D_{\Gamma'',\Sigma''}$. By (5.3) we may assume $|\Gamma'| = |\Gamma''|$. Applying [6] as in the argument that $|\Sigma'| = |\Sigma''|$ implies $D_{\varnothing,\Sigma'} = D_{\varnothing,\Sigma''}$, we may further assume that $\Gamma' = \Gamma''$. Call it Γ and enumerate $\Gamma = \{\alpha_1, \ldots, \alpha_t\}$. Following (5.2), the only open orbits whose closures contain $D_{\Gamma,\Sigma'}$ are the $D_{\varnothing,\Sigma}$ with $\Sigma = \Sigma' \cup \{\alpha_1,\dots,\alpha_i\}$ where $0 \leqq i \leqq t$. Similarly, the only open orbits whose closures contain $D_{\Gamma,\Sigma''}$ are the $D_{\varnothing,\Sigma}$ with $\Sigma = \Sigma'' \cup \{\alpha_1,\dots,\alpha_i\}$ where $0 \leqq i \leqq t$. These collections of open orbits are the same, so now $|\Sigma'| = |\Sigma''|$. We have proved

$$D_{\Gamma',\Sigma'} = D_{\Gamma'',\Sigma''} \text{ if and only if } |\Gamma'| = |\Gamma''| \text{ and } |\Sigma'| = |\Sigma''|. \tag{5.4}$$

Statements (i), (iii), (iv) and (v) of Theorem 3.8 are proved; (ii) remains. In the course of the proof of (5.4) we proved that

$$\begin{gathered} K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c_{\Sigma'}^2 z) = K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma''}c_{\Sigma''}^2 z) \\ \text{if and only if } |\Gamma'| = |\Gamma''| \text{ and } |\Sigma'| = |\Sigma''|. \end{gathered} \tag{5.5}$$

Fix a collection $\mathcal{C}$ of disjoint pairs (Γ,Σ) of subsets of Ψ^{m} such that $Z = \bigcup_{(\Gamma,\Sigma)\in\mathcal{C}} D_{\Gamma,\Sigma}$, disjoint union. Express

$$Z = K_0G[\Psi^{\mathrm{m}}](z) = \bigcup_{\Gamma',\Sigma'} K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c_{\Sigma'}^2 z)$$

where the union runs over all disjoint pairs (Γ',Σ') of subsets of Ψ^{m}. Each $K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c_{\Sigma'}^2 z)$ is contained in some $D_{\Gamma,\Sigma}$, $(\Gamma,\Sigma)\in\mathcal{C}$. Compare (5.4) and (5.5) to see that each $K_0G_0[\Psi^{\mathrm{m}}](c_{\Gamma'}c_{\Sigma'}^2 z)$ is equal to its ambient $D_{\Gamma,\Sigma}$. Now $D_{\Gamma,\Sigma} = K_0G_0[\Phi^{\mathrm{m}}](c_\Gamma c_\Sigma^2 z)$ whenever Γ and Σ are disjoint subsets of Ψ^{m}. That is statement (ii) of Theorem 3.8. It completes the proof. □

Appendix. The distance formula

We expand on Remark 3.11 and give some details. Recall from the discussion after (3.2) that $G_0[\Psi^{\mathrm{m}}](z_0)$ is a polysphere (product of Riemann spheres) holomorphically embedded in Z. We endow Z with a G_u-invariant metric, as follows. The factors of the polysphere are permuted transitively by certain elements of K_0. This allows us to scale a G_u-invariant metric so that the distance from pole to equator in each of the factors is 1. Using this metric to calculate distance we set

$$\operatorname{dist}_Z(Y,D) = \inf_{y\in Y, z\in D} \operatorname{dist}_Z(y,z) \tag{A.1}$$

where $Y = K_0(z_0)$ and D is an open orbit. The following corollary is a consequence of Theorem 3.8.

A.2. **Corollary.** $\operatorname{dist}_Z(Y,D_{\varnothing,\Sigma})^2 = |\Sigma|$.

Proof. Let $z \in D_{\varnothing,\Sigma}$. There is a minimizing geodesic from Y to z which is necessarily orthogonal to Y. Translating by some $k_0 \in K_0$ we obtain a minimizing geodesic $\gamma(t)$ from z_0 to k_0^{-1} of the same length and orthogonal to Y. Thus $\gamma(t) = \exp(t\xi)$ for some $\xi \in i(\mathfrak{s}_0 \cap \mathfrak{r}_0)$. But ξ is $(K_0 \cap L_0)$-conjugate to an element of $\mathfrak{a}_u \subset \mathfrak{g}_u[\Psi^{\mathrm{m}}]$. We translate by an element of $K_0 \cap L_0$ and obtain a minimizing geodesic from z_0 to a point of $G_u[\Psi^{\mathrm{m}}](z_0) \cap D_{\varnothing,\Sigma}$. Theorem 3.8 says, among other things, that

$$G_u[\Psi^{\mathrm{m}}](z_0) \cap D_{\varnothing,\Sigma} = \bigcup_{|\Sigma'|=|\Sigma|} G_0[\Psi^{\mathrm{m}}](c^2_{\Sigma'} z_0).$$

Thus $\mathrm{dist}_Z(Y, D_{\varnothing,\Sigma})^2 \geqq |\Sigma|$. But $\mathrm{dist}_Z(z_0, G_0[\Psi^{\mathrm{m}}](c^2_\Sigma z_0)) = |\Sigma|$, so we have equality. □

As mentioned in Remark 3.11, this distance formula may be proved directly in case $Z = G/KP_\pm$. In that case it provides a simple proof that $D_{\varnothing,\Sigma} = D_{\varnothing,\Sigma'}$ implies $|\Sigma| = |\Sigma'|$. This is done as follows, and it simplifies the argument of [10] and [11].

A.3. **Theorem.** *Let* $Z = G/KS_\pm$ *and* $\Sigma \subset \Psi^{\mathfrak{s}}$. *Then* $\mathrm{dist}_Z(z_0, D_{\varnothing,\Sigma})^2 = |\Sigma|$.

Proof. By Lemma 2.1 and [14, Theorem 4.3], $D_{\varnothing,\Sigma} = K_0 G_0[\Psi^{\mathrm{m}}](c^2_\Sigma z_0)$. Let $\gamma(t)$ be a minimizing geodesic from z_0 to $k_0 z$ for some $z \in G_0[\Psi^{\mathrm{m}}](c^2_\Sigma z_0)$. Translating by k_0^{-1} we get a minimizing geodesic from z_0 to z of the form $\exp(t\xi)$ with $\xi \in i(\mathfrak{s}_0 \cap \mathfrak{g}_0[\Psi^{\mathfrak{s}}])$. The polysphere is totally geodesic in Z, so the length of the minimizing geodesic is the distance from z_0 to z in the polysphere. Thus the minimum distance from $D_{\varnothing,\Sigma}$ to z_0 is the minimum distance in the polysphere, which is $|\Sigma|$. □

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