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Compact Subvarieties in Flag Domains

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to Bert Kostant on the occasion of his sixty–fifth birthday

Abstract. A real reductive Lie group G_0 acts on complex flag manifolds G/Q, where Q is a parabolic subgroup of the complexification G of G_0. The open orbits $D = G_0(x)$ include the homogeneous complex manifolds of the form G_0/V_0 where $V_0 = G_0 \cap Q_x$ is the centralizer of a torus; those are the G_0–homogeneous pseudo–kähler manifolds. For an appropriate choice K_0 of maximal compact subgroup of G_0, the orbit $Y = K_0(x)$ is a maximal compact complex submanifold of D. The "cycle space" $M_D = \{gY \mid g \in G \text{ and } gY \subset D\}$, space of maximal compact linear subvarieties of D, has a natural complex structure. M_D plays important roles in the theory of moduli of compact kähler manifolds and in automorphic cohomology theory. Here we sketch a brief exposition of this interesting mathematical topic.

0. Introduction

Some of the most interesting homogeneous spaces in algebraic and differential geometry are of the form $D = G_0(x) \cong G_0/V_0$ where G_0 is a real semisimple group acting naturally on a flag manifold $X = G/Q$. Here G is the complexification of G_0 and Q is a "parabolic subgroup" of G. The complex flag manifolds themselves are examples of these real group orbits $D = G_0(x)$, corresponding to the case where G_0 is a compact real form of G. Grassmann manifolds and the other hermitian symmetric spaces of compact type are complex flag manifolds. The class of real group orbits $D = G_0(x) \subset G/Q = X$ on complex flag manifolds includes the noncompact "real forms" of complex flag manifolds, such as bounded symmetric domains, and also moduli spaces for polarized Hodge structures. These homogeneous spaces also play important roles in algebraic topology, in harmonic analysis (specifically in the representation theory for semisimple Lie groups), and in related theories of automorphic cohomology.

The theory is especially rich in the case of "flag domains", the case of measurable open orbits $D = G_0(x) \subset X$. In this note I'll sketch the structure of those open orbits. They are important in harmonic analysis and representation theory for semisimple Lie groups (see [12], [13], [14], [16] and [24]), for the corresponding automorphic function theory (see [7],

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[8], [9], [20], [21] and [25]), and for their intrinsic geometric interest. The connection with representation theory is the realization of discrete series (or limits of discrete series) representations of G_0 as the natural action of G_0 on cohomologies $H^s(D;\mathbb{E})$ of homogeneous holomorphic vector bundles $\mathbb{E} \to D$. The connection with automorphic function theory, as well as other aspects of complex geometry and complex harmonic analysis, comes through a certain "linear cycle space" M_D of compact subvarieties $Y \subset D$. If $\phi \in H^s(D;\mathbb{E})$ with $s = \dim_{\mathbb{C}} Y$ then integration over compact subvarieties carries ϕ to a section of a certain vector bundle over M_D. This carries a number of analytic problems from cohomology to the more accessible setting of sections of vector bundles. That sort of consideration plays a key role in the proof of Fréchet convergence of certain Poincaré series $\sum_\Gamma \gamma^*(\phi)$.

It is a pleasure to indicate some aspects of the theory of flag manifolds and flag domains in a volume celebrating this milestone occasion for Prof. Bertram Kostant. For he introduced me to the theory of real semisimple Lie groups and guided my first explorations of the orbit theory.

In §1 we indicate the structure of parabolic subgroups Q in a complex semisimple Lie group G and the corresponding flag manifold $X = G/Q$. This material was developed by Jacques Tits [17] in the 1950's. In §§2 and 3 we indicate a few generalities on orbits $G_0(x) \subset X$ where G_0 is a real form of X, and I'll look at the case where $G_0(x)$ is open in X. In §4 we sketch the situation ([22], [23]) for the hermitian symmetric case. It is fairly concrete and is needed later in §8. Then §5 returns to the case of open orbits, specializing to the measurable case, the case where $D = G_0(x)$ carries a G_0–invariant pseudo–kähler metric. In that case we say that D is a "flag domain." The material of §§2, 3 and 5 represents joint work with Bert Kostant in the 1960's and is published in [22]. In §6 I sketch the construction of an exhaustion function for a flag domain D and the corresponding vanishing theorem for cohomologies of coherent analytic sheaves $\mathcal{F} \to D$. This construction, due to Wilfried Schmid and myself [15] and based on earlier work of Schmid [12], comes into the construction of representations of G_0 on cohomologies of G_0–homogeneous holomorphic vector bundles over D ([12], [13], [16], [20], [21]). It also comes into analysis ([20], [26]) of the space M_D of maximal compact subvarieties of D. The coset space structure of M_D is described in §7, and in §8 I indicate my proof [26] that M_D is a Stein manifold.

1. Parabolic subgroups

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a positive root system $\Delta^+ = \Delta^+(\mathfrak{g},\mathfrak{h})$. Then Δ is the disjoint union of Δ^+ with $\Delta^- = -\Delta^+$. Let Ψ denote the corresponding simple root system. Thus every root $\alpha \in \Delta$ has unique expression $\alpha = \sum_{\psi\in\Psi} n_\psi \psi$, the n_ψ are integers, every $n_\psi \geqq 0$ if $\alpha \in \Delta^+$, and every $n_\psi \leqq 0$ if $\alpha \in \Delta^-$.

Every subset $\Phi \in \Psi$ defines a subalgebra $\mathfrak{q}_\Phi = \mathfrak{q}_\Phi^{-n} + \mathfrak{q}_\Phi^r \subset \mathfrak{g}$ where

$$\begin{aligned} \mathfrak{q}_\Phi^r &= \mathfrak{h} + \textstyle\sum_{\alpha\in\Phi^r} \mathfrak{g}_\alpha \text{ with } \Phi^r = \{\alpha \in \Delta \mid \alpha = \textstyle\sum_{\psi\in\Phi} n_\psi\psi\}, \\ \mathfrak{q}_\Phi^n &= \textstyle\sum_{\beta\in\Phi^{-n}} \mathfrak{g}_\beta \quad \text{with } \Phi^n = \{\alpha \in \Delta^+ \mid \alpha \notin \Phi^r\}, \\ \mathfrak{q}_\Phi^{-n} &= \textstyle\sum_{\beta\in\Phi^{-n}} \mathfrak{g}_\beta \quad \text{with } \Phi^{-n} = \{\alpha \in \Delta^- \mid \alpha \notin \Phi^r\}. \end{aligned} \tag{1.1}$$

Thus $\mathfrak{q}_\Phi^{-n}$ is the nilradical of $\mathfrak{q}_\Phi$ and $\mathfrak{q}_\Phi^r$ is a reductive (Levi) complement. The $2^{|\Psi|}$ subalgebras $\mathfrak{q}_\Phi \subset \mathfrak{g}$ are the **standard parabolic subalgebras** of $\mathfrak{g}$. The subalgebra $\mathfrak{q}_\Phi^n + \mathfrak{q}_\Phi^r \subset \mathfrak{g}$ is called the **opposite** of $\mathfrak{q}_\Phi$.

Note that we have set things up so that the nilradical $\mathfrak{q}_\Phi^{-n}$ is spanned by negative root spaces. When we go to complex flag manifolds, this will mean that the holomorphic tangent space $\mathfrak{q}_\Phi^n$ is spanned by positive root spaces, so positive bundles will correspond to positive linear functionals on $\mathfrak{h}$.

The extreme cases are $\mathfrak{q}_\Psi = \mathfrak{g}$, the entire algebra, and $\mathfrak{q}_\emptyset = \mathfrak{h} + \sum_{\beta\in\Delta^-} \mathfrak{g}_\beta$, a Borel subalgebra. More generally, the maximal solvable subalgebras of $\mathfrak{g}$ are called **Borel subalgebras** and are $\mathrm{Int}(\mathfrak{g})$–conjugate to $\mathfrak{q}_\emptyset$, where $\mathrm{Int}(\mathfrak{g})$ denotes the group of inner automorphisms of $\mathfrak{g}$. A subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is called a **parabolic subalgebra** if it contains a Borel subalgebra, and every parabolic subalgebra of $\mathfrak{g}$ is $\mathrm{Int}(\mathfrak{g})$–conjugate to exactly one standard parabolic subalgebra.

Let G be a reductive complex Lie group. In other words, its Lie algebra $\mathfrak{g}$ is reductive. If $\mathfrak{q} = \mathfrak{q}_\Phi \subset \mathfrak{g}$ is a parabolic subalgebra, then the corresponding **parabolic subgroup** of G is its normalizer

$$Q = Q_\Phi = N_G(\mathfrak{q}) = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{q} = \mathfrak{q}\}. \tag{1.2}$$

Let G^0 denote the topological component of the identity in G. Then $Q \cap G^0$ is connected, that is, $Q \cap G^0 = Q^0$, for every parabolic subgroup $Q \subset G$. This is a consequence of simple transitivity of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ on the collection of simple subsystems of $\Delta(\mathfrak{g},\mathfrak{h})$. On the other hand, if $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}$ then Q meets the component gG^0 of G. But if $\mathrm{Ad}(g)$ is outer this depends on Φ.

Let $L \subset G$ be a complex Lie subgroup. If G is connected then the following conditions are equivalent:

(i) L is a parabolic subgroup of G,
(ii) the Lie algebra $\mathfrak{l}$ of L is a parabolic subalgebra of $\mathfrak{g}$, and
(iii) the complex homogeneous space G/L is compact.

Under these conditions G/L is called a **complex flag manifold**. Note here that L is connected and contains a Cartan subgroup, so G/L is simply connected.

More generally, whenever G is a reductive complex Lie group and Q is a parabolic subgroup, we say that $X = G/Q$ is a **complex flag manifold**. Each topological component of X is simply connected. We view X as a complex manifold and G as a group of biholomorphic transformations of X. Since Q is the G–normalizer of $\mathfrak{q}$ we sometimes view X as the space of all $\mathrm{Ad}(G)$–conjugates of $\mathfrak{q}$. When G is connected this means that we sometimes view X as the space of all $\mathrm{Int}(\mathfrak{g})$–conjugates of $\mathfrak{q}$. In other words we sometimes identify $x \in X$ with the isotropy algebra

$$\mathfrak{q}_x : \text{ Lie algebra of } Q_x = \{g \in G \mid g(x) = x\}. \tag{1.3}$$

2. Real group orbits

Let G_0 be a reductive real Lie group, $\mathfrak{g}_0$ its real Lie algebra, and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ its complexified Lie algebra. We fix

$$\mathfrak{q} : \qquad \text{parabolic subalgebra of } \mathfrak{g} \tag{2.1a}$$

such that

$$\text{if } g \in G_0 \text{ then } \mathrm{Ad}(g)\mathfrak{q} \text{ is } \mathrm{Int}(\mathfrak{g})\text{–conjugate to } \mathfrak{q}. \tag{2.1b}$$

This last condition is automatic if G_0 is connected. Let G be any connected complex Lie group with Lie algebra $\mathfrak{g}$. Then G_0 acts on the complex flag manifold

$$X = G/Q : \qquad \text{all } \mathrm{Int}(\mathfrak{g})\text{–conjugates of } \mathfrak{q} \tag{2.2}$$

where Q is the parabolic subgroup of G that is the analytic subgroup for $\mathfrak{q}$. It acts through its adjoint action on $\mathfrak{g}$. Since we will only be interested in the G_0-orbits and their structure, it will be convenient to assume, and we will assume, that

$$\begin{aligned} &G \text{ is connected, simply connected and semisimple,} \\ &G_0 \subset G \text{ is the analytic subgroup for } \mathfrak{g}_0 . \end{aligned} \tag{2.3}$$

Thus G_0 is a real form of G.

G_0 has isotropy subgroup $G_0 \cap Q_x$ at $x \in X$. That subgroup has Lie algebra $\mathfrak{g}_0 \cap \mathfrak{q}_x$. The latter can be described as a real form of $\mathfrak{q}_x \cap \tau\mathfrak{q}_x$ where τ denotes complex conjugation of G over G_0, of $\mathfrak{g}$ over $\mathfrak{g}_0$. If $\mathfrak{h} \subset \mathfrak{g}$ is a τ–stable Cartan subalgebra, then τ acts on the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ by $(\tau\alpha)(\xi) = \alpha(\tau^{-1}\xi) = \alpha(\tau\xi)$ for $\alpha \in \Delta$ and $\xi \in \mathfrak{h}$. Since the intersection of two Borel subalgebras contains a Cartan subalgebra, these considerations lead to

2.4. Theorem. *$\mathfrak{q}_x \cap \tau\mathfrak{q}_x$ contains a τ–stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and there is a choice of positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and subset $\Phi \subset \Psi$ of simple roots such that $\mathfrak{q}_x = \mathfrak{q}_\Phi$. That choice made, $\mathfrak{q}_x \cap \tau\mathfrak{q}_x$ is the semidirect sum of its nilradical*

$$\begin{aligned} (\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^{-n} &= (\mathfrak{q}_\Phi^r \cap \tau\mathfrak{q}_\Phi^{)} + (\mathfrak{q}_\Phi^{-n} \cap \tau\mathfrak{q}_\Phi^r) + (\mathfrak{q}_\Phi^{-n} \cap \tau\mathfrak{q}_\Phi^{-n}) \\ &= \sum_{\Phi^r \cap \tau\Phi^{-n}} \mathfrak{g}_\beta + \sum_{\Phi^{-n} \cap \tau\Phi^r} \mathfrak{g}_\beta + \sum_{\Phi^{-n} \cap \tau\Phi^{-n}} \mathfrak{g}_\gamma \end{aligned} \tag{2.5a}$$

with its reductive (Levi) complement

$$(\mathfrak{q}_x \cap \tau\mathfrak{q}_x)^r = \mathfrak{q}_\Phi^r \cap \tau\mathfrak{q}_\Phi^r = \mathfrak{h} + \sum_{\Phi^r \cap \tau\Phi^r} \mathfrak{g}_\alpha . \tag{2.5b}$$

In particular

$$\dim_{\mathbb{R}}(\mathfrak{g}_0 \cap \mathfrak{q}_x) = \dim_{\mathbb{C}}(\mathfrak{q}_x \cap \tau\mathfrak{q}_x) = \dim_{\mathbb{C}} \mathfrak{q}_\Phi^r + |\Phi^{-n} \cap \tau\Phi^{-n}|. \tag{2.5c}$$

2.6. Corollary. *The orbit $G_0(x)$ has real codimension $|\Phi^{-n} \cap \tau\Phi^{-n}|$ in X. In particular $G_0(x)$ is open in X if and only if $\Phi^{-n} \cap \tau\Phi^{-n}$ is empty.*

Each G_0–orbit comes from a choice of G_0–conjugacy class of pairs $(H_0, \Delta^+(\mathfrak{g}, \mathfrak{h}))$ where H_0 is a Cartan subgroup of G_0 and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is a positive root system. The positive root system is determined only up to conjugacy within the G_0–normalizer of H_0 and the Weyl group $W(\mathfrak{q}_\Phi^r, \mathfrak{h})$. Thus

2.7. Corollary. *Let $\{H_{1,0}, \dots, H_{m,0}\}$ be a complete system of conjugacy classes of Cartan subgroups of G_0. Let H_j denote the Cartan subgroup of G corresponding to $\mathfrak{h}_j = \mathfrak{h}_{j,0} \otimes \mathbb{C}$. Let Q_j be a G–conjugate of Q that contains H_j. In general let $W(\cdot, \cdot)$ denote the corresponding Weyl group. Then there are at most*

$$\sum_{1 \leqq j \leqq m} |W(G_0, H_{j,0}) \backslash W(G, H_j) / W(Q_j^r, H)| \tag{2.8}$$

distinct G_0–orbits on X. In particular, the number of G_0–orbits on X is finite, so there are closed G_0–orbits and there are open G_0–orbits.

Here the closed orbit turns out to be unique. There is a good structure theory for the closed orbits, but it has not yet been exploited. We turn to the structure of the open orbits.

3. Open orbits

Fix a Cartan involution θ of $\mathfrak{g}_0$ and G_0. In other words θ is an automorphism of square 1 and, using (2.3) so that $\mathfrak{g}_0$ is semisimple and G_0 has finite center, the fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0. Thus $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of K_0 and is the (+1)–eigenspace of θ on $\mathfrak{g}_0$, $\mathfrak{s}_0$ is the (-1)–eigenspace, $\mathfrak{k}_0 \perp \mathfrak{s}_0$ under the Killing form of $\mathfrak{g}_0$, and that Killing form is negative definite on $\mathfrak{k}_0$, positive definite on $\mathfrak{s}_0$.

Every Cartan subalgebra of $\mathfrak{g}_0$ is $\mathrm{Ad}(G_0)$–conjugate to a θ–stable Cartan subalgebra. A θ–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is called **fundamental** if it maximizes $\dim(\mathfrak{h}_0 \cap \mathfrak{k}_0)$, **compact** if it is contained in $\mathfrak{k}_0$, which is a more stringent condition. More generally, a Cartan subalgebra of $\mathfrak{g}_0$ is called **fundamental** if it is conjugate to a θ–stable fundamental Cartan subalgebra.

3.1. Lemma. *The following conditions on a θ–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ are equivalent:*

(i) *$\mathfrak{h}_0$ is a fundamental Cartan subalgebra of $\mathfrak{g}_0$,*
(ii) *$\mathfrak{h}_0 \cap \mathfrak{k}_0$ contains a regular element of $\mathfrak{g}_0$, and*
(iii) *there is a positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$, such that $\tau\Delta^+ = \Delta^-$.*

A θ–stable Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is compact if and only if $\tau\Delta^+ = \Delta^-$ for every positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

3.2. Theorem. *Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. The orbit $G_0(x)$ is open in X if and only if $\mathfrak{q}_x = \mathfrak{q}_\Phi$ where*

(i) *$\mathfrak{h}_0 \subset \mathfrak{q}_x \cap \mathfrak{g}_0$ is a fundamental Cartan subgroup of $\mathfrak{g}_0$ and*
(ii) *Φ is a set of simple roots for a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$, such that $\tau\Delta^+ = \Delta^-$.*

Fix $\mathfrak{h}_0 = \theta\mathfrak{h}_0$, $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and Φ as above. Let $W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0}$ and $W(\mathfrak{q}_\Phi^r, \mathfrak{h})^{\mathfrak{h}_0}$ denote the respective subgroups of Weyl groups that stabilize $\mathfrak{h}_0$. Then the open G_0–orbits on X are parameterized by the double coset space $W(\mathfrak{k}, \mathfrak{h} \cap \mathfrak{k}) \backslash W(\mathfrak{g}, \mathfrak{h})^{\mathfrak{h}_0} / W(\mathfrak{q}_x, \mathfrak{h})^{\mathfrak{h}_0}$.

3.3. Corollary. *Suppose that G_0 has a compact Cartan subgroup, i.e. that $\mathfrak{k}_0$ contains a Cartan subalgebra of $\mathfrak{g}_0$. Then an orbit $G_0(x)$ is open in X if and only if $\mathfrak{g}_0 \cap \mathfrak{q}_x$ contains a compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and then, in the notation of* Theorem 3.2, *the open G_0–orbits on X are parameterized by $W(\mathfrak{k}, \mathfrak{h}) \backslash W(\mathfrak{g}, \mathfrak{h}) / W(\mathfrak{q}_x^r, \mathfrak{h})$.*

Classical methods of differential equations tell us when a homogeneous space carries an invariant complex structure [5]. The connection with semisimple structure theory was first sketched in the late 1950's ([2]. [3]), was connected to the theory of parabolic subgroups in the early 1960's [18], and was polished to an enumeration of invariant complex structures a few years later ([27], [28]). We apply this to $K_0(x) \subset G_0(x) \subset X$. A careful examination of the way $\mathfrak{k}_0$ sits in both $\mathfrak{k}$ and $\mathfrak{g}_0$ gives us

3.4. Theorem. *Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. Let $x \in X$ such that $G_0(x)$ is open in X, and let $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{q}_x$ be a θ–stable fundamental Cartan subalgebra of $\mathfrak{g}_0$. Then $K_0(x)$ is a compact complex submanifold of $G_0(x)$. Let K be the complexification of K_0, analytic subgroup of G with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$. Then $K_0(x) = K(x) \cong K/(K \cap Q_x)$, complex flag manifold of K.*

The compact subvariety $K_0(x)$ controls the topology of an open orbit $G_0(x) \subset X$, as follows. It follows from Corollary 3.3 that the compact real form $G_u \subset G$ is transitive on X. That gives us a realization $X = G_u/V_u$ where $V_u \subset G_u$ is the centralizer there of a torus subgroup. In particular V_u is connected. In view of (2.3) now X is compact and simply connected. In view of Theorem 3.4, one can apply this argument to the compact subvariety $K_0(x) \subset G_0(x)$; so it is simply connected. Now a deformation argument shows that the open orbit $G_0(x) \subset X$ has $K_0(x)$ as a deformation retract, so $G_0(x)$ is simply connected. Thus one obtains

3.5. Proposition. *Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. Let $x \in X$ such that $G_0(x)$ is open in X. Then $G_0(x)$ is simply connected and G_0 has connected isotropy subgroup $(Q_x \cap \tau Q_x)_0$ at x.*

The compact subvariety $Y = K_0(x)$ also has a strong influence on the function theory for an open orbit $D = G_0(x) \subset X$. The idea is that a holomorphic function on D must be constant on gY whenever $g \in G$ and $gY \subset D$, so if there are "too many" translates of Y inside D then that holomorphic function must be constant on D. But this has to be formulated carefully.

Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. Let $x \in X$ such that $G_0(x)$ is open in X. Then there are decompositions $G = G_1 \times \cdots \times G_m$ and $Q = Q_1 \times \cdots \times Q_m$ with $Q_i = Q \cap G_i$ and each G_i simple. Consider the corresponding decompositions $X = X_1 \times \cdots \times X_m$ with $X_i = G_i/Q_i$ and $x = (x_1, \ldots, x_m)$, $G_0 = G_{1,0} \times \cdots \times G_{m,0}$, $G_0(x) = G_{1,0}(x_1) \times \cdots \times G_{m,0}(x_m)$ and $K_0(x) = K_{1,0}(x_1) \times \cdots \times K_{m,0}(x_m)$. If

(i) $G_{i,0} \cap (Q_i)_{x_i} = ((Q_i)_{x_i} \cap \tau(Q_i)_{x_i})_0$ is compact, thus contained in $K_{i,0}$,

(ii) $G_{i,0}/K_{i,0}$ is an hermitian symmetric coset space, and
(iii) $G_{i,0}(x_i) \to G_{i,0}/K_{i,0}$ is holomorphic for one of the two invariant complex structures on $G_{i,0}/K_{i,0}$

then we set $L_i = K_i$ so $L_{i,0} = K_{i,0}$. Otherwise we set $L_i = G_i$ so $L_{i,0} = G_{i,0}$. Note that each $G_{i,0}/L_{i,0}$ is a bounded symmetric domain, irreducible or reduced to a point. Set $L = L_0 \times \cdots \times L_m$ so $L_0 = L_{1,0} \times \cdots \times L_{m,0}$. Then we say that

$$D(G_0, x) = G_0/L_0 = (G_{1,0}/L_{1,0}) \times \cdots \times (G_{m,0}/L_{m,0}) \tag{3.6}$$

is the **bounded symmetric domain subordinate to** $G_0(x)$. Now we can state a precise result for holomorphic functions on $G_0(x)$.

3.7. Theorem. *Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. Let $x \in X$ with $G_0(x)$ is open in X. Let $D(G_0, x)$ be the bounded symmetric domain subordinate to $G_0(x)$. Then $\pi : g(x) \mapsto gL_0$ is a holomorphic map of $G_0(x)$ onto $D(G_0, x)$, and the holomorphic functions on $G_0(x)$ are just the $\widetilde{f} = f \cdot \pi$ where $f : D(G_0, x) \to \mathbb{C}$ is holomorphic.*

Thus, in most cases there are no nonconstant holomorphic functions on $G_0(x)$, but in fact this depends on some delicate structure.

4. Example: Hermitian symmetric spaces

In this section, $X = G_u/K_0$ is an irreducible hermitian symmetric space of compact type. Thus $X = G/Q$ where G is a connected simply connected complex simple Lie group with a real form $G_0 \subset G$ of hermitian type, as follows. Fix a Cartan involution θ of G_0 and the corresponding eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ where $\mathfrak{k}_0$ is the Lie algebra of the fixed point set $K_0 = G_0^\theta$. Then $G_u \subset G$ is the compact real form of G that is the analytic subgroup for the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \mathfrak{s}_u$ of $\mathfrak{g}$ where $\mathfrak{s}_u = \sqrt{-1}\,\mathfrak{s}_0$ of $\mathfrak{g}$.

There is a compact Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ of $\mathfrak{g}_0$. If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ then either $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and we say that the root α is **compact**, or $\mathfrak{g}_\alpha \subset \mathfrak{s}$ and we say that α is **noncompact**. There is a simple root system $\Psi = \{\psi_0, \dots, \psi_m\}$ such that ψ_0 is noncompact and the other ψ_i are compact. Furthermore, every noncompact positive root is of the form $\psi_0 + \sum_{1 \leqq i \leqq m} n_i\psi_i$ with each integer $n_i \geqq 0$. If there are two root lengths then the noncompact roots are long; this is immediate from the classification. Thus

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}^+ + \mathfrak{s}^- \text{ where } \mathfrak{k} = \mathfrak{t} + \sum_{n_0=0} \mathfrak{g}_\alpha\,, \mathfrak{s}^+ = \sum_{n_0=1} \mathfrak{g}_\alpha\,, \text{ and } \mathfrak{s}^- = \sum_{n_0=-1} \mathfrak{g}_\alpha\,. \tag{4.1}$$

Here $\mathfrak{q} = \mathfrak{q}_{\{\psi_1,\dots,\psi_m\}}$, in other words

$$\mathfrak{q}^r = \mathfrak{k}\,,\ \mathfrak{q}^n = \mathfrak{s}^+\,,\ \text{and } \mathfrak{q}^{-n} = \mathfrak{s}^-\ ;\ \text{so } \mathfrak{q} = \mathfrak{k} + \mathfrak{s}^-\,. \tag{4.2}$$

The Cartan subalgebras of $\mathfrak{g}_0$ all are $\mathrm{Ad}(G_0)$–conjugate to one of the $\mathfrak{h}_{\Gamma,0}$ given as follows. Let $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ be a set of noncompact positive roots that is **strongly orthogonal** in the sense that

$$\text{if } 1 \leqq i < j \leqq r \text{ then none of } \pm\gamma_i \pm \gamma_j \text{ is a root.} \tag{4.3}$$

Then each $\mathfrak{g}[\gamma_i] = [\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{-\gamma_i}] + \mathfrak{g}_{\gamma_i} + \mathfrak{g}_{-\gamma_i} \cong \mathfrak{sl}(2,\mathbb{C})$, say with

$$[\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{-\gamma_i}] \ni h_{\gamma_i} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_{\gamma_i} \ni e_{\gamma_i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$\mathfrak{g}_{-\gamma_i} \ni f_{\gamma_i} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

such that $\mathfrak{g}_0[\gamma_i] = \mathfrak{g}_0 \cap \mathfrak{g}_{\gamma_i} \cong \mathfrak{su}(1,1)$ is spanned by $\sqrt{-1}\,h_{\gamma_i}$, $e_{\gamma_i} + f_{\gamma_i}$ and $\sqrt{-1}\,(e_{\gamma_i} - f_{\gamma_i})$. Thus $\sqrt{-1}\,h_{\gamma_i}$ spans the compact Cartan subalgebra $\mathfrak{t}_{\gamma_i} = \mathfrak{g}_0[\gamma_i] \cap \mathfrak{t}$ of $\mathfrak{g}_0[\gamma_i]$ and $e_{\gamma_i} + f_{\gamma_i}$ spans the noncompact Cartan subalgebra $\mathfrak{a}_{\gamma_i} = \mathfrak{g}_0[\gamma_i] \cap \mathfrak{s}$ of $\mathfrak{g}_0[\gamma_i]$. Strong orthogonality (4.3) says $[\mathfrak{g}_{\gamma_i}, \mathfrak{g}_{\gamma_j}] = 0$ for $1 \leqq i < j \leqq r$. Define

$$\mathfrak{t}_\Gamma = \sum_{1 \leqq i \leqq r} \mathfrak{t}_{\gamma_i} \text{ and } \mathfrak{a}_\Gamma = \sum_{1 \leqq i \leqq r} \mathfrak{a}_{\gamma_i}\,. \tag{4.4}$$

Then $\mathfrak{g}$ has Cartan subalgebras

$$\mathfrak{t} = \mathfrak{t}_\Gamma + (\mathfrak{t} \cap \mathfrak{t}_\Gamma^\perp) \quad \text{and} \quad \mathfrak{h}_\Gamma = \mathfrak{a}_\Gamma + (\mathfrak{t} \cap \mathfrak{t}_\Gamma^\perp) \tag{4.5}$$

They are $\mathrm{Int}(\mathfrak{g})$–conjugate, for the **partial Cayley transform**

$$c_\Gamma = \prod_{1 \leqq i \leqq r} \exp\left(\frac{\pi}{4}\sqrt{-1}\,(e_{\gamma_i} - f_{\gamma_i})\right) \text{ satisfies } \mathrm{Ad}(c_\Gamma)\mathfrak{t}_\Gamma = \mathfrak{a}_\Gamma\,. \tag{4.6}$$

However, their real forms

$$\mathfrak{t}_0 = \mathfrak{g}_0 \cap \mathfrak{t} \text{ and } \mathfrak{h}_{\Gamma,0} = \mathfrak{g}_0 \cap \mathfrak{h}_\Gamma \tag{4.7}$$

are not $\mathrm{Ad}(G_0)$–conjugate except in the trivial case where Γ is empty, for the Killing form has rank $m = \dim \mathfrak{t}_0$ and signature $2|\Gamma| - m$ on $\mathfrak{h}_{\Gamma,0}$. More precisely,

4.8. Proposition. *Every Cartan subalgebra of $\mathfrak{g}_0$ is $Ad(G_0)$–conjugate to one of the $\mathfrak{h}_{\Gamma,0}$, and Cartan subalgebras $\mathfrak{h}_{\Gamma,0}$ and $\mathfrak{h}_{\Gamma',0}$ are $Ad(G_0)$–conjugate if and only if the cardinalities $|\Gamma| = |\Gamma'|$.*

We recall Kostant's "cascade construction" of a maximal set of strongly orthogonal noncompact positive roots in $\Delta(\mathfrak{g}, \mathfrak{t})$. This set has cardinality $\ell = \operatorname{rank}_{\mathbb{R}} \mathfrak{g}_0$ and is given by

$$\begin{aligned} &\Xi = \{\xi_1, \dots, \xi_\ell\}, \text{ where} \\ &\quad \xi_1 \text{ is the maximal (necessarily noncompact positive) root and} \\ &\quad \xi_{m+1} \text{ is a maximal noncompact positive root } \perp \{\xi_1, \dots, \xi_m\}. \end{aligned} \tag{4.9}$$

Any set of strongly orthogonal noncompact positive roots in $\Delta(\mathfrak{g}, \mathfrak{t})$ is $W(G_0, T_0)$–conjugate to a subset of Ξ. Further, the Weyl group $W(G_0, T_0)$ induces every permutation of Ξ.

Let $x_0 = 1 \cdot Q \in G/Q = X$, the base point of our flag manifold X when X is viewed as a homogeneous space. The Cartan subalgebra $\mathfrak{h}_{\Gamma,0} \subset \mathfrak{g}_0$ leads to the orbits $G_0(c_\Gamma c_\Sigma^2 x_0) \subset X$ where $\Gamma \cup \Sigma$ is a set of strongly orthogonal noncompact positive roots in $\Delta(\mathfrak{g}, \mathfrak{t})$ with Γ and Σ disjoint. In view of the Weyl group equivalence just discussed, we may take $\Gamma = \{\xi_1, \dots, \xi_r\}$ and $\Sigma = \{\xi_{r+1}, \dots, \xi_{r+s}\}$, both inside Ξ. Using $G_0 = K_0 \exp(\mathfrak{a}_{\Xi,0}) K_0$ one arrives at

4.10. Theorem. *The G_0–orbits on X are just the orbits $D_{\Gamma,\Sigma} = G_0(c_\Gamma c_\Sigma^2 x_0)$ where Γ and Σ are disjoint subsets of Ξ. Two such orbits $D_{\Gamma,\Sigma} = D_{\Gamma',\Sigma'}$ if and only if cardinalities $|\Gamma| = |\Gamma'|$ and $|\Sigma| = |\Sigma'|$. An orbit $D_{\Gamma,\Sigma}$ is open if and only if Γ is empty, closed if and only if $(\Gamma, \Sigma) = (\Xi, \emptyset)$. An orbit $D_{\Gamma',\Sigma'}$ is in the closure of $D_{\Gamma,\Sigma}$ if and only if $|\Sigma'| \leqq |\Sigma|$ and $|\Sigma \cup \Gamma| \leqq |\Sigma' \cup \Gamma'|$.*

5. Measurable open orbits

There is a class of open orbits that currently are much better understood than the general case. That is the class of **measurable** open orbits — the open orbits $D = G_0(x) \subset X$ where D carries a G_0–invariant measure. They are characterized [22, Theorem 6.3] by

5.1. Theorem. *Let $D = G_0(x)$, open orbit in X. If D is measurable then its G_0–invariant measure is induced by the volume form of a G_0–invariant indefinite-kähler metric. Further, the following conditions are equivalent, and D is measurable if and only if they hold.*

(i) *$G_0 \cap Q_x$ is the centralizer of a torus subgroup Z of $K_0 \cap Q_x$,*
(ii) *$\mathfrak{q}_x \cap \tau\mathfrak{q}_x$ is reductive,*
(iii) *$\mathfrak{q}_x \cap \tau\mathfrak{q}_x = \mathfrak{q}_x^r$,*
(iv) *$\tau\mathfrak{q}_x^{-n} = \mathfrak{q}_x^n$.*

Under these circumstances, $\theta\mathfrak{q} = \mathfrak{q}$ where θ is the Cartan involution of $\mathfrak{g}_0$ with fixed point set $\mathfrak{k}_0$.

The conditions of Theorem 5.1 are automatic if K_0 contains a Cartan subgroup of G_0, that is, if $\operatorname{rank} K_0 = \operatorname{rank} G_0$, in particular if $G_0 \cap Q_x$ is compact. Thus in particular the open orbits $G_0(c_\Sigma^2 x_0)$ of §4 are measurable. They are also automatic if Q is a Borel subgroup of G. More generally, they are equivalent [22, Theorem 6.7] to the condition that $\tau\mathfrak{q}$ be $\operatorname{Int}(\mathfrak{g})$–conjugate to the parabolic subalgebra of $\mathfrak{g}$ that is opposite to $\mathfrak{q}$.

6. Exhaustion functions for flag domains

Bounded symmetric domains $D \subset \mathbb{C}^n$ are convex, and thus Stein, so cohomologies $H^k(D; \mathcal{F}) = 0$ for $k > 0$ whenever $\mathcal{F} \to D$ is a coherent analytic sheaf. This is useful for dealing with holomorphic discrete series representations. More generally for dealing with general discrete series representations and their analytic continuations one has

6.1. Theorem. *Let $X = G/Q$ be a complex flag manifold, G semisimple and simply connected, and let G_0 be a real form of G. Let $D = G_0(x) \subset X = G/Q$ be a measurable open orbit. Let $Y = K_0(x)$, maximal compact subvariety of D, and let $s = \dim_{\mathbb{C}} Y$. Then D is $(s+1)$–complete in the sense of Andreotti–Grauert [1]. In particular, if $\mathcal{F} \to D$ is a coherent analytic sheaf then $H^k(D; \mathcal{F}) = 0$ for $k > s$.*

The special case where Q is a Borel subgroup is due to Schmid [12], and the general case is due to Schmid and myself [15]. The arguments are similar: one examines the Levy form of an exhaustion function constructed from canonical line bundles. In this section I'll indicate the proof.

Let $\mathbb{K}_X \to X$ and $\mathbb{K}_D = K_X|_D \to D$ denote the canonical line bundles. Consider the dual bundles

$$\mathbb{L}_X = \mathbb{K}_X^* \to X \quad \text{and } \mathbb{L}_D = \mathbb{K}_D^* \to D. \tag{6.2}$$

They are the homogeneous holomorphic line bundle over X associated to the holomorphic character

$$e^\lambda : Q_x \to \mathbb{C} \text{ defined by } e^\lambda(q) = \operatorname{trace} \operatorname{Ad}(q)|_{\mathfrak{q}_x^n}. \tag{6.3}$$

Write $D = G_0/V_0$ where $V_0 = G_0 \cap Q_x$, real form of Q_x^r. It is convenient to write V for the complexification Q_x^r of V_0. Then, following standard terminology, $\rho_{G/V}$ is half the sum of the roots that occur in $\mathfrak{q}_x^n$ and $\lambda = 2\rho_{G/V}$. Thus, if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$,

$\langle\alpha,\lambda\rangle = 0$ and $\alpha\in\Phi^r$, or $\langle\alpha,\lambda\rangle > 0$ and $\alpha\in\Phi^n$, or $\langle\alpha,\lambda\rangle < 0$

and $\alpha\in\Phi^{-n}$.

Now $\tau\lambda = -\lambda$. Decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ under the Cartan involution with fixed point set $\mathfrak{k}_0$, thus decomposing the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0\cap\mathfrak{q}_x$ as $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ with $\mathfrak{t}_0 = \mathfrak{h}_0\cap\mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0\cap\mathfrak{s}_0$. Then $\lambda(\mathfrak{a}_0) = 0$.

View $D = G_0/V_0$ and $X = G_u/V_0$ where G_u is the analytic subgroup of G for the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$. Then e^λ is a unitary character on V_0. Now

$$\begin{aligned}&\mathbb{L}_X \to X = G_u/V_0 \text{ has a } G_u\text{–invariant hermitian metric } h_u\,,\\ &\mathbb{L}_D \to D = G_0/V_0 \text{ has a } G_0\text{–invariant hermitian metric } h_0\,.\end{aligned} \tag{6.4}$$

We now have enough information to carry out a computation that results in

6.5. Lemma. *The hermitian form $\sqrt{-1}\,\partial\bar\partial h_u$ on the holomorphic tangent bundle of X is negative definite. The hermitian form $\sqrt{-1}\,\partial\bar\partial h_0$ on the holomorphic tangent bundle of D has signature $n-2s$ where $n = \dim_{\mathbb{C}} D$.*

6.6. Corollary. *Define $\phi : D \to \mathbb{R}$ by $\phi = \log(h_0/h_u)$. Then the Levy form $\mathcal{L}(\phi)$ has at least $n-s$ positive eigenvalues at every point of D.*

The next point is to show that ϕ is an exhaustion function for D, in other words that

$$\{z\in D \mid \phi(z) \leqq c\} \text{ is compact for every } c\in\mathbb{R}.$$

It suffices to show that $e^{-\phi}$ has a continuous extension from D to the compact manifold X that vanishes on the topological boundary $\mathrm{bd}(D)$ of D in X. For that, choose a G_u–invariant metric h_u^* on $\mathbb{L}_X^* = \mathbb{K}_X$ normalized by $h_u h_u^* = 1$ on X, and a G_0–invariant metric h_0^* on $\mathbb{L}_D^* = \mathbb{K}_D$ normalized by $h_0 h_0^* = 1$ on D. Then $e^{-\phi} = h_0^*/h_u^*$. So it suffices to show that h_0^*/h_u^* has a continuous extension from D to X that vanishes on $\mathrm{bd}(D)$.

The holomorphic cotangent bundle $\mathbb{T}_X^* \to X$ has fibre $\mathrm{Ad}(g)(\mathfrak{q}_x^n)^* = \mathrm{Ad}(g)(\mathfrak{q}_x^{-n})$ at $g(x)$. Thus its G_u–invariant hermitian metric is given on the fibre $\mathrm{Ad}(g)(\mathfrak{q}_x^{-n})$ at $g(x)$ by $F_u(\xi,\eta) = -\langle\xi,\tau\theta\eta\rangle$ where $\langle,\rangle$ is the Killing form. Similarly the G_0–invariant indefinite–hermitian metric on $\mathbb{T}_D^* \to D$ is given on the fibre $\mathrm{Ad}(g)(\mathfrak{q}_x^{-n})$ at $g(x)$ by $F_0(\xi,\eta) = -\langle\xi,\tau\eta\rangle$. But $\mathbb{K}_X = \det\mathbb{T}_X^*$ and $\mathbb{K}_D = \det\mathbb{T}_D^*$, so

$$h_0^*/h_u^* = c\cdot(\text{determinant of } F_0 \text{ with respect to } F_u)$$

for some nonzero real constant c. This extends from D to a C^∞ function on X given by

$$f(g(x)) = c\cdot\left(\det F_0|_{\mathrm{Ad}(g)(\mathfrak{q}_x^{-n})} \text{ relative to } \det F_u|_{\mathrm{Ad}(g)(\mathfrak{q}_x^{-n})}\right). \tag{6.7}$$

It remains only to show that the function f of (6.7) vanishes on $\mathrm{bd}(D)$. If $g(x)\in\mathrm{bd}(D)$ then $G_0(g(x))$ is not open in X, so $\mathrm{Ad}(g)(\mathfrak{q}_x)+\tau\mathrm{Ad}(g)(\mathfrak{q}_x) \neq \mathfrak{g}$. Thus

$$\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)(\mathfrak{q}_x^{-n}) \text{ but } \mathfrak{g}_{-\alpha} \not\subset \mathrm{Ad}(g)(\mathfrak{q}_x)+\tau\mathrm{Ad}(g)(\mathfrak{q}_x)$$

for some $\alpha\in\Delta(\mathfrak{g},\mathrm{Ad}(g)\mathfrak{h})$.

If $\beta\in\Delta(\mathfrak{g},\mathrm{Ad}(g)\mathfrak{h})$ with $\mathfrak{g}_\beta \subset \mathrm{Ad}(g)(\mathfrak{q}_x^{-n})$ then $F_0(\mathfrak{g}_\alpha,\mathfrak{g}_\beta) = 0$, so $f(g(x)) = 0$. Thus ϕ is an exhaustion function for D in X. In view of Corollary 6.6 now D is $(s+1)$–complete. Theorem 6.1 follows.

7. Coset structure of the cycle space

In this section, $X = G/Q$ is a complex flag manifold, G semisimple, connected and simply connected, and G_0 is a real form of G. We fix an open orbit $D = G_0(x) \subset X = G/Q$ and assume that it is measurable. As before, $Y = K_0(x)$, maximal compact subvariety of D, and we write $n = \dim_{\mathbb{C}} X$ and $s = \dim_{\mathbb{C}} Y$.

The **linear cycle space** or the **space of maximal compact linear subvarieties** of D is, by definition,

$$M_D = \{gY \mid g\in G \text{ and } gY\subset D\}. \tag{7.1}$$

Since Y is compact and D is open in X, M_D is open in

$$M_X = \{gY \mid g\in G\} \cong G/L \tag{7.2a}$$

where

$$L = \{g\in G \mid gY = Y\}, \text{ closed complex subgroup of } G. \tag{7.2b}$$

We now look at the structure of the G–stabilizer L of the maximal compact linear subvariety Y in our open orbit $D = G_0(x) \cong G_0/V_0$. The starting point is the following lemma, which is obvious.

7.3. Lemma. *The kernel of the action of $L = \{g\in G \mid gY = Y\}$ on Y is*

$$E = \bigcap_{k\in K} kQ_xk^{-1} = \bigcap_{k\in K_0} kQ_xk^{-1} \tag{7.4}$$

and $KE \subset L \subset KQ_x$.

In general, G, G_0, Q, X, D, K, K_0 and Y break up as direct products according to any decomposition of $\mathfrak{g}_0$ as a direct sum of ideals, equivalently any decomposition of G_0 as a direct product. Here we use our assumption that G be connected and simply connected. So, for purposes of determining the group L specified in (7.2) and just above, we may and do assume that G_0 and $\mathfrak{g}_0$ are noncompact and simple. This is equivalent to the assumption that G_0/K_0 be an irreducible riemannian symmetric space of noncompact type.

We will say that G_0 is of **hermitian type** if the irreducible riemannian symmetric space G_0/K_0 carries the structure of an hermitian symmetric space.

As before, we write θ for the Cartan involution of G_0 with fixed point set K_0, for its holomorphic extension to G, and for its differential on $\mathfrak{g}_0$ and $\mathfrak{g}$; and we denote the θ–eigenspace decomposition by $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. Our irreducibility assumption says, exactly, that the adjoint action of K_0 on $\mathfrak{s}_0 = \mathfrak{g}_0 \cap \mathfrak{s}$ is irreducible. G_0 is of hermitian type if and only if this action fails to be absolutely irreducible. Let $S_\pm = \exp(\mathfrak{s}_\pm) \subset G$. Then G_0/K_0 is an open G_0–orbit on G/Q where $Q = KS_-$ as in §4.

As before we have the compact real form $G_u \subset G$, real analytic subgroup for $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$, and $K_0 = G_0 \cap G_u$. K_0 is its own normalizer in G_0, but its normalizer $N_{G_u}(K_0)$ in G_u can have several components.

7.5. Proposition. *Either G_0 is of hermitian type and $L = KE = KS_\pm$, connected, or*[1] *$L = KN_{G_u}(K_0)$ with identity component $L^0 = K$. In either case $G_0 \cap L = K_0$. In general, if $G_0 \cap Q_x$ is compact then $L = KE$ and L is connected.*

This Proposition is proved by running through cases, which are as follows.

(1) G_0 is of hermitian type with $Q_x \subset KS_-$.
(2) G_0 is of hermitian type with $Q_x \subset KS_+$.
(3) G_0 is of hermitian type with $Q_x \not\subset KS_-$, $Q_x \not\subset KS_+$, and $S_- \subset Q_x$.
(4) G_0 is of hermitian type with $Q_x \not\subset KS_-$, $Q_x \not\subset KS_+$, and $S_+ \subset Q_x$.
(5) G_0 is of hermitian type with $Q_x \not\subset KS_-$, $Q_x \not\subset KS_+$, $S_- \not\subset Q_x$, and $S_+ \not\subset Q_x$.
(6) G_0 is not of hermitian type.

An important consequence of Proposition 7.5 is

[1]This latter situation occurs both for G_0 of hermitian type and for G_0 not of hermitian type.

7.6. Corollary. *Either L is a parabolic subgroup $KS_\pm$ of G and $M_X = G/L$ is a projective algebraic variety, or L is a reductive subgroup of G with identity component K and $M_X = G/L$ is an affine algebraic variety.*

8. Holomorphic structure of the cycle space

Here I want to indicate the proof of

8.1. Theorem. *Let $D = G_0(x)$ be a measurable open G_0–orbit on a complex flag manifold $X = G/Q$. Then the linear cycle space M_D is a Stein manifold.*

Results of this sort were suggested by a vanishing theorem in Schmid's thesis [12] and by Griffiths' discussion [8] of moduli spaces for compact Kaehler manifolds. One case was worked out by R. O. Wells, Jr. using explicit matrix calculations [19]. Later Wells and I gave an argument for Theorem 8.1 in the case where $G_0 \cap Q_x$ is compact [20, Theorem 2.5.6], but there were problems with the combinatorics of the proof. I settled these problems in the general case of Theorem 8.1, as stated, in [26], and that's the argument that is indicated below.

Consider the first of the two cases of Corollary 7.6.

8.2. Theorem. *Suppose that M_X is a projective algebraic variety. Then every open orbit $D = G_0(x) \subset X$ is measurable and M_D is a bounded symmetric domain. In particular M_D is a Stein manifold.*

Here G_0 is of hermitian type and $L = KS_\pm$, maximal parabolic subgroup. We can replace Δ^+ by its negative if necessary and assume $L = KS_-$. Thus we are in Case 1 (when $V_0 = G_0 \cap Q_x$ is compact) or in Case 3 (when $V_0 = G_0 \cap Q_x$ is noncompact) of the proof of Proposition 7.5. Also, $M_X = G/L$ is the standard complex realization of the compact hermitian symmetric space G_u/K_0. Denote

$$G\{D\} = \{g \in G \mid gY \subset D\}. \tag{8.3}$$

It is an open subset of G, and $M_D \subset M_X \cong G/L$ consists of the cosets gL with $g \in G\{D\}$. Evidently M_D is stable under the action of G_0. Thus

$$G\{D\} \text{ is a union of double cosets } G_0 g L \text{ with } g \in G. \tag{8.4}$$

The proof of Theorem 8.2 consists of showing that only the identity double coset occurs in GD. That relies on the orbit structure of G_0 on M_X as described in §4.

The case where $V_0 = G_0 \cap Q_x$ is compact is easy. With V_0 compact, $Q_x \subset L$ and there is a holomorphic fibration $\pi : X \to M_X$ given by

$gQ_x \mapsto gL$. Here $\pi(D)$ is the bounded symmetric domain $\{gL \mid g \in G_0\}$ and the gY, $g \in G$, are the fibres of $\pi : X \to M_X$. Thus M_D is the bounded symmetric domain $\{gL \mid g \in G_0\}$.

Return to the general case, where V_0 may be noncompact. The double cosets G_0gL of (8.4) are in one to one correspondence with the G_0–orbits on M_X. Those orbits were described in Theorem 4.10 above, and we use the notation of §4.

$G\{D\}$ is open in G and the map $G \to G/L = M_X$ is open. So $G\{D\}(z)$ is open in M_X. Thus, if $c_{\Gamma'}c^2_{\Sigma'} \in G\{D\}$, and if $G_0(c_{\Gamma'}c^2_{\Sigma'}z)$ is in the closure of $G_0(c_\Gamma c^2_\Sigma z)$, then $c_\Gamma c^2_\Sigma \in G\{D\}$. Now (8.4) and Theorem 4.10 combine as follows.

8.5. Lemma. *There is a (necessarily finite) set $\mathcal{C}$ of transforms $c_\Gamma c^2_\Sigma$, where Γ and Σ are disjoint subsets of Ξ, such that (i) if $c_\Gamma c^2_\Sigma, c_{\Gamma'}c^2_{\Sigma'} \in \mathcal{C}$ with $|\Gamma| = |\Gamma'|$ and $|\Sigma| = |\Sigma'|$ then $\Gamma = \Gamma'$ and $\Sigma = \Sigma'$ and (ii) $G\{D\} = \bigcup_{c\in\mathcal{C}} G_0cL$. So if $c_\Gamma c^2_\Sigma \in \mathcal{C}$ then $c^2_{\Gamma\cup\Sigma'} \in \mathcal{C}$ for every subset $\Sigma' \subset \Sigma$. In particular, if $c^2_\Sigma \notin \mathcal{C}$ whenever $\emptyset \neq \Sigma \subset \Psi$ then $\mathcal{C} = \{1\}$ and $G\{D\} = G_0L$.*

This reduces the proof of Theorem 8.2 to the assertion $\emptyset \neq \Sigma \subset \Xi$ implies $c^2_\Sigma \notin \mathcal{C}$.

$R = L \cap Q_x$ is a parabolic subgroup of G and $W = G/R$ is a complex flag manifold because $S_- \subset L \cap Q_x$. So there are holomorphic projections

$$\begin{aligned}\pi' &: W \to X \text{ by } gR \mapsto gQ_x, \text{ fibre } F' = \pi'^{-1}(x) = V(w) \cong V/V \cap L,\\ \pi'' &: W \to M_X \text{ by } gR \mapsto gL, \text{ fibre } F'' = \pi''^{-1}(z) = K(w) \cong K/K \cap Q_x,\end{aligned} \tag{8.6}$$

where $g \in G$ and $w = 1 \cdot R$ in W. Set $\widetilde{D} = G_0(w)$. Then

$$\pi' : \widetilde{D} \to D \text{ by } g(w) \mapsto g(x), \text{ fibre } V_0(w) \cong V_0/K_0 \cap V_0, \text{ open in } F'. \tag{8.7}$$

F' is a complex flag manifold of $V = Q^r_x$, $V_0(w)$ is open in F', and $V_0 \cap K_0$ is a maximal compact subgroup of V_0; so $V_0(w)$ is a bounded symmetric domain and F' is its compact dual.

The usual positive definite hermitian inner product on $\mathfrak{g}_0$ is $\langle \xi, \eta\rangle = -b(\xi, \tau\theta\eta)$ where b is the Killing form. The associated length function defines

$$||\xi||_{\mathfrak{g}} : \text{ operator norm of } \mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g} \text{ for } \xi \in \mathfrak{g}. \tag{8.8a}$$

The Hermann Convexity Theorem says

$$G_0(z) = \pi''(\widetilde{D}) = \{\exp(\zeta)(z) \mid \zeta \in \mathfrak{s}_+ \text{ with } ||\xi||_{\mathfrak{g}} < 1\}. \tag{8.8b}$$

Recall $\mathfrak{v} = \mathfrak{q}^r_x$. A glance at the proof of the Harish–Chandra realization of $G_0(z)$ as a bounded symmetric domain, and of (8.8b), shows that every

$g \in G_0$ has expression

$$\begin{aligned}&g = \exp(\zeta_1 + \zeta_2) \cdot k \cdot \exp(\eta) \text{ where}\\ &\quad \eta \in \mathfrak{s}_-,\ k \in K,\ \zeta_2 \in \mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+), \text{ and } \zeta_1 \in \mathfrak{s}_+ \perp \mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+).\end{aligned} \tag{8.9}$$

There is a number $a = a_G > 0$ such that, in (8.9), $||\zeta_1||_{\mathfrak{g}} < a_G$.

The operator norm information pulls back from $G(z)$ to D. The result is

8.10. Lemma. *Decompose $g \in G_0$ as in (8.9). Define $\widetilde{f} : G_0 \to \mathbb{R}$ by $\widetilde{f}(g) = ||\zeta_1||_{\mathfrak{g}}$. Then $f(gx) = \widetilde{f}(g)$ is a well defined function $f : D \to \mathbb{R}$. If $gx \in D$ then $0 \leqq f(gx) < a_G$ where a_G is given as above.*

Now the proof of Theorem 8.2 proceeds as follows. Let $\emptyset \neq \Sigma \subset \Xi$ with $c^2_\Sigma \in \mathcal{C}$. In the notation of §4, we conjugate by an appropriate element of K_0 and may assume $\Sigma = \{\xi_1, \dots, \xi_m\} \subset \Xi$, with $1 \leqq m \leqq \ell$. Let $\mathfrak{g}[\Sigma] = \sum_{1\leqq i\leqq m} \mathfrak{g}[\xi_i]$ and let $G[\Sigma]$ be the corresponding analytic subgroup of G. Then $G\{D\}$ contains the diagonal subgroup $G^d[\Sigma] \cong SL(2;\mathbb{C})$ in $G[\Sigma]$. Since ξ_1 is not a root of $\mathfrak{q}^r_x = \mathfrak{v}$ the orbit $X^d[\Sigma] = G^d[\Sigma](x)$ is a Riemann sphere contained in D.

Let $\Sigma' = \{\sigma \in \Sigma \mid \mathfrak{g}_\sigma \notin \mathfrak{v} = \mathfrak{q}^r_x\}$, nonempty because it contains ξ_1. The diagonal subgroup $G^d[\Sigma'] \cong SL(2;\mathbb{C})$ in $G[\Sigma']$ has properties:

$$\begin{aligned}&G^d[\Sigma'](x) = X^d[\Sigma'] \text{ is the same Riemann sphere } X^d[\Sigma],\\ &(\mathfrak{g}^d[\Sigma'] \cap \mathfrak{s}_+) \perp (\mathfrak{v} \cap \mathfrak{s}_+), \text{ and}\\ &G^d[\Sigma'] \cap K \text{ is contained in the Cartan subgroup } H \text{ with Lie algebra } \mathfrak{h}.\end{aligned} \tag{8.11}$$

Now look at the corresponding orbits in the hermitian symmetric flag variety $M_X = G/KS_-$. The orbit $G^d[\Sigma'](z)$, call it $Z^d[\Sigma']$, is a Riemann sphere, the diagonal $Z[\Sigma'] = G[\Sigma'](z)$. Its intersection with the bounded symmetric domain $G_0(z)$ is the hemisphere $G^d_0[\Sigma'](z)$, where $G^d_0[\Sigma'] = G_0 \cap G^d[\Sigma']$.

Let $f^* = \widetilde{f}|_{G^d[\Sigma']}$ in the notation of Lemma 8.10. Then f^* is real analytic and has a unique real analytic extension $f^\dagger$ to $G^d[\Sigma'] \cap \exp(\mathfrak{s}_+)K\exp(\mathfrak{s}_-)$. Evidently $f^\dagger$ is unbounded. The function f of Lemma 8.10 is real analytic on the lower hemisphere of the Riemann sphere $X^d[\Sigma] = X^d[\Sigma']$ and its restriction to that hemisphere has unique real analytic extension h to the complement $X^d[\Sigma] \setminus c^2_\Sigma(x)$ of the pole opposite to x, extension defined by $f^\dagger$ just as f is defined by $\widetilde{f}$. In view of (8.11), $h = f|_{X^d[\Sigma]\setminus c^2_\Sigma(x)}$. Since $f^\dagger$ is unbounded, it follows that f is unbounded. This contradicts Lemma 8.10.

One concludes that $\mathcal{C}$ cannot contain any c^2_Σ with $\emptyset \neq \Sigma \subset \Xi$. As noted earlier, that completes the proof of Theorem 8.2.

The second case of Corollary 7.6 is addressed by

8.12. Theorem. *Suppose that the open orbit $D \subset X$ is measurable and that M_X is an affine algebraic variety. Then M_D is an open Stein subdomain of the Stein manifold M_X.*

Recall the exhaustion function $\phi : D \to \mathbb{R}$ defined in Corollary 6.6. It is real analytic and its Levy form

$$\mathcal{L}(\phi) = \sqrt{-1}\partial\bar{\partial}\phi \tag{8.13}$$

has at least $n-s$ positive eigenvalues at every point of D. Here $n = \dim_{\mathbb{C}} D$ and $s = \dim_{\mathbb{C}} Y$. Since ϕ is an exhaustion function, the subdomains $D_c = \{z \in D \mid \phi(z) < c\}$ are relatively compact in D.

Analyticity allows one to transfer ϕ to M_D. Define $\phi_M : M_D \to \mathbb{R}^+$ by

$$\phi_M(gY) = \sup_{y \in Y} \phi(g(y)) = \sup_{k \in K} \phi(gk(x)) \,. \tag{8.14}$$

The result is

8.15. Lemma. *ϕ_M is a real analytic plurisubharmonic[2] function on M_D. If Y_∞ is a point on the boundary of M_D in M_X and $\{Y_i\}$ is a sequence in M_D that tends to Y_∞ then $\lim_{Y_i \to Y_\infty} \phi_M(Y_i) = \infty$.*

The next step is to modify ϕ_M to obtain a strictly plurisubharmonic exhaustion function on M_D. Since M_X is affine, it is Stein, so there is a proper holomorphic embedding $F : M_X \to \mathbb{C}^{2m+1}$ as a closed analytic submanifold of $\mathbb{C}^{2m+1}$. The norm square function $N(m) = ||F(m)||^2$ has positive definite Levy form, and the sets $\{m \in M_X \mid N(m) < c\}$ are relatively compact. Now

$$\zeta : M_D \to \mathbb{R}^+ \text{ defined by } \zeta(m) = \phi_M(m) + N(m) \tag{8.16}$$

has positive definite Levy form, thus is strictly plurisubharmonic. Since N and ϕ are real analytic, so is ζ. And ζ tends to ∞ at every boundary point of M_D because ϕ has that property by hypothesis and N has values $\geqq 0$. So every set

$$M_{\zeta,c} = \{m \in M \mid \zeta(m) < c\} \tag{8.17}$$

has closure contained in M_D. But F is a proper embedding of $\widetilde{M}$ in $\mathbb{C}^{2m+1}$, so the sets $M_{\zeta,c}$ of (8.17) have compact closure in M_D.

We have proved that ζ is a real analytic strictly plurisubharmonic exhaustion function on M_D. Theorem 8.12 now follows using H. Grauert's solution [6] to the Levy Problem.

Note that this argument shows

[2] A C^2 function f on a complex manifold is called *plurisubharmonic* if the hermitian form $\mathcal{L}(f)$ is positive semidefinite at every point, *strictly plurisubharmonic* if $\mathcal{L}(f)$ is positive definite everywhere. See [4], [11] or the exposition [10, §2.6].

8.18. Lemma. *Let M be an open submanifold of a Stein manifold $\widetilde{M}$. Suppose that M carries a C^r plurisubharmonic function ξ, $r \in \{2, 3, \dots, \infty, \omega\}$, that blows up on the boundary of M in $\widetilde{M}$ in the sense: if $y_\infty \in \mathrm{bd}M$ and $\{y_i\} \subset M$ tends to y_∞ then $\lim_{i\to\infty} \xi(y_i) = \infty$. Then M carries a C^r strictly plurisubharmonic exhaustion function.*

Theorem 8.1 follows from Theorem 8.2 when M_X is a projective algebraic variety, from Theorem 8.12 when M_X is an affine algebraic variety. Corollary 7.6 says that these are the only cases.

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