Uncertainty Principles for Gelfand Pairs and Cayley Complexes

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Abstract: I'll sketch an extension of the classical Uncertainty Principle to the context of Gelfand pairs. The Gelfand pair setting includes riemannian symmetric spaces, compact topological groups, and locally compact abelian groups. If the locally compact abelian group is $\mathbb{R}^n$ one recovers a sharp form of the signal processing version of the classical Heisenberg uncertainty principle. Then I'll indicate applications to spherical functions on riemannian symmetric spaces and to Cayley complexes, and finally I'll describe the extension to hypergroups.

1 Introduction.

The classical uncertainty principle says that a function and its Fourier transform cannot both be mostly concentrated on short intervals: if $f(t)$ has most of its support in an interval of length ℓ, and its Fourier transform $\widehat{f}(\tau)$ has most of its support in an interval of length $\widehat{\ell}$ then $\ell \cdot \widehat{\ell} \geq 1 - \eta$ where η is specified by the precise meaning of "most of its support". In signal processing this says that instantaneous frequency cannot be measured precisely *[9]*.

In 1989, D. L. Donoho and P. B. Stark proved *[5]* a sharp classical extension of the uncertainty principle. Let T be a measurable set, 1_T its indicator (= characteristic) function, and $||\cdot|| = ||\cdot||_p$, an L_p norm. We say that

$$f \in L_p \text{ is } \epsilon - \textbf{concentrated} \text{ on } T \text{ if } ||f - 1_T f|| \leq \epsilon ||f||. \tag{1.1}$$

The precise form of Donoho and Stark's stronger L_2 version mentioned above is *[5]*

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Let $f, \widehat{f} \in L_2(\mathbb{R})$, let $\epsilon, \delta \geq 0$, and let $T, W \subset \mathbb{R}$ be measurable sets.
(1.2) Suppose that f is ϵ-concentrated on T and $\widehat{f}$ is δ-concentrated on W.
Then Lebesgue measures satisfy $|T| \cdot |W| \geq (1 - \epsilon - \delta)^2$.

The key point in the Donoho–Stark L_2 argument is the operator norm inequality

$$||QP||^2 \leq |T| \cdot |W| \text{ where } Pf = 1_T f \text{ and } Qf = (1_W \widehat{f})^{\check{}} \tag{1.3}$$

where $\hat{}$ denotes Fourier transform as usual and $\check{}$ is its inverse.

The formulation (1.2) of Donoho and Stark is due to K. T. Smith *[21]*, who extended it locally compact abelian groups by extending the inequality (1.3). Then in *[28]* I modified Smith's arguments so that they apply more generally to Gelfand pairs, thus to riemannian symmetric spaces and compact topological groups as well as locally compact abelian groups. It turns out *[27]* that this modification applies with no essential change to give an analogous uncertainty principle for commutative hypergroups.

There are two apparently different generalizations to Smith's extension of the uncertainty principle to Gelfand pairs. Fix a Gelfand pair (G, K). Theorem 2.23 extends the uncertainty principle to functions on $K\backslash G/K$. It depends on the spherical transform for the Gelfand pair (G, K) and the resulting decomposition of $L_2(K\backslash G/K)$ by positive definite zonal spherical functions. This is the result that extends with no essential change to hypergroups. Theorem 3.10 extends the uncertainty principle to functions on G/K; it depends on the vector valued transform corresponding to a direct integral decomposition of $L_2(G/K)$. That is the result that has implications for various classes of special functions. These two extensions are in fact equivalent because the underlying L_2 decompositions use the same Plancherel measure.

The uncertainty principles that we consider here are in the same spirit as the local uncertainty inequalities of Price, Racki, Sitaram and others. For those, see, for example, *[16]*, *[18]* and *[19]*. But our methods and ideas are quite different from the support theorems of Benedicks, Cowling, Price, Sitaram and others, which they call *local uncertainty principles* (*[1]*, *[17]*, *[2]*, *[13]*, *[7]*). The latter are of the form *if both f and $\widehat{f}$ are supported on sets of finite measure then $f = 0$ a.e.* Theorems of that sort do not apply to compact groups or compact Gelfand pairs; in fact they depend in an essential way on the appearance of a connected noncompact abelian group in the computation of the Fourier transform (*[13]*, *[2]*). In contrast, our considerations apply to both the compact and the noncompact settings.

In Section 2 we recall some function theory on $K\backslash G/K$ for a Gelfand pair (G, K) and indicate the associated scalar uncertainty principle. In Section 3 we describe the direct integral decomposition of $L_2(G/K)$ and the corresponding vector valued transform on G/K and verify the analogs of the operator norm inequalities of § 2 to derive the vector uncertainty principle for G/K.

In Section 4 we indicate some of the connection with various classes of special functions that occur as spherical functions on riemannian symmetric spaces.

In Section 5 we show how uncertainty principles for generalized Cayley graphs follow from uncertainty principles for Gelfand pairs. In Section 6 we show how those considerations extend to a class of simplicial complexes, for which generalized Cayley graphs are the 1–dimensional case.

Finally in Section 7 we describe just how the scalar uncertainty principle extends to commutative hypergroups.

2 The Scalar Uncertainty Principle and Commutative Function Theory of Gelfand Pairs.

Let G be a locally compact topological group and K a compact subgroup. Let m_G be Haar measure on G, subject to the usual convention: counting measure if G is infinite and discrete, total mass 1 if G is compact. Let m_K be Haar measure of total mass 1 on K.

Convolution on G is the action of functions under the left regular representation:

$$f_1 * f_2(y) = \int_G f_1(x) f_2(x^{-1}y)\, dm_G(x). \tag{2.1}$$

Young's Inequality is

$$||f * h||_p \leq ||f||_1 ||h||_p \text{ for } f \in L_1(G) \text{ and } h \in L_p(G). \tag{2.2}$$

A function f on G is called K–bi–invariant if $f(k_1 x k_2) = f(x)$ for all $x \in G$ and $k_i \in K$. Similarly a Borel measure μ on G is called K–bi–invariant if $\mu(k_1 A k_2) = \mu(A)$ for all measurable A and all $k_i \in K$. We say that a function $f : G \to \mathbb{C}$ vanishes at ∞ if, given $\epsilon > 0$, there is a compact subset $C_\epsilon \subset G$ such that $|f(x)| < \epsilon$ for $x \notin C_\epsilon$. Each of the spaces

$$\begin{array}{ll} M(G): & \text{finite Borel measures on } G \\ C_0(G): & \text{continuous functions } G \to \mathbb{C} \text{ with compact support} \\ C_\infty(G): & \text{continuous functions } G \to \mathbb{C} \text{ vanishing at } \infty \\ L_p(G): & \text{standard Lebesgue space of measurable functions } G \to \mathbb{C}, 1 \leq p \leq \infty \end{array} \tag{2.3}$$

projects onto its subspace

$$M(K\backslash G/K), \quad C_0(K\backslash G/K), \quad C_\infty(K\backslash G/K), \quad L_p(K\backslash G/K), 1 \le p \le \infty \tag{2.4}$$

of K–bi–invariant functions (or measures), by $f \mapsto f^\natural$ (or $\mu \mapsto \mu^\natural$) where

$$f^\natural(x) = \int_K \int_K f(k_1 x k_2) dm_K(k_1) dm_K(k_2) \text{ and } \mu^\natural(A) = \mu(KAK). \tag{2.5}$$

$C_0(G) \subset L_1(G) \subset M(G)$ are associative algebras under convolution, we have respective subalgebras $C_0(K\backslash G/K) \subset L_1(K\backslash G/K) \subset M(K\backslash G/K)$. The following conditions are equivalent

(1) the convolution algebra $C_0(K\backslash G/K)$ is commutative
(2) the convolution algebra $L_1(K\backslash G/K)$ is commutative
(3) the convolution algebra $M(K\backslash G/K)$ is commutative
(4) if $x, y \in G$ then $KxK \cdot KyK = KyK \cdot KxK$

and they imply that G is unimodular. If they hold, one says that (G, K) is **commutative** or equivalently that (G, K) is a **Gelfand pair**. See *[3]* and *[8]* for a general introduction.

If G is a locally compact abelian group, and we set $K = \{1\}$, then (G, K) is commutative. There are more interesting examples, riemannian symmetric spaces and compact groups, as follows.

The name "Gelfand pair" comes from a result of I. M. Gelfand: let G be a locally compact group and θ an involutive automorphism of G such that $G = SK$ where K is a compact subgroup of G, if $k \in K$ then $\theta(k) = k$, and if $s \in S$ then $\theta(s) = s^{-1}$. Then G is unimodular and (G, K) is commutative.

In particular, one has the famous result of Élie Cartan[2]: if M is a connected riemannian symmetric space, if G is any group of isometries of M that contains the identity component of the group of all isometries, and if K is the isotropy subgroup of G at some point of M, then (G, K) is commutative.

Another special case concerns compact groups: let M be a compact topological group, let $G = M \times M$ (so G acts on M by $(x, y) : m \mapsto xmy^{-1}$), and let K be the stabilizer of the identity element of M (so $K = \{(x, x^{-1}) \mid x \in M\} = \Delta M$, the diagonal M in G). Then (G, K) is commutative.

Fix a Gelfand pair (G, K). A nonzero Radon measure μ on G is called **spherical** if it is K–bi–invariant and if $\mu : C_0(K\backslash G/K) \to \mathbb{C}$ is an algebra

[2] In modern language, Cartan proved that the commuting algebra for the left regular representation of G on $L_2(G/K)$ is commutative, i.e. that this left regular representation is multiplicity–free.

homomorphism, i.e., $\mu(f * h) = \mu(f)\mu(h)$. Here $\mu(f)$ means $\int_G f(x)d\mu(x)$. A continuous function $\omega : G \to \mathbb{C}$ is called a **zonal spherical function** or **zsf** if $d\mu(x) = \omega(x^{-1})dm_G(x)$ defines a spherical measure. That is the analog of exponential function on $\mathbb{R}$ or quasi–character on a locally compact abelian group. The following are equivalent for a function $\omega : G \to \mathbb{C}$.

(1) ω is a zonal spherical function on G.

(2) ω is continuous, is K–bi–invariant, and satisfies $\omega(1) = 1$, and if $f \in C_0(K\backslash G/K)$ there is a constant $\lambda_f \in \mathbb{C}$ such that $f * \omega = \lambda_f \omega$.

(3) ω is not identically zero, and if $x, y \in G$ then $\omega(x)\omega(y) = \int_K \omega(xky)dm_K(k)$.

The Fourier transform, whether on $\mathbb{R}$ or an arbitrary locally compact abelian group, only uses unitary characters, that is, quasi–characters that, as functions, are positive definite.[3] Continuous positive definite functions $\phi : G \to \mathbb{C}$ with $\phi(1) = 1$ correspond to unitary representations π of G with cyclic unit vectors u in the representation space $\mathcal{H}_\pi$, such that $\phi(x) = \langle u, \pi(x)u\rangle$ for all $x \in G$. The pair (π, u) is unique up to unitary equivalence. The connection with Gelfand pairs is

Theorem 2.6: *Let ϕ be a positive definite zonal spherical function and (π, u) the corresponding cyclic unitary representation, $\phi(x) = \langle u, \pi(x)u\rangle$. Then π is irreducible and u spans the space $\mathcal{H}_\pi^K$ of $\pi(K)$–fixed vectors. Conversely, if π is an irreducible unitary representation of G and $\mathcal{H}_\pi^K$ is spanned by a unit vector u then $\phi(x) = \langle u, \pi(x)u\rangle$ is a positive definite zonal spherical function.*

Write $S = S(G, K)$ for the set of all zonal spherical functions $f : G \to \mathbb{C}$, and write $P = P(G, K)$ for the set of all positive definite zonal spherical functions. The **spherical transform** is the map $f \mapsto \widehat{f}$, from K–bi–invariant functions on G to functions on $S = S(G, K)$, given by

$$\widehat{f}(\omega) = \mu_\omega(f) = \int_G f(x)\omega(x^{-1})dm_G(x). \tag{2.7}$$

If $f \in L_1(K\backslash G/K)$, in particular if $f \in C_0(K\backslash G/K)$, then the integral is absolutely convergent. The map

$$S \to \prod \mathbb{C}_f \ \text{ by } \ \omega \mapsto (\widehat{f}(\omega)) \ \text{ as } f \text{ ranges over } C_0(K\backslash G/K). \tag{2.8}$$

is injective and we now view $S \subset \prod \mathbb{C}_f$. The subspace topology on S is the weak topology for the functions $\widehat{f}$ with $f \in C_0(K\backslash G/K)$. Since positive definite functions are bounded by their value at the identity element $1 \in G$,

$$P = P(G, K) \subset \prod D_f \ \text{ where } \ D_f = \{z_f \in \mathbb{C}_f \mid |z_f| \leq ||f||_1\}. \tag{2.9}$$

[3]A function $\phi : G \to \mathbb{C}$ is **positive definite** if $\sum \overline{c_i}c_j\phi(x_i^{-1}x_j) \geq 0$ whenever $n \geq 1$, $\{c_1, \ldots, c_n\} \subset \mathbb{C}$, and $\{x_1, \ldots, x_n\} \subset G$.

If $\omega \in S$ then $\omega \in P$ if and only if $\mu_\omega(f * f^*) \geq 0$ for all $f \in C_0(K\backslash G/K)$. Here $f^*(x) = \overline{f(x^{-1})}$ because G is unimodular. Combining this with (2.9) one proves that P has closure $\mathrm{cl}(P)$ consisting of all $z \in \prod D_f$ such that

$$\text{(2.10)} \qquad \begin{aligned} &a) \quad f \to z_f \text{ is a linear functional on } C_0(K\backslash G/K), \\ &b) \quad z_{f_1 * f_2} = z_{f_1} z_{f_2} \text{ for all } f_1, f_2 \in C_0(K\backslash G/K), \text{ and} \\ &c) \quad z_{f * f^*} \geq 0 \text{ for all } f \in C_0(K\backslash G/K). \end{aligned}$$

Thus P is locally compact. Either P is compact and equal to its closure in $\prod \mathbb{C}_f$, or $\mathrm{cl}P = P \cup 0$ is the 1-point compactification of P. As a corollary one has the Riemann–Lebesgue Lemma: If $f \in L_1(K\backslash G/K)$ then $\widehat{f} \in C_\infty(P)$, i.e., f is a continuous function on P vanishing at ∞.

Mautner *[15]* and Godement (*[10]*, *[11]*) extended the classical Pontrjagin–Plancherel Theorem from locally compact abelian groups to Gelfand pairs:

Theorem 2.11: *Let (G, K) be commutative. Then there is a unique positive Radon measure ν on P, concentrated on a certain subset M, such that*

$$\text{if } f \in C_0(K\backslash G/K) \text{ then } \widehat{f} \in L_2(P,\nu) \text{ and } ||\widehat{f}||_{L_2(P,\nu)} = ||f||_{L_2(G)} \,. \tag{2.12}$$

Moreover $f \to \widehat{f}$ extends by continuity to a Hilbert space isomorphism of $L_2(K\backslash G/K)$ onto $L_2(P,\nu)$ which intertwines the (left) convolution representation

$$\ell: \quad C_0(K\backslash G/K) \text{ on } L_2(K\backslash G/K) \text{ by } \ell(f)h = f * h \tag{2.13a}$$

with the (left) multiplication representation

$$\widehat{\ell}: \quad C_0(K\backslash G/K) \text{ on } L_2(P,\nu) \text{ by } \widehat{\ell}(f)q = \widehat{f}q \,. \tag{2.13b}$$

ν is called **Plancherel measure** for (G, K). The uniform closure of $\ell(C_0(K\backslash G/K))$ in the algebra of bounded linear operators on $L_2(K\backslash G/K)$ is a commutative C^* algebra, and M is its maximal ideal space. If $f \in C_0(K\backslash G/K)$ then $\ell(f)$ has dual $\ell(f)\check{}$, function on M given by $\ell(f)\check{}\,(m) = \ell(f)$ modulo m. If $m \in M$ there is a unique zonal spherical function ω_m such that

$$\ell(f)\check{}\,(m) = \int_G f(x)\omega_m(x^{-1})dm_G(x) \,. \tag{2.14}$$

Furthermore ω_m is positive definite, so M is identified with a subset of P, and $M \hookrightarrow P$ extends to a homeomorphism of $M \cup \{0\}$ onto a closed subset of $P \cup \{0\}$. If $f \in C_0(K\backslash G/K)$ then $\ell(f)\check{}\,(m) = \widehat{f}(\omega_m)$, i.e., $\ell(f)\check{} = \widehat{f}|_M$. With these identifications, the construction of Plancherel measure proceeds along the same lines as the standard construction of Haar measure. Along the way one gets the analog of Bochner's Theorem, $f(x) = \int_M \omega_m(x)\widehat{f}(m)d\nu(m)$, and the analog of the Fourier inversion formula, $\widehat{f} \in L_1(P,\nu)$ and $f(x) = \int_P \widehat{f}(\omega)\omega(x)d\nu(\omega)$ for f in the dense subspace $C_0(K\backslash G/K) * C_0(K\backslash G/K)$ of $L_2(K\backslash G/K)$.

To adapt (1.3) we use the spherical transform and its inverse,

$$\widehat{f}(\omega) = \int_G f(x)\omega(x^{-1})dm_G(x) \text{ and } \check{h}(x) = \int_P h(\omega)\omega(x)d\nu(\omega). \tag{2.15}$$

Straightforward computation gives $||\widehat{f}||_\infty \leq ||f||_1$ for $f \in L_1(K\backslash G/K)$ and $||\check{h}||_\infty \leq ||h||_1$ for $h \in L_1(P,\nu)$, so Riesz-Thorin interpolation gives

$$\begin{aligned} &||\widehat{f}||_{p'} \leq ||f||_p \quad \text{for} \quad f \in L_p(K\backslash G/K), \quad 1 \leq p \leq 2, \\ &||\check{h}||_{p'} \leq ||h||_p \quad \text{for} \quad h \in L_p(P,\nu), \quad 1 \leq p \leq 2, \end{aligned} \tag{2.16}$$

with the usual $\frac{1}{p} + \frac{1}{p'} = 1$. Compare *[2]*, § 51.

Fix subsets $T = KTK \subset G$ and $U \subset P$ of finite measure. Let 1_T and 1_U denote their respective indicator functions. Define operators $P = P_T$ and $Q = Q_U$ by

$$Pf = 1_T f \text{ and } Qf = (1_U\widehat{f})\check{}. \tag{2.17}$$

Proposition 2.18: *If* $1 \leq p \leq 2$ *and* $q \geq 1$ *then* $||PQf||_q \leq m_G(T)^{1/q}\nu(U)^{1/p}||f||_p$ *for* $f \in L_p(K\backslash G/K)$. *If* $p = q$ *the operator norm on* $L_p(K\backslash G/K)$ *satisfies* $||PQ||_p \leq m_G(T)^{1/p}\nu(U)^{1/p}$.

Proof: We compute

$$\begin{aligned} PQf(x) &= 1_T(x)\int_P 1_U(\omega)\left\{\int_G f(y)\omega(y^{-1})dm_G(y)\right\}\omega(x)d\nu(\omega) \\ &= 1_T(x)\int_G f(y)\left\{\int_P 1_U(\omega)\omega(y^{-1})\omega(x)d\nu(\omega)\right\}dm_G(y). \end{aligned}$$

As $\omega(y^{-1})\omega(x) = \int_K \omega(y^{-1}kx)dm_K(k)$ and f is K–bi–invariant,

$$PQf(x) = 1_T(x)\int_G f(y)\left\{\int_P 1_U(\omega)\omega(y^{-1}x)d\nu(\omega)\right\}dm_G(y) = \langle f, \overline{k_x}\rangle$$

where $k_x(y) = 1_T(x)\int_P 1_U(\omega)\omega(y^{-1}x)d\nu(\omega) = 1_T(x)\check{1_U}(y^{-1}x)$. Using Hölder,

$$|PQf(x)| = |\langle f, \overline{k_x}\rangle| \leq ||f||_p||k_x||_{p'} = ||f||_p||\check{1_U}||_{p'}|1_T(x)|.$$

Integration $\int_G |PQf(x)|^q dm_G(x)$ and (2.16) give

$$\begin{aligned} ||PQf||_q &\leq ||f||_p\,||\check{1_U}||_{p'}\,m_G(T)^{1/q} \leq ||f||_p\,||1_U||_p\,m_G(T)^{1/q} \\ &= m_G(T)^{1/q}\,\nu(U)^{1/p}\,||f||_p\,. \end{aligned}$$

That completes the proof of Proposition 2.18. ∎

Proposition 2.18 is sufficient for the uncertainty principle of Theorem 2.23 below. But there are some other useful operator norm estimates for Gelfand pairs. The estimate analogous to that of Proposition 2.18, but in the other order, is

Proposition 2.19: *If $1 \le p \le 2$, $q \ge p'$, and $f \in L_p(K\backslash G/K)$, then*[4]

$$||QPf||_q \le m_G(T)^{1/q}\nu(U)^{1/p}||f||_p \quad and \quad ||QPf||_q \le m_G(T)^{1/p}\nu(U)^{1/p}||Pf||_q \ .$$

If $K\backslash G/K$ has finite measure, then G is compact, and one has a stronger form of part of (2.16), analog of the Hausdorff–Young Inequality (again compare *[2]*, § 51):

$$||\widehat{f}||_q \le ||f||_p \text{ for } f \in L_p(K\backslash G/K) \text{ with } 1 \le p \le \infty \text{ and } \frac{1}{q} \le \min(\frac{1}{p'}, \frac{1}{2}) \ . \tag{2.20a}$$

Similarly, if P has finite measure, for example if P is compact, then

$$||h^{\urcorner}||_q \le ||h||_p \text{ for } h \in L_p(P,\nu) \text{ with } 1 \le p \le \infty \text{ and } \frac{1}{q} \le \min(\frac{1}{p'}, \frac{1}{2}) \ . \tag{2.20b}$$

These lead to slightly stronger forms of Propositions 2.18 and 2.19.

We have a Gelfand pair (G, K), subsets $T = KTK \subset G$ and $U \subset P$ of finite measure, and operators $P = P_T$ and $Q = Q_U$ as in (2.17). Given $\epsilon \ge 0$ we say that

$$f \in L_p(K\backslash G/K) \text{ is } L_p\ \epsilon-\textbf{ concentrated} \text{ on } T \text{ if } ||f - 1_T f||_p \le \epsilon ||f||_p \tag{2.21a}$$

and similarly, given $\delta \ge 0$,

$$h \in L_{p'}(P,\nu) \text{ is } L_{p'}\ \delta-\textbf{concentrated} \text{ on } U \text{ if } ||h - 1_U h||_{p'} \le \delta ||h||_{p'} \tag{2.21b}$$

Somewhat analogously, we say (compare *[5]* and *[21]*) that

$$f \in L_p(K\backslash G/K) \text{ is } L_p\ \delta-\textbf{ bandlimited} \text{ to } U \text{ if there exists}$$
$$f_U \in L_p(K\backslash G/K) \text{ with } \widehat{f_U} \text{ supported in } U \text{ and } ||f - f_U||_p \le \delta ||f||_p \ .$$

[4] There is a typographical error (a misplaced prime) in the published statement of *[21]*, Theorem 3.2. The correction is implicit in the argument below. The argument for the second inequality of Proposition 2.19 follows *[22]*.

Theorem 2.23: ***(Scalar Uncertainty Principle)*** *Suppose that* $0 \neq f \in L_p(K\backslash G/K)$ *with* $1 \leq p \leq 2$ *and* $\epsilon, \delta \geq 0$ *such that* f *is* ϵ*–concentrated on* T *and* δ*–bandlimited to* U*. Then*

$$m_G(T)^{1/p}\nu(U)^{1/p} \geq ||PQ||_p \geq \frac{1-\epsilon-\delta}{1+\delta}.$$

And if $p = 2$ *then, further,* $||PQ||_2 \geq 1 - \epsilon - \delta$.

Proof: The first inequality is the case $p = q$ of Proposition 2.18. Note $f_U = Qf_U$ because $\widehat{f_U}$ is supported in U, and of course $||P|| \leq 1$, to compute

$$\begin{aligned} ||f||_p - ||PQf||_p &\leq ||f - Pf||_p + ||Pf - Pf_U||_p + ||PQf_U - PQf||_p \\ &= (\epsilon + \delta + \delta||PQ||_p)||f||_p \ , \end{aligned}$$

so $||PQf||_p \geq (1 - \epsilon - \delta - \delta||PQ||_p)||f||_p$. Now $(1+\delta)||PQ||_p \geq 1 - \epsilon - \delta$, proving the general assertion. If $p = 2$ we can take $\widehat{f_U} = Qf$ so that $Qf = Qf_U$ and the $||PQf_U - PQf||_p$ term does not occur. ∎

3 The Vector Uncertainty Principle and Noncommutative Function Theory of Gelfand Pairs.

In this Section we describe the vector–valued transform that leads to an analysis of $L_2(G/K)$ where (G, K) is a Gelfand pair.

From Theorem 2.6, if $f \in L_1(G/K)$ and $x \in G$ then $(f * \omega)(x) = \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega}$. On the other hand, a calculation shows that if $f \in C(G/K) \cap L_1(G/K)$ and $x \in G$ then $f(x) = \int_P (f * \omega)(x)d\nu(\omega)$. So $f * \omega \in C(G/K) \cap L_1(G/K)$ corresponds to $\pi_\omega(f)u_\omega \in \mathcal{H}_\omega$. These facts combine to prove

Lemma 3.1: *If* $f \in C_0(G/K)$ *and* $x \in G$ *then*

$$f(x) = \int_P \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \ \ \textit{and}$$

$$||f||^2_{L_2(G/K)} = \int_P ||\pi_\omega(f)u_\omega||^2_{\mathcal{H}_\omega} d\nu(\omega) \ .$$

Recall the notion of direct integral. Let (Y, τ) be a measure space. For each $y \in Y$ let $\mathcal{H}_y$ be a Hilbert space. Let $\{s_\alpha\}$ be a countable family of maps $Y \to \bigcup_{y \in Y} \mathcal{H}_y$ such that $s_\alpha(y) \in \mathcal{H}_y$ and $\{s_\alpha(y)\}$ spans $\mathcal{H}_y$ for all $y \in Y$, and the functions $y \mapsto \langle s_\alpha(y), s_\alpha(y)\rangle_{\mathcal{H}_y}$ belong to $L_1(Y,\tau)$. Then the **direct integral** $\mathcal{H} = \int_Y \mathcal{H}_y d\tau(y)$ **modeled** on the $\{s_\alpha\}$ is the linear span of all the maps $s : Y \to \bigcup_{y\in Y} \mathcal{H}_y$ such that $s(y) \in \mathcal{H}_y$ a.e. (Y,τ) and the functions $y \mapsto \langle s(y), s_\alpha(y)\rangle_{\mathcal{H}_y}$ belong to $L_1(Y,\tau)$. $\mathcal{H}$ is a separable Hilbert space with inner product $\langle s, s'\rangle_{\mathcal{H}} = \int_Y \langle s(y), s'(y)\rangle_{\mathcal{H}_y} d\tau(y)$.

More generally, fix $1 \le p \le \infty$. Then the L_p **direct integral** $\mathcal{H}_p$ is the linear span of the maps $s : Y \to \bigcup_{y\in Y} \mathcal{H}_y$ such that $s(y) \in \mathcal{H}_y$ a.e. (Y,τ) and the functions $y \mapsto \langle s(y), s(y)\rangle_{\mathcal{H}_y}^{1/2}$ belong to $L_p(Y,\tau)$. $\mathcal{H}_p$ is a Banach space with norm $||s||_p = \left(\int_P \langle s(y), s(y)\rangle_{\mathcal{H}_y}^{p/2}\right)^{1/p}$ for $1 \le p < \infty$, norm $||s||_\infty =$ ess sup$_{(P,\nu)}\langle s(y), s(y)\rangle_{\mathcal{H}_y}^{1/2}$. Of course $\mathcal{H}_2$ is the Banach structure underlying the Hilbert space structure of $\mathcal{H}$.

Let $\mathcal{B}(\mathcal{H}_y)$ denote the algebra of bounded linear operators on the Hilbert space $\mathcal{H}_y$. Let $T : Y \to \bigcup_{y\in Y} \mathcal{B}(\mathcal{H}_y)$ such that $T(y) \in \mathcal{B}(\mathcal{H}_y)$ a.e. (Y,τ), and if $s, s' \in \mathcal{H}$ then $y \mapsto \langle T(y)s(y), s'(y)\rangle$ belongs to $L_1(Y,\tau)$. Then $s \mapsto Ts \in \mathcal{H}$, $Ts(y) = T(y)s(y)$, defines an element $T \in \mathcal{B}(\mathcal{H})$. This element is denoted $T = \int_Y T(y)d\nu(y)$ and is called the **direct integral** of the $T(y)$.

In our case, (P,ν) is the measure space; for $\omega \in P$ we have the Hilbert space $\mathcal{H}_\omega$; for $f \in C_0(G/K)$ we have $s_f(\omega) = \pi_\omega(f)u_\omega$, and $\{s_\alpha\} = \{s_{f_\alpha}\}$ where $\{f_\alpha\}$ is a countable dense subset of $C_0(G/K)$. We define the **Fourier transform** on G/K to be the map $\mathcal{F} : L_1(G/K) \to \int_P \mathcal{H}_\omega d\nu(\omega)$ given by $\mathcal{F}(f)(\omega) = s_f(\omega) = \pi_\omega(f)u_\omega$.

Lemma 3.1 gives us the Godement–Plancherel Theorem for G/K:

Theorem 3.2: *Let (G,K) be commutative and let ν be its Plancherel measure. Then the Fourier transform satisfies*

$$f(x) = \int_P \langle \mathcal{F}(f), \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \;\; \textit{and} \;\; ||\mathcal{F}(f)||_{\mathcal{H}} = ||f||_{L_2(G/K)}. \tag{3.3}$$

Moreover $f \mapsto \mathcal{F}(f)$ extends by continuity to a Hilbert space isomorphism of $L_2(G/K)$ onto $\mathcal{H}$ which intertwines the (left) regular representation

$$\lambda : \quad L_1(G) \;\; \textit{on} \;\; L_2(G/K) \;\; \textit{by} \;\; \lambda(\psi)f = \psi * f \tag{3.4a}$$

with the direct integral representation

$$\pi = \int_P \pi_\omega \, d\nu(\omega) : L_1(G) \;\; \textit{on} \;\; \mathcal{H} = \int_P \mathcal{H}_\omega d\nu(\omega) \;\; \textit{by} \;\; \pi(\psi) = \int_P \pi_\omega(\psi)d\nu(\omega). \tag{3.4b}$$

The standard Plancherel formula is of the form $f(x) = \int_{\widehat{G}}$ trace $\pi(\ell(x^{-1})f)d\mu(\pi)$, where $\widehat{G}$ is the unitary dual of G and $\ell(y)f(g) = f(y^{-1}g)$. The connection with Theorem 3.2 is given by

$$\text{trace } \pi_\omega(\ell(x^{-1})f) = \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle. \tag{3.5}$$

That formula depends on the fact that if $v_\omega \perp u_\omega$ in $\mathcal{H}_\omega$ and $f \in L_1(G/K)$ then $\pi_\omega(f)v_\omega = 0$. The same calculation shows that if $\pi \in \widehat{G}$ has no K–fixed vector and $f \in L_1(G/K)$ then $\pi(f) = 0$. So for functions on G/K, only the π_ω contribute to the Plancherel formula.

We now carry the considerations of § 2 from $K\backslash G/K$ to G/K, starting with the formulae for the Fourier transform and its inverse,

$$\mathcal{F}(f)(\omega) = \pi_\omega(f)u_\omega \text{ and } \mathcal{F}^{-1}(v)(x) = \int_P \langle v_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega). \tag{3.6}$$

As in the classical case,

$$\begin{aligned} ||\mathcal{F}(f)||_\infty &= \text{ess sup}_{(P,\omega)} \, ||\mathcal{F}(f)(\omega)||_{\mathcal{H}_\omega} \\ &= \text{ess sup}_{(P,\omega)} \, ||\pi_\omega(f)u_\omega||_{\mathcal{H}_\omega} \le \text{ess sup}_{(P,\omega)} \, ||f||_1 = ||f||_1 \end{aligned}$$

for $f \in L_1(G/K)$, and

$$||\mathcal{F}^{-1}(v)||_\infty = \text{ess sup} \int_P \langle v(\omega), \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \le \int_P ||v(\omega)||_{\mathcal{H}_\omega} d\nu(\omega) = ||v||_1$$

for $v \in \mathcal{H}_1$. So as before, Riesz-Thorin interpolation results in a G/K analog of the Titchmarch Inequality

$$\begin{array}{llll} ||\mathcal{F}f||_{p'} \le ||f||_p & \text{for} & f \in L_p(G/K), & 1 \le p \le 2, \\ ||\mathcal{F}^{-1}(v)||_{p'} \le ||v||_p & \text{for} & v \in \mathcal{H}_p\ , & 1 \le p \le 2, \end{array} \tag{3.7}$$

with the usual $\frac{1}{p} + \frac{1}{p'} = 1$.

Fix sets $T = TK \subset G$ and $U \subset P$ of finite measure. As in (2.17) define $Pf = 1_T f$ and $Qf = \mathcal{F}^{-1}(1_U \mathcal{F}(f))$.

Proposition 3.8: *If $1 \le p \le 2$ and $q \ge 1$ then $||PQf||_q \le m_G(T)^{1/q}\nu(U)^{1/p}||f||_p$ for $f \in L_p(G/K)$. If $p = q$ the operator norm on $L_p(G/K)$ satisfies $||PQ||_p \le m_G(T)^{1/p}\nu(U)^{1/p}$.*

Proof: We compute essentially as in the proof of Proposition 2.18:

$$\begin{aligned} PQf(x) &= 1_T(x)\int_P 1_U(\omega)\left\{\int_G f(y)\langle\pi_\omega(y)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} dm_G(y)\right\} d\nu(\omega) \\ &= 1_T(x)\int_G f(y)\left\{\int_P 1_U(\omega)\langle u_\omega, \pi_\omega(y^{-1}x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega)\right\} dm_G(y) \\ &= \langle f, \overline{k_x}\rangle_{L_2(G/K)} \end{aligned}$$

where

$$\begin{aligned} k_x(y) &= 1_T(x)\int_P 1_U(\omega)\langle u_\omega, \pi_\omega(y^{-1}x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \\ &= 1_T(x)\mathcal{F}^{-1}(\omega \mapsto 1_U(\omega)u_\omega)(y^{-1}x). \end{aligned}$$

Using Hölder, integration $\int_G |PQf(x)|^q dm_G(x)$, and (3.7), we see as before that

$$\begin{aligned} ||PQf||_q &\le ||f||_p\, ||\mathcal{F}^{-1}(\omega \mapsto 1_U(\omega)u_\omega)||_{p'}\, m_G(T)^{1/q} \\ &\le ||f||_p\, ||1_U||_p\, m_G(T)^{1/q} = m_G(T)^{1/q}\, \nu(U)^{1/p}\, ||f||_p. \end{aligned}$$

That completes the proof. ∎

Given $\epsilon \ge 0$ we say that $f \in L_p(G/K)$ is L_p ϵ**–concentrated** on T if $||f - 1_T f||_p \le \epsilon ||f||_p$ and similarly, given $\delta \ge 0$, $v \in \mathcal{H}_{p'}$ is $L_{p'}$ δ**–concentrated** on U if $||v - 1_U\, v||_{p'} \le \delta ||v||_{p'}$. Analogously,

(3.9) $f \in L_p(G/K)$ is L_p $\delta-$ **bandlimited** to U if there exists $f_U \in L_p(G/K)$ with $\mathcal{F}(f_U)$ supported in U and $||f - f_U||_p \le \delta ||f||_p$.

We proved Theorem 2.23, the scalar uncertainty principle for $K\backslash G/K$, as a formal consequence of Proposition 2.18. Exactly the same argument proves the vector uncertainty principle for G/K as a formal consequence of Proposition 3.8:

Theorem 3.10: ***(Vector Uncertainty Principle)*** *Suppose that* $0 \ne f \in L_p(G/K)$ *with* $1 \le p \le 2$ *and* $\epsilon, \delta \ge 0$ *such that* f *is* ϵ*-concentrated on* T *and* δ*-bandlimited to* U*. Then*

$$m_G(T)^{1/p}\nu(U)^{1/p} \ge ||PQ||_p \ge \frac{1-\epsilon-\delta}{1+\delta}.$$

And if $p = 2$ *then, further,* $||PQ||_2 \ge 1 - \epsilon - \delta$.

There is an equivalent uncertainty principle for the operator–valued transform $f \mapsto \mathbb{F}(f)$ where $\mathbb{F}(f)(\omega) = \pi_\omega(f)$ for $f \in L_p(G/K)$ and $\omega \in P$. See the discussion surrounding (3.5).

4 Application to Special Functions.

The results described in § § 2 and 3 apply in particular to riemannian symmetric spaces $X = G/K$ of noncompact type. There the space P of positive definite zonal spherical functions, the zonal spherical functions, and the Plancherel measure on P, all are known explicitly from the work of Harish–Chandra and others. See *[12]*, Chapter 4 for an exposition and *[12]*, pp. 492–493 for extensive references. The zonal spherical functions usually turn out to be interesting, well known special functions. Thus the uncertainty principles described in § § 2 and 3 give results on some classes of special functions that are usually studied in other settings. Of course many cases had been worked out using specific properties of those functions. Here we'll illustrate this phenomenon by looking at the case $G/K = SU(\ell, \ell+q)/S(U(\ell) \times U(\ell+q))$ of *[20]*.

Fix a riemannian symmetric space $X = G/K$ of noncompact type. Then G is a connected semisimple Lie group with finite center and no nontrivial compact normal subgroup, K is a maximal compact subgroup, and we have an involutive automorphism θ of G with fixed point set K. The Lie algebra $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$, decomposition into ±1–eigenspaces of $d\theta$. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, let A be the corresponding analytic subgroup of G, let M denote the centralizer of A in K, and let M' denote the normalizer of A in K. Then M'/M is the Weyl group $W = W(G, A)$ which acts simply transitively on the collection of positive subsystems of Φ. The group MA is the centralizer of A in G, and the Lie algebra decomposes as the direct sum $\mathfrak{g} = (\mathfrak{m}+\mathfrak{a}) + \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$ of the centralizer of $\mathfrak{a}$ and the other $\mathfrak{a}$–root spaces. Choose a positive subsystem $\Phi^+ \subset \Phi$, let $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, and let N be the corresponding analytic subgroup of G. MA normalizes N and we have the Iwasawa decomposition $G = KAN$. Any two such Iwasawa decomposition differ by a G–conjugacy.

Here is the connection with the spherical transform of § § 2 and 3. The space of positive definite spherical functions is parameterized by

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \mid \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \Phi^+\}. \tag{4.1}$$

The Plancherel measure ν for (G, K) is given by $d\nu(\lambda) = |\mathbf{c}(\lambda)|^{-2} d\lambda$ where $\mathbf{c}(\lambda)$ is the value of Harish–Chandra's $\mathbf{c}$–function for X. The irreducible unitary representations with K–fixed unit vectors, π_λ corresponding to $\lambda \in \mathfrak{a}^*_+$, are the representations unitarily induced from the minimal parabolic subgroup as follows:

$$\pi_\lambda = \operatorname{Ind}_Q^G(\chi_\lambda) \text{ where } Q = MAN \text{ and } \chi_\lambda(man) = e^{i\lambda(\log a)}. \tag{4.2}$$

Thus the representation space of π_λ is

$$\mathcal{H}_\lambda = \{\phi : G \to \mathbb{C} \mid \phi(gman) = e^{(i\lambda-\rho)(\log a)}\phi(g) \text{ and } \int_{G/Q} |\phi(g)|^2 dm_G(g) < \infty\} \tag{4.3}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}(\dim \mathfrak{g}_\alpha)\alpha$, that is, $\rho(H) = \frac{1}{2}\,\mathrm{trace}\,(\,\mathrm{ad}\,(H)|_{\mathfrak{n}})$ for $H \in \mathfrak{a}$. Since K is transitive on the Firstenberg boundary $\mathcal{B} = G/Q$ and $K \cap Q = M$, we identify the representation space (4.3) with $L_2(\mathcal{B})$ and view first, $\mathcal{B} = K/M$ with K–invariant measure $m_{\mathcal{B}}$ induced from m_K and, second, $\mathcal{H}_\lambda \cong L_2(\mathcal{B})$ under $\phi \mapsto \phi|_K$. Thus the Fourier transform $\mathcal{F} : L_1(G/K) \to \int_{\mathfrak{a}} \mathcal{H}_\lambda d\nu(\lambda)$, given by $\mathcal{F}(f)(\lambda) = s_f(\lambda) = \pi_\lambda(f)u_\lambda$, becomes

$$\mathcal{F} : L_1(G/K) \to L_2(\mathfrak{a}_+^* \times \mathcal{B}, \nu \times m_{\mathcal{B}})$$

by

$$\widetilde{f}(\lambda, kM) = \mathcal{F}(f)(\lambda, kM) = \int_X f(gK)e^{(i\lambda-\rho)(H(g^{-1}k))}dm_X(gK). \tag{4.4}$$

Here m_X is the measure on $X = G/K$ induced by m_G and we use the Iwasawa decomposition $G = KAN$ to define $H : G \to \mathfrak{a}$ by $g = k\exp(H(g))n$.

We need two more ingredients to make the Plancherel formula explicit for $X = G/K$: the formula for the zonal spherical function ϕ_λ that corresponds to $\lambda \in \mathfrak{a}_+^*$ and the formula for the Harish–Chandra c–function $\mathbf{c}\,(\lambda)$.

The general expression for spherical functions on $X = G/K$ is $\phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(H(gk))}dm_K(k)$. In the case $G = SU(\ell, \ell+q)$ this is expressed using hypergeometric functions

$$F(a, b; c; z) = \frac{1}{2\pi i}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_C \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s)(-z)^s ds \tag{4.5}$$

where z is negative real and the contour C of integration is obtained from the imaginary axis by making dents so that the poles of $\Gamma(-s)$ are to the right of C and the poles of $\Gamma(a+s)\Gamma(b+s)$ are to the left of C. (4.5) is Barnes' integral representation. Let t_j be the coordinates on A given by

$$a = \begin{bmatrix} 0 & 0 & d(a) \\ 0 & I & 0 \\ d(a) & 0 & 0 \end{bmatrix}, \text{ where } d(a) = \mathrm{diag}\{e^{t_1}, \ldots, e^{t_\ell}\} \text{ and } I \text{ is } q \times q.$$

Then the positive $\mathfrak{a}$–roots are given by $\alpha(\log a) = t_i$ with multiplicity $m_\alpha = 2q$, by $\alpha(\log a) = 2t_i$ with multiplicity $m_\alpha = 1$, and by $\alpha(\log a) = t_i \pm t_j$, $i < j$, with multiplicity $m_\alpha = 2$. Set

$$\Phi_{\lambda_j}(t_j) = \sinh(t_j)^{i\lambda_j-(q+1)}F(\frac{1}{2}(-q+1-i\lambda_j), \frac{1}{2}(q+1-i\lambda_j); 1-i\lambda_j; -(\sinh(t_j)^{-2})) \tag{4.6a}$$

and

$$\Phi_\lambda(a) = \left(\prod_{j=1}^{\ell}\Phi_{\lambda_j}(t_j(a))\right) \Big/ \left(\prod_{i<j}(\cosh 2t_i - \cosh 2t_j)\right) \tag{4.6b}$$

We have $G = KA_+K$ where $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha_j(H) \geq 0 \text{ for } 1 \leq j \leq \ell\}$ and $A_+ = \exp(\mathfrak{a}_+)$. The zonal spherical functions are given by K–bi–invariance and the formula

$$\phi_\lambda(a) = c_G \sum_{w \in W} \mathbf{c}\,(w\lambda)\Phi_{w\lambda}(a) \tag{4.7}$$

where $\lambda \in \mathfrak{a}_+^*$, $a \in A_+$, c_G is a nonzero constant and $\mathbf{c}\,(w\lambda)$ is the value of the Harish–Chandra c–function.

The general formula for the Harish–Chandra c–function is due to Harish–Chandra in real rank 1 and for complex simple G, to Bhanu–Murthy for all but one case of the split classical groups G, and then in general in the form of the product formula (4.8) by Gindikin and Karpelevič. Let $\Phi_0^+ = \{\alpha \in \Phi^+ \mid k\alpha \in \Phi^+ \text{ implies } k \geq 1\}$, the set of primitive positive $\mathfrak{a}$–roots. Then

$$\mathbf{c}\,(\lambda) = c_0 \prod_{\alpha \in \Phi_0^+} \frac{2^{-\langle i\lambda, \alpha_0\rangle}\Gamma(\langle i\lambda, \alpha_0\rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0\rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0\rangle))} \tag{4.8}$$

where $\alpha_0 = ||\alpha||^{-2}\alpha$ and the constant c_0 is specified by $\mathbf{c}\,(-i\rho) = 1$.

In the case of $G = SU(\ell, \ell+q)$ we have now expressed all the ingredients of the uncertainty principles of § § 2 and 3 in terms of gamma functions and a few hyperbolic trig functions. Shashahani *[20]* used these explicit expressions to prove an uncertainty principle that he needed for application to scattering theory. That principle is as follows. For $\epsilon > 0$ let $N(\epsilon) = \{\lambda \in \mathfrak{a}_+^* \mid ||\lambda||^2 \leq \epsilon\}$. For $r > 0$ let B_r denote the ball of radius r centered at the base point $1K \in G/K = X$ of our riemannian symmetric space. Then there are constants $s, t > 0$ that depend only on X and r such that, if $f \in L_2(X)$ with support in B_r , then

$$\int_B \int_{N(\epsilon)} |\widetilde{f}(\lambda, b)|^2 \frac{d\lambda}{|\mathbf{c}\,(\lambda)|^2} dm_B(b) \leq s\epsilon^t \int_X |f(x)|^2 dx. \tag{4.9}$$

Here $N(\epsilon)$ has finite volume just when $\ell = 1$. The inequality (4.9) corresponds to a bound on the operator Q of (2.17) and its analog in § 3.

This indicates how particular results involving special functions can be closely related to the uncertainty principles discussed in § § 2 and 3.

5 Application to Cayley Graphs

Recall the definition of Cayley graph based on a group G. The vertices of the graph are just the elements of G, and a subset $S \subset G$ determines the edges as $\{e = [g, gs] \mid g \in G \text{ and } s \in S\}$. It is best if $1 \notin S = S^{-1}$ and $[g, gs]$ is interpreted as the oriented edge from g to gs.

If G is discrete, there are no topological considerations, and we can use counting measure when we want to sum a function on the graph. In our case, however, we want to use the topological structure of G in a serious way, imposing both topology and measure.

We now suppose that G is a locally compact group. Then the space of vertices carries the locally compact topology of G and left Haar measure m_G. S carries the subspace topology and the induced Borel structure. Fix a Radon probability measure σ on S. One should view σ as the branching probability from a vertex g along the oriented edges $\{[g, gs] \mid s \in S\}$. The best situation is that in which S generates a locally closed (and thus locally compact) subgroup $N \subset G$ and σ is the S–restriction of Haar measure m_N. But of course that situation is exceptional. Instead we will assume that σ is absolutely continuous with respect to m_N, say $d\sigma(s) = \gamma(s)m_N(s)$ for some measurable function γ on N positive on S and zero on $N \setminus S$. In any case, the space of edges carries the topology of $G \times S$ and measure $m_G \times \sigma$.

Let G be abelian and $f \in L_p(G \times S, m_G \times \sigma)$, $1 \leq p \leq 2$. We view $f^\dagger = f\gamma^{1/p}$ as an element of $L_p(G \times S, m_G \times m_N)$, so that we have the spherical transform (here the Pontrjagin–Fourier transform) $\widehat{f^\dagger} : \widehat{G} \times \widehat{N} \to \mathbb{C}$. Then Scalar Uncertainty Principle of Theorem 2.23 holds. Thus, if $f^\dagger$ is ϵ–concentrated on $T \subset G \times S$ with respect to Haar measure $m_G \times m_N$, and if $f^\dagger$ is δ–bandlimited to $U \subset \widehat{G} \times \widehat{N}$, again with respect to Haar measure $m_G \times m_N$, then $(m_G \times m_N)(T)^{1/p}(\nu_G \times \nu_N)(U)^{1/p} \geq ||PQ||_p \geq \frac{1-\epsilon-\delta}{1+\delta}$. And if $p = 2$ then, further, $||PQ||_2 \geq 1 - \epsilon - \delta$. Thus the Scalar Uncertainty Principle for the locally compact abelian group $G \times N$ gives an uncertainty principle for the edge space of the Cayley graph with branching probability, defined by (G, S, σ).

We will now carry these considerations from locally compact abelian groups G to Gelfand pairs (G, K).

Let K be a subgroup of G and suppose that S normalizes K. Suppose that $S \cap K = \emptyset$, in other words that $gK \neq gsK$ for any $s \in S$. Then we have the directed graph $\Gamma = \Gamma(G/K, SK/K)$ whose vertices are the elements of G/K and whose edges are the $[gK, gsK], g \in G$ and $s \in S$. In the topological setting K should be a closed subgroup of G so that the space of vertices will inherit a good topology. This extends the notion of Cayley graph, which is the case $K = \{1\}$.

Suppose that (G, K) is a Gelfand pair. S normalizes K so N normalizes K and NK is a locally compact subgroup of G that contains K. Now

$(G \times NK,\ K \times K)$ is a Gelfand pair corresponding to the coset space $G/K \times NK/K$, and the subset $(G/K \times SK/K) \subset (G/K \times NK/K)$ is the edge space of the generalized Cayley graph $\Gamma(G/K, SK/K)$. Thus the Vector Uncertainty Principle of Theorem 3.10 applies to this edge space as follows. Let $f \in L_p(G/K \times SK/K,\ m_G \times \sigma)$ with $1 \le p \le 2$. So as before, $f^\dagger = f\gamma^{1/p} \in L_p(G/K \times SK/K, m_G \times m_{NK})$. Let $\epsilon, \delta \ge 0$ and let $T \subset G \times S$ and $U \subset P_{G/K \times NK/K}$ be sets of finite measure such that $f^\dagger$ is ϵ–concentrated on T relative to $m_G \times m_{NK}$ and δ–bandlimited to U relative to $m_G \times m_{NK}$ and $\nu_{G/K} \times \nu_{NK/K}$. Then

$$(m_G \times m_N)(T)^{1/p}(\nu_{G/K} \times \nu_{NK/K})(U)^{1/p} \ge ||PQ||_p \ge \frac{1-\epsilon-\delta}{1+\delta}.$$

And if $p = 2$ then, further, $||PQ||_2 \ge 1 - \epsilon - \delta$. Thus the Vector Uncertainty Principle for the Gelfand pair $(G \times NK,\ K \times K)$ implies an uncertainty principle for the edge space of the generalized Cayley graph with branching probabilities, based on (G, K, S, σ).

Using somewhat different methods, Velasquez proved an uncertainty principle for 1–coboundaries on Cayley graphs $\Gamma(G, S)$ where G is a finite group *[25]*. A finite (or even compact) group G can be viewed as $(G \times G)/\delta G$ where $G \times G$ acts on G by left and right translations, and δG is the diagonal in $G \times G$. There $(G \times G, \delta G)$ is a Gelfand pair. Thus one might expect a direct connection between the just–described uncertainty principle for generalized Cayley graphs based on Gelfand pairs, and Velasquez' uncertainty principle for 1–coboundaries on finite Cayley graphs *[25]*. Velasquez and I are looking into that now *[26]*.

6 Application to Cayley Complexes.

We now view (generalized) Cayley graphs as the 1–dimensional case of (generalized) Cayley complexes. This does not seem to be in the literature, but the idea is hardly novel.

Definition 6.1: *Let G be a group and fix a subset $S = S^{-1} \subset G$ with $1 \notin S$.*

*A 0–**simplex** for (G, S) is an element of G. A 1–**simplex** for (G, S) is an ordered pair $\Delta_1 = [g, gs]$ with $g \in G$ and $s \in S$. We recursively define an n–**simplex** for (G, S) to be an ordered $(n + 1)$–tuple $\Delta_n = [t_0, t_1, \ldots, t_n]$ modulo even permutations, such that each $\partial_i \Delta_n = [t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n]$ is an $(n - 1)$–simplex for (G, S).*

*The **Cayley complex** associated to (G, S) is $\Gamma(G, S) = \bigcup_{n \ge 0} \Gamma_n(G, S)$, where $\Gamma_n(G, S)$ consists of the n–simplexes for (G, S), and where the boundary map is given by $\partial\Delta_n = \sum_i (-1)^i \partial_i \Delta_n$.*

Note that $\Gamma_0(G, S) \cup \Gamma_1(G, S)$ is the Cayley graph associated to (G, S).

Here Δ_n is viewed as an oriented n–simplex with vertices t_i and the $\partial_i\Delta_n$ are viewed as its faces. Δ_n specifies the t_i up to an even permutation. Since each $[t_i, t_j] \in \Gamma_1(G,S)$ for $i \neq j$, each $t_i^{-1}t_j \in S$. Let Σ_{n+1} denote the group of even permutations of $\{0,1,2,\ldots,n\}$. Its action $p : (t_0, t_1, \ldots, t_n) \mapsto (t_{p(0)}, t_{p(1)}, \ldots, t_{p(n)})$ carries over to

$$'p : (t_0, t_0^{-1}t_1, t_1^{-1}t_2, \ldots, t_{n-1}^{-1}t_n) \mapsto (t_{p(0)}, t_{p(0)}^{-1}t_{p(1)}, t_{p(1)}^{-1}t_{p(2)}, \ldots, t_{p(n-1)}^{-1}t_{p(n)}). \tag{6.2}$$

The left action $g : [t_0, t_1, \ldots, t_n] \mapsto [gt_0, gt_1, \ldots, gt_n]$ of G on $\Gamma_n(G,S)$ carries over to

$$'g : (t_0, t_0^{-1}t_1, t_1^{-1}t_2, \ldots, t_{n-1}^{-1}t_n) \mapsto (gt_0, t_0^{-1}t_1, t_1^{-1}t_2, \ldots, t_{n-1}^{-1}t_n). \tag{6.3}$$

$\Gamma_n(G,S)$ is parameterized by $(G \times S_n)/\Sigma_{n+1}$ where S_n is a certain subset of $S^n = S \times \cdots \times S$ and where Σ_{n+1} acts on $(G \times S_n)$ as in (6.2). In this parameterization, G acts as in (6.3).

As in § 5, suppose that G is a locally compact group. Then $\Gamma_0(G,S) = G$ carries the locally compact topology of G and left Haar measure m_G. As before, S carries the subspace topology and the induced Borel structure, and we fix a Radon probability measure σ on S to represent branching probabilities on the Cayley graph $\Gamma_0(G,S) \cup \Gamma_1(G,S)$. Suppose that S_n generates a locally compact subgroup $N_n \subset G^n = G \times \cdots \times G$. Suppose also that S_n has positive measure with respect to $\sigma^n = \sigma \times \cdots \times \sigma$ and that σ^n is absolutely continuous with respect to left Haar measure m_{N_n}. Let τ_n denote the probability measure $\sigma_n(S_n)^{-1}\sigma^n$ on S_n. Then $d\tau_n(s_1, \ldots, s_n) = \gamma_n(s_1, \ldots, s_n)m_{N_n}(s_1, \ldots, s_n)$ for some measurable function γ_n on N_n positive on S_n and zero on $N_n \setminus S_n$. We will write $m_G \times \tau_n$ both for the product measure on $G \times S_n$ and for its push–down to $\Gamma_n(G,S) \cong (G \times S_n)/\Sigma_{n+1}$. Then the space $\Gamma_n(G,S)$ of n–simplexes carries both the topology of $(G \times S_n)/\Sigma_{n+1}$ and the measure $m_G \times \tau^n$.

We proceed as in the case of Cayley graphs. Let G be abelian and $f \in L_p(\Gamma_n(G,S), m_G \times \tau_n), 1 \leq p \leq 2$. View $f^\dagger = f\gamma^{1/p}$ as a Σ_{n+1}–invariant element of $L_p(G \times S_n, m_G \times m_{N_n})$, so that we have the spherical transform $\widehat{f^\dagger} : \widehat{G} \times \widehat{N_n} \to \mathbb{C}$. Then Theorem 2.23 tells us: If $f^\dagger$ is ϵ–concentrated on a Σ_{n+1}–invariant subset $T \subset G \times S_n$ with respect to Haar measure $m_G \times m_{N_n}$, and if $f^\dagger$ is δ–bandlimited to $U \subset \widehat{G} \times \widehat{N_n}$, again with respect to Haar measure $m_G \times m_{N_n}$, then

$$(m_G \times m_{N_n})(T)^{1/p}(\nu_G \times \nu_{N_n})(U)^{1/p} \geq ||PQ||_p \geq \frac{1-\epsilon-\delta}{1+\delta}. \tag{6.4}$$

And if $p = 2$ then, further, $||PQ||_2 \geq 1 - \epsilon - \delta$. Thus the Scalar Uncertainty Principle for the locally compact abelian group $G \times N_n$ gives an uncertainty principle for the space $\Gamma_n(G,S)$ of n–simplexes in the Cayley complex $\Gamma(G,S)$.

These considerations carry over from locally compact abelian groups G to Gelfand pairs (G, K) as in § 5. K is a subgroup of G and that S normalizes K, so $L_n = N_n K^n$ is a subgroup of G^n. Suppose that $S \cap K = \emptyset$, so $gK \neq gsK$ for any $s \in S$. Then

$$\pi : [t_0, t_1, \ldots, t_n] \mapsto [t_0K, t_1K, \ldots, t_nK] \tag{6.5a}$$

maps the Cayley complex $\Gamma(G, S)$ onto a **generalized Cayley complex**

$$\Gamma(G, K, S) = \bigcup_{n \geq 0} \Gamma_n(G, K, S). \tag{6.5b}$$

Note that

$$\pi\Gamma_n(G, S) = \Gamma_n(G, K, S), \text{ and } \pi\partial\Delta_n = \partial\pi\Delta_n \text{ for all } \Delta_n \in \Gamma_n(G, S). \tag{6.5c}$$

In the topological setting K should be a closed subgroup of G so that the quotient topology on $\Gamma_n(G, K, S)$ is locally compact. This extends the notion of Cayley complex, which is the case $K = \{1\}$.

Suppose that (G, K) is a Gelfand pair. S normalizes K so N_n normalizes $K^n = K \times \cdots \times K$, and $L_n = N_n K^n$ is closed in G^n. Now L_n is a locally compact subgroup of G^n that contains K^n, and $(G \times L_n,\ K^{n+1})$ is a Gelfand pair corresponding to the coset space $G/K \times L_n/K^n$. The subset $(G/K \times S_nK^n/K^n) \subset (G/K \times L_n/K^n)$ represents the space $\Gamma_n(G, K, S) \cong (G/K \times S_nK^n/K^n)/\Sigma_{n+1}$ of n–simplexes. Thus the Vector Uncertainty Principle of Theorem 3.10 gives us an uncertainty principle for $\Gamma_n(G, K, S)$ as follows. Let $f \in L_p(G/K \times S_nK^n/K^n,\ m_G \times \tau_n)$ with $1 \leq p \leq 2$. So $f^\dagger = f\gamma^{1/p} \in L_p(G/K \times S_nK^n/K^n, m_G \times m_{N_n})$. Let $\epsilon, \delta \geq 0$ and let $T = \Sigma_{n+1}(T) \subset G \times S_n$ and $U \subset P_{G/K \times L_n/K^n}$ be sets of finite measure such that $f^\dagger$ is ϵ–concentrated on T relative to $m_G \times m_{L_n}$ and δ–bandlimited to U relative to $m_G \times m_{L_n}$ and $\nu_{G/K} \times \nu_{L_n/K^n}$. Then

$$(m_G \times m_{N_n})(T)^{1/p}(\nu_{G/K} \times \nu_{L_n/K^n})(U)^{1/p} \geq ||PQ||_p \geq \frac{1 - \epsilon - \delta}{1 + \delta}. \tag{6.6}$$

And if $p = 2$ then, further, $||PQ||_2 \geq 1 - \epsilon - \delta$. Thus the Vector Uncertainty Principle for the Gelfand pair $(G \times L_n,\ K \times K^n)$ implies an uncertainty principle for the space of n–simplexes on the generalized Cayley complex $\Gamma(G, K, S)$.

7 Application to Commutative Hypergroups.

We recall some basic facts about hypergroups (*[6]*, *[14]*, *[23]*, *[24]*) and indicate the extension *[27]* of the scalar uncertainty principle (Theorem 2.23) above from Gelfand pairs to commutative hypergroups.

Let X be a locally compact Hausdorff topological space and $M(X)$ the space of regular complex–valued Borel measures. $M^+(X)$ denotes the space of non-negative measures in $M(X)$ with the weak topology for the maps $\mu \mapsto \mu(f), f \in C_0^+(X)$. This topology on $M^+(X)$ is called the **cone topology**. Given $x \in X$ we have the point mass $\delta_x \in M^+(X)$ and this gives a homeomorphism of X onto a closed subset of $M^+(X)$.

The space $\mathcal{C}(X)$, consisting of all compact subsets of X, carries a locally compact hausdorff topology generated by the sets $\mathcal{C}_U(V) = \{S \in \mathcal{C}(X) \mid S \text{ meets } U \text{ and } S \subset V\}$ for U, V open in X. If Y is closed in X then $\mathcal{C}(Y)$ is closed in $\mathcal{C}(X)$, and $x \mapsto \{x\}$ is a homeomorphism of X onto a closed subset of $\mathcal{C}(X)$.

Definition 7.1: *Let $* : M(X) \times M(X) \to M(X)$ be an associative algebra product such that*

1. *$M^+(X) * M^+(X) \subset M^+(X)$ and $* : M^+(X) \times M^+(X) \to M^+(X)$ is continuous,*
2. *if $x, y \in X$ then $\delta_x * \delta_y$ is a compactly supported probability measure on X,*
3. *the map $X \times X \to \mathcal{C}(X)$, given by $(x, y) \mapsto \delta_x * \delta_y$, is continuous,*
4. *there is an element $e \in X$ such that $\delta_x * \delta_e = \delta_x = \delta_e * \delta_x$ for all $x \in X$, and*
5. *there is an involution $x \mapsto \overline{x}$ of X such that $e \in Supp(\delta_x * \delta_y)$ if and only if $x = \overline{y}$.*

Then $(X, *)$ is a ***hypergroup****.*

The locally compact group case is the case where each $\delta_x * \delta_y$ is of the form δ_z; then the group composition is $xy = z$ and the involution is $x \mapsto x^{-1}$.

The double coset space case, $X = K\backslash G/K$ where G is a locally compact group and K is a compact subgroup, is the case where $*$ is inherited from convolution on G. Then $\delta_{KaK} * \delta_{KbK}$ is supported in $K(aKb)K$. In the double coset space case $X = K\backslash G/K$, $(X, *)$ is commutative if and only if (G, K) is a Gelfand pair. When G/K is a symmetric space, the symmetry gives the hypergroup involution.

Voit *[27]* observed that all the tools used in my argument *[28]* for the scalar uncertainty principle for Gelfand pairs are available for commutative hypergroups, and that my proof of Theorem 2.23 goes through with no essential change for commutative hypergroups. The result is the same uncertainty principle.

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