Riemannian Exponential Maps and Decompositions of Reductive Lie Groups

Joseph A. Wolf and Roger Zierau

ABSTRACT. Let X be a complete connected riemannian manifold, Y a closed submanifold, and $\mathbb{N}_{Y,X} \to Y$ the normal bundle of Y in X. Then the exponential map $\exp_{Y,X} : \mathbb{N}_{Y,X} \to X$ is surjective. When X is a riemannian symmetric space $X = G/K$, G reductive, this extends a number of decomposition theorems of the form $G = H \cdot \exp_G(\mathfrak{s} \cap \mathfrak{r}) \cdot K$, and when Y is totally geodesic in X it extends a number of "Euler angle type" formulae of the form $G = HAK$. The principal new features here are that H can be any reductive subgroup of G and the symmetric space X may have compact and/or euclidean factors. There are also some consequences for pseudo-riemannian manifolds and for open G-orbits on complex flag manifolds $G_{\mathbb{C}}/Q$. The papers [11] and [12] use the result with compact factors, and [3] uses the pseudo-riemannian result.

1. Riemannian Exponential Map

Let X be a complete connected riemannian manifold. Fix a closed submanifold $Y \subset X$. The **normal bundle** $\mathbb{N}_{Y,X} \to Y$ is the sub-bundle of the restriction $\mathbb{T}(X)|_Y$ of the tangent bundle of X, whose fibre over $y \in Y$ is the orthocomplement $T_y(Y)^{\perp} \subset T_y(X)$ of the tangent space to Y at y in the tangent space to X at y. The **exponential map** $\exp_{Y,X} : \mathbb{N}_{Y,X} \to X$ is just the corresponding restriction of the usual riemannian exponential map $\exp_X : \mathbb{T}(X) \to X$. In this note we will see that the rather easy theorem

Theorem 1.1. *The exponential map* $\exp_{Y,X} : \mathbb{N}_{Y,X} \to X$ *is surjective.*

has a number of interesting consequences for the structure of real reductive Lie groups. Some of these consequences were known through rather delicate results of Mostow [4]. Others are new and are needed in [3], [11], and [12].

The case of Theorem 1.1 where Y is a single point $\{y\}$, is part of the classical Hopf–Rinow Theorem: every point $x \in X$ can be joined to y by a geodesic. Our argument relies on that case. The case where X has sectional curvature $\leqq 0$ and $Y \subset X$ is a totally geodesic submanifold, was studied by Hermann [2]; there $\exp_{Y,X} : \mathbb{N}_{Y,X} \to X$ is a covering map.

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Proof. Let $x \in X$. Choose $w \in Y$ and let $m = d(x,w)$ where $d(\cdot,\cdot)$ denotes riemannian distance. Then $E = \{v \in Y \mid d(x,v) \leqq m\}$ is compact, so we have $y \in E$ minimizing the distance from x to any point of Y. Now the minimizing geodesic arc from x to y has tangent vector at y that is orthogonal to $T_y(Y)$ inside $T_y(X)$. In other words, there is a tangent vector $\xi \in T_y(Y)^\perp$ such that $\exp_{Y,X}(\xi) = x$. We have proved that $x \in \exp_{Y,X}(\mathbb{N}_{Y,X})$. □

2. Reductive Group Decomposition

In order to extract some structural results on Lie groups from Theorem 1.1 we fix

$$\begin{aligned} &G : \text{reductive Lie group,}\\ &\theta : \text{involutive automorphism of } G,\\ &K : \text{open subgroup of the fixed point set } G^\theta \\ &\qquad \text{with } X = G/K \text{connected, and}\\ &ds^2 : G\text{–invariant } \theta\text{–invariant riemannian metric on } X = G/K. \end{aligned} \tag{2.1}$$

Let $\mathfrak{g}$ denote the Lie algebra of G. In (2.1) there is no restriction on how the center of $\mathfrak{g}$ is allocated between the ± 1 eigenspaces of θ. Compare [8]. In any case, (X, ds^2) is a connected riemannian symmetric space. The usual case is when G is a connected semisimple Lie group with no compact factors, θ is a Cartan involution of G, and $K = G^\theta$. Here however X could have compact or euclidean factors, in particular could be compact. Now fix

$$H : \text{closed } \theta\text{–invariant subgroup of } G \tag{2.2}$$

and denote

$$\begin{aligned} &Y = H(x_0) \subset X \text{ where } x_0 = 1K,\\ &\qquad \text{identity coset in } G/K \text{ and base point in } X. \end{aligned} \tag{2.3}$$

In view of (2.1), a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra of G is reductive in $\mathfrak{g}$ if and only if some conjugate $\mathrm{Ad}(g)\mathfrak{h}$ is θ–invariant. See [6, §12.1] for the case where θ is a Cartan involution; the general case follows. Let $\mathfrak{h}$ be the Lie algebra of H. Then (2.2) is essentially (up to conjugacy of θ in the group of automorphisms of G) equivalent to the condition that H be a reductive subgroup of G.

Decompose the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ into ± 1 eigenspaces of θ,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s} \text{ and } \mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{s} \cap \mathfrak{h}) \tag{2.4}$$

where $\mathfrak{k}$ is both the $+1$ eigenspace of θ and the Lie algebra of K. In view of (2.2),

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{r} \tag{2.5a}$$

where

$$\mathrm{Ad}(H)\mathfrak{r} = \mathfrak{r}\ , \mathfrak{k} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{k} \cap \mathfrak{r})\ ,\ \text{and } \mathfrak{s} = (\mathfrak{s} \cap \mathfrak{h}) + (\mathfrak{s} \cap \mathfrak{r}). \tag{2.5b}$$

If β denotes the positive definite bilinear form on $\mathfrak{s}$ that corresponds to ds^2 then we may assume that the decomposition (2.5) of $\mathfrak{s}$ is an orthogonal direct sum.

The tangent space $T_{gx_0}(X)$ is represented by $\mathrm{Ad}(g)\mathfrak{s}$ for $g \in G$. The subspace $T_{hx_0}(Y) \subset T_{hx_0}(X)$ is represented by $\mathrm{Ad}(h)(\mathfrak{s} \cap \mathfrak{h})$ whenever $h \in H$, and the normal space $T_{hx_0}(Y)^{\perp}$ is represented by $\mathrm{Ad}(h)(\mathfrak{s} \cap \mathfrak{r})$. Since X is a riemannian symmetric space, the riemannian and Lie group exponential maps are related by $\exp_X(g_*\xi) = \exp_G(\mathrm{Ad}(g)\xi)K = \exp_G(\mathrm{Ad}(g)\xi)x_0$ whenever $g \in G$ and $\xi \in \mathfrak{s} = T_{x_0}(X)$. Thus

Lemma 2.6. *Let $h \in H$. Then the exponential map* $\exp_{Y,X} : \mathbb{N}_{Y,X} \to X$ *is given on the fibre $T_{hx_0}(Y)^{\perp}$ at hx_0 by*

$$\begin{aligned}\exp_{Y,X}(\mathrm{Ad}(h)\xi) &= \exp_G(\mathrm{Ad}(h)\xi)hx_0 \\ &= h\exp_G(\xi)x_0 \ \textit{for}\ \xi \in (\mathfrak{s} \cap \mathfrak{r}).\end{aligned}$$

Theorem 1.1 and Lemma 2.6 combine to give the first statement of Theorem 2.7 below, and the second statement follows from the first by $g \mapsto g^{-1}$.

Theorem 2.7. *$G = H \cdot \exp_G(\mathfrak{s} \cap \mathfrak{r}) \cdot K$ in the sense that $\phi : (h, \xi, k) \mapsto h\ \exp_G(\xi)\ k$ is a real analytic map of $H \times (\mathfrak{s} \cap \mathfrak{r}) \times K$ onto G. Similarly $G = K \cdot \exp_G(\mathfrak{s} \cap \mathfrak{r}) \cdot H$.*
{Of course ϕ cannot be injective: if $\ell \in H \cap K$ then $\phi(\ell, 0, 1) = \phi(1, 0, \ell)$.}

3. Pseudo–Riemannian Exponential Map

As G is reductive and $X = G/K$ is riemannian symmetric, the riemannian metric ds^2 comes from a nondegenerate $\mathrm{Ad}(G)$–invariant symmetric bilinear form (again call it β) on $\mathfrak{g}$. The restriction of β to $\mathfrak{r}$ is nondegenerate because H is reductive in G. Now we have a pseudo–riemannian manifold

$$D = G/H \text{ with metric } d\sigma^2 \text{ defined by } \beta|_{\mathfrak{r}}\ . \tag{3.1}$$

$(D, d\sigma^2)$ has a compact totally geodesic submanifold

$$E = K(d_0) \subset D \text{ where } d_0 = 1H \in G/H \text{ is the base point in } D. \tag{3.2}$$

This situation is especially interesting when D is an open G–orbit on a complex flag manifold $G_{\mathbb{C}}/Q$; then E is a maximal compact subvariety and its $G_{\mathbb{C}}$–translates inside D carry a lot of geometric and analytic information on both G and D. Compare [3], [7], [8], [9], [10] and [12].

As before, we have the normal bundle $\mathbb{N}_{E,D} \to D$, sub–bundle of the restriction $\mathbb{T}(D)|_E$ of the tangent bundle of D, whose fibre over $d \in D$ is the orthocomplement $T_d(E)^\perp \subset T_d(D)$ of the tangent space to E at d in the tangent space to D at d. Here it is important to notice that $T_d(E)$ is a $d\sigma^2$–nondegenerate subspace of $T_d(D)$. The exponential map $\exp_{E,D} : \mathbb{N}_{E,D} \to D$ again is just the corresponding restriction of the usual exponential map $\exp_D : \mathbb{T}(D) \to D$. As in Lemma 2.6,

Lemma 3.3. *Let $k \in K$. Then the exponential map* $\exp_{E,D} : \mathbb{N}_{E,D} \to D$ *is given on the fibre $T_{kd_0}(E)^\perp$ at kd_0 by*

$$\exp_{E,D}(\mathrm{Ad}(k)\xi) = \exp_G(\mathrm{Ad}(k)\xi)kd_0 = k\exp_G(\xi)d_0 \text{ for } \xi \in (\mathfrak{s} \cap \mathfrak{r}).$$

Lemma 3.3 combines with the second statement of Theorem 2.7 to yield

Theorem 3.4. *The exponential map* $\exp_{E,D} : \mathbb{N}_{E,D} \to D$ *is surjective.*

4. Symmetric Space Case and Euler Angle Decompositions

Now consider the case where $D = G/H$ is a pseudo–riemannian symmetric space. In other words, there is an involutive automorphism τ of G such that H is an open subgroup of the fixed point set G^τ. Then τ and θ commute because $\theta(H) = H$, and $\mathfrak{r}$ is the -1 eigenspace of τ on $\mathfrak{g}$.

Decompose the Lie algebra $\mathfrak{g}$ into ± 1 eigenspaces of θ and τ,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s} = \mathfrak{h} + \mathfrak{r} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{k} \cap \mathfrak{r}) + (\mathfrak{s} \cap \mathfrak{h}) + (\mathfrak{s} \cap \mathfrak{r}). \tag{4.1}$$

Let $L \subset G^{\tau\theta}$ be the identity component of the fixed point set of $\tau\theta$. Its Lie algebra $\mathfrak{l} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{s} \cap \mathfrak{r})$ and $L(x_0) \cong L/(K \cap L)$ is riemannian symmetric. Denote

$$\mathfrak{a} : \text{ maximal abelian subspace of } \mathfrak{s} \cap \mathfrak{r} \text{ and } A = \exp_G(\mathfrak{a}) \tag{4.2}$$

Then it is standard that $\mathfrak{a}$ is unique up to $(K \cap L)$–conjugacy and $L = (K \cap L)A(K \cap L)$. But $K \cap L$ is a maximal compactly embedded subgroup of L, hence connected because L is connected, so $(K \cap L) \subset (K \cap H)$. As $\exp_G(\mathfrak{s} \cap \mathfrak{r}) \subset L$, this combines with Theorem 2.7 to yield

Theorem 4.3. $G = HAK = KAH$ *as in* Theorem 2.7.

In case $K = H$ this is the classical "Cartan decomposition", generalizing the Euler angle decomposition of $SO(3)$. In case G is a connected semisimple group of noncompact type and with finite center, decompositions of this sort derive from results of Mostow [4] and have been used extensively in representation theory. See [1] and [5]. When G is compact, the decomposition seems to be new.

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