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New Classes of Infinite-Dimensional Lie Groups

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ABSTRACT. We describe some new constructions of infinite-dimensional Lie groups based on direct limits in various categories of linear spaces. In each case the limit group takes its topology and analytic structure from the limit Lie algebra. In all of the situations we investigate, the limit of the Lie algebras is a good topological Lie algebra. We prove that the result is essentially the same for limits of Lie groups, but the situation is much more delicate. In the case of the locally convex direct limit, the limit is a Lie group, except that in some cases the group composition is only separately continuous.

1. Preliminaries

In this note we summarize our work in progress on Lie group structures for various classes of direct limits of finite-dimensional groups. In effect, by taking the direct limit of the Lie algebras in various categories of topological spaces, and using the exponential map, we carry a topology and an analytic structure to the limit group.

The case of the usual direct limit, in the category of topological vector spaces, was published in [7]. The cases of direct limits in the categories of normed linear spaces and of locally convex topological vector spaces will appear in [8].

Fix a directed set A. Thus A is a partially ordered set, say with order relation $\leqq$, such that if α, $\beta \in A$ then one has $\gamma \in A$ with $\alpha, \beta \leqq \gamma$. We consider directed systems

$$\{G_\alpha, \phi_{\beta,\alpha};\ V_\alpha, \eta_{\beta,\alpha};\ \pi_\alpha\}. \tag{1.1}$$

First, by definition, α and β run over A, each G_α is a (finite-dimensional real) Lie group, say with real Lie algebra $\mathfrak{g}_\alpha$, and if $\alpha \leqq \beta$ then

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$\phi_{\beta,\alpha} : G_\alpha \to G_\beta$ is an analytic homomorphism. We require the standard $\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \cdot \phi_{\beta,\alpha}$ for $\alpha \leqq \beta \leqq \gamma$ and $\phi_{\alpha,\alpha} = \mathrm{ident}_{G_\alpha}$ for all α. Second, each of the V_α is a finite dimensional complex vector space, and if $\alpha \leqq \beta$ then $\eta_{\beta,\alpha} : V_\alpha \to V_\beta$ is a linear transformation. As above we require the standard $\eta_{\gamma,\alpha} = \eta_{\gamma,\beta} \cdot \eta_{\beta,\alpha}$ for $\alpha \leqq \beta \leqq \gamma$ and $\eta_{\alpha,\alpha} = \mathrm{ident}_{V_\alpha}$ for all α. Third, π_α is a continuous representation of G_α on V_α, and one has the consistency condition that for $\alpha \leqq \beta$ the left-hand diagram diagram of

$$(1.2)\qquad \begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ \phi_{\beta,\alpha}\big\downarrow \eta_{\beta,\alpha} & & \big\downarrow \eta_{\beta,\alpha} \\ G_\beta \times V_\beta & \xrightarrow{\pi_\beta} & V_\beta \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{d\pi_\alpha} & V_\alpha \\ d\phi_{\beta,\alpha}\big\downarrow \eta_{\beta,\alpha} & & \big\downarrow \eta_{\beta,\alpha} \\ \mathfrak{g}_\beta \times V_\beta & \xrightarrow{d\pi_\beta} & V_\beta \end{array}$$

is commutative. If $\alpha \leqq \beta$ then $\phi_{\beta,\alpha}$ defines a Lie algebra homomorphism $d\phi_{\beta,\alpha} : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$. The Lie algebra representations $d\pi_\alpha$ satisfy the consistency condition that comes out of the condition for the π_α. So the right-hand diagram of (1.2) is commutative.

The *direct limit* or *injective limit* group $G = \varinjlim G_\alpha$ consists of the equivalence classes $[g_\alpha]$ of sets $\{g_\alpha\}$ where each $g_\alpha \in G_\alpha$ and, for some $\beta \in A$, if $\beta \leqq \gamma$ then $g_\gamma = \phi_{\gamma,\beta}(g_\beta)$. The equivalence relation is such that $[g_\alpha]$ is determined by the eventual behavior of $\{g_\alpha\}$: $\{g_\alpha\} \sim \{'g_\alpha\}$ when, for some $\beta \in A$, if $\beta \leqq \gamma$ then $g_\gamma = 'g_\gamma$. G is a group with the operations $[g_\alpha] \cdot [' g_\alpha] = [h_\alpha]$ where each $h_\alpha = g_\alpha \cdot 'g_\alpha$ and $[g_\alpha]^{-1} = [g_\alpha^{-1}]$. We have homomorphisms

(1.3)

$$\phi_\beta : G_\beta \to G \text{ by } \phi_\beta(x) = [g_\gamma] \text{ where } g_\gamma = \phi_{\gamma,\beta}(x) \text{ for } \beta \leqq \gamma,\ g_\gamma = 1_{G_\gamma}$$

otherwise. Those homomorphisms define the naïve *direct limit* (DL) topology on G: A subset $U \subset G$ is DL-open in G if and only if $\phi_\beta^{-1}(U)$ is open in G_β for every $\beta \in A$. $G_{DL} = \varinjlim_{\mathrm{DL}} G_\alpha$ consists of G with this DL topology. G_{DL} is a (Hausdorff) topological group.

Similarly the *direct limit* Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ consists of the equivalence classes $[\xi_\alpha]$ of sets $\{\xi_\alpha\}$ where each $\xi_\alpha \in \mathfrak{g}_\alpha$ and, for some $\beta \in A$, if $\beta \leqq \gamma$ then $\xi_\gamma = d\phi_{\gamma,\beta}(\xi_\beta)$. The equivalence relation is the Lie algebra version of the relation for G. Now $\mathfrak{g}$ is a Lie algebra, one has Lie algebra homomorphisms $d\phi_\beta : \mathfrak{g}_\beta \to \mathfrak{g}$, and these homomorphisms define a naïve direct limit (DL) topology on $\mathfrak{g}$ as above. $\mathfrak{g}_{\mathrm{DL}} = \varinjlim_{\mathrm{DL}} \mathfrak{g}_\alpha$ consists of $\mathfrak{g}$ with this DL topology. It is a topological Lie algebra, and the exponential map

$$(1.4)\qquad \exp_G : \mathfrak{g}_{\mathrm{DL}} \to G_{\mathrm{DL}} \text{ defined by } \exp_G([\xi_\alpha]) = [\exp_{G_\alpha}(\xi_\alpha)]$$

is well defined and continuous.

The *direct limit* vector space $V = \varinjlim V_\alpha$ is defined as above. One has linear transformations $\eta_\beta : V_\beta \to V$ as in (1.3), they define a naïve direct limit (DL) topology on V, and V together with that topology is a topological vector space $V_{\mathrm{DL}} = \varinjlim_{\mathrm{DL}} V_\alpha$.

The *direct limit* representation $\pi = \varinjlim \pi_\alpha$ is the representation of G on V given by $\pi([g_\alpha])([v_\alpha]) = [\pi_\alpha(g_\alpha)(v_\alpha)]$. Here π is well defined and is a continuous representation of G_{DL} on V_{DL}. Similarly $d\pi([\xi_\alpha])([v_\alpha]) = [d\pi_\alpha(\xi_\alpha)(v_\alpha)]$ is a continuous representation of $\mathfrak{g}_{\mathrm{DL}}$ on V_{DL}, the series $\exp(d\pi([\xi_\alpha]))([v_\alpha])$ converges in V_{DL}, and

$$\pi(\exp_G([\xi_\alpha]))([v_\alpha]) = \exp(d\pi([\xi_\alpha]))([v_\alpha]).$$

The diagrams

$$(1.5)\qquad \begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ \phi_\alpha\big\downarrow \eta_\alpha & & \big\downarrow \eta_\alpha \\ G_{\mathrm{DL}} \times V_{\mathrm{DL}} & \xrightarrow{\pi} & V_{\mathrm{DL}} \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{d\pi_\alpha} & V_\alpha \\ d\phi_\alpha\big\downarrow \eta_\alpha & & \big\downarrow \eta_\alpha \\ \mathfrak{g}_{\mathrm{DL}} \times V_{\mathrm{DL}} & \xrightarrow{d\pi} & V_{\mathrm{DL}} \end{array}$$

are commutative.

EXAMPLE 1. DIRECT LIMITS OF CLASSICAL ALGEBRAS. Let $\mathfrak{g}_n \subset \mathfrak{gl}(V_n)$, where V_n is a finite-dimensional vector space. For instance, $\mathfrak{g}_n$ could be one of the classical algebras $\mathfrak{u}(n)$, $\mathfrak{o}(n)$, or $\mathfrak{u}(q, n)$. Here $d\phi_{n+k,n} : \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+k}$ is given by $\xi \mapsto \left(\begin{smallmatrix} \xi & 0 \\ 0 & 0 \end{smallmatrix}\right)$, obtained by adding $\dim V_{n+k} - \dim V_n$ zero columns and rows to ξ.

$\mathfrak{g}_{\mathrm{DL}} = \varinjlim_{\mathrm{DL}} \mathfrak{g}_n$ is a locally convex Lie algebra since for countable direct limits of topological vector spaces the direct limit and the locally convex direct limits coincide.

It is convenient to work in the case where the maps of the directed system are injective. This implies no loss of generality. In fact, we proved [7, Proposition 3.1] that there is a unique quotient directed system with the same limits and with all maps $\phi_{\beta,\alpha}$, $\eta_{\beta,\alpha}$, ϕ_α, and η_α injective. In [7] we denoted this injective quotient system with overlines, but here we just pass to the injective quotient system and drop the extra overline notation.

The spectral growth condition of [7] is

$$(1.6)\qquad \iota(\xi) = \sup\{|\mathrm{Im}\ \lambda| \mid \lambda \text{ is an eigenvalue of } d\pi(\xi)\} < \infty \quad \text{for all } \xi \in \mathfrak{g}.$$

This condition (1.6) has strong consequences. It forces

$$(1.7)\qquad \mathcal{O} = \{\xi \in \mathfrak{g} \mid \iota(\xi) < \pi\}$$

to contain an open neighborhood $\mathcal{O}_1$ of 0 in $\mathfrak{g}_{\mathrm{DL}}$ [7, Proposition 5.5]. If $d\pi : \mathfrak{g} \to \mathrm{End}(V)$ is injective, it says that $\exp_G : \mathcal{O}_1 \to G$ is injective, and that

(1.8) $U = U_{\mathrm{DL}} = \exp_G(\mathcal{O}_1)$ is an open subset of G_{DL};

(1.9) $\exp_G : \mathcal{O}_1 \to U$ is a homeomorphism; and

(1.10) $\exp_{G_\alpha} : d\phi_\alpha^{-1}(\mathcal{O}_1) \to \phi_\alpha^{-1}(U)$ is a diffeomorphism for all $\alpha \in A$.

Write $C^\omega(\cdot)$ for the sheaf of germs of real analytic functions on a finite-dimensional real analytic manifold. The direct limit sheaf is $C^\omega(G_{\rm DL}) = \varinjlim_{\rm DL} C^\omega(G_\alpha)$. If S is open in $G_{\rm DL}$, then $f : S \to \mathbb{C}$ is a section of $C^\omega(G_{\rm DL})$ over S just when each $f \cdot \phi_\alpha : \phi_\alpha^{-1}(S) \to \mathbb{C}$ is C^ω. That specifies the presheaf and thus specifies $C^\omega(G_{\rm DL})$. Since the ingredients of this construction are invariant under group operations, the sheaf $C^\omega(G_{\rm DL})$ is invariant under left and right group translations and group inversion.

The sheaf $C^\omega(\mathfrak{g}_{\rm DL})$ is defined similarly. It also has a standard definition in the context of linear spaces. If B is open in $\mathfrak{g}_{\rm DL}$, then $f : B \to \mathbb{C}$ is analytic just when $f \cdot \ell$ is analytic in the usual sense for every affine $\ell : \mathbf{R} \to \mathfrak{g}$. The two definitions are equivalent because ℓ must have image in one of the finite-dimensional spaces $\mathfrak{g}_\alpha$.

The main result of [7] says that, given the spectral growth condition, $G_{\rm DL}$ has a C^ω Lie group structure modelled on the topological vector space $\mathfrak{g}_{DL}$, such that

$$\text{(1.11)}\qquad \text{the } (\exp_G|_{\mathcal{O}_1})^{-1} \cdot L_{g^{-1}} : gU \to \mathcal{O}_1 \hookrightarrow \mathfrak{g}_{\rm DL} \text{ form a local coordinate cover on } G_{\rm DL}$$

and

$$\text{(1.12)}\qquad C^\omega(G_{\rm DL}) \text{ is the sheaf of germs of } C^\omega \text{ functions on } G_{\rm DL}.$$

Furthermore, $\exp_G : \mathcal{O}_1 \to U$ is an analytic diffeomorphism and the $\phi_\alpha : G_\alpha \to G_{\rm DL}$ are analytic.

In [8] we consider the locally convex direct limit of a directed system of Lie algebras, and also the norm direct limit and the Banach direct limit. These concepts are defined below, and can be thought of as the direct limits in different categories of Lie algebras and continuous Lie algebra homomorphisms. We then proceed to obtain results similar to those of [7] for each of these types of limits.

In order to construct the groups associated to these types of limit Lie algebras, we need to assume that the operator norms on $\{\mathfrak{g}_\alpha\}$ are of bounded growth, in the sense specified below. This condition implies the spectral growth condition (1.6). It is a condition which is automatically satisfied by the examples that occur most naturally.

For each $\alpha \in A$, let ν_α be the operator norm defined on $\mathfrak{g}_\alpha$ by the representation $d\pi_\alpha$ on V_α. The family $\{\nu_\alpha\}_{\alpha\in A}$ is said to be of *bounded growth* if for each $\xi = [\xi_\alpha] \in \mathfrak{g}$, there exists an index $\alpha \in A$ such that $\lim\sup\{\nu_\beta(\phi_{\beta,\alpha}\xi_\alpha) \mid \alpha \leqq \beta \in A\} < \infty$.

In the case of the norm direct limit, we also need to assume that the norm we are dealing with be "not too different" from an operator norm of bounded growth, in the sense made precise by condition (3.2) below.

2. Convexity and direct limits. The norm- and Banach direct limits

We focus on direct limits of finite-dimensional Lie algebras, but several of the results are valid more generally for direct limits of vector spaces.

Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system of topological vector spaces and continuous linear transformations. The directed system is *strict* if $V_\alpha \cong \psi_{\beta,\alpha}(V_\beta)$, topological isomorphism, whenever $\alpha \leqq \beta$. Note that the index set A may be uncountable.

Let $V = \varinjlim_{\alpha\in A} V_\alpha$, algebraic direct limit. We will deal with several topologies on V:

1. The (naïve) *direct limit* (DL) topology, described in §1. It is the strongest topology on V for which the inclusion maps $\psi_\alpha : V_\alpha \longrightarrow V$ are continuous for all $\alpha \in A$. A linear transformation $F : V_{\rm DL} \longrightarrow W$ into any topological vector space W, is continuous if and only if the transformations $F \cdot \psi_\alpha : V_\alpha \longrightarrow W$ are continuous for all $\alpha \in A$. Such an F is said to be *DL-continuous*.

2. Assume that each V_α is locally convex. The *locally convex direct limit* (LCDL) topology is defined to be the strongest locally convex topology on V such that all the $\psi_\alpha : V_\alpha \longrightarrow V$ are continuous. A *convex* subset U of V is LCDL open if and only if $\psi_\alpha^{-1}(U)$ is open for all $\alpha \in A$. A linear transformation $F : V_{\rm LCDL} \longrightarrow W$ into a locally convex vector space W, is continuous if and only if the $F \cdot \psi_\alpha : V_\alpha \longrightarrow W$ are all continuous. We will then say that F is *LCDL-continuous.*

In the theory of linear spaces, the LCDL topology is used more than the DL topology, and is often simply called the direct limit topology.

3. Assume that the V_α have *compatible* norms $\|\cdot\|_\alpha$ in the sense that

$$\text{(2.1)}\qquad \|\psi_{\beta,\alpha}(v)\|_\beta = \|v\|_\alpha \quad \text{for } \alpha \leqq \beta \quad \text{and} \quad v \in V_\alpha.$$

The *norm direct limit* or *NDL* topology on V is the topology given by the norm $\|\cdot\| = \varinjlim \|\cdot\|_\alpha$, defined as follows. Let $v = [v_\alpha] \in V$, and take $\beta \in A$ large enough so that $\psi_\beta^{-1}(v)$ is not empty. Then

$$\text{(2.2)}\qquad \|v\| = \|\psi_\beta^{-1}(v)\|_\beta.$$

It follows from condition (2.1) that this norm on V is well defined. The normed vector space thus obtained is denoted by $V_{\rm NDL}$, or by $(V, \|\cdot\|)$.

4. The *Banach direct limit* $V_{\rm BDL}$ of $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is defined to be the completion of V with respect to the NDL topology.

The topological vector spaces $V_{\rm DL}$, $V_{\rm LCDL}$, $V_{\rm NDL}$ and $V_{\rm BDL}$ can be conveniently considered as the limits of the same $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in different categories, and that is the approach we take in [8].

EXAMPLE 2. THE SPACES ℓ_p. Let A be the set of natural numbers, and let $V_n = \mathbf{R}^n$. Let $\phi_{m,n} : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be the natural inclusion map

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 0, \dots, 0) \in \mathbf{R}^m \qquad (n \leqq m).$$

The direct limit space V, often denoted $\mathbf{R}^\infty$, is the vector space of all sequences, with coefficients in $\mathbf{R}$, which are eventually zero. Now consider each of the $\mathbf{R}^n$ as a Banach space for the norm ℓ_p, and observe that the compatibility condition (2.1) is satisfied, so that the ℓ_p NDL makes sense. Note that V is not NDL-complete. Its completion V_{BDL} is the usual space ℓ_p.

Example 3. Norm- and Banach Direct Limits of Classical Lie Algebras. We can also take a norm limit of the algebras in Example 1, as follows. Let $1 \leqq p \leqq \infty$ and define $\| \ \|_{n,p}$ on $\mathfrak{g}_n$ to be the ℓ_p norm, namely $\|\xi\|_{n,p} = (\mathrm{Tr}(\xi^*\xi)^{\frac{p}{2}})^{\frac{1}{p}}$ for $1 \leqq p < \infty$ and $\|\xi\|_{n,\infty} = \sup_{\|v\|=1}(\|\xi v\|)$. Then, for each p, $\{\| \cdot \|_{n,p}\}_{n\in\mathbf{N}}$ is a compatible family of norms and hence we have a well-defined norm, $\| \ \|_p = \varinjlim \| \cdot \|_{n,p}$ on $\mathfrak{g}$. We will refer to this situation as $(\mathfrak{g}, \| \ \|_p)$. Our limit Lie algebras provide more examples of these constructions.

Lemma 2.3. *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system of vector spaces and linear transformations. Then the natural inclusion maps $V_{DL} \hookrightarrow V_{LCDL} \hookrightarrow V_{NDL}$ are continuous, for the NDL topology given by any family $\{\| \cdot \|_\alpha\}$ of norms satisfying the compatibility condition* (2.1).

We say that the system $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *eventually stabilizes* if there exists an index α_0 such that $\psi_{\beta,\alpha}$ is an isomorphism onto V_β whenever $\beta \geqq \alpha \geqq \alpha_0$. In that case V_{DL} and V_{LCDL} are both isomorphic to V_{α_0}.

Lemma 2.4. *Let $\{V_n, \phi_{m,n}\}$ be a countable, strict directed system. If the system never stabilizes, then no locally convex topology makes V into a Baire space.*

Corollary 2.5. *No NDL topology on V is complete, unless the system eventually stabilizes.*

Lemma 2.6. *Let T be any one of the topologies DL, LCDL, NDL or BDL. If $B \subset A$ is a directed set under the partial ordering "$\leqq$" it inherits from A, then*

(a) $\varinjlim_{T,\beta\in B} V_\beta \hookrightarrow \varinjlim_{T,\alpha\in A} V_\alpha$ *is continuous, and*

(b) *if B is cofinal in A, then* $\varinjlim_{T,\beta\in B} V_\beta = \varinjlim_{T,\alpha\in A} V_\alpha$.

Several useful properties hold for the case when the index set $A = \mathbf{N}$ and the directed system is strict. For example, then V_{LCDL} coincides with V_{DL} and is complete and Hausdorff, and V_α is isomorphic to its image $\psi_\alpha(V_\alpha)$ where the latter has the topology induced by V_{LCDL}. See **[11]**. However, these properties do not hold in general for uncountable directed systems; see **[6]**. A result of Komura there shows that, even if every V_α is Hausdorff, V_{LCDL} can have the property that no two points are separated.

Of course, if the index set A is uncountable, but has a cofinal subsequence, then it follows from Lemma 2.6 that V_{LCDL} has all of the desirable properties known to hold for countable locally convex direct limits.

Unless we state otherwise, we do not assume that the directed system is countable. On the other hand, for the remainder of this section and most of the rest of this paper, we assume that each of the spaces V_α is finite dimensional. In many situations, it turns out that this finite-dimensionality makes up for uncountability of the index set.

Recall our assumption that each of the $\psi_{\beta,\alpha}$ is injective. Together with the finite dimensionality of the V_α, it implies that the direct limit is strict.

We use a Zorn's lemma argument to prove

Proposition 2.7. *There exists a compatible system of norms on*

$$\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$$

and hence an NDL topology on V.

In particular the LCDL topology is stronger than a Hausdorff topology, so

Corollary 2.6. *The LCDL topology on V is Hausdorff.*

The following corollary is immediate from the Hausdorff property of V_{LCDL}. Its importance stems from these facts: For a barrelled vector space, the conclusion of the Banach-Steinhaus Theorem (Principle of Uniform Boundedness) holds. For a bornological vector space, a linear map is bounded (maps bounded sets into bounded sets) if and only if it is continuous.

Corollary 2.9. *V_{LCDL} is bornological and barrelled.*

We also have

Proposition 2.10. *If the index set A is linearly ordered, then the locally convex vector space V_{LCDL} is sequentially complete.*

As indicated above, the LCDL behaves nicely with respect to continuous linear maps between locally convex spaces. The projective tensor product $V \otimes_\varpi W$ provides us with a device allowing us to turn questions dealing with continuous bilinear maps into questions about continuous linear transformations. If V and W are locally convex spaces, then (see **[10,** Chapter 43–46]) the projective tensor product topology is the unique locally convex topology on $V \otimes W$ such that: For every locally convex space Z, the canonical isomorphism of the space of bilinear mappings of $V \times W$ into Z, onto the space of linear mappings of $V \otimes W$ into Z, induces an isomorphism of the space of continuous bilinear mappings of $V \times W$ into Z, onto the space of continuous linear mappings of $V \otimes W$ into Z.

Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha, \chi_{\beta,\alpha}\}_{\alpha,\beta\in\hat{A}}$ be two directed systems of locally convex vector spaces and continuous linear transformations. There is no loss of generality in assuming that $A = \hat{A}$, since we could if necessary consider both the given systems as having the index set $A \times \hat{A}$ (the product directed set, which by definition has the lexicographic order). Both $\varinjlim_{LCDL}(V_\alpha \otimes_\varpi W_\beta)$ and $(\varinjlim_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim_{LCDL} W_\alpha)$ have as underlying vector space $(\varinjlim V_\alpha) \otimes (\varinjlim W_\alpha)$. It is known that the natural inclusion map $\varinjlim_{LCDL}(V_\alpha \otimes_\varpi W_\beta) \hookrightarrow (\varinjlim_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim_{LCDL} W_\alpha)$ is continuous. We show that under our hypotheses, this result can be strengthened.

PROPOSITION 2.11. *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha, \chi_{\beta,\alpha}\}_{\alpha,\beta\in \hat{A}}$ be two strict directed systems of finite-dimensional vector spaces and continuous linear maps. Then*

$$\varinjlim{}_{LCDL}(V_\alpha \otimes_\varpi W_\beta) = (\varinjlim{}_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim{}_{LCDL} W_\beta).$$

Direct limits of finite-dimensional Lie algebras. We now consider a directed system $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ of finite-dimensional Lie algebras $\mathfrak{g}_\alpha$ and Lie algebra homomorphisms $\psi_{\beta,\alpha}$. Let $\mathfrak{g} = \varinjlim_{\alpha\in A} \mathfrak{g}_\alpha$, as described in §1. Since $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is in particular a directed system of locally convex vector spaces and continuous linear maps, $\mathfrak{g}_{DL}$, $\mathfrak{g}_{LCDL}$, $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ all make sense. Each of them has a topological vector space structure, and also a Lie algebra structure. We next investigate the question of when these two structures are compatible.

PROPOSITION 2.12. (a) *If $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is a strict directed system of Lie algebras, then the Lie bracket is a continuous map $\mathfrak{g}_{DL} \times \mathfrak{g}_{DL} \longrightarrow \mathfrak{g}_{DL}$.*

(b) *Let $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a strict directed system of locally convex Lie algebras. If the index set A is countable, or each $\mathfrak{g}_\alpha$ is finite-dimensional, then the Lie bracket is a continuous map $\mathfrak{g}_{LCDL} \times \mathfrak{g}_{LCDL} \longrightarrow \mathfrak{g}_{LCDL}$.*

To prove statement (a) we need some results on direct limits in the category of topological spaces and continuous maps. These are given in [8]. Statement (b) is a consequence of Proposition 2.11.

Consider the NDL case. Given a compatible family $\{\|\cdot\|_\alpha\}$ of norms for $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$, and given $\xi_\alpha \in \mathfrak{g}_\alpha$, for each index $\beta \geqq \alpha$ we have an operator norm for the adjoint action of $\psi_{\beta,\alpha}\xi_\alpha$ on $\mathfrak{g}_\beta$,

$$\|\mathrm{ad}(\psi_{\beta,\alpha}\xi_\alpha)\|_{\beta,\infty} := \sup\left\{\frac{\|[\psi_{\beta,\alpha}\xi_\alpha, \xi_\beta]\|_\beta}{\|\xi_\beta\|_\beta} \;\middle|\; \xi_\beta \in \mathfrak{g}_\beta\right\}.$$

For any $\alpha \leqq \beta \leqq \gamma$ we then have $\|\mathrm{ad}(\psi_{\beta,\alpha}\xi_\alpha)\|_{\beta,\infty} \leqq \|\mathrm{ad}(\psi_{\gamma,\alpha}\xi_\alpha)\|_{\gamma,\infty}$.

PROPOSITION 2.13. *Let $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system of Lie algebras, with a compatible family of norms $\{\|\cdot\|_\alpha\}$. Suppose that*

(2.14) *if $\xi = [\xi_\alpha] \in \mathfrak{g}$ and $\alpha \in A$ then $\limsup_{\beta\geqq\alpha}\{\|ad(\xi_\alpha)\|_{\beta,\infty}\} < \infty$.*

Then there exists $M > 0$ such that $\|[\xi, \zeta]\| \leqq M \cdot \|\xi\| \cdot \|\zeta\|$ for all $\xi, \zeta \in \mathfrak{g}$ where $\|\cdot\| = \varinjlim \|\cdot\|_\alpha$. Hence the corresponding $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ are normed Lie algebras.

When we speak of the limit Lie algebras $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ we shall always assume that the directed system in question satisfies the hypothesis (2.14) of Proposition 2.13.

3. The basic local coordinate system

Let T be one of the topologies DL, LCDL, NDL or BDL. In this section we discuss the question of finding sufficient conditions for the existence of a "good" neighborhood $\mathcal{O}_1$ of 0 in $\mathfrak{g}_T$, i.e. of a T-open neighborhood $\mathcal{O}_1$ of 0, such that the restriction of the exponential map to $\mathcal{O}_1$ is one-to-one. This neighborhood is essential for construction of analytic manifold structure on G_T in §§4 and 5. The set $U = \exp(\mathcal{O}_1)$ will turn out to be a chart of G_T at 1, and conditions (1.8) to (1.10) of §1 will hold with T in place of DL.

We already know that the restriction of $\exp_G$ to the set $\mathcal{O}$, given by (1.7), is injective. So we need only show that $\mathcal{O}$ contains a T-open neighborhood $\mathcal{O}_1$ of 0.

The NDL topology for the operator norm. Assume that $\{d\pi_\alpha, V_\alpha\}_{\alpha\in A}$ is a compatible family of representations, and that ν_α is the corresponding operator norm on $\mathfrak{g}_\alpha$. Assume that

$$\limsup\{\nu_\alpha(\xi_\alpha)\} < \infty$$

for each $[\xi_\alpha] \in \mathfrak{g}$. The spectral growth condition follows. We have an operator norm $\nu = \limsup \nu_\alpha$ on $\mathfrak{g}$. Let ι and $\mathcal{O}$ be as defined in (1.6) and (1.7), using this system $\{d\pi_\alpha\}$. We will now show that $\mathcal{O}$ is a neighborhood of 0 in the NDL Lie algebra $(\mathfrak{g}, \nu)$, where $\nu = \limsup \nu_\alpha$.

For each $\alpha \in A$ we set $\iota_\alpha(\xi) = \sup\{|\mathrm{Im}\,\lambda| \mid \lambda \text{ is an eigenvalue of } d\pi_\alpha(\xi)\}$ and $\mathcal{O}_\alpha = \{\xi \in \mathfrak{g}_\alpha \mid \iota_\alpha(\xi) < \pi\}$. Then $\iota_\alpha(\xi_\alpha) \leqq \nu_\alpha(\xi_\alpha)$. Thus $\mathcal{O}_\alpha$ contains the open neighborhood $\mathcal{N}_\alpha = \{\xi_\alpha \in \mathfrak{g}_\alpha \mid \nu_\alpha(\xi_\alpha) < \pi\}$ and hence it is itself a neighborhood of 0. Since this holds for each index α, and $\mathcal{O}_\alpha = \psi_\alpha^{-1}(\mathcal{O})$, it follows that $\mathcal{O}$ is a neighborhood of 0 in $(\mathfrak{g}, \nu)$. We now take as our $\mathcal{O}_1 = \mathcal{O}_{1,\nu}$ any ν-open neighborhood of 0 which is contained in $\mathcal{O}$. We could for example take

$$\mathcal{O}_1 = \mathcal{O}_{1,\nu} := \{\xi \in \mathfrak{g} \mid \nu(\xi) < \pi\}. \tag{3.1}$$

The DL and LCDL topologies. Let $\mathcal{O}_1$ be as in (3.1), and ν be an operator norm, as above. Since the natural inclusion maps $\mathfrak{g}_{DL} \hookrightarrow \mathfrak{g}_{LCDL} \hookrightarrow (\mathfrak{g}, \nu)$ are continuous, $\mathcal{O}_1$ is an open neighborhood of 0 for both the LCDL and the DL topologies.

The NDL topology for the general case. Let $\{\|\cdot\|_\alpha\}$ be a compatible family of norms for the directed system and let $\|\cdot\| = \varinjlim \|\cdot\|_\alpha$. We prove that the NDL Lie algebra $(\mathfrak{g}, \|\cdot\|)$ has a neighborhood on which $\exp_G$ is injective if the norm is "not too different" from an operator norm.

More precisely, let $\{\nu_\alpha\}$ be a family of operator norms that come from some representation with bounded spectral growth, as in §1. For each α, let $c_\alpha(\nu)$ be a positive number such that $\nu_\alpha(\xi) \leqq c_\alpha(\nu)\|\xi\|_\alpha$ for all $\xi \in \mathfrak{g}_\alpha$. Assume that $c_\alpha(\nu)$ is minimal for this condition. We prove that if

(3.2) the operator norm $\nu = \varinjlim \nu_\alpha$ satisfies $\limsup c_\alpha(\nu) < \infty$,

then $\mathcal{O}_{1,\nu}$ is $\|\cdot\|$-open. In effect, (3.2) implies $\nu(\xi) \leqq \limsup c_\alpha(\nu)\|\xi\|$ for all $\xi \in \mathfrak{g}$, so the natural inclusion $(\mathfrak{g}, \|\cdot\|) \hookrightarrow (\mathfrak{g}, \nu)$ is continuous.

4. The topologies on the limit groups

In this section we indicate how the topologized Lie algebras $\mathfrak{g}_{\mathrm{DL}}$, $\mathfrak{g}_{\mathrm{LCDL}}$, $\mathfrak{g}_{\mathrm{NDL}}$ and $\mathfrak{g}_{\mathrm{BDL}}$ define topological structures $\mathfrak{G}_{\mathrm{DL}}$, $\mathfrak{G}_{\mathrm{LCDL}}$, $\mathfrak{G}_{\mathrm{NDL}}$ and $\mathfrak{G}_{\mathrm{BDL}}$ on the limit group $G = \varinjlim G_\alpha$. Specifically, we indicate how each G_T carries the structure of a topological manifold modelled on the corresponding $\mathfrak{g}_T$, where T is DL, LCDL, NDL or BDL. The C^ω differentiable manifold structures are described in §5.

The topologies of $\mathfrak{G}_{\mathrm{DL}}$, $\mathfrak{G}_{\mathrm{NDL}}$ and $\mathfrak{G}_{\mathrm{BDL}}$ have the property that the group operations are continuous. So each of them is a topological group in the usual sense. For $\mathfrak{G}_{\mathrm{LCDL}}$, we only obtain a group with a topology that makes $x \mapsto x^{-1}$ continuous and $(x, y) \mapsto xy$ separately continuous.

The key to this process is the following result in [2, Chapter II, §II]: A family $\mathcal{V}$ of subsets of a group G is a fundamental system of neighborhoods of 1 in G, for some Hausdorff topology under which the group operations of G are continuous, if and only if $\mathcal{V}$ satisfies the five conditions:

1. If $U_1, U_2 \in \mathcal{V}$, then there exists a set $U_3 \in \mathcal{V}$ such that $U_3 \subset U_1 \cap U_2$,
2. The intersection of all sets of $\mathcal{V}$ is $\{1\}$,
3. If $U \in \mathcal{V}$, there exists a set $U_1 \in \mathcal{V}$ such that $U_1^{-1} \subset U$,
4. If $U \in \mathcal{V}$ and $g \in G$, there exists a set $U_1 \in \mathcal{V}$ such that $gU_1g^{-1} \subset U$,
5. If $U \in \mathcal{V}$, there exists a set $U_1 \in \mathcal{V}$ such that $U_1U_1 \subset U$.

Let $\mathcal{O}_1 \subset \mathcal{O}$ be an open neighborhood (for topology to be specified) of 0 and set

$$\tilde{\mathcal{V}} = \{\tilde{U} \subset \mathcal{O}_1 \mid \tilde{U} \text{ is T-open}\} \text{ and}$$
$$\mathcal{V} = \{U \in G \mid U = \exp(\tilde{U}) \text{ for some } \tilde{U} \in \tilde{\mathcal{V}}\}. \tag{4.1}$$

The desired properties of $\mathcal{V}$ follow from a list of analogous properties of $\tilde{\mathcal{V}}$. We denote by H the Campbell-Hausdorff-Dynkin series in $\mathfrak{g}$. Consider the five conditions:

$\tilde{1}$. If $\tilde{U}_1, \tilde{U}_2 \in \tilde{\mathcal{V}}$, then there exists a set $\tilde{U}_3 \in \tilde{\mathcal{V}}$ such that $\tilde{U}_3 \subset \tilde{U}_1 \cap \tilde{U}_2$,

$\tilde{2}$. The intersection of all sets of $\tilde{\mathcal{V}}$ is $\{0\}$,

$\tilde{3}$. If $\tilde{U} \in \tilde{\mathcal{V}}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that $-\tilde{U}_1 \subset \tilde{U}$,

$\tilde{4}$. If $\tilde{U} \in \tilde{\mathcal{V}}$ and $\xi \in \mathfrak{g}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that

$$H(\xi, H(\tilde{U}_1, -\xi)) \subset \tilde{U},$$

$\tilde{5}$. If $\tilde{U} \in \tilde{\mathcal{V}}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that $H(\tilde{U}_1, \tilde{U}_1) \subset \tilde{U}$.

By manipulating of the Campbell-Hausdorff-Baker series, we show that conditions $\tilde{1}$ through $\tilde{5}$ all hold for $\mathfrak{g}_{\mathrm{NDL}}$, consequently that conditions 1 through 5 all hold for $\mathfrak{G}_{\mathrm{NDL}}$. We have conditions $\tilde{1}$ through $\tilde{4}$ for $\mathfrak{g}_{\mathrm{LCDL}}$, and therefore conditions 1 through 4 for $\mathfrak{G}_{\mathrm{LCDL}}$; but in the LCDL case condition $\tilde{5}$ does not generally hold.

Definition 4.2. Let T = NDL or LCDL. Define $G_{\mathrm{NDL}} = \varinjlim_{\mathrm{NDL}} G_\alpha$ to be the abstract group $G = \varinjlim G_\alpha$ with the topology defined by the family $\mathcal{V}$ of (4.1) for the NDL topology. Define $\mathfrak{G}_{\mathrm{LCDL}} = \varinjlim_{\mathrm{LCDL}} G_\alpha$ to be the abstract group $G = \varinjlim G_\alpha$ with the topology defined by the family $\mathcal{V}$ of (4.1) for the LCDL topology. In other words, a subset $V \subset \mathfrak{G}_{\mathrm{NDL}}$ resp. $\mathfrak{G}_{\mathrm{LCDL}}$ is open if and only if for each $g \in V$ there exists $U \in \mathcal{V}$ such that $gU \subset V$.

The topological group $\mathfrak{G}_{\mathrm{BDL}}$ is defined as the completion of $\mathfrak{G}_{\mathrm{NDL}}$ with respect to its two-sided group uniformity.

There are (at least) two natural ways in which to consider completing the group $\mathfrak{G}_{\mathrm{NDL}}$: one might use the two-sided group uniformity, or, on the other hand, one might first complete $\mathfrak{g}_{\mathrm{NDL}}$ to $\mathfrak{g}_{\mathrm{BDL}}$ and then use a process such as the one used above to construct G_T. Proposition 4.3 below says that both $\exp : \mathcal{O}_1 \to U_1$ and its inverse $\log : U_1 \to \mathcal{O}_1$ are uniformly continuous in the NDL topology, so both paths lead to the same complete topological group $\mathfrak{G}_{\mathrm{BDL}}$.

A net $\{g_i\}_{i\in I} \subset \mathfrak{G}_{\mathrm{NDL}}$ is (*two-sided*) *Cauchy* if for every neighborhood U of 1 in $\mathfrak{G}_{\mathrm{NDL}}$ there exists an $i_0 \in I$ such that both $g_ig_j^{-1}$ and $g_i^{-1}g_j$ are in U, whenever $i, j \geqq i_0$.

Proposition 4.3. *Let* $(\mathfrak{g}, \|\cdot\|) = \mathfrak{g}_{NDL} = \varinjlim_{NDL} \mathfrak{g}_\alpha$ *and let* M *be a constant such that* $\|[\xi, \zeta]\| \leqq M\|\xi\| \cdot \|\zeta\|$ *for all* $\xi, \zeta \in \mathfrak{g}$. *Fix* $r > 0$ *and a net* $\{\xi_i\} \subset \mathcal{O}_1$ *with each* $\|\xi_i\| \leqq r/2M$. *Then* $\{\xi_i\}$ *is* $\|\cdot\|$-*Cauchy in* $\mathfrak{g}$ *if and only if* $\{\exp(\xi_i)\}$ *is Cauchy in* G_{NDL}.

In summary,

Theorem 4.4. *The DL, LCDL and NDL topologies on* $\mathfrak{g}$ *define topologies on* G *such that, for the resulting topological spaces* G_{DL}, G_{LCDL} *and* G_{NDL}, *the map* $\exp : \mathcal{O}_1 \to U_1$ *is a homeomorphism onto an open set.* G_{DL} *and* G_{NDL} *are topological groups, as is the completion* G_{BDL} *of* G_{NDL}. G_{LCDL} *is essentially a topological group, except that group multiplication may be only separately continuous.*

Now consider the BDL case. Assume (3.2) and define $\mathcal{O}_1$ by the same inequality (3.1) as in the NDL case. Then Proposition 4.3 and Theorem 4.4 give us

Corollary 4.5. *Let* $\mathfrak{g}_{BDL}$ *be the completion of the norm direct limit algebra* $\mathfrak{g}_{NDL}$, *let* G_{BDL} *be the completion of the corresponding norm direct limit group* G_{NDL}, *with respect to the two-sided group uniformity, and* *assume* (3.2). *Then* G_{BDL} *is a topological Lie group and the restriction to* $\mathcal{O}_1$ *of the exponential map* $\exp : \mathfrak{g}_{BDL} \longrightarrow G_{BDL}$ *is a homeomorphism onto an open set.*

5. Structure sheaves

In this section we carry the sheaves $C^{\omega}(G_{\mathrm{DL}})$ and $C^{\omega}(\mathfrak{g}_{\mathrm{DL}})$ of germs of analytic functions for the DL topology to corresponding sheaves $C^{\omega}(G_{\mathrm{NDL}})$ and $C^{\omega}(\mathfrak{g}_{\mathrm{NDL}})$ for the NDL topologies, to $C^{\omega}(G_{\mathrm{BDL}})$ and $C^{\omega}(\mathfrak{g}_{\mathrm{BDL}})$ for the BDL topologies, and to $C^{\omega}(G_{\mathrm{LCDL}})$ and $C^{\omega}(\mathfrak{g}_{\mathrm{LCDL}})$ for the LCDL topology.

Recall the standard construction [1, p. 9] of direct image sheaves. Let X and Y be topological spaces and $\psi : X \to Y$ a continuous map. If $\mathcal{F} \to X$ is a sheaf one has a presheaf over Y, which assigns to an open set $W_Y \subset Y$ the abelian group of all sections of $\mathcal{F}$ over $\psi^{-1}(W_Y)$. This presheaf is complete. That defines the *direct image sheaf* $\psi_* \mathcal{F} \to Y$. The assignment ψ_* is a left exact covariant functor.

The natural maps $G_{\mathrm{DL}} \to G_{\mathrm{LCDL}} \to G_{\mathrm{NDL}}$, by $x \mapsto x$, and $\mathfrak{g}_{\mathrm{DL}} \to \mathfrak{g}_{\mathrm{LCDL}} \to \mathfrak{g}_{\mathrm{NDL}}$ by $\xi \mapsto \xi$, are continuous. The sections of the direct image sheaves $C^{\omega}(G_{\mathrm{NDL}}) \to G_{\mathrm{NDL}}$, $C^{\omega}(\mathfrak{g}_{\mathrm{NDL}}) \to \mathfrak{g}_{\mathrm{NDL}}$, $C^{\omega}(G_{\mathrm{LCDL}}) \to G_{\mathrm{LCDL}}$, and $C^{\omega}(\mathfrak{g}_{\mathrm{LCDL}}) \to \mathfrak{g}_{\mathrm{LCDL}}$ are just those sections of the corresponding DL sheaf whose domains are NDL- or LCDL-open. So they are the analytic function germ sheaves where we define

DEFINITION 5.1. Let W be an open set in G_{NDL}, $\mathfrak{g}_{\mathrm{NDL}}$, G_{LCDL}, or $\mathfrak{g}_{\mathrm{LCDL}}$, respectively. Then a function $f : W \to \mathbf{C}$ is *real analytic* if (a) f is continuous and (b) f is DL-analytic.

The natural maps $G_{\mathrm{NDL}} \hookrightarrow G_{\mathrm{BDL}}$ and $\mathfrak{g}_{\mathrm{NDL}} \hookrightarrow \mathfrak{g}_{\mathrm{BDL}}$ are continuous. The sections of the direct image sheaves $C^{\omega}(G_{\mathrm{BDL}}) \to G_{\mathrm{BDL}}$ and $C^{\omega}(\mathfrak{g}_{\mathrm{BDL}}) \to \mathfrak{g}_{\mathrm{BDL}}$ are just those continuous functions whose restrictions to G_{NDL} or $\mathfrak{g}_{\mathrm{NDL}}$ are sections of the corresponding NDL sheaf. So they are the analytic function germ sheaves where we define

DEFINITION 5.2. let W be an open set in G_{BDL} or $\mathfrak{g}_{\mathrm{BDL}}$, respectively. Then a function $f : W \to \mathbf{C}$ is *real analytic* if (a) f is continuous and (b) the restriction of f to the dense subset $W \cap G_{\mathrm{NDL}}$, respectively $W \cap \mathfrak{g}_{\mathrm{NDL}}$, is NDL-analytic.

6. Conclusion and examples

We now combine the material of §§4 and 5. Let T be one of the topologies DL, LCDL, NDL or BDL. In the DL case, assume the spectral growth condition (1.6). In the LCDL, NDL and BDL cases assume the stronger conditions (2.14) and (3.2). In a group, L_x denotes the left translation $y \mapsto xy$. Our results are summarized in

THEOREM 6.1. *The real analytic structures on* $\mathfrak{g} = \varinjlim \mathfrak{g}_{\alpha}$, *corresponding to the topologies T = DL, LCDL or NDL, respectively, define structures* G_T

of C^{ω} *differentiable manifold on* $G = \varinjlim G_{\alpha}$ *based on the topogical vector space* $\mathfrak{g}_T$. *A* C^{ω} *local coordinate cover on* G_T, *corresponding to the topology T, is given by the* $(\exp|_{\mathcal{O}_1})^{-1} \cdot L_{g^{-1}} : gU_1 \to \mathcal{O}_1 \hookrightarrow \mathfrak{g}_T$; *and* $C^{\omega}(G_T)$ *is the sheaf of germs of* C^{ω} *functions on* G_{DL} *whose domains are open in* G_T. G_{DL} *and* G_{NDL} *are* C^{ω} *Lie groups, as is the completion* G_{BDL} *of* G_{NDL}. G_{LCDL} *is essentially a* C^{ω} *Lie group, except that group multiplication may be only separately analytic.*

EXAMPLE 4. DIRECT LIMITS OF CLASSICAL LIE GROUPS. Let $\mathfrak{g}_n$ be as in Example 1, and let G_n be their respective Lie groups. For the instances mentioned explicitly in Example 1, the Lie groups are $G_n = U(n)$, $O(n)$ and $U(q, n)$ respectively. The embedding $\phi_{n+k,n}$ is given by $\xi \mapsto \left(\begin{smallmatrix} \xi & 0 \\ 0 & I \end{smallmatrix}\right)$ where I is the identity matrix of size $\dim V_{n+k} - \dim V_n$. The group $\mathfrak{G}_{\mathrm{DL}} = \mathfrak{G}_{\mathrm{LCDL}} = \varinjlim_{\mathrm{LCDL}} G_n$ is now seen to be a locally convex Lie group modelled on its Lie algebra $\varinjlim_{\mathrm{LCDL}} \mathfrak{g}_n$.

EXAMPLE 5. NORM- AND BANACH DIRECT LIMITS OF CLASSICAL LIE GROUPS. Using Example 3, each sequences of classical Lie groups in Example 4, and for each p with $1 \leqq p \leqq \infty$, gives group $G = \varinjlim G_n$.That group G has the structure of normed Lie group $\mathfrak{G}_{\mathrm{NDL}}$ modelled on the norm direct limit Lie algebra $(\mathfrak{g}, \|\cdot\|_p)$. The compatibility condition (2.1) is satisfied in these cases, so each of the normed Lie groups $\mathfrak{G}_{\mathrm{NDL}}$ can be completed to a Banach Lie group $\mathfrak{G}_{\mathrm{BDL}}$ modelled on $\mathfrak{g}_{\mathrm{BDL}}$.

The groups of Examples 4 and 5 have been studied as topological groups by Kolomytsev, Semoilenko, Ol′shanskii and others. For example see [4, 5, and 9]. Here [5] contains some bibliography on the subject.

EXAMPLE 6. C^{∞} FUNCTIONS. Let Ω be a separable C^{∞} manifold, e.g. an open subset of $\mathbf{R}^n$, $\mathbf{G}$ a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$. Then $C^{\infty}(\Omega, \mathfrak{g})$ and $C^{\infty}(\Omega, \mathbf{G})$ are a topological Lie algebra and group respectively, with the topology of uniform convergence of the functions and their derivatives on compact sets. Here the algebra and group operations are specified pointwise. It is standard that $C^{\infty}(\Omega, \mathfrak{g})$ is complete with respect to this topology.

For $K \subset \Omega$, K compact, define $C^{\infty}_K(\Omega, \mathfrak{g}) = \{f \in C^{\infty}(\Omega, \mathfrak{g}) \mid \operatorname{supp}(f) \subset K\}$ where $\operatorname{supp}(f)$ is the support of the function f. Define $C^{\infty}_c(\Omega, \mathfrak{g}) = \bigcup_{K \text{ compact}} C^{\infty}_K(\Omega, \mathfrak{g})$. As Ω is locally compact with a countable basis for open sets, we have a sequence $B_1 \subset B_2 \subset \cdots$ of open sets with $\Omega = \bigcup B_n$ and each $K_i =$ closure B_i compact. Thus, $C^{\infty}_c(\Omega, \mathfrak{g}) = \bigcup_{K_j} C^{\infty}_{K_j}(\Omega, \mathfrak{g})$. Give $C^{\infty}_{K_j}(\Omega, \mathfrak{g})$ the subspace topology inherited from $C^{\infty}(\Omega, \mathfrak{g})$. Then the inclusion map $C^{\infty}_{K_j}(\Omega, \mathfrak{g}) \to C^{\infty}_{K_{j+1}}(\Omega\mathfrak{g})$ is an isomorphism onto its image. Thus, we can view $C^{\infty}_c(\Omega, \mathfrak{g})$ as the strict countable direct limit, thus the locally convex direct limit, of the $C^{\infty}_{K_j}(\Omega, \mathfrak{g})$. The sheaf of analytic functions on $C^{\infty}_c(\Omega, \mathfrak{g})$ is the direct limit sheaf. Note that this topology on $C^{\infty}_c(\Omega, \mathfrak{g})$

is in general strictly finer than the topology of uniform convergence on compact sets.

Let $C^\infty_{K_j}(\Omega, \mathbf{G})$ be C^∞ functions on Ω with values in $\mathbf{G}$, whose support lies in K_j. Then, $C^\infty_{K_j}(\Omega, \mathbf{G})$ is a Lie group with Lie algebra $C^\infty_{K_j}(K_j, \mathfrak{g})$. We can now define a differentiable structure on the direct limit group, $C^\infty_c(\Omega, \mathbf{G}) = \bigcup C^\infty_{K_j}(\Omega, \mathbf{G})$, modelled on $C^\infty_c(\Omega, \mathfrak{g})$. Let O and U be neighborhoods of 0 and 1 respectively in $\mathfrak{g}$ and $\mathbf{G}$, on which the exponential map is a diffeomorphism. Then, the exponential map from

$$\tilde{O} = \{f \in C^\infty_c(\Omega, \mathfrak{g}) | \ f \text{ has compact support and } f(\Omega) \subset O\}$$

to

$$\tilde{U} = \{f \in C^\infty_c(\Omega, \mathbf{G}) | \ f \text{ has compact support and } f(\Omega) \subset U\}$$

is a homeomorphism. The sheaf of analytic functions on $C^\infty_c(\Omega, \mathbf{G})$ is defined as the direct limit sheaf. The existence of local sections is ensured by the fact that the topology on $C^\infty_c(\Omega, \mathbf{G})$ is given locally by the Lie algebra $C^\infty_c(\Omega, \mathfrak{g})$.

EXAMPLE 7. LIE GROUPS AND LIE ALGEBRAS OF OPERATORS ON A HILBERT SPACE. Let $\mathcal{H}$ be a Hilbert space, not necessarily separable. Let $\{e_i\}_{i\in I}$ be a complete orthonormal set in $\mathcal{H}$. Our indexing set in this example will be the directed set A of all finite subsets of I, with the partial order $\alpha \leqq \beta \iff \alpha \subseteq \beta$.

For each $\alpha \in A$, denote by $\mathcal{H}_\alpha$ the finite-dimensional subspace of $\mathcal{H}$ with basis $\{e_i\}_{i\in\alpha}$.

Let $GL(\mathcal{H}_\alpha)$ the group of invertible linear operators on $\mathcal{H}_\alpha$. If $\alpha \leqq \beta$ in A, then $\phi_{\beta,\alpha} : GL(\mathcal{H}_\alpha) \longrightarrow GL(\mathcal{H}_\beta)$ is the natural inclusion map which identifies $GL(\mathcal{H}_\alpha)$ with the subgroup $\{g \in GL(\mathcal{H}_\beta) \mid g(e_i) = e_i$ whenever $i \in \beta$ but $i \notin \alpha\}$. Thus the directed system $\{GL(\mathcal{H}_\alpha), \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is strict.

The inclusion map $\phi_\alpha : GL(\mathcal{H}_\alpha) \longrightarrow \mathbf{G}(\mathcal{H}) := \varinjlim GL(\mathcal{H}_\alpha)$ is an isomorphism of $GL(\mathcal{H}_\alpha)$ onto its image, which is the subgroup $\{g \in GL(\mathcal{H}) \mid g(e_i) = e_i$ for $i \notin \alpha\}$. The limit $\mathbf{G}(\mathcal{H})$ is a subgroup of the group $GL(\mathcal{H})$ of all bounded invertible linear operators on $\mathcal{H}$. It consists of those operators which have the form $1 +$ (finite rank).

The choice of embedding makes it clear that the spectral growth condition is satisfied. Hence $\mathbf{G}(\mathcal{H})$ with the direct limit topology becomes a Lie group, $\mathbf{G}_{\mathrm{DL}}(\mathcal{H})$ with Lie algebra $\mathfrak{g}_{\mathrm{DL}}(\mathcal{H}) := \varinjlim_{\mathrm{DL}} \mathfrak{gl}(\mathcal{H}_\alpha)$. We can also retopologize $\mathbf{G}(\mathcal{H})$ using the locally convex Lie algebra $\mathfrak{g}_{\mathrm{LCDL}}(\mathcal{H})$. Then $\mathbf{G}_{\mathrm{LCDL}}(\mathcal{H})$ is a separately continuous locally convex Lie group modelled on its Lie algebra $\mathfrak{g}_{\mathrm{LCDL}}(\mathcal{H})$.

For $1 \leqq p \leqq \infty$, we have algebras $(\mathfrak{g}(\mathcal{H}), \|\cdot\|_p)$ as in Examples 2 and 5. Each of them gives us a normed Lie group $(\mathbf{G}_{\mathrm{NDL}}(\mathcal{H}), \|\cdot\|_p)$. Their completions give us Banach Lie groups $(\mathbf{G}_{\mathrm{BDL}}(\mathcal{H}), \|\cdot\|_p)$ with Banach Lie algebras $\mathfrak{g}_{\mathrm{BDL}}(\mathcal{H})$.

We have as a special case using the uniform operator norm on $\mathfrak{g}(\mathcal{H})$ that the completion is the algebra of compact operators and the completed group is the group of bounded operators of the form $1 +$ (compact).

EXAMPLE 8. SUBGROUPS AND SUBALGEBRAS OF SPACES OF BOUNDED LINEAR OPERATORS. Retain the notation of the previous example. We look at certain subalgebras $\mathfrak{k}_\alpha \subset \mathfrak{gl}(\mathcal{H}_\alpha)$. For example consider $\mathfrak{k}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the Lie algebra of skew-Hermitian operators on $\mathcal{H}_\alpha$. The corresponding Lie group is $\mathbf{K}_\alpha = \mathcal{U}(\mathcal{H}_\alpha)$, the group of unitary operators on $\mathcal{H}_\alpha$. If the $\{\mathfrak{k}_\alpha, d\phi_{\beta,\alpha}|_{\mathfrak{k}_\alpha}\}$ and $\{\mathbf{K}_\alpha, \phi_{\beta,\alpha}|_{\mathbf{K}_\alpha}\}$ form directed systems, then the direct limit group $\mathbf{K}$ is a Lie group with Lie algebra, $\mathfrak{k}$, the direct limit Lie algebra in the direct limit and various norm topologies. Again with the locally convex topology, $\mathbf{K}$ has a differentiable structure modeled on $\mathfrak{k}$, but the group operations are only separately continuous. The normed Lie algebras can be completed to yield Banach Lie algebras and Banach Lie groups. In general, $(\mathbf{G}_{\mathrm{BDL}}(\mathcal{H}), \|\cdot\|_p) = \{g \in GL(\mathcal{H}) \mid g$ preserves a nondegenerate form, and $\|g - 1\|_p < \infty\}$.

For the special case that $\mathfrak{k}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the limit $\mathfrak{k}$ is the Lie algebra of skew-Hermitian finite rank operators on $\mathcal{H}$ and $\mathbf{K}$ is the group of unitary operators of the form $1+$(finite rank). The completions of the above with respect to the norm topology yield, respectively, $\mathfrak{k}_{\mathrm{BDL}}$, the Lie algebra of skew-Hermitian compact operators and $\mathbf{K}_{\mathrm{BDL}}$, the Lie group of unitary operators of the form $1 +$ (compact).

EXAMPLE 9. THE SEPARABLE CASE. If $\mathcal{H}$ is separable, the limit Lie algebras and Lie groups of Examples 7 and 8 can be considered as countable direct limits. Indeed, let $\{e_i\}_{i\in\mathbf{N}}$ be a complete orthonormal system for $\mathcal{H}$ and define for each $n \in \mathbf{N}$, $\alpha_n = \{i \in \mathbf{N} \mid i \leqq n\}$. Thus the directed set A has a cofinal subsequence $\{\alpha_n\}_{n\in\mathbf{N}}$ and it follows from Lemma 2.6 (b) that the direct limit over $\{\alpha_n\}_{n\in\mathbf{N}}$ equals the direct limit over A in each of the direct limit topologies considered here. The classical Banach-Lie algebras and groups of operators studied by de la Harpe [3] all fit into this scheme.

REFERENCES

1. G. E. Bredon, *Sheaf theory*, McGraw-Hill, 1967.
2. C. Chevalley, *Theory of Lie groups*. I, Princeton Univ. Press, Princeton, NJ, 1946.
3. P. de la Harpe, *Classical Banach-Lie algebras and Banach-Lie groups of operators on Hilbert space*, Lecture Notes in Math., vol. 285, Springer-Verlag, Berlin and New York, 1972.
4. A. A. Kirillov, *Infinite dimensional Lie groups; their orbits, invariants and representations*, The Geometry of Moments, Lecture Notes in Math., vol. 970, Springer-Verlag, Berlin and New York, 1982, pp. 101–123.
5. V. I. Kolomytsev and Yu. S. Semoilenko, *Irreducible representations of inductive limits of groups*, Ukrainski Matematicheskii Zhurnal **29** (1977), 526–531.
6. G. Köthe, *Topological vector spaces*, **237**, Springer Grundlehren Mathematischen Wissenschaften, 1979.
7. L. Natarajan, E. Rodríguez-Carrington, and J. A. Wolf, *Differentiable structure for direct limit groups*, Letters Math. Physics **23** (1991), 99–109.
8. ——, *Locally convex Lie groups*, Nova J. Algebra and Geometry, to appear in 1993.

9. G. I. Ol'shanskii, *Unitary representations of infinite dimensional pairs* (G, K) *and the formalism of R. Howe*, Representations of Lie Groups and Related Topics (A. Vershik and D. Zhelobenko, eds.), Advanced Studies in Contemporary Mathematics 7 (1990), 269–463, Gordon and Breach.
10. F. Treves, *Topological vector spaces, distributions and kernels*, Pure and Applied Mathematics, vol. 25, Academic Press, N.Y., 1967.
11. A. Wilansky, *Modern methods in topological vector spaces*, McGraw-Hill International Book Co., N.Y., 1978.

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