# FLAG MANIFOLDS AND REPRESENTATION THEORY

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This article is an expanded version of lectures at the "Fifth Workshop on Representation Theory of Lie Groups and Its Applications," Córdoba, Argentina, August 1995. The topics were complex flag manifolds, real group orbits, and linear cycle spaces, with applications to the geometric construction of representations of semisimple Lie groups. These topics come up in many aspects of complex differential geometry and harmonic analysis.

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## PART 1. COMPLEX FLAG MANIFOLDS.

In this part we indicate the basic facts for real group orbits on complex flag manifolds.

### §1. PARABOLIC SUBALGEBRAS AND COMPLEX FLAGS.

Fix a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{h})$ denote the corresponding root system, and fix a positive subsystem $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$. The corresponding **Borel subalgebra**

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha} \subset \mathfrak{g} \tag{1.1}$$

has its nilradical[1] $\mathfrak{b}^{-n} = \sum \mathfrak{g}_{-\alpha}$ and a Levi complement $\mathfrak{h}$.

In general a subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is called a **Borel subalgebra** if it is $\mathrm{Int}\,(\mathfrak{g})$–conjugate to a subalgebra of the form (1.1), in other words if there exist choices of $\mathfrak{h}$ and $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{s}$ is given by (1.1).

Let $G$ denote the (unique) connected simply connected Lie group with Lie algebra $\mathfrak{g}$. The Cartan subgroup of $G$ corresponding to $\mathfrak{h}$ is $H = Z_G(\mathfrak{h})$. It has Lie algebra $\mathfrak{h}$, and it is connected because $G$ is connected, complex and semisimple. The **Borel subgroup** $B \subset G$ corresponding to a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is defined to be the $G$–normalizer of $\mathfrak{b}$, that is,

$$B = \{g \in G \mid \mathrm{Ad}\,(g)\mathfrak{b} = \mathfrak{b}\}\ . \tag{1.2}$$

Here are the basic facts on these Borel subgroups.

**1.3. Lemma.** *$B$ has Lie algebra $\mathfrak{b}$, $B$ is a closed connected subgroup of $G$, and $B$ is its own normalizer in $G$.*

*Proof.* $B$ is closed in $G$ by definition (1.2). It follows that the normalizer $E = N_G(B)$ is closed in $G$, so $E$ is a Lie subgroup. Let $\mathfrak{e}$ denote the Lie algebra of $E$. Then $\mathfrak{b} \subset \mathfrak{e}$ and $[\mathfrak{e}, \mathfrak{b}] \subset \mathfrak{b}$. Any subalgebra of $\mathfrak{g}$ that properly contains $\mathfrak{b}$ must be of the form $\mathfrak{b} + \sum_{\sigma \in S} \mathfrak{g}_\alpha$ with $S \subset \Sigma^+$, because $\mathfrak{h} \subset \mathfrak{b}$. Thus it would contain a 3–dimensional simple subalgebra and could not normalize $\mathfrak{b}$. Now $\mathfrak{e} = \mathfrak{b}$, in particular $E$ normalizes $\mathfrak{b}$, so $E = B$. This shows both that $B$ is its own normalizer and that $B$ has Lie algebra $\mathfrak{b}$. Finally, $B$ is connected because the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ is **simply** transitive on the set of all positive subsystems of $\Sigma(\mathfrak{g}, \mathfrak{h})$. □

The other basic facts are not quite as obvious.

[1] Here we describe the nilradical as a sum of negative root spaces, rather than positive, so that, in applications, positive functionals on $\mathfrak{h}$ will correspond to positive bundles (instead of negative bundles), and holomorphic discrete series representations will be highest weight (instead of lowest weight) representations.

**1.4. Lemma.** *Let $G_u \subset G$ be a compact real form. Then $G_u$ is transitive on $X = G/B$, and $X$ has a $G_u$–invariant Kaehler metric. In particular $X$ has the structure of compact Kaehler manifold.*

*Proof.* It suffices to consider a $G_u$ constructed by means of a Weyl basis of $\mathfrak{g}$ using $\mathfrak{h}$ and $\Sigma^+$. This yields a real form $\mathfrak{g}_u \subset \mathfrak{g}$ on which the Killing form is negative definite. Then the $G$–normalizer of $\mathfrak{g}_u$ coincides with the real analytic subgroup of $G$ for $\mathfrak{g}_u$, that is, $G_u$. By construction $\mathfrak{h}_u = \mathfrak{g}_u \cap \mathfrak{h}$ is the real form of $\mathfrak{h}$ on which the roots take pure imaginary values, and $\mathfrak{g} \cap \mathfrak{b} = \mathfrak{h}_u$. Now a dimension count shows that the $G_u$–orbit of the identity coset $x_0 = 1B \in G/B = X$ is open in $X$. It is also closed in $X$ because $G_u$ is compact. This proves the transitivity, and thus proves that $X$ is compact.

Let $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha \rangle > 0$ for every $\alpha \in \Sigma^+$. Extend $\lambda$ to a linear functional on $\mathfrak{g}$ by $\lambda(\mathfrak{g}_\gamma) = 0$ for every $\gamma \in \Sigma$, and view it as a 1–cochain for Lie algebra cohomology of $(\mathfrak{g}, \mathfrak{h})$. Then $d\lambda$ is a 2–cocycle on $G_u/H_u = X$, and as a 2–form it combines with the complex structure to define a Kaehler metric. Thus, for every $\lambda$ in the positive Weyl chamber of $(\mathfrak{g}, \mathfrak{h}, \Sigma^+)$, we have a $G_u$–invariant Kaehler metric on $X$. □

**1.5. Lemma.** *There is a finite dimensional irreducible representation $\pi$ of $G$ with the following property: Let $[v]$ be the image of a lowest weight vector in the projective space $\mathbb{P}(V_\pi)$ corresponding to the representation space of $\pi$. Then the action of $G$ on $V_\pi$ induces a holomorphic action of $G$ on $\mathbb{P}(V_\pi)$, and $B$ is the $G$–stabilizer of $[v]$. In particular $X = G/B$ is a complete projective variety.*

*Proof.* For example, let $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} \alpha$ as usual and let $\pi$ be the irreducible representation of highest weight $\rho$. The lowest weight is $-\rho$ and the assertions are immediate. □

**1.6. Lemma.** *$B$ is a maximal solvable subgroup of $G$.*

*Proof.* The argument of Lemma 1.3 shows that $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$. If $E \subset G$ is a solvable subgroup, and $B \subsetneq E$ then the closure of $E$ in $G$ has those same properties, so we may assume $E$ closed in $G$. But then $E$ has a Lie algebra that is not solvable, so $E$ is not solvable. We conclude that $B$ is maximal solvable. □

A theorem of Borel says that any solvable subgroup of $G$ has a fixed point on the complete projective variety $X$; this is conjugate to a subgroup of $B$. This gives another proof of Lemma 1.6, in fact it shows that the Borel subgroups are exactly the maximal solvable subgroups of $G$. That's how Borel originally defined them. The Borel subalgebras and subgroups given by (1.1) and (1.2) are the **standard** Borels.

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called **parabolic** if it contains a Borel subalgebra. For example, let $\Psi$ be the simple root system corresponding to $\Sigma^+$ and let $\Phi$ be an arbitrary subset of $\Psi$. Every root $\alpha \in \Sigma$ has a unique expression

$$\alpha = \sum_{\psi \in \Psi} n_\psi(\alpha)\psi \tag{1.7}$$

where the $n_\psi(\alpha)$ are integers, all $\geqq 0$ if $\alpha \in \Sigma^+$ and all $\leqq 0$ if $\alpha \in \Sigma^- = -\Sigma^+$. Set

$$\Phi^r = \{\alpha \in \Sigma \mid n_\psi(\alpha) = 0 \text{ whenever } \psi \notin \Phi\} \tag{1.8}$$

and

$$\Phi^n = \{\alpha \in \Sigma^+ \mid \alpha \notin \Phi^r\} = \{\alpha \in \Sigma \mid n_\psi(\alpha) > 0 \text{ for some } \psi \notin \Phi\}. \tag{1.9}$$

Now define

$$\mathfrak{p}_\Phi = \mathfrak{p}_\Phi^r + \mathfrak{p}_\Phi^{-n} \text{ with } \mathfrak{p}_\Phi^r = \mathfrak{h} + \sum_{\alpha\in\Phi^r} g_\alpha \text{ and } \mathfrak{p}_\Phi^{-n} = \sum_{\alpha\in\Phi^n} \mathfrak{g}_{-\alpha}\,. \tag{1.10}$$

Then $\mathfrak{p}_\Phi$ is a subalgebra of $\mathfrak{g}$ that contains the Borel subalgebra (1.1), so it is a parabolic subalgebra of $\mathfrak{g}$.

**1.11. Proposition.** *Let $\mathfrak{p} \subset \mathfrak{g}$ be a subalgebra that contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$. Then there is a set $\Phi$ of simple roots such that $\mathfrak{p} = \mathfrak{p}_\Phi$.*

*Proof.* Define $\Phi = \{\psi \in \Psi \mid \mathfrak{g}_\psi \subset \mathfrak{p}\}$. Then $\mathfrak{p}_\Phi \subset \mathfrak{p}$, and we must prove $\mathfrak{p} \subset \mathfrak{p}_\Phi$. Both contain $\mathfrak{b}$, so this comes down to showing that $\alpha \in \Sigma^+, \mathfrak{g}_\alpha \subset \mathfrak{p}$ implies $n_\psi(\alpha) = 0$ whenever $\psi \in \Psi \setminus \Phi$. We will prove this by induction on the level $\ell(\alpha) = \sum n_\psi(\alpha)$.

If $\ell(\alpha) = 1$ then $\alpha$ is simple, so $\mathfrak{g}_\alpha \subset \mathfrak{p}$ implies $\alpha \in \Phi$. Then $\psi \notin \Phi$ implies $\psi \neq \alpha$ so $n_\psi(\alpha) = 0$.

Now let $\ell(\alpha) = \ell_0 > 1$ and suppose that $n_{\psi'}(\gamma) = 0$ for all $\psi' \in \Psi \setminus \Phi$, whenever $\gamma \in \Sigma^+$ and $\mathfrak{g}_\gamma \subset \mathfrak{p}$ with $\ell(\gamma) < \ell_0$. Suppose first that we can (and do) choose $\psi \in \Phi$ such that $\gamma = \alpha - \psi$ is a root. Then

$$\mathfrak{g}_\gamma = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\psi}] \subset [\mathfrak{p}, \mathfrak{b}^{-n}] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}.$$

If $\psi' \in \Psi \setminus \Phi$, then $n_{\psi'}(\alpha) = n_{\psi'}(\gamma)$, which is zero by the induction hypothesis. Suppose second that we cannot (and do not) choose $\psi$ from among the elements of $\Phi$. Then

$$\mathfrak{g}_\psi = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\gamma}] \subset [\mathfrak{p}, \mathfrak{b}^{-n}] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p},$$

so $\psi \in \Phi$, a contradiction. We have proved $n_{\psi'}(\gamma) = 0$ for all $\psi' \in \Psi \setminus \Phi$. Proposition 1.11 is proved. □

The **parabolic subgroup** $P \subset G$ corresponding to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is defined to be the $G$–normalizer of $\mathfrak{p}$, that is,

$$P = \{g \in G \mid \mathrm{Ad}\,(g)\mathfrak{p} = \mathfrak{p}\}\,. \tag{1.12}$$

The basic facts on parabolic subgroups are most easily derived from the corresponding results for Borel subgroups. However, the two notions were developed separately, and from different viewpoints, in the 1950s.

**1.13. Lemma.** *The parabolic subgroup $P \subset G$ defined by* (1.12) *has Lie algebra $\mathfrak{p}$. That group $P$ is a closed connected subgroup of $G$, and $P$ is its own normalizer in $G$. In particular, a Lie subgroup of $G$ is parabolic if and only if it contains a Borel subgroup.*

*Proof.* The argument of Lemma 1.3 shows that $P$ has Lie algebra $\mathfrak{p}$, is closed and connected, and is equal to its own $G$–normalizer. Let $S \subset G$ be a Lie subgroup that contains a Borel subgroup $B$. Then its Lie algebra $\mathfrak{s}$ contains $\mathfrak{b}$, hence is parabolic. Because $S$ is pinched between the analytic subgroup of $G$ for $\mathfrak{s}$ and the $G$–normalizer of $\mathfrak{s}$, which coincide because parabolic subgroups are closed and connected, $S$ is the parabolic subgroup of $G$ for $\mathfrak{s}$. □

Let $B \subset P \subset G$ consist of a Borel subgroup contained in a parabolic subgroup. Then we have complex homogeneous quotient spaces $X = G/B$ and $Z = G/P$ and a $G$–equivariant holomorphic projection $X \to Z$ given by $gB \mapsto gP$. In particular, transitivity of $G_u$ on $X$ gives transitivity of $G_u$ on $Z$ in

**1.14. Lemma.** *Let $G_u \subset G$ be a compact real form. Then $G_u$ is transitive on $Z = G/P$, and $Z$ has a $G_u$–invariant Kaehler metric. In particular $Z$ has the structure of compact Kaehler manifold.*

The argument of Lemma 1.4 is easily modified to prove the Kaehler statement in Lemma 1.14. Just take $\lambda$ in the dual space of the center of $\mathfrak{p}^r$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi^n$.

**1.15. Lemma.** *Fix a standard parabolic subgroup $P = P_\Phi$ in $G$. Then there is a finite dimensional irreducible representation $\pi$ of $G$ with the following property: Let $[v]$ be the image of a lowest weight vector in the projective space $\mathbb{P}(V_\pi)$ corresponding to the representation space of $\pi$. Then the action of $G$ on $V_\pi$ induces a holomorphic action of $G$ on $\mathbb{P}(V_\pi)$, and $P$ is the $G$–stabilizer of $[v]$. In particular $Z = G/P$ is a complete projective variety.*

*Proof.* We use the argument of Lemma 1.5, with a different choice of highest weight. Recall $\rho = \frac{1}{2}\sum_{\Sigma^+} \alpha$ and set $\rho_\Phi = \frac{1}{2}\sum_{\Phi^r \cap \Sigma^+} \alpha$. If $\psi \in \Psi$ now $\frac{2\langle \rho_\Phi, \psi \rangle}{\langle \psi, \psi \rangle}$ is 1 if $\psi \in \Phi$, is 0 if $\psi \notin \Phi$. Now let $\pi$ be the irreducible representation of $G$ with lowest weight $-(\rho - \rho_\Phi)$, in other words highest weight $w(\rho - \rho_\Phi)$ where $w$ is the element of the Weyl group that sends $\Sigma^+$ to its negative. Then the assertions are immediate. □

At this point we summarize, as follows.

**1.16. Proposition.** *Let $P$ be a complex Lie subgroup of $G$. Then the following conditions are equivalent.* (1) *$G/P$ is a compact complex manifold.* (2) *$G/P$ is a complete projective variety.* (3) *If $G_u$ denotes a compact real form of $G$ then $G/P$ is a $G_u$–homogeneous compact Kaehler manifold.* (4) *$G/P$ is the projective space orbit of an extremal weight vector in an irreducible finite dimensional representation of $G$.* (5) *$G/P$ is a $G$–equivariant quotient manifold of $G/B$, for some Borel subgroup $B \subset G$.* (6) *$P$ is a parabolic subgroup of $G$.*

We will simply refer to these spaces $Z = G/P$ as **complex flag manifolds**.

**References for §1.**


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## §2. Intersections of Parabolics.

In order to examine the orbit structure of a complex flag manifold $Z = G/P$ under the action of a real form $G_0$ of $G$, we need to know that the intersection of any two parabolic subgroups of $G$ contains a Cartan subgroup.

The Bruhat Lemma for the complex flag manifold $X = G/B$ is as follows. We may assume $B$ given by (1.1) and (1.2). Consider the Weyl group $W = W(\mathfrak{g}, \mathfrak{h}) = N_G(H)/H$. Given $w \in W$ choose a representative $s_w \in N_G(H)$. Let $x_0 = 1B \in G/B = X$. The crudest form of the Bruhat decomposition is sufficient for our needs. Here is the statement; I won't give a proof.

**2.1. Lemma.** *$X$ is the disjoint union of the $B$–orbits $B(s_w x_0), w \in W$.*

In fact this decomposes $X$ as a union of cells. To see that, one first notes that the isotropy subgroup of $B$ at $s_w x_0$ is the analytic subgroup $B_w$ of $G$ with Lie algebra $\mathfrak{b}_w = \mathfrak{h} + \sum_{\beta \in w(\Sigma^+)} \mathfrak{g}_{-\beta}$. One then checks that this decomposes $B = N_w(B \cap B_w)$ where $N_w$ is the unipotent analytic subgroup of $G$ with Lie algebra

$$\mathfrak{n}_w = \sum_{\alpha \in \Sigma^+ \cap w(\Sigma^-)} \mathfrak{g}_{-\alpha} \, .$$

Thus the map $\xi \mapsto \exp(\xi) s_w x_0$ gives a diffeomorphism of the real vector space $\mathfrak{n}_w$ onto the orbit $B(s_w x_0)$.

**2.2. Lemma.** *If $P_1$ and $P_2$ are parabolic subgroups of $G$ then $P_1 \cap P_2$ contains a Cartan subgroup of $G$.*

*Proof.* Let $\mathfrak{b}$ and $\mathfrak{b}'$ be Borel subalgebras of $\mathfrak{g}$. We will show that $\mathfrak{b} \cap \mathfrak{b}'$ contains a Cartan subalgebra of $\mathfrak{g}$. For this, we may assume that $\mathfrak{b}$ is our standard Borel $\mathfrak{h} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}$. Let $B$ and $B'$ be the corresponding Borel subgroups of $G$. Then $B'$ is the $G$–stabilizer of a point $x' \in X = G/B$. Following the Bruhat Lemma 2.1 we may take $x' = b s_w x_0$ for some $b \in B$ and $w \in W$. Without loss of generality we conjugate by $b^{-1}$. Now we may assume $x' = s_w x_0$. Then $B' = \mathrm{Ad}\,(s_w) B$ so $\mathfrak{b}' = \mathrm{ad}\,(s_w)\mathfrak{b}$, which contains $\mathfrak{h}$.

If $h \in H$ then $h$ normalizes both $\mathfrak{b}$ and $\mathfrak{b}'$, so $h \in B \cap B'$. Thus the intersection of two Borel subgroups contains a Cartan subgroup. The lemma follows. □

**2.3. Corollary.** *Let $\tau$ denote complex conjugation of $\mathfrak{g}$ over a real form $\mathfrak{g}_0$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{p} \cap \tau\mathfrak{p}$ contains a $\tau$–stable Cartan subalgebra of $\mathfrak{g}$.*

*Proof.* Set $\mathfrak{q} = \mathfrak{p} \cap \tau\mathfrak{p}$. It is a $\tau$–stable complex subalgebra of $\mathfrak{g}$, so $\mathfrak{q}_0 = \mathfrak{g}_0 \cap \mathfrak{q}$ is a real form of $\mathfrak{q}$ and $\tau$ induces the complex conjugation of $\mathfrak{q}$ over $\mathfrak{q}_0$. Choose a Cartan subalgebra $\mathfrak{j}_0$ of $\mathfrak{q}_0$. Its complexification $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{q}$. Lemma 2.2 says that $\mathfrak{q}$ contains Cartan subalgebras of $\mathfrak{g}$. Thus $\mathfrak{j}$ is a $\tau$–stable Cartan subalgebra of $\mathfrak{g}$. □

**References for §2.**


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## §3. Real Group Actions.

Let $G_0$ be a real form of $G$. In other words, $G_0$ is a Lie subgroup of $G$ whose Lie algebra $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. Although $G$ is connected, $G_0$ does not have to be connected. We write $\tau$ both for the complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$ and for the corresponding conjugation of $G$ over $G_0$.

Fix a parabolic subgroup $P \subset G$ and let $Z$ denote the corresponding complex flag manifold. Since $P$ is its own normalizer in $G$, we may view $Z$ as the space of all $G$–conjugates of $\mathfrak{p}$, by the correspondence $gP \leftrightarrow \mathrm{Ad}\,(g)\mathfrak{p}$. We will write $p_z$ for the parabolic subalgebra of $\mathfrak{g}$ corresponding to $z \in Z$, and will write $P_z$ for the corresponding parabolic subgroup of $G$.

Here is the principal trick for dealing with $G_0$–orbits on $Z$. We will use it constantly. Consider the orbit $G_0(z)$. The isotropy subgroup of $G_0$ at $z$ is $G_0 \cap P_z$. That isotropy subgroup has Lie algebra $\mathfrak{g}_0 \cap \mathfrak{p}_z$, which is a real form of $\mathfrak{p}_z \cap \tau\mathfrak{p}_z$. Lemma 2.3 says that $\mathfrak{p}_z \cap \tau\mathfrak{p}_z$ contains a $\tau$–stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Now $\mathfrak{p}_z$ contains a Borel subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$. Express that Borel as $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ for an appropriate choice of positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$. We have proved

**3.1. Theorem.** *Let $G_0$ be a real form of the complex semisimple Lie group $G$, let $\tau$ denote complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$, and consider an orbit $G_0(z)$ on a complex flag manifold $Z = G/P$. Then there exist a $\tau$–stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{p}_z$ of $\mathfrak{g}$, a positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$, and a set $\Phi$ of simple roots, such that $\mathfrak{p}_z = \mathfrak{p}_\Phi$ and $P_z = P_\Phi$.*

**3.2. Corollary.** *In the notation of* Theorem 3.1, $\mathfrak{p}_z \cap \tau\mathfrak{p}_z$ *is the semidirect sum of its nilpotent radical*

$$(\mathfrak{p}_\Phi^{-n} \cap \tau\mathfrak{p}_\Phi^{-n}) + (\mathfrak{p}_\Phi^{r} \cap \tau\mathfrak{p}_\Phi^{-n}) + (\mathfrak{p}_\Phi^{-n} \cap \tau\mathfrak{p}_\Phi^{r})$$

*with the Levi complement*

$$\mathfrak{p}_\Phi^r \cap \tau\mathfrak{p}_\Phi^r = \mathfrak{h} + \sum_{\Phi^r \cap \tau\Phi^r} \mathfrak{g}_\alpha \; .$$

*In particular,* $\dim_\mathbb{R} \mathfrak{g}_0 \cap \mathfrak{p}_z = \dim_\mathbb{C} \mathfrak{p}_\Phi^r + |\Phi^n \cap \tau\Phi^n|$.

*Proof.* The subspace $(\mathfrak{p}_\Phi^{-n} \cap \tau\mathfrak{p}_\Phi^{-n}) + (\mathfrak{p}_\Phi^{r} \cap \tau\mathfrak{p}_\Phi^{-n}) + (\mathfrak{p}_\Phi^{-n} \cap \tau\mathfrak{p}_\Phi^{r})$ of $\mathfrak{p}_\Phi \cap \tau\mathfrak{p}_\Phi$ is the sum of all root spaces $\mathfrak{g}_\alpha \subset \mathfrak{p}_\Phi \cap \tau\mathfrak{p}_\Phi$ such that $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}_\Phi \cap \tau\mathfrak{p}_\Phi$. So it is the nilradical of $\mathfrak{p}_\Phi \cap \tau\mathfrak{p}_\Phi$ . The subspace $\mathfrak{p}_\Phi^r \cap \tau\mathfrak{p}_\Phi^r = \mathfrak{h} + \sum_{\Phi^r \cap \tau\Phi^r} \mathfrak{g}_\alpha$ is a reductive subalgebra that is a vector space complement, so it is a Levi complement. Now compute

$$\begin{aligned}
&\dim_\mathbb{R} \mathfrak{g}_0 \cap \mathfrak{p}_z \\
&\qquad = \dim_\mathbb{C} \mathfrak{p}_\Phi \cap \tau\mathfrak{p}_\Phi = \dim_\mathbb{C} \mathfrak{h} + |(\Phi^r \cup \Phi^n) \cap \tau(\Phi^r \cup \Phi^n)| \\
&\qquad = \big(\dim_\mathbb{C} \mathfrak{h} + |\Phi^r \cap \tau\Phi^r| + |\Phi^n \cap \tau\Phi^r| + |\Phi^r \cap \tau\Phi^n|\big) + |\Phi^n \cap \tau\Phi^n| \\
&\qquad = \dim_\mathbb{C} \mathfrak{p}_\Phi^r + |\Phi^n \cap \tau\Phi^n|
\end{aligned}$$

as asserted. □

**3.3. Corollary.** *In the notation of* Theorem 3.1, $\operatorname{codim}_\mathbb{R}(G_0(z) \subset Z) = |\Phi^n \cap \tau\Phi^n|$. *In particular, $G_0(z)$ is open in $Z$ if and only if $\Phi^n \cap \tau\Phi^n$ is empty.*

*Proof.* In view of Corollary 3.2, the codimension in question is given by

$$\begin{aligned}
&\operatorname{codim}_\mathbb{R}(G_0(z) \subset Z) \\
&\qquad = \dim_\mathbb{R} Z - \dim_\mathbb{R} G_0(z) \\
&\qquad = 2|\Phi^n| - [\dim_\mathbb{R} G_0 - \dim_\mathbb{R}(G_0 \cap P_z)] \\
&\qquad = 2|\Phi^n| - [(\dim_\mathbb{R} \mathfrak{h} + |\Phi^r| + 2|\Phi^n|) - (\dim_\mathbb{R} \mathfrak{h} + |\Phi^r| + |\Phi^n \cap \tau\Phi^n|)] \\
&\qquad = |\Phi^n \cap \tau\Phi^n|
\end{aligned}$$

as asserted. □

**3.4. Corollary.** *The number of $G_0$–orbits on $Z$ is finite. The maximal–dimensional orbits are open and the minimal–dimensional orbits are closed.*

*Proof.* The number of $G_0$–conjugacy classes of Cartan subalgebras $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is finite. So the number of $G_0$–conjugacy classes of $\tau$–stable Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ is finite. Given such an $\mathfrak{h}$, the number of positive root systems $\Sigma^+$ is finite. Given $(\mathfrak{h}, \Sigma^+)$, the number of sets $\Phi$ of simple roots is finite. Thus the number of possibilities for $P_\Phi$ is finite up to $G_0$–conjugacy. This proves that the number of $G_0$–orbits on $Z$ is finite. It also gives a (very) rough upper bound on the number. The other statements follow because the closure of an orbit is a union of orbits. □

**References for §3.**


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## §4. Open Orbits.

Fix a Cartan involution $\theta$ of $\mathfrak{g}_0$ and $G_0$. In other words $\theta$ is an automorphism of square 1 and, using $G_0 \subset G$ so that $\mathfrak{g}_0$ is semisimple and $G_0$ has finite center, the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of $G_0$. Thus $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of $K_0$ and is the $(+1)$–eigenspace of $\theta$ on $\mathfrak{g}_0$, and $\mathfrak{s}_0$ is the $(-1)$–eigenspace. The Killing form of $\mathfrak{g}_0$ is negative definite on $\mathfrak{k}_0$ and is positive definite on $\mathfrak{s}_0$, and $\mathfrak{k}_0 \perp \mathfrak{s}_0$ under the Killing form.

Every Cartan subalgebra of $\mathfrak{g}_0$ is $\mathrm{Ad}(G_0)$–conjugate to a $\theta$–stable Cartan subalgebra. A $\theta$–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is called **fundamental** if it maximizes $\dim(\mathfrak{h}_0 \cap \mathfrak{k}_0)$, **compact** if it is contained in $\mathfrak{k}_0$, which is a more stringent condition. More generally, a Cartan subalgebra of $\mathfrak{g}_0$ is called **fundamental** if it is conjugate to a $\theta$–stable fundamental Cartan subalgebra.

**4.1. Lemma.** *The following conditions are equivalent for a $\theta$–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$.*

(i) $\mathfrak{h}_0$ *is a fundamental Cartan subalgebra of* $\mathfrak{g}_0$,
(ii) $\mathfrak{h}_0 \cap \mathfrak{k}_0$ *contains a regular element of* $\mathfrak{g}_0$, *and*
(iii) *there is a positive root system* $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h}), \mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$, *such that* $\tau\Sigma^+ = \Sigma^-$.

*A $\theta$–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is compact if and only if $\tau\Sigma^+ = \Sigma^-$ for every positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{h})$.*

**4.2. Theorem.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. The orbit $G_0(z)$ is open in $Z$ if and only if $\mathfrak{p}_z = \mathfrak{p}_\Phi$ where*

(i) $\mathfrak{p}_z \cap \mathfrak{g}_0$ *contains a fundamental Cartan subalgebra* $\mathfrak{h}_0 \subset \mathfrak{g}_0$ *and*
(ii) $\Phi$ *is a set of simple roots for a positive root system* $\Sigma^+(\mathfrak{g}, \mathfrak{h}), \mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$, *such that* $\tau\Sigma^+ = \Sigma^-$.

*Fix $\mathfrak{h}_0 = \theta\mathfrak{h}_0$, $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ and $\Phi$ as above. Let $W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0}$ and $W(\mathfrak{p}_\Phi^r, \mathfrak{h})^{\mathfrak{h}_0}$ denote the respective subgroups of Weyl groups that stabilize $\mathfrak{h}_0$. Then the open $G_0$–orbits on $Z$ are parameterized by the double coset space $W(\mathfrak{k}, \mathfrak{h} \cap \mathfrak{k})\backslash W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0} / W(\mathfrak{p}_z, \mathfrak{h})^{\mathfrak{h}_0}$.*

**4.3. Corollary.** *Suppose that $G_0$ has a compact Cartan subgroup, i.e. that $\mathfrak{k}_0$ contains a Cartan subalgebra of $\mathfrak{g}_0$. Then an orbit $G_0(z)$ is open in $Z$ if and only if $\mathfrak{g}_0 \cap \mathfrak{p}_z$ contains a compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and then,* *in the notation of* Theorem 4.2, *the open $G_0$–orbits on $Z$ are parameterized by* $W(\mathfrak{k},\mathfrak{h})\backslash W(\mathfrak{g},\mathfrak{h})/W(\mathfrak{p}_z^r,\mathfrak{h})$.

A careful examination of the way $\mathfrak{k}_0$ sits in both $\mathfrak{k}$ and $\mathfrak{g}_0$ gives us

**4.4. Theorem.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $z \in Z$ such that $G_0(z)$ is open in $Z$, and let $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{p}_z$ be a $\theta$–stable fundamental Cartan subalgebra of $\mathfrak{g}_0$. Then $K_0(z)$ is a compact complex submanifold of $G_0(z)$. Let $K$ be the complexification of $K_0$, analytic subgroup of $G$ with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$. Then $K_0(z) = K(z) \cong K/(K \cap P_z)$, complex flag manifold of $K$.*

The compact subvariety $K_0(z)$ controls the topology of an open orbit $G_0(z) \subset Z$, as follows. As we saw before, or by Corollary 4.3, the compact real form $G_u \subset G$ is transitive on $Z$. This gives us a realization $Z = G_u/V_u$ where $V_u \subset G_u$ is the centralizer of a torus subgroup. In particular, $V_u$ is connected. Since $G_0 \subset G$, $Z$ is compact and simply connected. In view of Theorem 4.4, one can apply this argument to the compact subvariety $K_0(z) \subset G_0(z)$, so it is simply connected. Now a deformation argument shows that the open orbit $G_0(z) \subset Z$ has $K_0(z)$ as a deformation retract, so $G_0(z)$ is simply connected. Thus one obtains

**4.5. Proposition.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $z \in Z$ such that $G_0(z)$ is open in $Z$. Then $G_0(z)$ is simply connected and $G_0$ has connected isotropy subgroup $(P_z \cap \tau P_z)_0$ at $z$.*

The compact subvariety $Y = K_0(z)$ also has a strong influence on the function theory for an open orbit $D = G_0(z) \subset Z$. The idea is that a holomorphic function on $D$ must be constant on $gY$ whenever $g \in G$ and $gY \subset D$, so if there are "too many" translates of $Y$ inside $D$ then that holomorphic function must be constant on $D$. But this has to be formulated carefully.

Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $z \in Z$ such that $G_0(z)$ is open in $Z$. Then there are decompositions $G = G_1 \times \cdots \times G_m$ and $P = P_1 \times \cdots \times P_m$ with $P_i = P \cap G_i$ and each $G_i$ simple. Consider the corresponding decompositions $Z = Z_1 \times \cdots \times Z_m$ with $Z_i = G_i/P_i$ and $z = (z_1, \dots, z_m)$, $G_0 = G_{1,0} \times \cdots \times G_{m,0}$, $G_0(z) = G_{1,0}(z_1) \times \cdots \times G_{m,0}(z_m)$ and $K_0(z) = K_{1,0}(z_1) \times \cdots \times K_{m,0}(z_m)$. If

(i) $G_{i,0} \cap (P_i)_{z_i} = ((P_i)_{z_i} \cap \tau(P_i)_{z_i})_0$ is compact, thus contained in $K_{i,0}$,
(ii) $G_{i,0}/K_{i,0}$ is an hermitian symmetric coset space, and
(iii) $G_{i,0}(z_i) \to G_{i,0}/K_{i,0}$ is holomorphic for one of the two invariant complex structures on $G_{i,0}/K_{i,0}$,

then we set $L_i = K_i$ so $L_{i,0} = K_{i,0}$. Otherwise we set $L_i = G_i$ so $L_{i,0} = G_{i,0}$. Note that each $G_{i,0}/L_{i,0}$ is a bounded symmetric domain, irreducible or reduced to a point. Set $L = L_0 \times \cdots \times L_m$ so $L_0 = L_{1,0} \times \cdots \times L_{m,0}$. Then we say that

$$D(G_0, z) = G_0/L_0 = (G_{1,0}/L_{1,0}) \times \cdots \times (G_{m,0}/L_{m,0}) \tag{4.6}$$

is the **bounded symmetric domain subordinate to** $G_0(z)$. Now we can state a precise result for holomorphic functions on $G_0(z)$.

**4.7. Theorem.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $z \in Z$ with $G_0(z)$ be open in $Z$. Let $D(G_0, z)$ be the bounded symmetric domain subordinate to $G_0(z)$. Then $\pi : g(z) \mapsto gL_0$ is a holomorphic map of $G_0(z)$ onto $D(G_0, z)$, and the holomorphic functions on $G_0(z)$ are just the $\widetilde{f} = f \cdot \pi$ where $f : D(G_0, z) \to \mathbb{C}$ is holomorphic.*

Thus, in most cases there are no nonconstant holomorphic functions on $G_0(z)$, but in fact this depends on some delicate structure.

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## §5. Example: Hermitian Symmetric Spaces.

In this section, $Z = G_u/K_0$ is an irreducible hermitian symmetric space of compact type. Thus $Z = G/P$ where $G$ is a connected simply connected complex simple Lie group with a real form $G_0 \subset G$ of hermitian type, as follows. Fix a Cartan involution $\theta$ of $G_0$ and the corresponding eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of the fixed point set $K_0 = G_0^\theta$. Then $G_u \subset G$ is the compact real form of $G$ that is the analytic subgroup for the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \mathfrak{s}_u$ of $\mathfrak{g}$ where $\mathfrak{s}_u = \sqrt{-1}\,\mathfrak{s}_0$ of $\mathfrak{g}$.

There is a compact Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ of $\mathfrak{g}_0$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{t})$ then either $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and we say that the root $\alpha$ is **compact**, or $\mathfrak{g}_\alpha \subset \mathfrak{s}$ and we say that $\alpha$ is **noncompact**. There is a simple root system $\Psi = \{\psi_0, \dots, \psi_m\}$ such that $\psi_0$ is noncompact and the other $\psi_i$ are compact. Furthermore, $\psi_0$ is a long root, and every noncompact positive root is of the form $\psi_0 + \sum_{1 \leqq i \leqq m} n_i \psi_i$ with each integer $n_i \geqq 0$. Thus $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}^+ + \mathfrak{s}^-$ where

$$\mathfrak{k} = \mathfrak{t} + \sum_{n_0=0} \mathfrak{g}_\alpha\,, \mathfrak{s}^+ = \sum_{n_0=1} \mathfrak{g}_\alpha\,, \text{ and } \mathfrak{s}^- = \sum_{n_0=-1} \mathfrak{g}_\alpha\,. \tag{5.1}$$

Here $\mathfrak{p} = \mathfrak{p}_{\{\psi_1,\dots,\psi_m\}}$, in other words

$$\mathfrak{p}^r = \mathfrak{k}\ ,\ \mathfrak{p}^n = \mathfrak{s}^+\ ,\ \text{and } \mathfrak{p}^{-n} = \mathfrak{s}^-\ ;\ \text{so } \mathfrak{p} = \mathfrak{k} + \mathfrak{s}^-\ . \tag{5.2}$$

The Cartan subalgebras of $\mathfrak{g}_0$ all are $\mathrm{Ad}(G_0)$–conjugate to one of the $\mathfrak{h}_{\Gamma,0}$ given as follows. Let $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ be a set of noncompact positive roots that is **strongly orthogonal** in the sense that

$$\text{if } 1 \leqq i < j \leqq r \text{ then none of } \pm\gamma_i \pm \gamma_j \text{ is a root.} \tag{5.3}$$

Then each $\mathfrak{g}[\gamma_i] = [\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{-\gamma_i}] + \mathfrak{g}_{\gamma_i} + \mathfrak{g}_{-\gamma_i} \cong \mathfrak{sl}(2,\mathbb{C})$, say with

$$h_{\gamma_i} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad e_{\gamma_i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad f_{\gamma_i} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $h_{\gamma_i} \in [\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{-\gamma_i}]$, $e_{\gamma_i} \in \mathfrak{g}_{\gamma_i}$ and $f_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$ as usual, and such that $\mathfrak{g}_0[\gamma_i] = \mathfrak{g}_0 \cap \mathfrak{g}_{\gamma_i} \cong \mathfrak{su}(1,1)$ is spanned by $\sqrt{-1}\,h_{\gamma_i}$, $e_{\gamma_i} + f_{\gamma_i}$ and $\sqrt{-1}\,(e_{\gamma_i} - f_{\gamma_i})$. Thus $\sqrt{-1}\,h_{\gamma_i}$ spans the compact Cartan subalgebra $\mathfrak{t}_{\gamma_i} = \mathfrak{g}_0[\gamma_i] \cap \mathfrak{t}$ of $\mathfrak{g}_0[\gamma_i]$ and $e_{\gamma_i} + f_{\gamma_i}$ spans the noncompact Cartan subalgebra $\mathfrak{a}_{\gamma_i} = \mathfrak{g}_0[\gamma_i] \cap \mathfrak{s}$ of $\mathfrak{g}_0[\gamma_i]$. Strong orthogonality (5.3) says $[\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{\gamma_j}] = 0$ for $1 \leqq i < j \leqq r$. Define

$$\mathfrak{t}_\Gamma = \sum_{1 \leqq i \leqq r} \mathfrak{t}_{\gamma_i} \text{ and } \mathfrak{a}_\Gamma = \sum_{1 \leqq i \leqq r} \mathfrak{a}_{\gamma_i}\ . \tag{5.4}$$

Then $\mathfrak{g}$ has Cartan subalgebras

$$\mathfrak{t} = \mathfrak{t}_\Gamma + (\mathfrak{t} \cap \mathfrak{t}_\Gamma^\perp) \quad \text{and} \quad \mathfrak{h}_\Gamma = \mathfrak{a}_\Gamma + (\mathfrak{t} \cap \mathfrak{t}_\Gamma^\perp) \tag{5.5}$$

They are $\mathrm{Int}(\mathfrak{g})$–conjugate, for the **partial Cayley transform**

$$c_\Gamma = \prod_{1 \leqq i \leqq r} \exp\left(\frac{\pi}{4}\sqrt{-1}\,(e_{\gamma_i} - f_{\gamma_i})\right) \text{ satisfies } \mathrm{Ad}(c_\Gamma)\mathfrak{t}_\Gamma = \mathfrak{a}_\Gamma\ . \tag{5.5}$$

However, their real forms

$$\mathfrak{t}_0 = \mathfrak{g}_0 \cap \mathfrak{t} \text{ and } \mathfrak{h}_{\Gamma,0} = \mathfrak{g}_0 \cap \mathfrak{h}_\Gamma \tag{5.6}$$

are not $\mathrm{Ad}(G_0)$–conjugate except in the trivial case where $\Gamma$ is empty, for the Killing form has rank $m = \dim \mathfrak{t}_0$ and signature $2|\Gamma| - m$ on $\mathfrak{h}_{\Gamma,0}$. More precisely,

**5.7. Proposition.** *Every Cartan subalgebra of $\mathfrak{g}_0$ is $\mathrm{Ad}(G_0)$–conjugate to one of the $\mathfrak{h}_{\Gamma,0}$, and Cartan subalgebras $\mathfrak{h}_{\Gamma,0}$ and $\mathfrak{h}_{\Gamma',0}$ are $\mathrm{Ad}(G_0)$–conjugate if and only if the cardinalities $|\Gamma| = |\Gamma'|$.*

We recall Kostant's "cascade construction" of a maximal set of strongly orthogonal noncompact positive roots in $\Sigma(\mathfrak{g}, \mathfrak{t})$. This set has cardinality $\ell = \mathrm{rank}_{\mathbb{R}}\mathfrak{g}_0$ and is given by

$$\begin{aligned} &\Xi = \{\xi_1, \dots, \xi_\ell\}, \text{ where} \\ &\xi_1 \text{ is the maximal (necessarily noncompact positive) root and} \\ &\xi_{m+1} \text{ is a maximal noncompact positive root } \perp \{\xi_1, \dots, \xi_m\}. \end{aligned} \tag{5.8}$$

The roots $\xi_i$ are long, and any set of strongly orthogonal noncompact positive long roots in $\Sigma(\mathfrak{g},\mathfrak{t})$ is $W(G_0,T_0)$–conjugate to a subset of $\Xi$. Further, the Weyl group $W(G_0,T_0)$ induces every permutation of $\Xi$.

Let $z_0 = 1 \cdot P \in G/P = Z$, the base point of our flag manifold $Z$ when $Z$ is viewed as a homogeneous space. The Cartan subalgebra $\mathfrak{h}_{\Gamma,0} \subset \mathfrak{g}_0$ leads to the orbits $G_0(c_\Gamma c_\Delta^2 z_0) \subset Z$ where $\Gamma \cup \Delta$ is a set of strongly orthogonal noncompact positive roots in $\Sigma(\mathfrak{g},\mathfrak{t})$ with $\Gamma$ and $\Delta$ disjoint. In view of the Weyl group equivalence just discussed, we may take $\Gamma = \{\xi_1,\dots,\xi_r\}$ and $\Delta = \{\xi_{r+1},\dots,\xi_{r+s}\}$, both inside $\Xi$. Using $G_0 = K_0\exp(\mathfrak{a}_{\Xi,0})K_0$ one arrives at

**5.9. Theorem.** *The $G_0$–orbits on $Z$ are just the orbits $D_{\Gamma,\Delta} = G_0(c_\Gamma c_\Delta^2 z_0)$ where $\Gamma$ and $\Delta$ are disjoint subsets of $\Xi$. Two such orbits $D_{\Gamma,\Delta} = D_{\Gamma',\Delta'}$ if and only if cardinalities $|\Gamma| = |\Gamma'|$ and $|\Delta| = |\Delta'|$. An orbit $D_{\Gamma,\Delta}$ is open if and only if $\Gamma$ is empty, closed if and only if $(\Gamma,\Delta) = (\Xi,\emptyset)$. An orbit $D_{\Gamma',\Delta'}$ is in the closure of $D_{\Gamma,\Delta}$ if and only if $|\Delta'| \leqq |\Delta|$ and $|\Gamma \cup \Delta| \leqq |\Gamma' \cup \Delta'|$.*

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## §6. The Closed Orbit.

There must be at least one closed $G_0$–orbit on $Z$, by Corollary 3.4. In the examples of §5 it is unique. We will see that it is unique in general and that it has some interesting structure.

First look at the case where $G = SL(2;\mathbb{C}), G_0 = SU(1,1)$, and $X$ is the Riemann sphere. $G$ acts as usual by linear fractional transformations. Then

$$G_0 = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,\middle|\, |a|^2 - |b|^2 = 1 \right\}. \tag{6.1}$$

and are three $G_0$–orbits, as follows.

The interior of the unit disk $G_0(0)$ :

$$P_0 = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \right\} \text{ and } H_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \, \theta \text{ real} \right\}.$$

The exterior of the unit disk $G_0(\infty)$ :

$$P_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\} \text{ and } H_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \, \theta \text{ real} \right\}.$$

(6.2) The unit circle $G_0(1)$ :

$$P_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1, a + b = c + d \right\}$$

$$\text{so } \mathfrak{p}_1^r = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\}, \mathfrak{p}_1^{-n} = \left\{ \begin{pmatrix} -b & b \\ -b & b \end{pmatrix} \right\}$$

$$\text{and } H_0 = \left\{ \pm \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \middle| \, t \text{ real} \right\}.$$

The first two give the open orbits, with $H_0$ compact, and the third gives the closed orbit, where $H_0$ is the $T_0A_0$ of an Iwasawa decomposition of $G_0$. This mirrors the general case for closed orbits:

**6.3. Theorem.** *Let $X = G/P$ be a complex flag manifold and let $G_0$ be a real form of $G$. Then there is a unique closed orbit $G_0(z) \subset Z$. Further, there is an Iwasawa decomposition $G_0 = K_0A_0N_0$ such that $G_0 \cap P_z$ contains $H_0N_0$ whenever $H_0$ is a Cartan subgroup of $G_0$ that contains $A_0$. (In other words, whenever $H_0 = T_0A_0$ where $T_0$ is a Cartan subgroup of the $K_0$–centralizer $M_0$ of $A_0$.)*

*Proof.* We first consider the case where $P = B$, Borel subgroup of $G$. Fix a closed orbit $G_0(x) \subset X$. Then $G_0(x)$ is compact. I claim that $G_0 \cap B_x$ contains the $A_0N_0$ of an Iwasawa decomposition $G_0 = K_0A_0N_0$. Let $H_0' \subset G_0 \cap B_x$ be a Cartan subgroup. Suppose that it is not conjugate to the $T_0A_0$ of a fixed minimal parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0 \subset \mathfrak{g}_0$. Replacing $\mathfrak{q}_0$ by a $G_0$–conjugate we then have $H_0' = T_0'A_0'$ with $T_0 \subsetneqq T_0' \subset K_0$ and $A_0' \subsetneqq A_0$. Then we have a root $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{h})$ that vanishes on $\mathfrak{t}$, and such that the intersection of

$$\mathfrak{g}[\alpha] = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \tag{6.4}$$

with $\mathfrak{h}$ is contained in $\mathfrak{a}$ while the intersection with $\mathfrak{h}'$ is contained in $\mathfrak{t}'$. This is exactly the example of (6.1) and (6.2). Now $G_0[\alpha](x)$ is an open hemisphere in $G[\alpha](x)$. But $G_0[\alpha]$ is closed in $G_0$ and has compact isotropy at $x$, so $G_0[\alpha](x)$ is closed in $G_0(x)$. With $G_0(x)$ closed in $X$ now $G_0[\alpha](x)$ is closed in $X$, thus closed in $G[\alpha](x)$, where in fact it is an open hemisphere. That contradicts our hypothesis that $G_0(x)$ is closed in $X$. We have proved[2] that $G_0 \cap B_x$ contains the $T_0 A_0$ of an Iwasawa decomposition of $G_0$.

Denote complexification by dropping the subscript 0. Since $T_0 A_0 \subset B_x$ now $B_x = B_{M,x} AN$ where $B_{M,x}$ is a Borel subgroup of $M = Z_K(A)$. It follows that $G_0 \cap B_x = T_0 A_0 N_0$. Now let $G_0(x')$ be another closed orbit on $X$. Then $B_{x'} = B_{M',x'} A' N'$ for another Iwasawa decomposition $G_0 = K_0' A_0' N_0'$ and a choice of Borel subgroup $B_{M',x'} \subset M'$. But any two Iwasawa decompositions of $G_0$ are conjugate by an element of $G_0$, and using compactness of $M_0$ we have that any two Borel subalgebras of $M$ are conjugate by an element of $M_0$. Thus $x' \in G_0(x)$ and $G_0(x) = G_0(x')$.

We have proved uniqueness of the closed orbit when $P$ is a Borel subgroup of $G$. For the general case, choose a Borel subgroup $B \subset P$ and note that the $G$–equivariant holomorphic fibration $\pi : X = G/B \to G/P = Z$ has compact fibres. Now the closed $G$–orbits in $Z$ are just the $\pi(G_0(x))$ where $G_0(x)$ is a closed $G$–orbit in $X$. The latter is unique. This completes the proof. □

Another interesting fact about the structure and geometry of closed orbits is

**6.5. Theorem.** *Let $Z = G/P$ be a complex flag manifold and let $G_0$ be a real form of $G$. Let $G_0(z)$ be the unique closed $G_0$–orbit on $Z$. Then $\dim_{\mathbb{R}} G_0(z) \geqq \dim_{\mathbb{C}} Z$, and the following conditions are equivalent.*

*1.* $\dim_{\mathbb{R}} G_0(z) = \dim_{\mathbb{C}} Z$.

*2.* $\tau\Phi^n = \Phi^n$.

*3. View $G$ as the group of complex points, and $G_0$ as an open subgroup in the group of real points, of a linear algebraic group defined over $\mathbb{R}$. Then $P_z$ is the group of complex points in an algebraic subgroup defined over $\mathbb{R}$.*

*4. $Z$ is the set of complex points in a projective variety defined over $\mathbb{R}$, and $G_0(z)$ is the set of real points.*

**References for §6.**


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## Part 2. The Bott–Borel–Weil Theorem and the Principal Series.

In this part we combine the Bott–Borel–Weil Theorem with unitary induction, realizing the unitary principal series on the closed orbit, in order to indicate the pattern used later for geometric realization of the standard tempered representations.

### §7. Principal Series and the Closed Orbit.

In order to introduce the connection between unitary representations of

[2]Here is a shorter, but less elementary, proof. $A_0 N_0$ is a solvable group acting birationally on the complete variety $G_0(x)$, so it has a fixed point by a theorem of Borel. If $g(x)$ is that fixed point then $\mathrm{Ad}(g^{-1})(A_0 N_0)$ fixes $x$.

$G_0$ and $G_0$–orbits on the complex flag manifold $Z = G/P$, we look at the **principal series** of $G_0$.

A subalgebra $\mathfrak{q}_0 \subset \mathfrak{g}_0$ is a **parabolic subalgebra** of $\mathfrak{g}_0$ if it is a real form of a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$. A subgroup $Q_0 \subset G_0$ is a **parabolic subgroup** of $G_0$ if it is a real form of a parabolic subalgebra $Q \subset G$, that is, if $Q_0 = G_0 \cap Q$ and its Lie algebra $\mathfrak{q}_0$ is a parabolic subalgebra of $\mathfrak{g}_0$. For example, fix an Iwasawa decomposition $G_0 = K_0 A_0 N_0$, and let $M_0 = Z_{K_0}(A_0)$, as usual. Then $Q_0 = M_0 A_0 N_0$ is minimal among the parabolic subgroups of $G_0$ and is called a **minimal parabolic subgroup**. From the construction, any two minimal parabolic subgroups of $G_0$ are conjugate. Now fix a minimal parabolic subgroup $Q_0 = M_0 A_0 N_0$.

Whenever $E$ is a topological group we write $\widehat{E}$ for its unitary dual. Thus $\widehat{E}$ consists of the unitary equivalence classes of (strongly continuous) topologically irreducible unitary representations of $E$. Now $[\eta] \in \widehat{M_0}$ and $\sigma \in \mathfrak{a}_0^*$ determine $[\alpha_{\eta,\sigma}] \in \widehat{Q_0}$ by

$$\alpha_{\eta,\sigma}(man) = \eta(m) e^{i\sigma(\log a)}. \tag{7.1}$$

The corresponding **principal series representation of** $G_0$ is

$$\pi_{\eta,\sigma} = \operatorname{Ind}_{Q_0}^{G_0}(\alpha_{\eta,\sigma}), \quad \text{unitarily induced representation.} \tag{7.2}$$

The **principal series** of $G_0$ consists of the unitary equivalence classes of these representations. A famous result of Bruhat says that if $\sigma$ satisfies a certain nonsingularity condition then $\pi_{\eta,\sigma}$ is irreducible.

In order to realize the principal series of $G_0$ on closed orbits, we need the Bott–Borel–Weil Theorem for $M_0$. We have to be careful here because the compact group $M_0$ need not be connected. We will first decompose $M_0$ as the product $Z_{M_0}(M_0^0)M_0^0$ where $M_0^0$ is its identity component, then indicate the analog of the Cartan highest weight description for $\widehat{M_0}$. That done, the standard Bott–Borel–Weil Theorem for $M_0^0$ will carry over to $M_0$.

Choose a Cartan subgroup $T_0 \subset M_0$. It specifies a Cartan subgroup $H_0 = T_0 A_0 \cong T_0 \times A_0$ in $G_0$. Our choice of $Q_0$ specifies a choice of positive restricted root system $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$: The Lie algebra of $N_0$ is given by $\mathfrak{n}_0 = \sum_{\alpha \in \Sigma^+(\mathfrak{g}_0,\mathfrak{a}_0)} (\mathfrak{g}_0)_{-\alpha}$. Now any positive root system $\Sigma^+(\mathfrak{m}, \mathfrak{t})$ specifies a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ by

$$\begin{aligned} \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{h}) \text{ if and only if either } \alpha|_{\mathfrak{a}_0} = 0 \text{ and } \alpha|_{\mathfrak{t}} \in \Sigma^+(\mathfrak{m}, \mathfrak{t}) \\ \text{or } \alpha|_{\mathfrak{a}_0} \neq 0 \text{ and } \alpha|_{\mathfrak{a}_0} \in \Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0). \end{aligned} \tag{7.3}$$

**7.4. Lemma.** $M_0 = Z_{M_0}(M_0^0)M_0^0$. *Given a unitary representation class* $[\eta] \in \widehat{M_0}$, *there exist unique classes* $[\chi] \in \widehat{Z_{M_0}(M_0^0)}$ *and* $[\eta^0] \in \widehat{M_0^0}$ *such that* $[\eta] = [\chi \otimes \eta^0]$, *and* $[\chi]$ *and* $[\eta^0]$ *restrict to multiples of the same unitary character on the center of* $M_0^0$.

*Remark. The argument will show that $T_0$ meets every topological component of $M_0$ .*

*Proof.* The first assertion is equivalent to the statement that if $m \in M_0$ then the coset $mM_0^0$ meets $Z_{M_0}(M_0^0)$. Replacing $m$ by some $mm'$ with $m' \in M_0^0$ we may assume that $\mathrm{Ad}(m)$ preserves both $T_0$ and a positive root system $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. By the definition of $M_0$, $\mathrm{Ad}(m)$ acts trivially on $A_0$, so it preserves the positive restricted root system $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Now $\mathrm{Ad}(m)$ preserves the positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{h})$ defined in (7.3). Thus $m$ centralizes $\mathfrak{h}$, that is, $m \in H$. Now $m \in M_0 \cap H = T_0$. In particular $\mathrm{Ad}(m)$ induces an inner automorphism on $M_0^0$. Thus $mM_0^0$ meets $Z_{M_0}(M_0^0)$, as claimed.

The second assertion follows from the first. □

Let $\Psi_{\mathfrak{m}}$ denote the set of simple roots in $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. Every subset $\Phi \subset \Psi_{\mathfrak{m}}$ defines

$$\begin{aligned}
&\mathfrak{z}_\Phi = \{\xi \in \mathfrak{t} \mid \phi(\xi) = 0 \text{ for all } \phi \in \Phi\} \\
&\qquad \text{and } \mathfrak{z}_{\Phi,0} = \mathfrak{m}_0 \cap \mathfrak{r}_\Phi, \text{ real form of } \mathfrak{z}_\Phi , \\
&U_\Phi = Z_M(\mathfrak{z}_\Phi), U_{\Phi,0} = M_0 \cap U_\Phi, \text{ and their Lie algebras } \mathfrak{u}_\Phi \text{ and } \mathfrak{u}_{\Phi,0} , \\
(7.5)\quad &\mathfrak{r}_\Phi = \mathfrak{u}_\Phi + \sum_{\gamma \in \Sigma^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_{-\gamma}, \text{ parabolic subalgebra of } \mathfrak{m} , \\
&R_\Phi = N_M(\mathfrak{r}_\Phi), \text{ corresponding parabolic subgroup of } M , \text{ and} \\
&S_\Phi = M/R_\Phi, \text{ associated complex flag manifold.}
\end{aligned}$$

Lemma 7.4 holds for $U_{\Phi,0}$ By Lemma 1.14, $M_0$ acts transitively on $S_\Phi$ , so $M_0 \cap R_\Phi = U_{\Phi,0}$ implies

**7.6. Lemma.** *$S_\Phi$ is a compact homogeneous Kaehler manifold under the action of $M_0$, and $S_\Phi = M_0/U_{\Phi,0}$ as coset space. Furthermore $U_{\Phi,0} = Z_{M_0}(M_0^0)U_{\Phi,0}^0$ , so $\widehat{U_{\Phi,0}}$ decomposes as does $\widehat{M_0}$ in* Lemma 7.4.

An irreducible unitary representation $\mu$ of $U_{\Phi,0}$ , say with representation space $V_\mu$ , gives us

$$\begin{aligned}
&\mathbb{V}_\mu \to S_\Phi : U_{\Phi,0}\text{–homogeneous, hermitian, holomorphic vector bundle,} \\
(7.7)\quad &A^{p,q}(S_\Phi; \mathbb{V}_\mu) : \text{ space of } C^\infty \mathbb{V}_\mu\text{–valued } (p,q)\text{–forms on } S_\Phi , \\
&\mathcal{O}(\mathbb{V}_\mu) : \text{ sheaf of germs of holomorphic sections of } \mathbb{V}_\mu \to S_\Phi .
\end{aligned}$$

If $\mathbb{T} \to S_\Phi$ is the holomorphic tangent bundle then $A^{p,q}(S_\Phi; \mathbb{V}_\mu)$ is the space of $C^\infty$ sections of

$$(7.8)\qquad \mathbb{V}_\mu^{p,q} = \mathbb{V}_\mu \otimes \Lambda^p(\mathbb{T}^*) \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to S_\Phi .$$

As $M_0$ is compact, $\mathbb{V}_\mu^{p,q}$ has an $M_0$–invariant hermitian metric, so we also have the Hodge–Kodaira orthocomplementation operators

$$(7.9)\qquad \begin{aligned}
&\sharp : A^{p,q}(S_\Phi; \mathbb{V}_\mu) \to A^{n-p,n-q}(S_\Phi; \mathbb{V}_\mu^*) \\
&\qquad\qquad \text{and } \tilde{\sharp} : A^{n-p,n-q}(S_\Phi; \mathbb{V}_\mu^*) \to A^{p,q}(S_\Phi; \mathbb{V}_\mu)
\end{aligned}$$

where $n = \dim_{\mathbb{C}} S_\Phi$ . The global $M_0$–invariant hermitian inner product on $A^{p,q}(S_\Phi;\mathbb{V}_\mu)$ is given by taking the inner product in each fibre of $\mathbb{V}_\mu^{p,q}$ and integrating over $S_\Phi$ . It can also be expressed in terms of the $\sharp$ operator,

$$(7.10) \qquad \langle F_1,F_2\rangle_{S_\Phi} = \int_{M_0} \langle F_1,F_2\rangle_{mU_{\Phi,0}} d(mU_{\Phi,0}) = \int_{S_\Phi} F_1 \bar{\wedge} \sharp F_2$$

where $\bar{\wedge}$ means exterior product followed by contraction of $V_\mu$ against $V_\mu^*$ . The last equality of (7.10) is essentially the definition of $\sharp$. Now the Cauchy–Riemann operator

$$\overline{\partial} : A^{p,q}(S_\Phi;\mathbb{V}_\mu) \to A^{p,q+1}(S_\Phi;\mathbb{V}_\mu)$$

has formal adjoint

$$(7.11) \qquad \overline{\partial}^* : A^{p,q+1}(S_\Phi;\mathbb{V}_\mu) \to A^{p,q}(S_\Phi;\mathbb{V}_\mu) \text{ given by } \overline{\partial}^* = -\tilde{\sharp}\overline{\partial}\sharp \ .$$

That in turn defines an operator that is elliptic $S_\Phi$ , the Kodaira–Hodge–Laplace operator

$$(7.12) \qquad \square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^{p,q}(S_\Phi;\mathbb{V}_\mu) \to A^{p,q}(S_\Phi;\mathbb{V}_\mu).$$

We have the space of square integrable $\mathbb{V}_\mu$–valued $(p,q)$–forms on $S_\Phi$ ,

$$(7.13) \qquad \begin{aligned} &L_2^{p,q}(S_\Phi;\mathbb{V}_\mu) : \\ &\qquad L_2 \text{ completion of } A^{p,q}(S_\Phi;\mathbb{V}_\mu) \text{ for the inner product (7.10)} \ . \end{aligned}$$

Weyl's Lemma says that the closure of $\widetilde{\square}$ of $\square$, as a densely defined operator on $L_2^{p,q}(S_\Phi;\mathbb{V}_\mu)$ from the domain $A^{p,q}(S_\Phi;\mathbb{V}_\mu)$, is essentially self–adjoint. Its kernel

$$(7.14) \qquad \mathcal{H}_2^{p,q}(S_\Phi;\mathbb{V}_\mu) = \{\omega \in \text{ Domain}(\widetilde{\square}) \mid \widetilde{\square}\omega = 0\}$$

is the space of **square integrable harmonic** $(p,q)$–forms on $S_\Phi$ with values in $\mathbb{V}_\mu$ . Harmonic forms are smooth by elliptic regularity, i.e., $\mathcal{H}_2^{p,q}(S_\Phi;\mathbb{V}_\mu) \subset A^{p,q}(S_\Phi;\mathbb{V}_\mu)$. Everything is invariant under the action of $M_0$, and the natural action of the group $M_0$ on $\mathcal{H}_2^{p,q}(S_\Phi;\mathbb{V}_\mu)$ is a unitary representation.

We write $\mathcal{H}_2^{q}(S_\Phi;\mathbb{V}_\mu)$ for $\mathcal{H}_2^{0,q}(S_\Phi;\mathbb{V}_\mu)$, because those are the only harmonic spaces that we will use, and because $\mathcal{H}_2^{q}(S_\Phi;\mathbb{V}_\mu)$ is naturally isomorphic to the sheaf cohomology $H^q(S_\Phi,\mathcal{O}(V_\mu))$.

Just to avoid confusion, we state some conventions explicitly. We will use (unless we state otherwise) $\chi$ for representations of $Z_{M_0}(M_0^0)$. We will use $\mu$ for representations of $U_{\Phi,0}$ and $\mu^0$ for representations of its identity component $U_{\Phi,0}^0$, $\rho_{\mathfrak{u}_\Phi}$ for half the sum of the roots in $\Sigma^+(\mathfrak{u}_\Phi,\mathfrak{t})$, and $\mu_\beta^0$ for the irreducible representation of $U_{\Phi,0}^0$ of highest weight $\beta - \rho_{\mathfrak{u}_\Phi}$ (corresponding to infinitesimal character $\beta$). Similarly, we will use $\eta$ for representations of $M_0$ and $\eta^0$ for representations of its identity component $M_0^0$, $\rho_{\mathfrak{m}}$ for half the sum of the roots in $\Sigma^+(\mathfrak{m},\mathfrak{t})$, and $\eta_\nu^0$ for the irreducible representation of $M_0^0$ of highest weight $\nu - \rho_{\mathfrak{m}}$ (corresponding to infinitesimal character $\nu$). With these conventions, the Bott–Borel–Weil Theorem for $M_0$ is

**7.15. Theorem.** *Let* $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ *and fix an integer* $q \geqq 0$.
1. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle = 0$ *for some* $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$ *then* $\mathcal{H}_2^q(S_\Phi; \mathbb{V}_\mu) = 0$.
2. *If* $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$, *let* $w$ *be the unique element in* $W(\mathfrak{m}, \mathfrak{t})$ *such that* $\nu = w(\beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}})$ *is in the positive Weyl chamber, i.e. satisfies* $\langle \nu, \alpha \rangle > 0$ *for all* $\alpha \in \Sigma^+(\mathfrak{m}, \mathfrak{t})$. *So* $q_0 = \text{length}(w) = |\{\alpha \in \Sigma^+(\mathfrak{m}, \mathfrak{t}) \mid \langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle < 0\}|$. *Then* $\mathcal{H}_2^q(S_\Phi; \mathbb{V}_\mu) = 0$ *for* $q \neq q_0$, *and* $M_0$ *acts irreducibly on* $\mathcal{H}_2^{q_0}(S_\Phi; \mathbb{V}_\mu)$ *by* $[\chi \otimes \eta_\nu^0]$.

Fix $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ as before. Given $\sigma \in \mathfrak{a}_0^*$ we will use the Bott–Borel–Weil Theorem to find the principal series representation $\pi_{\chi \otimes \eta_\nu^0, \sigma}$ on a cohomology space related to the closed orbit in the complex flag manifold $Z_\Phi = G/P_\Phi$. Here the simple root system $\Psi_{\mathfrak{m}} \subset \Psi$ by the coherence in our choice of $\Sigma^+(\mathfrak{g}, \mathfrak{h})$, so $\Phi \subset \Psi$ and $\Phi$ defines a parabolic subgroup $P_\Phi \subset G$.

Let $z_\Phi = 1P_\Phi \in G/P_\Phi = Z_\Phi$. As $A_0N_0 \subset G_0 \cap P_\Phi$ we have $G_0 \cap P_\Phi = U_{\Phi,0}A_0N_0$. Thus $Y_\Phi = G_0(z_\Phi)$ is the closed $G_0$–orbit on $Z_\Phi$, and $S_\Phi$ sits in $Y_\Phi$ as the orbit $M_0(z_\Phi)$. Here note that $Q_0 = M_0A_0N_0 = \{g \in G_0 \mid gS_\Phi = S_\Phi\}$.

**7.16. Lemma.** *The map* $Y_\Phi \to G_0/Q_0$, *given by* $g(z_\Phi) \mapsto gQ_0$, *defines a* $G_0$*–equivariant fibre bundle with structure group* $M_0$ *and whose fibres* $gS_\Phi$ *are the maximal complex analytic submanifolds of* $Y_\Phi$.

The data $(\mu, \sigma)$ defines a representation $\gamma_{\mu,\sigma}$ of $U_{\Phi,0}A_0N_0$ by

$$\text{(7.17a)} \qquad \gamma_{\mu,\sigma}(uan) = e^{(\rho_{\mathfrak{g}} + i\sigma)(\log a)}\mu(u) \quad \text{where } \rho_{\mathfrak{g}} = \tfrac{1}{2}\textstyle\sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha .$$

That defines a $G_0$–homogeneous complex vector bundle

$$\text{(7.17b)} \qquad \mathbb{V}_{\mu,\sigma} \to G_0/U_{\Phi,0}A_0N_0 = Y_\Phi \text{ such that } \mathbb{V}_{\mu,\sigma}|_{S_\Phi} = \mathbb{V}_\mu .$$

Each $\mathbb{V}_{\mu,\sigma}|_{gS_\Phi}$ is an $\mathrm{Ad}(g)Q_0$–homogeneous holomorphic vector bundle.

Since $[\mu]$ is unitary and $K_0$ acts transitively on $G_0/Q_0$ we have a $K_0$–invariant hermitian metric on $\mathbb{V}_{\mu,\sigma}$. We will use it without explicit reference.

Consider the subbundle $\mathbb{T} \to Y_\Phi$ of the complexified tangent bundle of $Y_\Phi$, defined by

$$\text{(7.18a)} \qquad \mathbb{T}|_{gS_\Phi} \to gS_\Phi \text{ is the holomorphic tangent bundle of } gS_\Phi .$$

It defines

$$\text{(7.18b)} \qquad \begin{aligned} &\mathbb{V}_{\mu,\sigma}^{p,q} = \mathbb{V}_{\mu,\sigma} \otimes \Lambda^p(\mathbb{T}^*) \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to Y_\Phi , \\ &A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) : C^\infty \text{ sections of } \mathbb{V}_{\mu,\sigma}^{p,q} \to Y_\Phi , \text{ and} \\ &\mathcal{O}(\mathbb{V}_{\mu,\sigma}) : \text{ sheaf of germs of } C^\infty \text{ sections of } \mathbb{V}_{\mu,\sigma} \to Y_\Phi \\ &\qquad\qquad\qquad \text{that are holomorphic over every } gS_\Phi . \end{aligned}$$

$A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is the space of $\mathbb{V}_{\mu,\sigma}$–valued partially $(p,q)$–forms on $Y_\Phi$.

The fibre $V_\mu$ of $\mathbb{V}_\mu \to S_\Phi$ has a positive definite $U_{\Phi,0}$–invariant hermitian inner product because $\mu$ is unitary; we translate this around by $K_0$ to obtain a

$K_0$–invariant hermitian structure on the vector bundle $\mathbb{V}^{p,q}_{\mu,\sigma} \to Y_\Phi$. Similarly $\mathbb{T} \to Y_\Phi$ carries a $K_0$–invariant hermitian metric. Using these hermitian metrics we have $K_0$–invariant Hodge–Kodaira orthocomplementation operators

$$(7.19) \qquad \begin{aligned} &\sharp : A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \to A^{n-p,n-q}(Y_\Phi; \mathbb{V}^*_{\mu,\sigma}) \\ &\tilde{\sharp} : A^{n-p,n-q}(Y_\Phi; \mathbb{V}^*_{\mu,\sigma}) \to A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \end{aligned}$$

where $n = \dim_{\mathbb{C}} S_\Phi$. The global $G_0$–invariant hermitian inner product on $A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is given by taking the $M_0$–invariant inner product along each fibre of $Y_\Phi \to G_0/Q_0$ and integrating over $G_0/Q_0$,

$$(7.20) \qquad \langle F_1, F_2 \rangle_{Y_\Phi} = \int_{K_0/M_0} \left( \int_{kS_\Phi} F_1 \bar{\wedge} \sharp F_2 \right) d(kM_0).$$

where $\bar{\wedge}$ means exterior product followed by contraction of $V_\mu$ against $V^*_\mu$.

The $\overline{\partial}$ operator of $Z_\Phi$ induces the $\overline{\partial}$ operators on each of the $gS_\Phi$, so they fit together to give us an operator

$$(7.21a) \qquad \overline{\partial} : A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \to A^{p,q+1}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$$

that has formal adjoint

$$(7.21b) \qquad \overline{\partial}^* : A^{p,q+1}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \to A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \text{ given by } \overline{\partial}^* = -\tilde{\sharp}\overline{\partial}\sharp .$$

This in turn defines an elliptic operator, the "partial Kodaira–Hodge–Laplace operator"

$$(7.21c) \qquad \square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}) \to A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}).$$

$A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is a pre Hilbert space with the global inner product (7.20). Denote

$$(7.22) \qquad L^{p,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma}) : \text{Hilbert space completion of } A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma}).$$

Apply Weyl's Lemma along each $gS_\Phi$ to see that the closure of $\widetilde{\square}$ of $\square$, as a densely defined operator on $L^{p,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ from the domain $A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$, is essentially self–adjoint. Its kernel

$$(7.23) \qquad \mathcal{H}^{p,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma}) = \{\omega \in \text{ Domain}(\widetilde{\square}) \mid \widetilde{\square}\omega = 0\}$$

is the space of **square integrable partially harmonic** $(p,q)$–forms on $Y_\Phi$ with values in $\mathbb{V}_{\mu,\sigma}$.

The factor $e^{\rho_{\mathfrak{s}}}$ in the representation $\gamma_{\mu,\sigma}$ that defines $\mathbb{V}_{\mu,\sigma}$ insures that the global inner product on $A^{p,q}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is invariant under the action of $G_0$. The other ingredients in the construction of $\mathcal{H}^{p,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ are invariant as well, so $G_0$ acts naturally on $\mathcal{H}^{p,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ by isometries. This action is a unitary representation of $G_0$.

Essentially as before, we write $\mathcal{H}^q_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ for $\mathcal{H}^{0,q}_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$, because those are the only harmonic spaces that we will use, and because $\mathcal{H}^q_2(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is closely related to the sheaf cohomology $H^q(Y_\Phi, \mathcal{O}(V_{\mu,\sigma}))$. The relation, which we will see later, is that they have the same underlying Harish–Chandra module.

We can now combine the Bott–Borel–Weil Theorem 7.15 with the definition ((7.1) and (7.2)) of the principal series, obtaining

**7.24. Theorem.** *Let $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ and $\sigma \in \mathfrak{a}_0^*$, and fix an integer $q \geqq 0$.*
1. *If $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle = 0$ for some $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$ then $\mathcal{H}_2^q(Y_\Phi; \mathbb{V}_{\mu,\sigma}) = 0$.*
2. *If $\langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$, let $w$ be the unique element in $W(\mathfrak{m}, \mathfrak{t})$ such that $\nu = w(\beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}})$ is in the positive Weyl chamber, i.e. satisfies $\langle \nu, \alpha \rangle > 0$ for all $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$. So $q_0 =$ length$(w) = |\{\alpha \in \Sigma^+(\mathfrak{m}, \mathfrak{t}) \mid \langle \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \alpha \rangle < 0\}|$. Then $\mathcal{H}_2^q(Y_\Phi; \mathbb{V}_{\mu,\sigma}) = 0$ for $q \neq q_0$, and the natural action of $G_0$ on $\mathcal{H}_2^{q_0}(Y_\Phi; \mathbb{V}_{\mu,\sigma})$ is the principal series representation $\pi_{\chi \otimes \eta_\nu^0, \sigma}$.*

**References for §7.**


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## PART 3. TEMPERED SERIES AND THE PLANCHEREL FORMULA.

In this part we indicate the basic facts on tempered representations and see just how the tempered series suffice for harmonic analysis on the real group.

### §8. THE DISCRETE SERIES.

We recall the definition and Harish–Chandra parametrization of the discrete series for reductive Lie groups. This can be viewed as a noncompact group version of Cartan's theory of the highest weight for representations of compact Lie groups.

The **discrete series** of a unimodular locally compact group $G_0$ is the subset $\widehat{G_{0,d}} \subset \widehat{G_0}$ consisting of all classes $[\pi]$ for which $\pi$ is equivalent to a subrepresentation of the left regular representation of $G_0$. The following are equivalent: (i) $\pi$ is a discrete series representation of $G_0$, (ii) every coefficient $f_{u,v}(x) = \langle u, \pi(x)v \rangle$ belongs to $L^2(G_0)$, (iii) for some nonzero $u, v$ in the representation space $H_\pi$, the coefficient $f_{u,v} \in L^2(G_0)$. Then one has orthogonality relations much as in the case of finite groups: There is a real number $\deg(\pi) > 0$

such that the $L^2(G_0)$-inner product of coefficients of $\pi$ is given by

$$(8.1a) \qquad \langle f_{u,v}, f_{s,t}\rangle = \frac{1}{\deg(\pi)}\langle u,s\rangle\overline{\langle v,t\rangle} \text{ for } s,t,u,v \in H_\pi.$$

Furthermore, if $\pi'$ is another discrete series representation of $G_0$, and is not equivalent to $\pi$, then

$$(8.1b) \qquad \langle f_{u,v}, f_{u',v'}\rangle = 0 \text{ for } u,v \in H_\pi \text{ and } u',v' \in H_{\pi'}.$$

In fact these orthogonality relations come out of convolution formulae. With the usual

$$f * h(x) = [L(f)h](x) = \int_G f(y)h(y^{-1}x)\, dy$$

we have

$$(8.2a) \qquad f_{u,v} * f_{s,t} = \frac{1}{\deg(\pi)}\langle u,t\rangle f_{s,v} \quad \text{for} \quad s,t,u,v \in H_\pi$$

and

$$(8.2b) \qquad f_{u,v} * f_{u',v'} = 0 \text{ for } u,v \in H_\pi \text{ and } u',v' \in H_{\pi'}$$

whenever $\pi$ and $\pi'$ are inequivalent discrete series representations of $G_0$.

If $G_0$ is compact, then every class in $\widehat{G_0}$ belongs to the discrete series, and if Haar measure is normalized as usual to total volume 1 then $\deg(\pi)$ has the usual meaning, the dimension of $H_\pi$. The orthogonality relations for irreducible unitary representations of compact groups are more or less equivalent to the Peter–Weyl Theorem.

More generally, if $G_0$ is a unimodular locally compact group then $L^2(G_0) = {}^0L^2(G_0) \oplus {}'L^2(G_0)$, orthogonal direct sum, where ${}^0L^2(G_0) = \sum_{[\pi]\in\widehat{G_{0,d}}} H_\pi \otimes H_\pi^*$, the "discrete" part, and ${}'L^2(G_0) = {}^0L^2(G_0)^\perp$, the "continuous" part. If, further, $G_0$ is a group of type I then ${}'L^2(G_0)$ is a continuous direct sum (direct integral) over $\widehat{G_0} \setminus \widehat{G_{0,d}}$ of the Hilbert spaces $H_\pi \otimes H_\pi^*$.

We will need the discrete series, not only for $G_0$ but for certain reductive subgroups as well. (A Lie group is called **reductive** if its Lie algebra is the direct sum of a semisimple Lie algebra and a commutative Lie algebra.) These reductive subgroups generally will not be semisimple, and even if $G_0$ is connected they will generally not be connected. So we want to work with a class of groups that is hereditary in the sense that it includes all the connected semisimple Lie groups of finite center, and also includes the above-mentioned subgroups of groups in the class. This is the **Harish–Chandra class**, or **class** $\mathcal{H}$.

While I'll state results for Harish–Chandra class, I'll set things up so that the statements remain valid without essential change for the larger hereditary class that contains all connected semisimple groups, whether of finite or of infinite center.

Let $G_0$ be a reductive Lie group, $G_0^0$ its identity component, $\mathfrak{g}_0$ its Lie algebra, and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Suppose that $[G_0^0, G_0^0]$ has finite center, that $G_0/G_0^0$ is finite, and that if $x \in G_0$ then $\mathrm{Ad}(x)$ is an inner automorphism of $\mathfrak{g}$. Then we say that $G_0$ belongs to class $\mathcal{H}$. From now on we will assume that $G_0$ belongs to class $\mathcal{H}$.

If $\pi$ is a unitary representation of $G_0$, and if $f \in L^1(G_0)$, we have the bounded operator $\pi(f) = \int_G f(x)\pi(x)dx$ on $H_\pi$. Now suppose that $\pi$ has finite composition series, i.e., is a finite sum of irreducible representations. If $f \in C_c^\infty(G_0)$ then $\pi(f)$ is of trace class. Furthermore, the map

$$(8.3) \qquad \Theta_\pi : C_c^\infty(G_0) \to \mathbb{C} \text{ defined by } \Theta_\pi(f) = \text{ trace } \pi(f)$$

is a distribution on $G_0$. $\Theta_\pi$ is called the **character**, the **distribution character** or the **global character** of $\pi$.

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. If we interpret $\mathcal{U}(\mathfrak{g})$ as the algebra of all left-invariant differential operators on $G_0$ then $\mathcal{Z}(\mathfrak{g})$ is the subalgebra of those that are also invariant under right translations. If $\pi$ is irreducible then $d\pi|_{\mathcal{Z}(\mathfrak{g})}$ is an associative algebra homomorphism $\chi_\pi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ called the **infinitesimal character** of $\pi$. We say that $\pi$ is **quasi-simple** if it has an infinitesimal character, i.e. if it is a direct sum of irreducible representations that have the same infinitesimal character.

Let $\pi$ be quasi-simple. Then the distribution character $\Theta_\pi$ satisfies a system of differential equations

$$(8.4) \qquad z \cdot \Theta_\pi = \chi_\pi(z)\Theta_\pi \text{ for all } z \in \mathcal{Z}(\mathfrak{g}).$$

The regular set

$$G_0' = \{x \in G_0 : \mathfrak{g}^{\mathrm{Ad}(x)} \text{ is a Cartan subalgebra of } \mathfrak{g}\}$$

is a dense open subset whose complement has codimension $\geqq 2$. Every $x \in G_0'$ has a neighborhood on which at least one of the operators $z \in \mathcal{Z}(\mathfrak{g})$ is elliptic. It follows that $\Theta_\pi|_{G_0'}$ is integration against a real analytic function $T_\pi$ on $G_0'$. A much deeper result of Harish–Chandra says that $\Theta_\pi$ has only finite jump singularities across the singular set $G_0 \setminus G_0'$, so $T_\pi$ is locally $L^1$ and $\Theta_\pi$ is integration against it,

$$(8.5) \qquad \Theta_\pi(f) = \int_{G_0} f(x)T_\pi(x)dx \text{ for all } f \in C_c^\infty(G_0).$$

So we may (and do) identify $\Theta_\pi$ with the function $T_\pi$. This key element of Harish–Chandra's theory allows the possibility of *a priori* estimates on characters and coefficients as well as explicit character formulae.

Fix a Cartan involution $\theta$ of $G_0$. In other words, $\theta$ is an automorphism of $G_0$, $\theta^2$ is the identity, and the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of $G_0$. The choice is essentially unique, because the Cartan involutions of $G_0$ are just the $\mathrm{Ad}(x) \cdot \theta \cdot \mathrm{Ad}(x)^{-1}$, $x \in G_0^0$. If $G_0 = U(p,q)$ then $\theta(x) = {}^t x^{-1}$ and $K_0 = U(p) \times U(q)$.

Every Cartan subgroup of $G_0$ is $\mathrm{Ad}(G_0^0)$-conjugate to a $\theta$-stable Cartan subgroup. In particular, $G_0$ has compact Cartan subgroups if and only if $K_0$ contains a Cartan subgroup of $G_0$.

Harish–Chandra proved that $G_0$ has discrete series representations if and only if it has a compact Cartan subgroup. Suppose that this is the case and fix a compact Cartan subgroup $T_0 \subset K_0$ of $G_0$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{t})$ be the root system, $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{t})$ a choice of positive root system, and let $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} \alpha$. If $\xi \in \mathfrak{t}$ then $\rho(\xi)$ is half the trace of $ad(\xi)$ on $\sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.

If $\pi$ is a discrete series representation of $G_0$ and $\Theta_\pi$ is its distribution character, then the equivalence class of $\pi$ is determined by the restriction of $\Theta_\pi$ to $T_0 \cap G_0'$. Harish–Chandra parameterizes the discrete series of $G_0$ by parameterizing those restrictions.

Let $G_0^\dagger$ denote the finite index subgroup $T_0 G_0^0 = Z_{G_0}(G_0^0) G_0^0$ of $G_0$. In fact the argument of Lemma 7.4 is easily modified here to prove $T_0 = Z_{G_0}(G_0^0) T_0^0$, so $T_0 = T_0^\dagger$. Lemma 7.4 says that the group $M_0$ of a minimal parabolic subgroup of $G_0$ satisfies $M_0 = M_0^\dagger$, and similarly, we have $U_{\Phi,0} = U_{\Phi,0}^\dagger$. In general, where $M_0$ may be noncompact, this need not hold.

The Weyl group $W^\dagger = W(G_0^\dagger, T_0)$ coincides with $W^0 = W(G_0^0, T_0^0)$ and is a normal subgroup of $W = W(G_0, T_0)$.

Every irreducible unitary representation of $T_0 = Z_{G_0}(G_0^0) T_0^0$ is of the form $\chi \otimes e^{i(\lambda - \rho)}$ where $\lambda \in i\mathfrak{t}_0^*$ and $\lambda - \rho$ satisfies an integrality condition, where $\chi \in \widehat{Z_{G_0}(G_0^0)}$, and where $\chi$ and $e^{i(\lambda-\rho)}$ restrict to (multiples of) the same unitary character on the center of $G_0^0$.

Let $\chi \otimes e^{i(\lambda-\rho)} \in \widehat{T_0}$ as above. Suppose that $\lambda$ is regular, i.e., that $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma$. Then there are unique discrete series representations $\pi_\lambda^0$ of $G_0^0$ and $\pi_{\chi,\lambda}^\dagger$ of $G_0^\dagger$ whose distribution characters satisfy

$$\begin{gathered}\Theta_{\pi_\lambda^0}(x) = (-1)^{q(\lambda)} \frac{\sum_{w \in W^0} \mathrm{sign}(w) e^{w(\lambda)}}{\prod_{\alpha \in \Sigma^+} (e^{\alpha/2} - e^{-\alpha/2})} \\ \text{and } \Theta_{\pi_{\chi,\lambda}^\dagger}(zx) = \mathrm{trace}\, \chi(z) \Theta_{\pi_\lambda^0}(x)\end{gathered} \tag{8.6a}$$

for $z \in Z_{G_0}(G_0^0)$ and $x \in T_0^0 \cap G_0'$, where

$$\begin{aligned}q(\lambda) = {} & |\{\alpha \in \Sigma^+(\mathfrak{k}, \mathfrak{t}) \mid \langle \alpha, \lambda \rangle < 0\}| \\ & + |\{\beta \in \Sigma^+(\mathfrak{g}, \mathfrak{t}) \setminus \Sigma^+(\mathfrak{k}, \mathfrak{t}) \mid \langle \beta, \lambda \rangle > 0\}|.\end{aligned} \tag{8.6b}$$

Here note that $\pi_{\chi,\lambda}^\dagger = \chi \otimes \pi_\lambda^0$.

The same datum $(\chi, \lambda)$ specifies a discrete series representation $\pi_{\chi,\lambda}$ of $G_0$, by the formula $\pi_{\chi,\lambda} = \mathrm{Ind}_{G_0^\dagger}^{G_0}(\pi_{\chi,\lambda}^\dagger)$. This induced representation is irreducible because its conjugates by elements of $G_0/G_0^\dagger$ are mutually inequivalent, consequence of regularity of $\lambda$. $\pi_{\chi,\lambda}$ is characterized by the fact that its distribution character is supported in $G_0^\dagger$ and is given on $G_0^\dagger$ by

$$\Theta_{\pi_{\chi,\lambda}} = \sum\nolimits_{1 \leqq i \leqq r} \Theta_{\pi^\dagger_{\chi,\lambda}} \cdot \gamma_i^{-1} \tag{8.7}$$

with $\gamma_i = \mathrm{Ad}(g_i)|_{G_0^\dagger}$ where $\{g_1, \dots, g_r\}$ is any system of coset representatives of $G_0$ modulo $G_0^\dagger$. To combine these into a single formula one chooses the $g_i$ so that they normalize $T_0$, i.e. chooses the $\gamma_i$ to be a system of coset representatives of $W$ modulo $W^\dagger$.

Every discrete series representation of $G_0$ is equivalent to a representation $\pi_{\chi,\lambda}$ as just described. Discrete series representations $\pi_{\chi,\lambda}$ and $\pi_{\chi',\lambda'}$ are equivalent if and only if $\chi' \otimes e^{i\lambda'} = (\chi \otimes e^{i\lambda}) \cdot w^{-1}$ for some $w \in W$. And $\lambda$ is both the infinitesimal character and the Harish–Chandra parameter for the discrete series representation $\pi_{\chi,\lambda}$.

**References for §8.**


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## §9. The Tempered Series.

The representations of $G_0$ that enter into its Plancherel formula are the **tempered representations**. They are constructed from a class of real parabolic subgroups of $G_0$ called the **cuspidal parabolic subgroups**. One constructs a standard tempered representation by first constructing a relative discrete series representation for the reductive part of cuspidal parabolic subgroup, and then by unitary induction from the parabolic subgroup up to $G_0$. We start by recalling the definitions.

Let $H_0$ be a Cartan subgroup of $G_0$. Fix a Cartan involution $\theta$ of $G_0$ such that $\theta(H_0) = H_0$. We write $K_0$ for the fixed point set $G_0^\theta$, which is a maximal compact subgroup of $G_0$. Decompose

$$\begin{aligned}&\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ and } H_0 = T_0 \times A_0 \\ &\qquad \text{where} \quad T_0 = H_0 \cap K_0,\ \theta(\xi) = -\xi \text{ on } \mathfrak{a}_0, \text{ and } A_0 = \exp_G(\mathfrak{a}_0).\end{aligned} \tag{9.1}$$

Then the centralizer $Z_{G_0}(A_0)$ of $A_0$ in $G_0$ has form $M_0 \times A_0$ where $\theta(M_0) = M_0$. The group $M_0$ is a reductive Lie group of Harish–Chandra class. $T_0$ is a compact Cartan subgroup of $M_0$, so $M_0$ has discrete series representations.

Suppose that our positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$ is defined by positive root systems $\Sigma^+(\mathfrak{m}, \mathfrak{t})$ and $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$ as in (7.3).

A (real) parabolic subgroup $P_0 \subset G_0$ is called **cuspidal** if the commutator subgroup of the Levy component (reductive part) has a compact Cartan subgroup.

The Cartan subgroup $H_0 \subset G_0$ defines a cuspidal parabolic subgroup $P_0 = M_0A_0N_0$ of $G_0$ as follows. The Lie algebra of $N_0$ is $\mathfrak{n}_0 = \sum_{\alpha \in \Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_{-\alpha}$, $M_0$ and $A_0$ are as above, and $M_0A_0 = M_0 \times A_0$ is the Levi component of $P_0$. One extreme is the case where $\dim \mathfrak{a}_0$ is maximal; then $P_0$ is a minimal parabolic subgroup of $G_0$. The other extreme is where $\dim \mathfrak{a}_0$ is minimal; if $\mathfrak{a}_0 = 0$ then $P_0 = G_0$.

Every cuspidal parabolic subgroup of $G_0$ is produced by the construction just described, as $H_0$ varies. Two cuspidal parabolic subgroups of $G_0$ are **associated** if they are constructed as above from $G_0$–conjugate Cartan subgroups; then we say that the $G_0$–conjugacy class of Cartan subgroups is **associated** to the $G_0$–association class of cuspidal parabolic subgroups.

As in (7.1),

$$\begin{aligned}&[\eta] \in \widehat{M_0} \text{ and } \sigma \in \mathfrak{a}_0^* \\ &\qquad \text{determine } [\alpha_{\eta,\sigma}] \in \widehat{P_0} \text{ by } \alpha_{\eta,\sigma}(man) = \eta(m)e^{i\sigma(\log a)}.\end{aligned} \tag{9.2}$$

Then we have

$$\pi_{\eta,\sigma} = \operatorname{Ind}_{P_0}^{G_0}(\alpha_{\eta,\sigma}) , \qquad \text{unitarily induced representation.} \tag{9.3}$$

The $H_0$**–series** or **principal** $H_0$**–series** of $G_0$ consists of the unitary equivalence classes of the representations (9.3) for which $\eta$ is a discrete series representation of $M_0$. Harish–Chandra extended Bruhat's irreducibility results to all the $H_0$–series.

As the terminology indicates, $\pi_{\eta,\sigma} = \operatorname{Ind}_{P_0}^{G_0}(\alpha_{\eta,\sigma})$ is independent of the choice of $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$. In fact this is the case even if $\eta$ does not belong to the discrete series of $M_0$, and is a consequence of the character formula, which we now describe.

If $J_0$ is a Cartan subgroup of $G_0$ we write $G'_{J_0}$ for the set of $G_0$–regular elements that are $G_0$–conjugate to an element of $J_0$. If further we fix a positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{j})$ then we write $\Delta_{G_0,J_0} = \prod_{\gamma\in\Sigma^+(\mathfrak{g},\mathfrak{j})}(e^{\gamma/2} - e^{-\gamma/2})$. Passing to a 2–sheeted cover if necessary (it is not necessary if $G_0 \subset G$ with $G$ complex and simply connected), $e^\rho$ and $\Delta_{G_0,J_0}$ are well defined functions on $J_0$.

When dealing with both $G_0$ and $M_0$, we write $M''_0$ for the $M_0$–regular subset of $M_0$. If $L_0$ is a Cartan subgroup of $M_0$ we write $M''_{L_0}$ for the set of elements of $M''_0$ that are $M_0$–conjugate to an element of $L_0$.

**9.4. Theorem.** *Fix a cuspidal parabolic subgroup $P_0 = M_0A_0N_0$ of $G_0$, let $[\eta] \in \widehat{M_0}$, and let $\sigma \in \mathfrak{a}_0^*$. Let $\chi_\nu$, with $\nu \in \mathfrak{t}^*$, be the infinitesimal character of $\eta$ and let $\Psi_\eta$ be the distribution character.*

1. *$[\pi_{\eta,\sigma}]$ has infinitesimal character $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}$.*

2. *$[\pi_{\eta,\sigma}]$ is a finite sum of classes from $\widehat{G_0}$. So it has well defined distribution character $\Theta_{\pi_{\eta,\sigma}}$ that is a locally summable function analytic on the regular set $G'_0$.*

3. *$\Theta_{\pi_{\eta,\sigma}}$ has support in the closure of $\bigcup G'_{J_0}$ where $J_0$ runs over a system of representatives of the $G_0$-conjugacy classes of Cartan subgroups of $M_0A_0$.*

4. *Fix a Cartan subgroup $J_0 = J_{M,0} \times A_0$ of $M_0A_0$. Let $\{J_{i,0} = g_iJ_0g_i^{-1} \mid 1 \leqq i \leqq \ell(J_0)\}$ be a system of representatives of the $M_0A_0$-conjugacy classes of Cartan subgroups of $M_0A_0$ that are $G_0$-conjugate to $J_0$. For each index $i$ let $N_{G_0}(J_{i,0})$ and $N_{M_0A_0}(J_{i,0})$ denote normalizers in $G_0$ and $M_0A_0$. Let $h \in J_0 \cap G'_0$ and define $h_i = g_ihg_i^{-1} \in J_{i,0}$ Then the sets $N_{G_0}(J_{i,0})(h_i)$ and $N_{M_0A_0}(J_{i,0})(h_i)$ are finite, and $\Theta_{\pi_{\eta,\sigma}}(h)$ is given by*

$$\begin{aligned}&\sum_{i=1}^{\ell(J_0)} \frac{1}{|\Delta_{G_0,J_{i,0}}(h_i)|}\times \\ &\times \sum_{N_{G_0}(J_{i,0})(h_i)} \frac{|\Delta_{M_0A_0,J_{i,0}}(wh_i)|}{|N_{M_0A_0}(J_{i,0})(wh_i)|}\Psi_\eta((wh_i)_{M_0})e^{i\sigma(\log(wh_i)_{A_0})} .\end{aligned} \tag{9.5}$$

*If $h \in J_0^0$, so each $h_i \in J_{i,0}^0$, then the second sum runs over the Weyl group $W(G_0, J_{i,0})$.*

5. *If $t \in T_0$ and $a \in A_0$ with $ta \in G_0'$ then* (9.5) *reduces to*

$$(9.6)\quad \Theta_{\pi_{\eta,\sigma}}(ta) = \frac{|\Delta_{M_0,T_0}(t)}{|\Delta_{G_0,H_0}(ta)|} \sum_{N_{G_0}(H_0)(ta)} \frac{1}{|N_{M_0}(T_0)(wt)|} \Psi_\eta(wt) e^{i\sigma(\log(wa))} .$$

The formula (9.5) shows in particular that the distribution $\Theta_{\pi_{\eta,\sigma}}$ is independent of the choice of cuspidal parabolic subgroup $P_0$ associated to the $G_0$–conjugacy class of $H_0$. As $[\pi_{\eta,\sigma}]$ is a finite sum from $\widehat{G_0}$, now $[\pi_{\eta,\sigma}]$ also is independent of choice of $P_0$ for the given $H_0$. So Theorem 9.4 implies

**9.7. Corollary.** *The class $[\pi_{\eta,\sigma}]$ is independent of choice of cuspidal parabolic subgroup $P_0 = M_0 A_0 N_0$ for the given Cartan subgroup $H_0 = T_0 \times A_0$.*

The proof of Theorem 9.4 is a bit technical. It is based on the Harish–Chandra transform $\mathcal{F}_{P_0} : C_0^\infty(G_0) \to C_0^\infty(M_0 A_0)$, given by

$$(9.8)\qquad \mathcal{F}_{P_0}(b)(ma) = e^{-\rho(\log a)} \int_{K_0} \left\{ \int_{N_0} b(kmank^{-1}) dn \right\} dk .$$

One first proves that $\pi_{\eta,\sigma}(b)$ is of trace class with

$$(9.9)\qquad \operatorname{trace} \pi_{\eta,\sigma}(b) = \int_{M_0 A_0} \mathcal{F}_{P_0}(b)(ma) \Psi_\eta(m) e^{i\sigma(\log a)} dm da .$$

Then one can calculate the infinitesimal character. From that, a look at $K_0$–types proves finiteness of the composition series. Then one has to extend the Weyl integration formula appropriately in order to compute the character formulae.

Theorem 9.4 specializes to the $H_0$–series as follows. Express

$$(9.10)\qquad \eta = \eta_{\chi,\nu} = \operatorname{Ind}_{M_0^\dagger}^{M_0} (\chi \otimes \eta_\nu^0)$$

corresponding to $\chi \in Z_{M_0}(M_0^0)$ and $e^{\nu - \rho_m} \in \widehat{T_0^0}$ that restrict to multiples of the same unitary character on the center of $M_0^0$. Choose coset representatives $\{x_1, \dots, x_\ell\}$ of $M_0$ modulo $M_0^\dagger$ that normalize $\mathfrak{t}_0$. They represent Weyl group elements $w_i \in W(M_0, T_0)$ that form a system of representatives of $W(M_0, T_0)$ modulo $W(M_0^0, T_0^0)$. Now, following (8.6) and (8.7), the distribution character of $\eta$ is supported on $M_0^\dagger$, and it satisfies

$$(9.11)\qquad \begin{aligned} \Psi_{\eta_{\chi,\nu}}(zt) = &\sum_{i=1}^{\ell} (-1)^{q_m(w_i\nu)} \operatorname{trace} \chi(x_i z x_i^{-1}) \times \\ &\times \frac{1}{\Delta_{M_0,T_0}(t)} \sum_{W(M_0^0, T_0^0)} \det(w w_i) e^{w w_i \nu}(t) \end{aligned}$$

for $z \in Z_{M_0}(M_0^0)$ and $t \in T_0^0 \cap G_0'$. The formula (9.11) characterizes $[\eta_{\chi,\nu}]$. With Theorem 9.4 it gives

**9.12. Theorem.** *Let* $[\eta_{\chi,\nu}] \in \widehat{M_{0,d}}$ *as in* (9.10) *and let* $\sigma \in \mathfrak{a}_0^*$. *Then* $[\pi_{\chi,\nu,\sigma}] = [\mathrm{Ind}_{P_0}^{G_0}(\eta_{\chi,\nu} \otimes e^{i\sigma})]$ *is the unique* $H_0$*–series representation class on* $G_0$ *whose distribution character satisfies*

$$\Theta_{\pi_{\eta_{\chi,\nu},\sigma}}(zta) =$$

$$\frac{|\Delta_{M_0,T_0}(zt)|}{|\Delta_{G_0,H_0}(zta)|} \sum_{w(zta)} \frac{1}{|N_{M_0}(T_0)(w(zt))|} \Psi_{\eta_{\chi,\nu}}(w(zt)) e^{i\sigma(\log(wa))} \tag{9.13}$$

*where* $w(zta)$ *runs over* $N_{G_0}(H_0)(zta)$, *the* $\Psi_{\eta_{\chi,\nu}}(w(zt))$ *are given by* (9.11), $z \in Z_{M_0}(M_0^0)$, $t \in T_0^0 \cap M_0''$ *and* $a \in A_0$.

*Two* $H_0$*–series representations* $[\pi_{\eta_{\chi,\nu},\sigma}]$, $[\pi_{\eta_{\chi',\nu'},\sigma'}]$ *of* $G_0$ *are equal if and only if* $([\chi'],\nu',\sigma')$ *is in the Weyl group orbit* $W(G_0,H_0)([\chi],\nu,\sigma)$.

*The* $H_0$*–series representations* $[\pi_{\eta_{\chi,\nu},\sigma}]$ *has dual* $[\pi^*_{\eta_{\chi,\nu},\sigma}] = [\pi_{\eta_{\overline{\chi},-\nu},-\sigma}]$ *and has infinitesimal character* $\chi_{\nu+i\sigma}$ *relative to* $\mathfrak{h}$, *In particular it sends the Casimir element of* $\mathcal{U}(\mathfrak{g})$ *to* $||\nu||^2 + ||\sigma||^2 - ||\rho||^2$.

Two complements to Theorem 9.12. First, one can check that if $H_0$ and $'H_0$ are non–conjugate Cartan subgroups of $G_0$ then every $H_0$–series representation is disjoint (no composition factors in common) from every $'H_0$–series representation. This is seen by examining the real and imaginary parts of the infinitesimal character. Second, the Harish–Chandra condition for irreducibility of $[\pi_{\eta_{\chi,\nu},\sigma}]$ is that $\sigma$ be regular for $(\mathfrak{g}_0,\mathfrak{a}_0)$.

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## §10. Indication of the Plancherel Formula.

We start with Kostant's "cascade construction" for the conjugacy classes of Cartan subgroups of $G_0$. Suppose first that $G_0$ has a compact Cartan subgroup $T_0$. Fix a Cartan involution $\theta$ of $G_0$ such that $\theta(T_0) = T_0$ and the corresponding $\pm 1$ eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of the maximal compact subgroup $K_0 = \{g \in G_0 \mid \theta(g) = g\}$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{t})$ then either $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and we say that $\alpha$ is **compact**, or $\mathfrak{g}_\alpha \subset \mathfrak{s}$ and we say that $\alpha$ is **noncompact**.

Let $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{t})$ be noncompact. Let $\mathfrak{g}[\alpha] = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ as in (6.4), let $G[\alpha]$ denote the corresponding analytic subgroup of $G$, and consider the corresponding real forms $\mathfrak{g}_0[\alpha] = \mathfrak{g}_0 \cap \mathfrak{g}[\alpha]$ and $G_0[\alpha] = G_0 \cap G[\alpha]$. Then $G_0[\alpha] \cap T_0$ is a compact Cartan subgroup, and we can simply replace it by the noncompact Cartan subgroup of $G_0[\alpha]$. Let $\mathfrak{a}_0[\alpha]$ denote the Lie algebra of that noncompact Cartan subgroup. Then we have a new Cartan subgroup

$$\text{(10.1a)} \qquad \mathfrak{h}_0\{\alpha\} = \left(\mathfrak{t}_0 \cap (\mathfrak{g}_0[\alpha] \cap \mathfrak{t}_0)^\perp\right) + \mathfrak{a}_0[\alpha]$$

and the corresponding Cartan subgroup

$$\text{(10.1b)} \qquad H_0\{\alpha\} = \{g \in G_0 \mid \text{ Ad}\,(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}_0\{\alpha\}.$$

The point is that $H_0\{\alpha\}$ has one compact dimension less than that of $T_0$ and one noncompact dimensions more.

Let $\alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{t})$ be noncompact. We can carry out the construction (10.1) for $\alpha$ and $\beta$ independently, one after the other, in $\alpha$ and $\beta$ are **strongly orthogonal** in the sense that $\alpha$ and $\beta$ are linearly independent and neither of $\alpha \pm \beta$ are roots. We write this relation as $\alpha \underline{\perp} \beta$. If $\alpha \underline{\perp} \beta$ then we have the new Cartan subgroup $H_0\{\alpha, \beta\}$ given by

$$\text{(10.2a)} \qquad \mathfrak{h}_0\{\alpha, \beta\} = \left(\mathfrak{t}_0 \cap ((\mathfrak{g}_0[\alpha] \oplus \mathfrak{g}_0[\beta]) \cap \mathfrak{t}_0)^\perp\right) + \left(\mathfrak{a}_0[\alpha] \oplus \mathfrak{a}_0[\beta]\right)$$

and

$$\text{(10.2b)} \qquad H_0\{\alpha, \beta\} = \{g \in G_0 \mid \text{ Ad}\,(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}_0\{\alpha, \beta\}.$$

Here $H_0\{\alpha, \beta\}$ has two compact dimensions less than that of $T_0$ and two non-compact dimension more.

We say that a set $S$ of noncompact roots is **strongly orthogonal** if it is linearly independent and if any two of its elements are strongly orthogonal. Then as above we have a Cartan subgroup $H_0\{S\}$ given by

$$\text{(10.3a)} \qquad \mathfrak{h}_0\{S\} = \left(\mathfrak{t}_0 \cap ((\textstyle\sum_{\alpha \in S} \mathfrak{g}_0[\alpha]) \cap \mathfrak{t}_0)^\perp\right) + \left(\textstyle\sum_{\alpha \in S} \mathfrak{a}_0[\alpha]\right)$$

and

$$\text{(10.3b)} \qquad H_0\{S\} = \{g \in G_0 \mid \text{ Ad}\,(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}_0\{S\}.$$

Here $H_0\{S\}$ has $|S|$ compact dimensions fewer than $T_0$ has, and $H_0\{S\}$ has $|S|$ noncompact dimensions more than $T_0$ has.

Cartan subgroups $H_0\{S_1\}$ and $H_0\{S_2\}$ are $G_0$–conjugate if and only if some $w \in W(G_0, T_0)$ sends $S_1$ to $S_2$. Kostant proved that every Cartan subgroup of $G_0$ is conjugate to $\mathfrak{h}_0\{S\}$ for some set $S$ of strongly orthogonal noncompact roots.

This sets up a hierarchy among the conjugacy classes of Cartan subgroups of $G_0$ : $H_0\{S_1\} \leqq H_0\{S_2\}$ if and only if some Weyl group element $w \in W(G_0, T_0)$ sends $S_2$ to a subset of $S_1$. That in turn sets up a hierarchy among parts of the regular set $G_0'$. If $H_0$ is any Cartan subgroup of $G_0$ we denote $G'_{H_0} = G'_0 \cap \mathrm{Ad}(G)H_0$, the set of all regular elements $G'_0$ that are conjugate to an element of $H_0$. Now $G'_{H_0\{S_1\}} \leqq G'_{H_0\{S_2\}}$ if and only if some Weyl group element $w \in W(G_0, T_0)$ sends $S_2$ to a subset of $S_1$. Here $G'_{T_0}$ sits at the top, the $G'_{H_0\{\alpha\}}$ sit just below, the $G'_{H_0\{\alpha,\beta\}}$ are on the next level down, and finally the part of $G'_0$ corresponding to the Cartan subgroup of the minimal parabolic subgroups sit at the bottom.

If $G_0$ does not have a compact Cartan subgroup, we reduce to that case as follows. Let $H_0 = T_0 \times A_0$ be a Cartan subgroup that is as compact as possible, i.e., $T_0$ is a Cartan subgroup of a maximal compact subgroup $K_0 \subset G_0$. Let $P_0 = M_0A_0N_0$ be an associated cuspidal parabolic subgroup. Then just do the cascade construction for $M_0$, obtaining a family of Cartan subgroups $H_{M,0}\{S\} \subset M_0$ as $S$ runs over the $W(M_0, T_0)$–conjugacy classes of strongly orthogonal sets $S \subset \Sigma(\mathfrak{m}, \mathfrak{t})$ of noncompact roots of $\mathfrak{m}$. Then the $H_0\{S\} = H_{M,0}\{S\} \times A_0$ give the conjugacy classes of Cartan subgroups of $G_0$.

A careful examination of the character formulae (8.6), (8.7), (9.11) and (9.13) shows that the various tempered series exhaust enough of $\widehat{G_0}$ for a decomposition of $L_2(G_0)$ essentially as

$$(10.4) \qquad \sum_{H_0 \in Car(G_0)} \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0}} \int_{\mathfrak{a}_0^*} H_{\pi_{\chi,\nu,\sigma}} \otimes H^*_{\pi_{\chi,\nu,\sigma}} m(H_0 : \chi : \nu : \sigma) d\sigma.$$

Here $Car(G_0)$ denotes the set of $G_0$–conjugacy classes of Cartan subgroups and the Borel measure $m(H_0 : \chi : \nu : \sigma)d\sigma$ is the **Plancherel measure** on $\widehat{G_0}$. In general the Plancherel density $m(H_0 : \chi : \nu : \sigma)$ has a formula that varies with the component of the regular set. This was worked out by Harish–Chandra for groups of Harish–Chandra class, and somewhat more generally by Herb and myself. Harish–Chandra's approach is based on an analysis of the structure of the Schwartz space, while Herb and I use explicit character formulae. These explicit formulae allow us to prove formula (10.4), as follows.

Start with $G'_{H_0}$ where $H_0$ represents the conjugacy class of Cartan subgroups of $G_0$ that are as compact as possible. A look at the character formulae cited above, shows that the $H_0$–series representations suffice to expand functions $f \in C_0^\infty(G'_{H_0})$. That expansion formula gives us the map

$$(10.5a) \qquad C_0^\infty(G_0) \to C^\infty(G_0 \setminus G'_{H_0}) \text{ by } f \mapsto f_1$$

where $r_x$ denotes right translation by $x \in G_0$ and

$$(10.5b) \qquad f_1(x) = f(x) - \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0}} \int_{\mathfrak{a}_0^*} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0 : \chi : \nu : \sigma) d\sigma.$$

Now let $\{H_0\{\alpha_1\}, \dots, H_0\{\alpha_{m_1}\}$ be a set of representatives of the conjugacy classes of Cartan subgroups just below $H_0$. A look at the character formulae cited above, shows that the $H_0\{\alpha_i\}$–series representations suffice to expand functions $f \in C_0^\infty(G'_{H_0\{\alpha_i\}})$. Those expansions do not interact, nor do they introduce nonzero values in $G'_{H_0}$, so they give us a map

$$\text{(10.6a)} \qquad C^\infty(G_0 \setminus G'_{H_0}) \to C^\infty\left(G_0 \setminus \left(G'_{H_0} \cup \bigcup G'_{H_0\{\alpha_i\}}\right)\right) \text{ by } f_1 \mapsto f_2$$

where

(10.6b)

$$f_2(x) - f_1(x) = \sum_{1 \leqq i \leqq m_1} \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0\{\alpha_i\}}} \int_{\mathfrak{a}_0^*\{\alpha_i\}} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0\{\alpha_i\} : \chi : \nu : \sigma) d\sigma.$$

Now simply proceed down one level at a time. The tricky point here is to know the character formulae completely, so that one knows $f_j$ well enough to compute $f_{j+1}$. Finally, one obtains the final form

$$\text{(10.7)} \quad f(x) = \sum_{H_0 \in Car(G_0)} \sum_{\chi \otimes e^{\nu - \rho_{\mathfrak{m}}} \in \widehat{T_0}} \int_{\mathfrak{a}_0^*} \Theta_{\pi_{\chi,\nu,\sigma}}(r_x f) m(H_0 : \chi : \nu : \sigma) d\sigma.$$

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## PART 4. GEOMETRIC REALIZATION OF THE TEMPERED SERIES.

In this part we show how the standard tempered representations occur as natural geometric objects over certain real group orbits.

### §11. MEASURABLE OPEN ORBITS AND THE DISCRETE SERIES.

Fix a complex flag manifold $Z = G/P$. An open orbit $G_0(z) \subset Z$ is called **measurable** if it carries a $G_0$–invariant volume element. If that is the case, then the invariant volume element is the volume element of a $G_0$–invariant, possibly indefinite, Kaehler metric on the orbit, and the isotropy subgroup $G_0 \cap P_z$ is the centralizer in $G_0$ of a (compact) torus subgroup of $G_0$. In more detail, measurable open orbits are characterized by

**11.1. Proposition.** *Let $D = G_0(z)$ be an open $G_0$–orbit on the complex flag manifold $Z = G/P$. Then the following conditions are equivalent.*
1. *The orbit $G_0(z)$ is measurable.*
2. *$G_0 \cap P_z$ is the $G_0$–centralizer of a (compact) torus subgroup of $G_0$.*
3. *$D$ has a $G_0$–invariant possibly–indefinite Kaehler metric, thus a $G_0$–invariant measure obtained from the volume form of that metric.*
4. *$\tau\Phi^r = \Phi^r$, and $\tau\Phi^n = -\Phi^n$ where $\mathfrak{p}_z = \mathfrak{p}_\Phi$.*
5. *$\mathfrak{p}_z \cap \tau\mathfrak{p}_z$ is reductive, i.e. $\mathfrak{p}_z \cap \tau\mathfrak{p}_z = \mathfrak{p}_z^r \cap \tau\mathfrak{p}_z^r$.*
6. *$\mathfrak{p}_z \cap \tau\mathfrak{p}_z = \mathfrak{p}_z^r$.*
7. *$\tau\mathfrak{p}$ is $\mathrm{Ad}\,(G)$–conjugate to the parabolic subalgebra $\mathfrak{p}^- = \mathfrak{p}^r + \mathfrak{p}^n$ opposite to $\mathfrak{p}$.*

*In particular, if one open $G_0$–orbit on $Z$ is measurable, then they all are measurable.*

Note that Condition 4 of Proposition 11.1 is automatic if the Cartan subalgebra $\mathfrak{h}_0$, relative to which $\mathfrak{p}_z = \mathfrak{p}_\Phi$, is the Lie algebra of a compact Cartan subgroup of $G_0$, for in that case $\tau\alpha = -\alpha$ for every $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{h})$. In particular, if $G_0$ has discrete series representations, so that by a result of Harish–Chandra it has a compact Cartan subgroup, then every open $G_0$–orbit on $Z$ is measurable.

Condition 4 is also automatic if $P$ is a Borel subgroup of $G$, and more generally Condition 7 provides a quick test for measurability.

Now suppose that $G_0$ has a compact Cartan subgroup $T_0 \subset K_0$. Let $Z = G/P$ be a complex flag manifold, let $z \in Z$, set $D = G_0(z)$, and suppose that

$$D \text{ is open in } Z \text{ and } G_0 \text{ has compact isotropy subgroup } U_0 \text{ at } z. \tag{11.2}$$

Passing to a conjugate, equivalently moving $z$ within $D$, we may suppose $T_0 \subset U_0$.

Let $\mu \in \widehat{U_0}$, let $E_\mu$ denote the representation space, and let $\mathbb{E}_\mu \to D \cong G_0/U_0$ denote the associated holomorphic homogeneous vector bundle. Then $\mu$ is finite dimensional and is constructed as follows. First, $U_0 \cap G_0^0$ is the identity component $U_0^0$, and $U_0 = Z_{G_0}(G_0^0)U_0^0$. Second there are irreducible representations $[\chi] \in \widehat{Z_{G_0}(G_0^0)}$ and $[\mu^0] \in \widehat{U_0^0}$ that agree on $Z_{G_0}$ such that $[\mu] = [\chi \otimes \mu^0]$.

Let $\beta - \rho_{\mathfrak{u}}$ denote the highest weight of $\mu^0$, corresponding to infinitesimal character $\beta$, and suppose that

$$\lambda = \beta - \rho_{\mathfrak{u}} + \rho_{\mathfrak{g}} \text{ is regular.} \tag{11.3}$$

Then $G_0$ has a discrete series representation $\pi_{\chi,\lambda}$, whose infinitesimal character has Harish–Chandra parameter $\lambda$.

Since $\mu$ is unitary, the bundle $\mathbb{E}_\mu \to D$ has a $G_0$–invariant hermitian metric. Essentially as in the compact case, we have the spaces

(11.4)

$$A_0^{(p,q)}(D; \mathbb{E}_\mu) : C^\infty \text{ compactly supported } \mathbb{E}_\mu\text{–valued } (p,q)\text{–forms on } D,$$

and the Kodaira–Hodge orthocomplementation operators

$$\begin{aligned} \sharp : & A_0^{(p,q)}(D; \mathbb{E}_\mu) \to A_0^{(n-p,n-q)}(D; \mathbb{E}_\mu^*) \\ \text{and } \tilde{\sharp} : & A_0^{(n-p,n-q)}(D; \mathbb{E}_\mu^*) \to A_0^{(p,q)}(D; \mathbb{E}_\mu) \end{aligned} \tag{11.5}$$

where $n = \dim_{\mathbb{C}} D$. Thus we have a positive definite inner product on $A_0^{(p,q)}(D; \mathbb{E}_\mu)$ given by

$$\langle F_1, F_2 \rangle_D = \int_{G_0} \langle F_1, F_2 \rangle_{gU_0} dg = \int_D F_1 \overline{\wedge} \sharp F_2 \tag{11.6}$$

and thus

$$L_2^{(p,q)}(D; \mathbb{E}_\mu) : \text{ Hilbert space completion of } \left(A_0^{(p,q)}(D; \mathbb{E}_\mu), \langle \cdot, \cdot \rangle_D\right). \tag{11.7}$$

Let $\square$ denote the Kodaira–Hodge–Laplace operator $\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ of $\mathbb{E}_\mu$. Then $\square$ is a hermitian–symmetric elliptic operator on $L_2^{(0,q)}(D; \mathbb{E}_\mu)$ with domain $A_0^{(p,q)}(D; \mathbb{E}_\mu)$, and a result of Andreotti and Vesentini allows one to conclude that its closure $\widetilde{\square}$ is self–adjoint. Accordingly, we have the Hilbert spaces

$$\mathcal{H}^{(p,q)}(D; \mathbb{E}_\mu) = \{\omega \in \text{ Domain}\,(\widetilde{\square}) \mid \widetilde{\square}(\omega) = 0\} \tag{11.8}$$

of square integrable harmonic $\mathbb{E}_\mu$–valued $(0,q)$–forms on $D$. The natural actions of $G_0$ on those spaces are unitary representations.

We write $\mathcal{H}^q(D; \mathbb{E}_\mu)$ for $\mathcal{H}^{(0,q)}(D; \mathbb{E}_\mu)$ and we write $\pi_\mu^q$ for the unitary representation of $G_0$ on $\mathcal{H}^q(D; \mathbb{E}_\mu)$.

**11.9. Theorem.** *Let* $[\mu] = [\chi \otimes \mu^0] \in \widehat{U_0}$ *where* $\mu^0$ *has highest weight* $\beta - \rho_{\mathfrak{u}}$ *and thus has infinitesimal character* $\beta$*. If* $\lambda = \beta - \rho_{\mathfrak{u}} + \rho_{\mathfrak{g}}$ (compare (11.3)) *is*

*$\Sigma(\mathfrak{g},\mathfrak{t})$-singular then every $\mathcal{H}^q(D;\mathbb{E}_\mu)=0$. Now suppose that $\lambda=\beta-\rho_{\mathfrak{u}}+\rho_{\mathfrak{g}}$ is $\Sigma(\mathfrak{g},\mathfrak{t})$-regular and define*

$$q_{\mathfrak{u}}(\lambda)=|\{\alpha\in\Sigma^+(\mathfrak{k},\mathfrak{t})\setminus\Sigma^+(\mathfrak{u},\mathfrak{t})\mid\langle\lambda,\alpha\rangle<0\}| + |\{\beta\in\Sigma^+(\mathfrak{g},\mathfrak{t})\setminus\Sigma^+(\mathfrak{k},\mathfrak{t})\mid\langle\lambda,\beta\rangle>0\}|. \tag{11.10}$$

*Then $\mathcal{H}^q(D;\mathbb{E}_\mu)=0$ for $q\neq q_{\mathfrak{u}}(\lambda)$, and $G_0$ acts irreducibly on $\mathcal{H}^{q_{\mathfrak{u}}(\lambda)}(D;\mathbb{E}_\mu)$ by the discrete series representation $\pi_{\chi,\lambda}$ of infinitesimal character $\lambda$.*

An interesting variation on this result realizes the discrete series on spaces of $L_2$ bundle-valued harmonic spinors.

*Indication of Proof.* The proof of Theorem 11.9 has three major components. The first is the alternating sum formula

$$\sum_{q\geqq 0}(-1)^q\Theta_{{}^0\pi_\mu^q}=(-1)^{|\Sigma^+|+q_{\mathfrak{u}}(\lambda)}\Theta_{\pi_{\chi,\lambda}} \tag{11.11a}$$

where ${}^0\pi_\mu^q$ is the discrete series component of the natural unitary representation $\pi_\mu^q$ of $G_0$ on $\mathcal{H}^q(D;\mathbb{E}_\mu)$, and $\Theta_{{}^0\pi_\mu^q}$ is its distribution character. It is implicit here that $\Theta_{{}^0\pi_\mu^q}$ exists. The second major component of the proof is the consequence

$$\pi_\mu^q={}^0\pi_\mu^q \tag{11.11b}$$

of the Plancherel formula (10.7). The third major component of the proof is the vanishing theorem

$$\mathcal{H}^q(D;\mathbb{E}_\mu)=0 \text{ for } q\neq q_{\mathfrak{u}}(\lambda). \tag{11.11c}$$

To simplify the argument one should carry out three reductions. First, one may assume that $G_0=G_0^\dagger$, for the discrete series representations of $G_0$ are induced from those of $G_0^\dagger$ and one has the character relation (8.7). Second, one may assume that $G_0$ is connected, $G_0=G_0^0$, for the discrete series characters of $G_0^\dagger$ are just products $\Theta_{\pi_{\chi,\lambda}}(zx)=\operatorname{trace}\chi(z)\Theta_{\pi_\lambda^0}(x)$, as in the second equation of (8.6a). Third, one may assume that $P$ is a Borel subgroup of $G$, so $U_0=T_0$, by using the Borel–Weil Theorem 7.15 on the fibres of $G_0/T_0\to G_0/U_0$ to make the Leray spectral sequence explicit.

We will assume that $G_0$ is connected and $U_0=T_0$ for the discussion of formulae (11.11).

We indicate the argument for the alternating sum formula (11.11a). Use the Plancherel formula to express

$$L_2^{(0,q)}(D;\mathbb{E}_\mu)=\int_{\widehat{G_0}}H_\pi\widehat{\otimes}\big(H_\pi^*\otimes\Lambda^q\mathfrak{n}^*\otimes E_\mu\big)^{U_0}dm(\pi) \tag{11.12}$$

where $m$ is Plancherel measure on $\widehat{G_0}$. Here $\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}_{-\alpha}$ is the nilradical $\mathfrak{p}^{-n}=\mathfrak{b}^{-n}$ as in (1.1), so $\mathfrak{n}$ represents the antiholomorphic tangent space and

$\mathfrak{n}^*$ represents the fibre for the bundle of (0, 1)–forms. Also, $\widehat{\otimes}$ denotes projective tensor product and the integral is a direct integral of Hilbert spaces. One now writes out the formulae for $\overline{\partial}$ and $\overline{\partial}^*$ and pushes them inside the integral of (11.12). They commute with the left action of $G_0$ and so act on the second projective tensor product factor $\left(H_\pi^* \otimes \Lambda^q \mathfrak{n}^* \otimes E_\mu\right)^{U_0}$. There they ignore the $E_\mu$ factor and act on $H_\pi^* \otimes \Lambda^q \mathfrak{n}^*$. The action on $H_\pi^* \otimes \Lambda^q \mathfrak{n}^*$ produces a certain finite dimensional Lie algebra cohomology $H^q(\pi)$ as follows.

Let $H_\pi^0$ denote the space of $K_0$–finite vectors in $H_\pi$. Then $\pi \otimes \mathrm{ad}^*$ gives a representation of $\mathfrak{t} = \mathfrak{p}^r$ on $H_\pi^0 \otimes \Lambda^\bullet \mathfrak{n}^*$. If $\{y_i\}$ is a basis of $\mathfrak{n}$ and $\{\omega^i\}$ is the dual basis of $\mathfrak{n}^*$ then the coboundary, for Lie algebra cohomology of $\mathfrak{t}$ relative to its representation on $H_\pi^0$, is

(11.13a)

$$\delta = \sum \left(d\pi(y_i) \otimes e(\omega^i) + \tfrac{1}{2} \otimes e(\omega^i)\, \mathrm{ad}^*(y_i)\right) : H_\pi^0 \otimes \Lambda^q \mathfrak{n}^* \to H_\pi^0 \otimes \Lambda^{q+1} \mathfrak{n}^*$$

where $e(\cdot)$ denotes exterior product. Let $i(\cdot)$ denote the dual operation, interior product. Then $\delta$ has adjoint

(11.13b)

$$\delta^* = \sum \left(-d\pi(\tau(y_i)) \otimes i(\omega^i) + \tfrac{1}{2} \otimes \mathrm{ad}^*(y_i)^* i(\omega^i)\right) : H_\pi^0 \otimes \Lambda^{q+1} \mathfrak{n}^* \to H_\pi^0 \otimes \Lambda^q \mathfrak{n}^* .$$

Then $\delta + \delta^*$ is essentially self–adjoint on $H_\pi^0 \otimes \Lambda^\bullet \mathfrak{n}^*$ and has finite dimensional kernel $H^q(\pi)$ on $H_\pi^0 \otimes \Lambda^q \mathfrak{n}^*$. One now combines (11.12) and (11.13) to obtain

$$\text{(11.14a)} \qquad \mathcal{H}_2^q(D; \mathbb{E}_\mu) = \int_{\widehat{G_0}} H_\pi \widehat{\otimes} \left(H^q(\pi^*) \otimes E_\mu\right)^{T_0} dm(\pi).$$

In particular, the discrete series part ${}^0\pi_\mu^q$ of $\pi_\mu^q$ is given by

$$\text{(11.14b)} \qquad {}^0\pi_\mu^q = \sum_{\pi \in \widehat{G_{0,d}}} \dim \left(H^q(\pi^*) \otimes E_\mu\right)^{T_0} \pi.$$

If $f \in C^\infty(K_0)$ then $\pi|_{K_0}(f) = \int_{K_0} f(k)\pi(k)dk$ is a trace class operator on $H_\pi$, $f \mapsto T_\pi(f) = \operatorname{trace} \pi|_{K_0}(f)$ is a distribution on $K_0$, and $T_\pi|_{K_0 \cap G_0'} = \Theta_\pi|_{K_0 \cap G_0'}$. These are delicate results of Harish–Chandra. The connection with (11.11a) and (11.14b) is that

$$\text{(11.15a)} \qquad f_\pi = \sum_{q \geqq 0} (-1)^q \left(\text{character of } T_0 = U_0 \text{ on } H^q(\pi)\right)$$

satisfies

$$\text{(11.15b)} \qquad f_\pi|_{T_0 \cap G_0'} = (-1)^{|\Sigma^+|}\, \Delta_{G_0, T_0}\, e^{\rho_\mathfrak{g}}\, T_\pi|_{T_0 \cap G_0} .$$

Now let $F_\lambda = \sum_{q \geqq 0} (-1)^q \Theta_{{}^0\pi^q_\lambda}$ and compute

$$\begin{aligned} F_\lambda &= \sum_{q \geqq 0} (-1)^q \sum_{\pi \in \widehat{G_{0,d}}} \dim\big(H^q(\pi^*) \otimes E_\mu)^{T_0}\big)\ \Theta_\pi \\ &= \sum_{\pi \in \widehat{G_{0,d}}} \left( \sum_{q \geqq 0} (-1)^q \dim\big(H^q(\pi^*) \otimes E_\mu\big)^{T_0} \right) \Theta_\pi \qquad (11.16a) \\ &= \sum_{\pi \in \widehat{G_{0,d}}} \big(\text{coefficient of } e^{-\lambda + \rho_{\mathfrak{g}}} \text{ in } f_{\pi^*}\big) \Theta_\pi \ . \end{aligned}$$

But

$$f_{\pi^*_\nu} = (-1)^{|\Sigma^+| + q_t(\nu)} \sum_{w \in W(G_0, T_0)} e^{-(w(\nu) - \rho_{\mathfrak{g}})} \ ,$$

in which the coefficient of $e^{-\lambda}$ is equal to 0 if $\lambda \notin W(G_0, T_0)(\nu)$, is equal to $(-1)^{|\Sigma^+| + q_t(\lambda)}$ if $\lambda \in W(G_0, Y_0)(\nu)$. Thus we have

$$F_\lambda = (-1)^{|\Sigma^+| + q_t(\lambda)} \Theta_{\pi_\lambda} \ . \qquad (11.16b)$$

This proves the alternating sum formula (11.11a).

The Plancherel Formula (10.7) implies that

$$\{\pi \in \widehat{G_0} \setminus \widehat{G_{0,d}} \mid T_{\pi^*}|_{K_0 \cap G_0'} = \Theta_{\pi^*}|_{K_0 \cap G_0'} \neq 0\} \qquad (11.17)$$

has Plancherel measure 0. It follows from (11.14) and (11.17) that ${}^0\pi^q_\mu = \pi^q_\mu$, and (11.11b) follows.

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## §12. Partially Measurable Orbits and Tempered Series.

Choose a Cartan subgroup $H_0 \subset G_0$. We are going to realize the $H_0$–series representations of $G_0$ in a way analogous to the way we realized the principle series in §7, with Theorem 11.9 in place of the Bott–Borel–Weil Theorem 7.15.

Let $\theta$ be the Cartan involution of $G_0$ that stabilizes $H_0$, split $H_0 = T_0 \times A_0$ and let $Z_{G_0}(A_0) = M_0 \times A_0$ as before. Fix a positive root system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{h})$ defined by positive root systems $\Sigma^+(\mathfrak{m}, \mathfrak{t})$ and $\Sigma^+(\mathfrak{g}_0, \mathfrak{a}_0)$ as in (7.3). Let $P_0 = M_0 A_0 N_0$ be the corresponding cuspidal parabolic subgroup of $G_0$ associated to $H_0$.

Following the idea of the geometric realization of the principal series, we fix a set $\Phi \subset \Psi_{\mathfrak{m}}$ where $\Psi_{\mathfrak{m}}$ is the simple root system for $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. Then as in (7.5) we have

$$\begin{aligned}
&\mathfrak{z}_\Phi = \{\xi \in \mathfrak{t} \mid \phi(\xi) = 0 \ \forall \phi \in \Phi\} \text{ and its real form } \mathfrak{z}_{\Phi,0} = \mathfrak{m}_0 \cap \mathfrak{x}_\Phi \mathfrak{z}_\Phi\ , \\
&U_\Phi = Z_M(\mathfrak{z}_\Phi), U_{\Phi,0} = M_0 \cap U_\Phi, \text{ and Lie algebras } \mathfrak{u}_\Phi \text{ and } \mathfrak{u}_{\Phi,0}\ , \\
(12.1)\quad &\mathfrak{r}_\Phi = \mathfrak{u}_\Phi + \sum_{\gamma \in \Sigma^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_{-\gamma}, \text{ parabolic subalgebra of } \mathfrak{m}\ , \\
&R_\Phi = N_M(\mathfrak{r}_\Phi), \text{ corresponding parabolic subgroup of } M\ , \text{ and} \\
&S_\Phi = M/R_\Phi, \text{ associated complex flag manifold.}
\end{aligned}$$

Let $r_\Phi$ denote the base point, $r_\Phi = 1R_\Phi \in R_\Phi$. Since $T_0$ is a compact Cartan subgroup of $M_0$ contained in $U_{\Phi,0}$,

$$(12.2a)\qquad D_\Phi = M_0(r_\Phi) \subset S_\Phi \text{ is a measurable open } M_0\text{–orbit on } R_\Phi$$

We now assume that

$$(12.2b)\qquad U_{\Phi,0} \text{ is compact, so the considerations of §11 apply to } D_\Phi \subset S_\Phi\ .$$

Fix $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ as before. Given $\sigma \in \mathfrak{a}_0^*$ we will use the Theorem 11.9 to find the $H_0$–series representation $\pi_{\chi \otimes \eta_\nu^0, \sigma}$ on a cohomology space related to a particular orbit in the complex flag manifold $Z_\Phi = G/P_\Phi$. Here as before, the simple root system $\Psi_{\mathfrak{m}} \subset \Psi$ by the coherence in our choice of $\Sigma^+(\mathfrak{g}, \mathfrak{h})$, so $\Phi \subset \Psi$ and $\Phi$ defines a parabolic subgroup $P_\Phi \subset G$.

Let $z_\Phi = 1P_\Phi \in G/P_\Phi = Z_\Phi$. As $A_0 N_0 \subset G_0 \cap P_\Phi$ we have $G_0 \cap P_\Phi = U_{\Phi,0} A_0 N_0$. Thus $Y_\Phi = G_0(z_\Phi)$ is a $G_0$–orbit on $Z_\Phi$, and $D_\Phi$ sits in $Y_\Phi$ as the orbit $M_0(z_\Phi)$. Here note that $P_0 = M_0 A_0 N_0 = \{g \in G_0 \mid gD_\Phi = D_\Phi\}$.

**12.3. Lemma.** *The map $Y_\Phi \to G_0/P_0$, given by $g(z_\Phi) \mapsto gP_0$, defines a $G_0$–equivariant fibre bundle with structure group $M_0$ and whose fibres $gD_\Phi$ are the maximal complex analytic submanifolds of $Y_\Phi$.*

The data $(\mu, \sigma)$ defines a representation $\gamma_{\mu,\sigma}$ of $U_{\Phi,0} A_0 N_0$ by

$$(12.4a)\qquad \gamma_{\mu,\sigma}(uan) = e^{(\rho_{\mathfrak{g}} + i\sigma)(\log a)} \mu(u) \quad \text{where } \rho_{\mathfrak{g}} = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha\ .$$

This defines a $G_0$–homogeneous vector bundle

$$\mathbb{E}_{\mu,\sigma} \to G_0/U_{\Phi,0}A_0N_0 = Y_\Phi \text{ such that } \mathbb{E}_{\mu,\sigma}|_{D_\Phi} = \mathbb{E}_\mu \ . \tag{12.4b}$$

Each $\mathbb{E}_{\mu,\sigma}|_{gD_\Phi}$ is an $\mathrm{Ad}(g)P_0$–homogeneous vector bundle.

Since $[\mu]$ is unitary and $K_0$ acts transitively on $G_0/P_0$ we have a $K_0$–invariant hermitian metric on $\mathbb{E}_{\mu,\sigma}$. We will use it without explicit reference.

Consider the subbundle of the complexified tangent bundle to $Y_\Phi$,

$$\begin{aligned} &\mathbb{T} \to Y_\Phi \text{ defined by:} \\ &\qquad \mathbb{T}|_{gD_\Phi} \to gD_\Phi \text{ is the holomorphic tangent bundle of } gD_\Phi \ . \end{aligned} \tag{12.5a}$$

It defines

$$\begin{aligned} &\mathbb{E}^{p,q}_{\mu,\sigma} = \mathbb{E}_{\mu,\sigma} \otimes \Lambda^p(\mathbb{T}^*) \otimes \Lambda^q(\overline{\mathbb{T}}^*) \to D_\Phi \ , \\ &A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) : C^\infty \text{ compactly supported sections of } \mathbb{E}^{p,q}_{\mu,\sigma} \to Y_\Phi \ , \\ &\mathcal{O}(\mathbb{E}_{\mu,\sigma}) : \text{ sheaf of germs of } C^\infty \text{ sections of } \mathbb{E}_{\mu,\sigma} \to Y_\Phi \\ &\qquad\qquad\qquad\qquad \text{holomorphic over every } gD_\Phi \ . \end{aligned} \tag{12.5b}$$

$A^{p,q}(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is the space of $\mathbb{E}_{\mu,\sigma}$–valued partially $(p,q)$–forms on $Y_\Phi$, and $A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is the subspace of compactly supported forms.

The fibre $E_\mu$ of $\mathbb{E}_\mu \to D_\Phi$ has a positive definite $U_{\Phi,0}$–invariant hermitian inner product because $\mu$ is unitary; we translate this around by $K_0$ to obtain a $K_0$–invariant hermitian structure on the vector bundle $\mathbb{E}^{p,q}_{\mu,\sigma} \to Y_\Phi$. Similarly $\mathbb{T} \to Y_\Phi$ carries a $K_0$–invariant hermitian metric. Using these hermitian metrics we have $K_0$–invariant Hodge–Kodaira orthocomplementation operators

$$\begin{aligned} &\sharp : A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{n-p,n-q}_0(Y_\Phi;\mathbb{E}^*_{\mu,\sigma}) \\ &\tilde{\sharp} : A^{n-p,n-q}_0(Y_\Phi;\mathbb{E}^*_{\mu,\sigma}) \to A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \end{aligned} \tag{12.6}$$

where $n = \dim_{\mathbb{C}} D_\Phi$. The global $G_0$–invariant hermitian inner product on $A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma})$ is given by taking the $M_0$–invariant inner product along each fibre of $Y_\Phi \to G_0/P_0$ and integrating over $G_0/P_0$,

$$\langle F_1, F_2\rangle_{Y_\Phi} = \int_{K_0/(K_0\cap M_0)} \left( \int_{kD_\Phi} F_1 \bar{\wedge} \sharp F_2 \right) d(k(K_0 \cap M_0)). \tag{12.7}$$

where $\bar{\wedge}$ means exterior product followed by contraction of $E_\mu$ against $E^*_\mu$.

The $\overline{\partial}$ operator of $Z_\Phi$ induces the $\overline{\partial}$ operators on each of the $gD_\Phi$, so they fit together to give us an operator

$$\overline{\partial} : A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{p,q+1}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \tag{12.8a}$$

that has formal adjoint

$$\overline{\partial}^* : A^{p,q+1}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \to A^{p,q}_0(Y_\Phi;\mathbb{E}_{\mu,\sigma}) \text{ given by } \overline{\partial}^* = -\tilde{\sharp}\overline{\partial}\sharp \ . \tag{12.8b}$$

That in turn defines a sub–elliptic operator, the "partial Kodaira–Hodge–Laplace operator"

$$\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma}) \to A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma}). \tag{12.8c}$$

$A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ is a pre Hilbert space with the global inner product (12.7). Denote

$$L_2^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma}) : \text{Hilbert space completion of } A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma}). \tag{12.9}$$

Apply Andreotti–Vesentini along each $gD_\Phi$ to see that the closure of $\widetilde{\square}$ of $\square$, as a densely defined operator on $L_2^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ from the domain $A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$, is essentially self–adjoint. Its kernel

$$\mathcal{H}_2^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma}) = \{\omega \in \text{ Domain}(\widetilde{\square}) \mid \widetilde{\square}\omega = 0\} \tag{12.10}$$

is the space of **square integrable partially harmonic** $(p,q)$–forms on $Y_\Phi$ with values in $\mathbb{E}_{\mu,\sigma}$ .

The factor $e^{\rho_{\mathfrak{g}}}$ in the representation $\gamma_{\mu,\sigma}$ that defines $\mathbb{E}_{\mu,\sigma}$ insures that the global inner product on $A_0^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ is invariant under the action of $G_0$ . The other ingredients in the construction of $\mathcal{H}_2^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ are invariant as well, so $G_0$ acts naturally on $\mathcal{H}_2^{p,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ by isometries. This action is a unitary representation of $G_0$ .

Essentially as before, we write $\mathcal{H}_2^{q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ for $\mathcal{H}_2^{0,q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$, because those are the only harmonic spaces that we will use, and because $\mathcal{H}_2^{q}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ is closely related to the sheaf cohomology $H^q(Y_\Phi, \mathcal{O}(E_{\mu,\sigma}))$. The relation, which we will see later, is that they have the same underlying Harish–Chandra module.

We can now combine Theorem 11.9 with the definition ((9.2) and (9.3)) of the $H_0$–series, obtaining

**12.11. Theorem.** *Let* $[\mu] = [\chi \otimes \mu_\beta^0] \in \widehat{U_{\Phi,0}}$ *where* $\mu^0$ *has highest weight* $\beta - \rho_{\mathfrak{u}}$ *and thus has infinitesimal character* $\beta$. *Let*

$$\nu = \beta - \rho_{\mathfrak{u}_\Phi} + \rho_{\mathfrak{m}}, \tag{12.12}$$

*suppose* $\sigma \in \mathfrak{a}_0^*$, *and fix an integer* $q \geqq 0$.
1. *If* $\langle \nu, \alpha \rangle = 0$ *for some* $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$ *then* $\mathcal{H}_2^q(Y_\Phi; \mathbb{E}_{\mu,\sigma}) = 0$.
2. *If* $\langle \nu, \alpha \rangle \neq 0$ *for all* $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$, *define*

$$\begin{aligned} q_{\mathfrak{u}_\Phi}(\nu) = &|\{\alpha \in \Sigma^+((\mathfrak{k} \cap \mathfrak{m}), \mathfrak{t}) \setminus \Sigma^+(\mathfrak{u}_\Phi, \mathfrak{t}) \mid \langle \nu, \alpha \rangle < 0\}| \\ &+ |\{\beta \in \Sigma^+(\mathfrak{m}, \mathfrak{t}) \setminus \Sigma^+((\mathfrak{k} \cap \mathfrak{m}), \mathfrak{t}) \mid \langle \nu, \beta \rangle > 0\}|. \end{aligned} \tag{12.13}$$

*Then* $\mathcal{H}^q(Y_\Phi; \mathbb{E}_{\mu,\sigma)} = 0$ *for* $q \neq q_{\mathfrak{u}_\Phi}(\nu)$, *and the action of* $G_0$ *on* $\mathcal{H}^{q_{\mathfrak{u}_\Phi}(\nu)}(Y_\Phi; \mathbb{E}_{\mu,\sigma})$ *is the* $H_0$*–series representation* $\pi_{\chi,\nu,\sigma}$ *of infinitesimal character* $\nu + i\sigma$.

A variation on this theorem realizes the tempered series on spaces of $L_2$ bundle-valued partially harmonic spinors.

**References for §12.**


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## PART 5. THE LINEAR CYCLE SPACE.

In this part we indicate the geometric setting for double fibration transforms, one of the current approaches to geometric construction of non–tempered representations.

### §13. EXHAUSTION FUNCTIONS ON MEASURABLE OPEN ORBITS.

Bounded symmetric domains $D \subset \mathbb{C}^n$ are convex, and thus Stein, so cohomologies $H^k(D;\mathcal{F}) = 0$ for $k > 0$ whenever $\mathcal{F} \to D$ is a coherent analytic sheaf. This is a key point in dealing with holomorphic discrete series representations. More generally, for general discrete series representations and their analytic continuations, one has

**13.1. Theorem.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $D = G_0(z) \subset Z = G/P$ be a measurable open orbit. Let $Y = K_0(z)$, maximal compact subvariety of $D$, and let $s = \dim_{\mathbb{C}} Y$. Then $D$ is $(s+1)$–complete in the sense of Andreotti–Grauert. In particular, if $\mathcal{F} \to D$ is a coherent analytic sheaf then $H^k(D;\mathcal{F}) = 0$ for $k > s$.*

**Indication of Proof.** Let $\mathbb{K}_Z \to Z$ and $\mathbb{K}_D = K_Z|_D \to D$ denote the canonical line bundles. Their dual bundles

$$\mathbb{L}_Z = \mathbb{K}_Z^* \to Z \quad \text{and } \mathbb{L}_D = \mathbb{K}_D^* \to D \tag{13.2}$$

are the homogeneous holomorphic line bundles over $Z$ associated to the character

$$e^\lambda : P_z \to \mathbb{C} \text{ defined by } e^\lambda(p) = \text{ trace}\,\mathrm{Ad}(p)|_{\mathfrak{p}_z^n}. \tag{13.3}$$

Write $D = G_0/V_0$ where $V_0$ is the real form $G_0 \cap P_z$ of $P_z^r$. Write $V$ for the complexification $P_z^r$ of $V_0$, $\rho_{G/V}$ for half the sum of the roots that occur in $\mathfrak{p}_z^n$, and $\lambda = 2\rho_{G/V}$. If $\alpha \in \Sigma(\mathfrak{g},\mathfrak{h})$ then (i) $\langle\alpha,\lambda\rangle = 0$ and $\alpha \in \Phi^r$, or (ii) $\langle\alpha,\lambda\rangle > 0$ and $\alpha \in \Phi^n$, or (iii) $\langle\alpha,\lambda\rangle < 0$ and $\alpha \in \Phi^{-n}$. Now $\tau\lambda = -\lambda$. Decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ under the Cartan involution with fixed point set $\mathfrak{k}_0$, thus decomposing the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{p}_z$ as $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ with $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{s}_0$. Then $\lambda(\mathfrak{a}_0) = 0$.

View $D = G_0/V_0$ and $Z = G_u/V_0$ where $G_u$ is the analytic subgroup of $G$ for the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$. Then $e^\lambda$ is a unitary character on $V_0$ so

$$(13.4)\qquad \begin{aligned} &\mathbb{L}_Z \to Z = G_u/V_0 \text{ has a } G_u\text{–invariant hermitian metric } h_u\,, \\ &\mathbb{L}_D \to D = G_0/V_0 \text{ has a } G_0\text{–invariant hermitian metric } h_0\,. \end{aligned}$$

We now have enough information to carry out a computation that results in

**13.5. Lemma.** *The hermitian form $\sqrt{-1}\,\partial\overline{\partial}h_u$ on the holomorphic tangent bundle of $Z$ is negative definite. The hermitian form $\sqrt{-1}\,\partial\overline{\partial}h_0$ on the holomorphic tangent bundle of $D$ has signature $n-2s$ where $n=\dim_{\mathbb{C}} D$.*

**13.6. Corollary.** *Define $\phi : D \to \mathbb{R}$ by $\phi = \log(h_0/h_u)$. Then the Levi form $\mathcal{L}(\phi)$ has at least $n-s$ positive eigenvalues at every point of $D$.*

The next point is to show that $\phi$ is an exhaustion function for $D$, in other words that

$$\{z \in D \mid \phi(z) \leqq c\} \text{ is compact for every } c \in \mathbb{R}.$$

It suffices to show that $e^{-\phi}$ has a continuous extension from $D$ to the compact manifold $Z$ that vanishes on the topological boundary $\mathrm{bd}(D)$ of $D$ in $Z$. For that, choose a $G_u$–invariant metric $h_u^*$ on $\mathbb{L}_Z^* = \mathbb{K}_Z$ normalized by $h_u h_u^* = 1$ on $Z$, and a $G_0$–invariant metric $h_0^*$ on $\mathbb{L}_D^* = \mathbb{K}_D$ normalized by $h_0 h_0^* = 1$ on $D$. Then $e^{-\phi} = h_0^*/h_u^*$. So it suffices to show that $h_0^*/h_u^*$ has a continuous extension from $D$ to $Z$ that vanishes on $\mathrm{bd}(D)$.

The holomorphic cotangent bundle $\mathbb{T}_Z^* \to Z$ has fibre $\mathrm{Ad}(g)(\mathfrak{p}_z^n)^* = \mathrm{Ad}(g)(\mathfrak{p}_z^{-n})$ at $g(z)$. Thus its $G_u$–invariant hermitian metric is given on the fibre $\mathrm{Ad}(g)(\mathfrak{p}_z^{-n})$ at $g(z)$ by $F_u(\xi,\eta) = -\langle \xi, \tau\theta\eta\rangle$ where $\langle , \rangle$ is the Killing form. Similarly the $G_0$–invariant indefinite–hermitian metric on $\mathbb{T}_D^* \to D$ is given on the fibre $\mathrm{Ad}(g)(\mathfrak{p}_z^{-n})$ at $g(z)$ by $F_0(\xi,\eta) = -\langle \xi, \tau\eta\rangle$. But $\mathbb{K}_Z = \det \mathbb{T}_Z^*$ and $\mathbb{K}_D = \det \mathbb{T}_D^*$, so

$$h_0^*/h_u^* = c \cdot (\text{determinant of } F_0 \text{ with respect to } F_u)$$

for some nonzero real constant $c$. This extends from $D$ to a $C^\infty$ function on $Z$ given by

$$(13.7)\qquad f(g(z)) = c \cdot (\det F_0|_{\mathrm{Ad}(g)(\mathfrak{p}_z^{-n})} \text{ relative to } \det F_u|_{\mathrm{Ad}(g)(\mathfrak{p}_z^{-n})})\,.$$

It remains only to show that the function $f$ of (13.7) vanishes on $\mathrm{bd}(D)$. If $g(z) \in \mathrm{bd}(D)$ then $G_0(g(z))$ is not open in $Z$, so $\mathrm{Ad}(g)(\mathfrak{p}_z) + \tau\mathrm{Ad}(g)(\mathfrak{p}_z) \neq \mathfrak{g}$. Thus $\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)(\mathfrak{p}_z^{-n})$ while there exists an $\alpha \in \Sigma(\mathfrak{g}, \mathrm{Ad}(g)\mathfrak{h})$ such that $\mathfrak{g}_{-\alpha} \not\subset \mathrm{Ad}(g)(\mathfrak{p}_z) + \tau\mathrm{Ad}(g)(\mathfrak{p}_z)$. If $\beta \in \Sigma(\mathfrak{g}, \mathrm{Ad}(g)\mathfrak{h})$ with $\mathfrak{g}_\beta \subset \mathrm{Ad}(g)(\mathfrak{p}_z^{-n})$ then $F_0(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$, so $f(g(z)) = 0$. Thus $\phi$ is an exhaustion function for $D$ in $Z$. In view of Corollary 13.6 now $D$ is $(s+1)$–complete. Theorem 13.1 follows.

**References for §13.**

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## §14. The Exhaustion Function on a General Open Orbit.

We extend Theorem 13.1 to arbitrary open orbits. The result is

**14.1. Theorem.** *Let $Z = G/P$ be a complex flag manifold, $G$ semisimple and simply connected, and let $G_0$ be a real form of $G$. Let $D = G_0(z) \subset Z = G/P$ be an open orbit. Let $Y = K_0(z)$, maximal compact subvariety of $D$, and let $s = \dim_{\mathbb{C}} Y$. Then $D$ is $(s+1)$–complete in the sense of Andreotti-Grauert. In particular, if $\mathcal{F} \to D$ is a coherent analytic sheaf then $H^k(D;\mathcal{F}) = 0$ for $k > s$.*

The idea of the proof is to show that the arbitrary open orbit $D = G_0(z) \subset Z$ is the base of a canonical holomorphic fibration $\pi_D : \widetilde{D} \to D$ where $\widetilde{D}$ is a measurable open $G_0$–orbit in a certain flag manifold $W$ that lies over $Z$. We then take a close look at that fibration and its relation to the maximal compact linear subvarieties.

Fix the open orbit $D = G_0(z) \subset Z = G/P$ and consider the parabolic subalgebra $\mathfrak{p}^+ = \mathfrak{p}^r + \mathfrak{p}^n \subset \mathfrak{g}$ opposite to $\mathfrak{p}_z = \mathfrak{p} = \mathfrak{p}^r + \mathfrak{p}^{-n}$. Denote

$$\mathfrak{q} = \mathfrak{p} \cap \tau\mathfrak{p}^+. \tag{14.2}$$

As $D$ is open, so $\mathfrak{p}^{-n} \cap \tau\mathfrak{p}^{-n} = 0$, $\mathfrak{q}$ is the sum of a nilpotent ideal $\mathfrak{q}^{-n}$ and a reductive subalgebra $\mathfrak{q}^r$ given by

(14.3)

$$\mathfrak{q}^r = \mathfrak{p}^r \cap \tau\mathfrak{p}^r \text{ and } \mathfrak{q}^{-n} = (\mathfrak{p}^r \cap \tau\mathfrak{p}^n) + (\mathfrak{p}^{-n} \cap \tau\mathfrak{p}^r) + (\mathfrak{p}^{-n} \cap \tau\mathfrak{p}^n) = (\mathfrak{p}^r \cap \tau\mathfrak{p}^n) + \mathfrak{p}^{-n}.$$

Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{q} \cap \tau\mathfrak{q} = \mathfrak{p}^r \cap \tau\mathfrak{p}^r$, which is reductive. Let $Q$ denote the parabolic subgroup of $G$ corresponding to $\mathfrak{q} \subset \mathfrak{g}$ and let $W$ denote the corresponding flag manifold $G/Q$. Our choice of $P$ was such that $\mathfrak{p} = \mathfrak{p}_z$ where $z \in Z$ and $D = G_0(z)$ is the open orbit under study. Note that we have implicitly made the corresponding choice on $W$:

**14.4. Lemma.** *Define $w \in W$ by $\mathfrak{q} = \mathfrak{q}_w$. Then $\widetilde{D} = G_0(w)$ is a measurable open $G_0$–orbit on $W$, and $gw \mapsto gz$ defines a surjective holomorphic projection $\pi_D : \widetilde{D} \to D$. Finally, the following are equivalent:* (i) *$D$ is measurable,* (ii) *$\widetilde{D} = D$,* (iii) *$\pi_D$ is one-to-one, and* (iv) *$Q = P$.*

The structure of the fibre of $\pi_D : \widetilde{D} \to D$ is given by

**14.5. Lemma.** *Let* $\mathfrak{u} = (\mathfrak{p}^r \cap \tau\mathfrak{p}^{-n}) + (\mathfrak{p}^{-n} \cap \tau\mathfrak{p}^r)$*, nilradical of* $\mathfrak{p} \cap \tau\mathfrak{p}$*, and let* $U$ *be the corresponding complex analytic subgroup of* $G$*. Then* $U$ *is unipotent,* $\mathfrak{u}_0 = \mathfrak{g}_0 \cap \mathfrak{u}$ *is a real form of* $\mathfrak{u}$*,* $U_0 = G_0 \cap U$ *is a real form of* $U$*,* $U(w) = U_0(w)$*, and* $\pi_D : \widetilde{D} \to D$ *is a holomorphic fibre bundle with structure group* $U$ *and affine fibres* $\pi_D^{-1}(gz) = gU_0(w)$*. If* $g \in G_0$ *then the holomorphic tangent space to* $gU_0(w)$ *at* $g(w)$ *is represented by* $\mathrm{Ad}\,(g)(\mathfrak{p} \cap \tau\mathfrak{p}^{-n})$ *and the antiholomorphic tangent space is represented by* $\mathrm{Ad}\,(g)(\mathfrak{p} \cap \tau\mathfrak{p}^{n})$*.*

*Proof.* Here $U$ is the nilradical of $P \cap \tau P$ so $U_0 = G_0 \cap U$ is the nilradical of the isotropy subgroup $G_0 \cap P$ and is a real form of $U$. Note $\mathfrak{u} = \mathfrak{v} + \tau\mathfrak{v}$ where $\mathfrak{v} = \mathfrak{p}^r \cap \tau\mathfrak{p}^{-n} = \mathfrak{u} \cap \mathfrak{q}^n$, and where $\tau\mathfrak{v} = \mathfrak{u} \cap \mathfrak{q}^{-n}$. Both are subalgebras; $\mathfrak{v}$ represents the holomorphic tangent space of $U_0(w)$ at $w$ and $\tau\mathfrak{v}$ represents the antiholomorphic tangent space. Note $[\mathfrak{v}, \tau\mathfrak{v}] = 0$.

Now $U(w) = V(w) = U_0(w)$ is the fibre over $z$ of $\pi_D : \widetilde{D} \to D$, and $G_0 \cap P$ is the semidirect product of its unipotent radical $U_0$ and a Levy complement $G_0 \cap Q$. Thus $\pi_D : \widetilde{D} \to D$ satisfies $\pi_D^{-1}(g \cdot (G_0 \cap P)) = gU_0 \cdot (G_0 \cap Q)$; in terms of the complex groups this is the same as $gV \cdot Q$. Now we can express $\pi_D$ as the quotient of $G_0/(G_0 \cap Q)$ by the action of $U_0$ on the right. Then the surjective holomorphic map $\pi_D$ is the projection of a principle $U_0$–bundle. The assertions follow. □

**14.6. Corollary.** *Denote* $\widetilde{Y} = K_0(w)$*. Then* $\widetilde{Y} = K(w)$*,* $\widetilde{Y}$ *is a maximal compact complex subvariety of* $\widetilde{D}$*, and* $\pi_D|_{\widetilde{Y}}$ *is a biholomorphic diffeomorphism of* $\widetilde{Y}$ *onto* $Y$*.*

Now we push down the exhaustion function $\widetilde{\phi}$ of Corollary 13.6 from the measurable open orbit $\widetilde{D} = G_0(w) \subset W$ to our given open orbit $D = G_0(z) \subset Z$. We keep the notation $h_0$ and $h_u$ of §13, but applied to $\widetilde{D}$ rather than to $D$.

**14.7. Lemma.** *If* $g \in G_0$ *then* $\sqrt{-1}\partial\overline{\partial}\log h_0|_{gU_0(w)} = 0$*.*

*Proof.* The holomorphic tangent space $\mathfrak{u} \cap \mathfrak{q}^n = \mathfrak{r}^r \cap \tau\mathfrak{r}^{-n}$ to $U_0(w)$ at $w$ has basis given by elements $\xi_\alpha \in \mathfrak{g}_\alpha$ as $\alpha$ runs over $\Gamma^n = \Phi^r \cap (-\tau\Phi^n)$. Let $\alpha, \beta \in \Gamma^n$. If $\tau\xi_\beta \in \mathfrak{g}_{-\alpha}$ then $\alpha \in \tau\Phi^r \cap \Phi^n$, so then $\alpha \in \Phi^r \cap (-\tau\Phi^n) \cap \tau\Phi^r \cap \Phi^n \subset \Gamma^r \cap \Gamma^n$, which is empty. The Lie algebra cohomology computation that leads to Lemma 13.5 shows $\sqrt{-1}\partial\overline{\partial}\log h_0(\xi_\alpha, \xi_\beta) = 0$. Take linear combinations to conclude that $\sqrt{-1}\partial\overline{\partial}\log h_0|_{U_0(w)}$ is identically zero at $w$. As $\sqrt{-1}\partial\overline{\partial}\log h_0$ is $G_0$–invariant, $\sqrt{-1}\partial\overline{\partial}\log h_0|_{gU_0(w)}$ is identically zero at $gw$, for every $g \in G_0$. □

**14.8. Lemma.** *If* $g \in G_0$ *then* $\mathcal{L}(\widetilde{\phi})|_{gU_0(w)}$ *is positive definite.*
This shows in particular that the fibres $gU_0(w)$ of $\pi_D : \widetilde{D} \to D$ are Stein manifolds. We already knew that for another reason: $U$ is unipotent, so those fibres are affine varieties.

*Proof.* $\sqrt{-1}\partial\overline{\partial}\log h_0|_{gU_0(w)}$ is identically zero, by Lemma 14.7. $\sqrt{-1}\partial\overline{\partial}\log h_u$ is negative definite, so $\sqrt{-1}\partial\overline{\partial}\log h_u|_{gU_0(w)}$ is negative definite, and the difference $\mathcal{L}(\widetilde{\phi})|_{gU_0(w)} = \sqrt{-1}\partial\overline{\partial}\log h_0|_{gU_0(w)} - \sqrt{-1}\partial\overline{\partial}\log h_u$ is positive definite. □

**14.9. Proposition.** *If $g \in G_0$ then $\widetilde{\phi}|_{gU_0(w)}$ has a unique minimum point $m(g)$, the function $\phi : D \to \mathbb{R}$ given by*

$$\phi(g(z)) = \widetilde{\phi}(m(g)) = \min\{\widetilde{\phi}(w') \mid w' \in \pi_D^{-1}(g(z))\} \tag{14.10}$$

*is well defined. Furthermore, $\phi$ is a real analytic exhaustion function on $D$.*

*Indication of Proof.* Let $g \in G_0$. If $c > 0$ then $\widetilde{D}_c = \{w' \in \widetilde{D} \mid \widetilde{\phi}(w') \leqq c\}$ is compact because $\widetilde{\phi}$ is an exhaustion function. Thus $\widetilde{D} \cap gU_0(w)$ is compact. In particular $\widetilde{\phi}|_{gU_0(w)}$ has an absolute minimum. Let $w_1 \neq w_2$ be relative minima of $\widetilde{\phi}|_{gU_0(w)}$. Choose a smooth curve $s$ in $gU_0(w)$ from $w_1$ to $w_2$, say $s(0) = w_1$ and $s(1) = w_2$, with $s'(t) \neq 0$ for $0 < t < 1$. Set $f(t) = d\widetilde{\phi}(s'(t)) = \frac{d}{dt}\widetilde{\phi}(s(t))$. Then $f$ has a relative maximum at some $t_0$ between 0 and 1. Here we use $w_1 \neq w_2$. But Lemma 14.8 says $f''(t) > 0$ for $0 < t < 1$. Thus $w_1 = w_2$. We have proved that $\widetilde{\phi}_{gU_0(w)}$ has a unique minimum point $m(g) \in gU_0(w)$.

Now $\phi : D \to \mathbb{R}$ is well defined by (14.10). Each $\pi_D(\widetilde{D}_c) = D_c$, compact, so $\phi : D \to \mathbb{R}$ is an exhaustion function. $\phi$ is $C^\omega$ because $M = \{m(g) \mid g \in G_0\}$, the minimum locus just described, is a $C^\omega$ subvariety of $\widetilde{D}$. □

**14.11. Remark.** The first part of the argument of Proposition 14.9 shows that $m(g)$ is the unique critical point of $\widetilde{\phi}|_{gU_0(w)}$. The second part of the argument shows that the minimum locus $M = \{m(g) \mid g \in G_0\}$ is a $C^\omega$ subvariety of $\widetilde{D}$.

Define $\zeta = \phi \cdot \pi_D$, so $\zeta : \widetilde{D} \to \mathbb{R}$ by $\zeta(g(w)) = \widetilde{\phi}(m(g)) = \phi(\pi_D(g(w))$. Then the holomorphic tangent spaces of the fibres of $\pi_D$ are in the kernel of the Levi form $\mathcal{L}(\zeta)$, and if $g \in G_0$ then $\mathcal{L}(\zeta)_{g(w)}$ has the same number of positive eigenvalues as $\mathcal{L}(\phi)_{g(z)}$.

Denote complex dimensions of our spaces by

$$n = \dim_{\mathbb{C}} D\ , \quad \tilde{n} = \dim_{\mathbb{C}} \widetilde{D}\ , \qquad s = \dim_{\mathbb{C}} Y\ , \quad \tilde{s} = \dim_{\mathbb{C}} \widetilde{Y} \tag{14.12}$$

where $Y = K_0(z) \subset D$ and $\widetilde{Y} = K_0(w) \subset \widetilde{D}$ are the maximal compact subvarieties. Lemma 14.6 implies $s = \tilde{s}$.

**14.13. Lemma.** *Recall the minimum locus $M \subset \widetilde{D}$ of* Proposition 14.9 *and* Remark 14.11. *Let $m \in M$ and let $T_m^{(1,0)}(M)$ denote the part of the holomorphic tangent space to $\widetilde{D}$ tangent to $M$ at $m$. Then $\mathcal{L}(\widetilde{\phi})|_{T_m^{(1,0)}(M)}$ has at least $n - s$ eigenvalues $> 0$.*

*Proof.* Proposition 13.6, applied to $\widetilde{D}$, says that $\mathcal{L}(\widetilde{\phi})$ has at least $\tilde{n} - \tilde{s}$ eigenvalues $> 0$ at $m$, and $\dim_{\mathbb{C}} \pi_D^{-1}\pi_D(m) = \tilde{n} - n$. So $\mathcal{L}(\widetilde{\phi})|_{T_m^{(1,0)}(M)}$ has at least $n - \tilde{s} = n - s$ eigenvalues $> 0$. □

**14.14. Corollary.** *$\mathcal{L}(\zeta)$ has at least $n - s$ eigenvalues greater than zero at every point of $\widetilde{D}$.*

Theorem 14.1 follows.

**References for §14.**


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## §15. The Stein Property.

Theorem 4.4 says that

$$Y = K_0(z) \cong K_0/(K_0 \cap P_z) \cong K/(K \cap P_z) \tag{15.1}$$

is a complex submanifold of $D$. Furthermore, $Y$ is not contained in any compact complex submanifold of $D$ of greater dimension. So $Y$ is a maximal compact subvariety of $D$. We will refer to

$$M_D = \{gY \mid g \in G \text{ and } gY \subset D\} \tag{15.2}$$

as the **linear cycle space** or the **space of maximal compact linear subvarieties** of $D$. Since $Y$ is compact and $D$ is open in $Z$, $M_D$ is open in

$$M_Z = \{gY \mid g \in G\} \cong G/L \tag{15.3a}$$

where

$$L = \{g \in G \mid gY = Y\}, \text{ closed complex subgroup of } G. \tag{15.3b}$$

Thus $M_D$ has a natural structure of complex manifold. Its structure is given by

**15.4. Theorem.** *Let $D$ be an open $G_0$-orbit on a complex flag manifold $Z = G/P$. Then the linear cycle space $M_D$ is a Stein manifold.*

The first step is Proposition 15.5 below, which gives the structure of $L$. Note that the kernel of the action of $L$ on $Y$ is $E = \bigcap_{k\in K_0} kP_zk^{-1} = \bigcap_{k\in K} kP_zk^{-1}$ and that $KE \subset L \subset KP_z$.

In general, $G, P, Z, D, K$ and $Y$ break up as direct products according to any decomposition of $\mathfrak{g}_0$ as a direct sum of ideals, equivalently any decomposition of $G_0$ as a direct product. Here we use our assumption that $G$ is connected and simply connected. So, for purposes of determining $L$ we may, and do, assume that $G_0$ is noncompact and simple, in other words that $G_0/K_0$ is an irreducible riemannian symmetric space of noncompact type.

As before, we say that $G_0$ is of **hermitian type** if the irreducible riemannian symmetric space $G_0/K_0$ is an hermitian symmetric space.

Let $\theta$ be the Cartan involution of $G_0$ with fixed point set $K_0$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ under $\theta$, as usual. By the irreducibility of $G_0/K_0$, the adjoint action of $K_0$ on $\mathfrak{s}_0 = \mathfrak{g}_0 \cap \mathfrak{s}$ is irreducible. $G_0$ is of hermitian type if and only if this action fails to be absolutely irreducible. Then there is a positive root system $'\Sigma^+ = {}'\Sigma^+(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$ where $\mathfrak{s}_+$ is a sum of $'\Sigma^+$–positive root spaces and represents the holomorphic tangent space of $G_0/K_0$, and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ is a sum of $'\Sigma^+$–negative root spaces and represents the antiholomorphic tangent space. Write $S_\pm = exp(\mathfrak{s}_\pm) \subset G$. Then $G_0/K_0$ is an open $G_0$–orbit on $G/KS_-$.

Recall the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\ \mathfrak{s}_0$ of $\mathfrak{g}$. The corresponding real analytic subgroup $G_u$ of $G$ is a compact real form, thus is a maximal compact subgroup, and $K_0 = G_0 \cap G_u$. $K_0$ is its own normalizer in $G_0$, but its normalizer $N_{G_u}(K_0)$ in $G_u$ can have several components.

**15.5. Proposition.** *Either $G_0$ is of hermitian type and $L = KE = KS_\pm$, connected, or*[3] *$L \subset KN_{G_u}(K)$ with identity component $L^0 = K$. In either case $G_0 \cap L = K_0$.*

The proof is a run through the structural possibilities for $G_0$ and $\mathfrak{p}_z$. The group $V = G \cap P_z$ is compact in Cases 1 and 2 below, is noncompact in Cases 3 and 4, and can be either compact or noncompact in Cases 5 and 6. The cases are

(1) $G_0$ is of hermitian type with $P_z \subset KS_-$. In this case $L = KE = KS_-$ and $G_0 \cap L = K_0$.
(2) $G_0$ is of hermitian type with $P_z \subset KS_+$. As in Case 1, $L = KE = KS_+$ and $G_0 \cap L = K_0$.
(3) $G_0$ is of hermitian type with $P_z \not\subset KS_-$, $P_z \not\subset KS_+$ and $S_- \subset P_z$. In this case $L = KE = KS_-$ and $G_0 \cap L = K_0$.
(4) $G_0$ is of hermitian type with $P_z \not\subset KS_-$, $P_z \not\subset KS_+$ and $S_+ \subset P_z$. Arguing as in Case 3, we conclude that $L = KE = KS_+$ and $G_0 \cap L = K_0$.
(5) $G_0$ is of hermitian type with $P_z \not\subset KS_-$, $P_z \not\subset KS_+$, $S_- \not\subset P_z$ and $S_+ \not\subset P_z$. In this case $L^0 = K$. and $G_0 \cap L = K_0$.
(6) $G_0$ is not of hermitian type. In this case $L^0 = K$ and $G_0 \cap L = K_0$.

**15.6. Corollary.** *Either $L$ is a parabolic subgroup $KS_\pm$ of $G$ and $M_Z = G/L$ is a projective algebraic variety, or $L$ is a reductive subgroup of $G$ with identity component $K$ and $M_Z = G/L$ is an affine algebraic variety.*

Consider the first of the two cases of Corollary 15.6. There the result is

**15.7. Proposition.** *Suppose that $M_Z$ is a projective algebraic variety. Then the open orbit $D \subset Z$ is measurable and $M_D$ is a bounded symmetric domain. In particular $M_D$ is a Stein manifold.*

[3]This latter situation occurs both for $G_0$ of hermitian type and for $G_0$ not of hermitian type.

*Indication of Proof.* $G_0$ is of hermitian type and we may assume $L = KS_-$. Also, $M_Z = G/L$ is the standard complex realization of the compact hermitian symmetric space $G_u/K$. Now

$$G\{D\} = \{g \in G \mid gY \subset D\}. \tag{15.8}$$

is open in $G$ and $M_D = \{gY \mid g \in G\{D\}\}$. $M_D$ is stable under the action of $G_0$ so

$$G\{D\} \text{ is a union of double cosets } G_0 gL \text{ with } g \in G. \tag{15.9}$$

The proof of Proposition 15.7 consists of showing that only the identity double coset occurs in $G\{D\}$. The double cosets $G_0gL$ of (15.9) are in one-to-one correspondence with the $G_0$–orbits on $M_Z$. Those orbits are completely understood, as described in §5. Following Theorem 5.9 there is a (necessarily finite) set $\mathcal{C}$ of transforms $c_\Gamma c_\Delta^2$, where $\Gamma$ and $\Delta$ are disjoint subsets of $\Xi$ (see (5.8), such that (i) if $c_\Gamma c_\Delta^2, c_{\Gamma'} c_{\Delta'}^2 \in \mathcal{C}$ with $|\Gamma| = |\Gamma'|$ and $|\Delta| = |\Delta'|$ then $\Gamma = \Gamma'$ and $\Delta = \Delta'$ and (ii) $G_{\mathbb{C}}\{D\} = \bigcup_{c \in \mathcal{C}} GcL$. So if $c_\Gamma c_\Delta^2 \in \mathcal{C}$ then $c_{\Gamma \cup \Delta'}^2 \in \mathcal{C}$ for every subset $\Delta' \subset \Delta$. In particular, if $c_\Delta^2 \notin \mathcal{C}$ whenever $\emptyset \neq \Delta \subset \Xi$ then $\mathcal{C} = \{1\}$ and $G_{\mathbb{C}}\{D\} = GL$.

Now the proof of Proposition 15.7 is reduced to the proof that $c_\Delta^2 \notin \mathcal{C}$ for all non–empty subsets $\Delta \subset \Xi$. That is seen by an analysis of the boundary of $G_0(1 \cdot L)$ in terms of some natural norms on $\mathfrak{g}$ and certain of its subspaces.

Next consider the second of the two cases of Corollary 15.6. There the result is

**15.10. Proposition.** *Suppose that $M_Z$ is an affine algebraic variety. Then $M_D$ is an open Stein subdomain of the Stein manifold $M_Z$.*

*Indication of Proof.* Recall the real analytic exhaustion function $\phi : D \to \mathbb{R}$ of Proposition 14.9. We use it to define $\phi_M : M_D \to \mathbb{R}^+$ by

$$\phi_M(gY) = \sup_{y \in Y} \phi(g(y)) = \sup_{k \in K} \phi(gk(z)) . \tag{15.11}$$

$W = G\{D\} = \{g \in G \mid gY \subset D\}$ is open in $G$, so

$$\psi : W \times K_0 \to \mathbb{R}^+ \text{ by } \psi(g,k) = \phi(gk(z))$$

is a $C^\omega$ function on the $C^\omega$ manifold $W \times K_0$. Thus the set defined by vanishing of the differential in the $K_0$–variable,

$$\widetilde{U} = \{(g,k) \in W \times K_0 \mid d_{K_0}\psi(g,k) = 0\},$$

is a $C^\omega$ subvariety of $W \times K_0$. $\widetilde{U}$ is a union of $C^\omega$ subvarieties, one of which is

$$U = \{(g,k) \in W \times K_0 \mid \psi(g,k) = \sup_{k' \in K_0} \phi(gk(z))\}.$$

The map $f : U \to M_D$ given by $f(g,k) = gY$ is $C^\omega$. If $(g,k) \in U$ then $\phi(gk(z)) = \psi(g,k) = \phi_M(gY)$. Since $f : U \to M_D$ is $C^\omega$ and surjective, and since $\psi|_U$ is $C^\omega$, now $\phi_M$ is $C^\omega$.

By construction, $\psi(g,k)$ is constant in the second variable $k \in K_0$. The Levi form $\mathcal{L}(\phi)$ has its positive eigenvalues in directions transversal to the $gY = gK_0(z)$, so the Levi form $\mathcal{L}(\phi_M)$ on $M_D$ is positive semidefinite and $\phi_M$ is plurisubharmonic. Now, using the fact that $\phi$ is an exhaustion function on $D$,

**15.12. Lemma.** *$\phi_M$ is a real analytic plurisubharmonic function on $M_D$. If $Y_\infty$ is a point on the boundary of $M_D$ in $M_Z$ and $\{Y_i\}$ is a sequence in $M_D$ that tends to $Y_\infty$ then $\lim_{Y_i \to Y_\infty} \phi_M(Y_i) = \infty$.*

The last step is to modify $\phi_M$ to obtain a strictly plurisubharmonic exhaustion function on $M_D$. Since $M_Z$ an affine algebraic variety, it is Stein. Now $M_Z$ carries a strictly plurisubharmonic exhaustion function $N$, and $\zeta = \phi_M + N : M_D \to \mathbb{R}$ is a strictly plurisubharmonic exhaustion function on $M_D$. It follows that $M_D$ is Stein.

*Proof of Theorem 15.4.* Theorem 15.4 follows from Proposition 15.7 when $M_Z$ is a projective algebraic variety, from Proposition 15.10 when $M_Z$ is an affine algebraic variety. Proposition 15.5 says that these are the only cases.

**References for §15.**


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## §16. Connection with Double Fibration Transforms.

In general, let $D = G_0(z)$ be an open orbit in the complex flag manifold $Z = G/P$, let $Y$ be the maximal compact linear subvariety $K_0(z)$, and consider the linear cycle space $M_D = \{gY \mid g \in G \text{ and } gY \subset D\}$. Then we have a double fibration

$$\begin{array}{ccc} \mathcal{Y}_D & \xrightarrow{p_M} & M_D \\ {\scriptstyle p_D}\downarrow & & \\ D & & \end{array} \qquad (16.1)$$

where $\mathcal{Y}_D = \{(Y', y') \mid y' \in Y' \in M_D\}$, and the projections $p_M(Y', y') = Y'$ and $p_D(Y', y') = y'$.

Let $n = \dim_{\mathbb{C}} D$ and $s = \dim_{\mathbb{C}} Y$ as before. Consider a **negative** homogeneous holomorphic vector bundle $\mathbb{E} \to D$. Then we can expect nonzero cohomology only in degree $s$. For many purposes, for example for making estimates of one sort or another, it would be preferable to have representations of

$G_0$ occur on spaces of functions rather than on cohomology spaces, and here we use a double fibration transform to carry $H^s(D;\mathcal{O}(\mathbb{E}))$ to a space of functions on $M_D$. For this, one first considers the pullback $p_D^*\mathcal{O}(\mathbb{E}) \to \mathcal{Y}_D$ and then the $G_0$–homogeneous $s^{\text{th}}$ Leray direct image sheaf $\mathcal{F} = \mathcal{R}^s(p_D^*\mathcal{O}(\mathbb{E})) \to M_D$. Here $\mathcal{F}$ is locally free so it corresponds to a $G_0$–homogeneous holomorphic vector bundle $\mathbb{F} \to M_D$, and $\mathbb{F} \to M_D$ is holomorphically trivial because $M_D$ is Stein. In this way one carries the $G_0$–module $H^s(D;\mathcal{O}(\mathbb{E}))$ to a space of sections of $\mathbb{F}$ and thus to a space of functions with values in the typical fibre $F = H^s(Y; p_D^*\mathcal{O}(\mathbb{E}))$ of $\mathbb{F}$. Of course, if $M_Z$ is a projective algebraic variety then, by Proposition 15.7, $G_0$ is of hermitian type and $M_D$ is the bounded symmetric domain $G_0/K_0$.

Consider the special case where $G_0 = SU(2,2)$ and $Z$ is the complex projective space $P^3(\mathbb{C})$ and $\mathbb{E} \to D$ is a negative line bundle. Here there are two open orbits, the positive definite lines in $\mathbb{C}^{2,2}$ and the negative definite lines, and $s = 1$ for each of them. The maps of $H^s(D;\mathcal{O}(\mathbb{E}))$ to a space of $F$–valued functions of $G_0/K_0$ are the classical Penrose Transforms.

In the general case, in order to make the double fibration transform explicit one needs to know the exact structure of $M_D$ and the differential equations that pick out the functions $f_\sigma : M_D \to F$ that correspond to cohomologies $\sigma \in H^s(D;\mathcal{O}(\mathbb{E}))$. The second item here is relatively straightforward. There is some recent progress on the first item by Dunne, Novak, Zierau and myself. In almost all cases Zierau and I have shown that if $G_0$ is of hermitian type and if $M_Z$ is not a projective algebraic variety then $M_D \cong (G_0/K_0) \times \overline{(G_0/K_0)}$.

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