New Classes of Infinite-Dimensional Lie Groups

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ABSTRACT. We describe some new constructions of infinite-dimensional Lie groups based on direct limits in various categories of linear spaces. In each case the limit group takes its topology and analytic structure from the limit Lie algebra. In all of the situations we investigate, the limit of the Lie algebras is a good topological Lie algebra. We prove that the result is essentially the same for limits of Lie groups, but the situation is much more delicate. In the case of the locally convex direct limit, the limit is a Lie group, except that in some cases the group composition is only separately continuous.

1. Preliminaries

In this note we summarize our work in progress on Lie group structures for various classes of direct limits of finite-dimensional groups. In effect, by taking the direct limit of the Lie algebras in various categories of topological spaces, and using the exponential map, we carry a topology and an analytic structure to the limit group.

The case of the usual direct limit, in the category of topological vector spaces, was published in [7]. The cases of direct limits in the categories of normed linear spaces and of locally convex topological vector spaces will appear in [8].

Fix a directed set $A$. Thus $A$ is a partially ordered set, say with order relation $\leq$, such that if $\alpha, \beta \in A$ then one has $\gamma \in A$ with $\alpha, \beta \leq \gamma$. We consider directed systems

\begin{equation}
\{G_\alpha, \phi_{\beta,\alpha}; V_\alpha, \eta_{\beta,\alpha}; \pi_\alpha\}.
\end{equation}

First, by definition, $\alpha$ and $\beta$ run over $A$, each $G_\alpha$ is a (finite-dimensional real) Lie group, say with real Lie algebra $g_\alpha$, and if $\alpha \leq \beta$ then

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The direct limit vector space $V = \lim DL V$ is defined as above. One has linear transformations $\eta_\beta : V_\beta \to V$ as in (1.3), they define a naive direct limit (DL) topology on $V$, and $V$ together with that topology is a topological vector space $\lim DL V = \lim V_\alpha$.

The direct limit representation $\pi = \lim DL \pi_\alpha$ is the representation of $G$ on $V$ given by $\pi (g_\infty) (v_\infty) = [\pi_\alpha (g_\infty) (v_\infty)]$. Here $\pi$ is well defined and is a continuous representation of $G_{DL}$ on $V_{DL}$. Similarly $d \pi (\xi_\infty) (v_\infty) = [d \pi_\alpha (\xi_\infty) (v_\infty)]$ is a continuous representation of $g_{DL}$ on $V_{DL}$. The series $\exp (d \pi (\xi_\infty)) (v_\infty)$ converges in $V_{DL}$, and

$$\pi (\exp G (\xi_\infty)) (v_\infty) = \exp (d \pi (\xi_\infty)) (v_\infty).$$

The diagrams

$$G_a \times V_a \xrightarrow{s_{\alpha}} V_a$$

(1.5)

$$G_{DL} \times V_{DL} \xrightarrow{\pi} V_{DL}$$

are commutative.

**Example 1. Direct Limits of Classical Algebras.** Let $g_\alpha \subset g l (V_\alpha)$, where $V_\alpha$ is a finite-dimensional vector space. For instance, $g_\alpha$ could be one of the classical algebras $u (n), o (n), or u (q, n)$. Here $d \phi_{n+k, n} : g_\alpha \to g_{n+k}$ is given by $\xi \mapsto (\xi \ 0)$, obtained by adding $V_{n+k} \times V_{n+k}$ zero columns and rows to $\xi$.

$g_{DL} = \lim DL g_\alpha$ is a locally convex Lie algebra since for countable direct limits of topological vector spaces the direct limit and the locally convex direct limits coincide.

It is convenient to work in the case where the maps of the directed system are injective. This implies no loss of generality. In fact, we proved [7, Proposition 3.1] that there is a unique quotient directed system with the same limits and with all maps $\phi_{\beta, \alpha}, \eta_{\beta, \alpha}, \phi_\alpha$, and $\eta_\alpha$ injective. In [7] we denoted this injective quotient system with overlines, but here we just pass to the injective quotient system and drop the extra overline notation.

The spectral growth condition of [7] is

$$\| \lambda \| = \sup \{ | \lambda | : | \lambda |$$

is an eigenvalue of $d \pi (\xi) < \infty$ for all $\xi \in g$.

This condition (1.6) has strong consequences. It forces

$$O = \{ \xi \in g : \| \lambda \| < \infty \}$$

(1.7)

to contain an open neighborhood $O_\infty$ of 0 in $g_{DL}$ [7, Proposition 5.5]. If $d \pi : g \to \text{End} (V)$ is injective, it says that $\exp G : O_\infty \to G$ is injective, and that

$$\exp G : O_\infty \to G$$

is an open subset of $G_{DL}$;

$$\exp G : O_\infty \to U$$

is a homeomorphism; and
2. Convexity and direct limits. The norm- and Banach direct limits

We focus on direct limits of finite-dimensional Lie algebras, but several of the results are valid more generally for direct limits of vector spaces.

Let \{V_\alpha, \psi_{\beta, \alpha}, \alpha, \beta \in A \} be a directed system of topological vector spaces and continuous linear transformations. The directed system is strict if \( V_\alpha \equiv \psi_{\beta, \alpha}(V_\beta) \), topological isomorphism, whenever \( \alpha \leq \beta \). Note that the index set \( A \) may be uncountable.

Let \( V = \lim_{\alpha \in A} V_\alpha \), algebraic direct limit. We will deal with several topologies on \( V \):

1. The (naive) direct limit (DL) topology, described in §1. It is the strongest topology on \( V \) for which the inclusion maps \( \psi_{\alpha, \alpha} : V_\alpha \rightarrow V \) are continuous for all \( \alpha \in A \). A linear transformation \( F : V_{\text{DL}} \rightarrow W \) into any topological vector space \( W \), is continuous if and only if the transformations \( F \cdot \psi_{\alpha, \beta} : V_\alpha \rightarrow W \) are continuous for all \( \alpha \in A \). Such an \( F \) is said to be DL-continuous.

2. Assume that each \( V_\alpha \) is locally convex. The locally convex direct limit (LCLD) topology is defined to be the strongest locally convex topology on \( V \) such that all the \( \psi_{\alpha, \beta} : V_\alpha \rightarrow V_\beta \) are continuous. A convex subset \( U \) of \( V \) is LCLD open if and only if \( \psi_{\alpha, \beta}^{-1}(U) \) is open for all \( \alpha \in A \). A linear transformation \( F : V_{\text{LCLD}} \rightarrow W \) into a locally convex vector space \( W \), is continuous if and only if the \( F \cdot \psi_{\alpha, \beta} : V_\alpha \rightarrow W \) are all continuous. We will then say that \( F \) is LCLD-continuous.

In the theory of linear spaces, the LCLD topology is used more than the DL topology, and is often simply called the direct limit topology.

3. Assume that the \( V_\alpha \) have compatible norms \( \| \cdot \|_\alpha \) in the sense that

\[ \| \psi_{\beta, \alpha}(v) \|_\beta = \| v \|_\alpha \quad \text{for} \quad \alpha \leq \beta \quad \text{and} \quad v \in V_\alpha. \]

The norm direct limit or NDL topology on \( V \) is the topology given by the norm

\[ \| \cdot \| = \lim_{\alpha \in A} \| \cdot \|_\alpha, \]

defined as follows. Let \( v = [v_\alpha] \in V \), and take \( \beta \in A \) large enough so that \( \psi_{\alpha, \beta}(v) \) is not empty. Then

\[ \| v \| = \| \psi_{\alpha, \beta}(v) \|_\beta. \]

It follows from condition (2.1) that this norm on \( V \) is well defined. The normed vector space thus obtained is denoted by \( V_{\text{NDL}} \), or by \( (V, \| \cdot \|) \).

4. The Banach direct limit \( V_{\text{BDL}} \) of \( \{ V_\alpha, \psi_{\beta, \alpha}, \alpha, \beta \in A \} \) is defined to be the completion of \( V \) with respect to the NDL topology.

The topological vector spaces \( V_{\text{DL}}, V_{\text{LCLD}}, V_{\text{NDL}} \) and \( V_{\text{BDL}} \) can be conveniently considered as the limits of the same \( \{ V_\alpha, \psi_{\beta, \alpha}, \alpha, \beta \in A \} \) in different categories, and that is the approach we take in [8].

EXAMPLE 2. The Spaces \( \ell_p \). Let \( A \) be the set of natural numbers, and let \( V_n = \mathbb{R}^n \). Let \( \phi_{m,n} : \mathbb{R}^m \rightarrow \mathbb{R}^n \) be the natural inclusion map.
The direct limit space \( V \), often denoted \( \mathbb{R}^{\infty} \), is the vector space of all sequences, with coefficients in \( \mathbb{R} \), which are eventually zero. Now consider each of the \( \mathbb{R}^n \) as a Banach space for the norm \( \ell_p \), and observe that the compatibility condition (2.1) is satisfied, so that the \( V_{\ell_p} \) makes sense. Note that \( V \) is not \( \ell_p \)-complete. Its completion \( V_{\text{BDL}} \) is the usual space \( \ell_p \).

Example 3. Norm- and Banach Direct Limits of Classical Lie Algebras. We can also take a norm limit of the algebras in Example 1, as follows. Let \( 1 \leq p \leq \infty \) and define \( \| \cdot \|_{\ell_p} \) on \( g_n \) to be the \( \ell_p \) norm, namely
\[
\| \xi \|_{\ell_p} = (\sum |\xi_i|^p)^{1/p} \quad \text{for } 1 \leq p < \infty \quad \text{and} \quad \| \xi \|_{\ell_{\infty}} = \sup_{\| \cdot \|_{\ell_p}} |\xi| \quad \text{for } p = \infty .
\]
Then, for each \( p \), \( \{ \| \cdot \|_{\ell_p} \}_{p \in \mathbb{N}} \) is a compatible family of norms and hence we have a well-defined norm, \( \| \cdot \|_{\ell_p} = \lim \| \cdot \|_{\ell_p} \) on \( g \). We will refer to this situation as \( (g, \| \cdot \|_{\ell_p}) \). Our limit Lie algebras provide more examples of these constructions.

Lemma 2.3. Let \( \{ V_\alpha, \psi_{\beta, \alpha} \}_{\alpha, \beta \in A} \) be a directed system of vector spaces and linear transformations. Then the natural inclusion maps \( V_{\text{DL}} \hookrightarrow V_{\text{LCDL}} \hookrightarrow V_{\text{NDL}} \) are continuous, for the NDL topology given by any family \( \{ \| \cdot \|_n \}_{n \in \mathbb{N}} \) of norms satisfying the compatibility condition (2.1).

We say that the system \( \{ V_\alpha, \psi_{\beta, \alpha} \}_{\alpha, \beta \in A} \) eventually stabilizes if there exists an index \( \alpha_0 \) such that \( \psi_{\beta, \alpha} \) is an isomorphism onto \( V_\beta \) whenever \( \beta \geq \alpha \geq \alpha_0 \). In that case \( V_{\text{DL}} \) and \( V_{\text{LCDL}} \) are both isomorphic to \( V_{\alpha_0} \).

Lemma 2.4. Let \( \{ V_\alpha, \phi_{m, \alpha} \} \) be a countable, strict directed system. If the system never stabilizes, then no locally convex topology makes \( V \) into a Baire space.

Corollary 2.5. No NDL topology on \( V \) is complete, unless the system eventually stabilizes.

Lemma 2.6. Let \( T \) be any one of the topologies DL, LCDL, NDL or BDL. If \( B \subset A \) is a directed set under the partial ordering \( \preceq \) it inherits from \( A \), then
(a) \( \lim_{T, \beta \in B} V_\beta \leftarrow \lim_{T, \alpha \in A} V_\alpha \) is continuous, and
(b) if \( B \) is cofinal in \( A \), then \( \lim_{T, \beta \in B} V_\beta = \lim_{T, \alpha \in A} V_\alpha \).

Several useful properties hold for the case when the index set \( A = \mathbb{N} \) and the directed system is strict. For example, then \( V_{\text{LCDL}} \) coincides with \( V_{\text{DL}} \) and is complete and Hausdorff, and \( V_\alpha \) is isomorphic to its image \( \psi_{\beta, \alpha}(V_\beta) \) where the latter has the topology induced by \( V_{\text{LCDL}} \). See [11]. However, these properties do not hold in general for uncountable directed systems; see [6]. A result of Komura there shows that, even if every \( V_\alpha \) is Hausdorff, \( V_{\text{LCDL}} \) can have the property that no two points are separated.

Of course, if the index set \( A \) is uncountable, but has a cofinal subsequence, then it follows from Lemma 2.6 that \( V_{\text{LCDL}} \) has all of the desirable properties known to hold for countable locally convex direct limits.

Unless we state otherwise, we do not assume that the directed system is countable. On the other hand, for the remainder of this section and most of the rest of this paper, we assume that each of the spaces \( V_\alpha \) is finite dimensional. In many situations, it turns out that this finite-dimensionality makes up for uncountability of the index set.

Recall our assumption that each of the \( \psi_{\beta, \alpha} \) is injective. Together with the finite dimensionality of the \( V_\alpha \), it implies that the direct limit is strict.

We use a Zorn's lemma argument to prove

Proposition 2.7. There exists a compatible system of norms on
\[
\{ V_\alpha, \psi_{\beta, \alpha} \}_{\alpha, \beta \in A}
\]
and hence an NDL topology on \( V \).

In particular the LCDL topology is stronger than a Hausdorff topology, so

Corollary 2.6. The LCDL topology on \( V \) is Hausdorff.

The following corollary is immediate from the Hausdorff property of \( V_{\text{LCDL}} \).

Corollary 2.7. \( V_{\text{LCDL}} \) is bornological and barrelled.

We also have

Proposition 2.10. If the index set \( A \) is linearly ordered, then the locally convex vector space \( V_{\text{LCDL}} \) is sequentially complete.

As indicated above, the LCDL behaves nicely with respect to continuous linear maps between locally convex spaces. The projective tensor product \( V \otimes W \) provides us with a device allowing us to turn questions dealing with continuous bilinear maps into questions about continuous linear transformations. If \( V \) and \( W \) are locally convex spaces, then (see [10, Chapter 43–46]) the projective tensor product topology is the unique locally convex topology on \( V \otimes W \) such that: For every locally convex space \( Z \), the canonical isomorphism of the space of bilinear mappings of \( V \times W \) into \( Z \), onto the space of linear mappings of \( V \otimes W \) into \( Z \), induces an isomorphism of the space of continuous bilinear mappings of \( V \times W \) into \( Z \), onto the space of continuous linear mappings of \( V \otimes W \) into \( Z \).

Let \( \{ V_\alpha, \psi_{\beta, \alpha} \}_{\alpha, \beta \in A} \) and \( \{ W_\alpha, \chi_{\beta, \alpha} \}_{\alpha, \beta \in A} \) be two directed systems of locally convex vector spaces and continuous linear transformations. There is no loss of generality in assuming that \( A = \tilde{A} \), since we could if necessary consider both the given systems as having the index set \( A \times \tilde{A} \) (the
product directed set, which by definition has the lexicographic order). Both \( \lim_{\text{LCDO}} (V_\alpha \otimes \omega \ W_\beta) \) and \( (\lim_{\text{LCDO}} V_\alpha) \otimes (\lim_{\text{LCDO}} W_\beta) \) have as underlying vector space \((\lim_{\text{LCDO}} V_\alpha) \otimes (\lim_{\text{LCDO}} W_\beta)\). It is known that the natural inclusion map \( \lim_{\text{LCDO}} (V_\alpha \otimes \omega W_\beta) \hookrightarrow (\lim_{\text{LCDO}} V_\alpha) \otimes (\lim_{\text{LCDO}} W_\beta) \) is continuous. We show that under our hypotheses, this result can be strengthened.

**Proposition 2.11.** Let \( \{V_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) and \( \{W_\alpha, x_{\beta, \alpha}\}_{\alpha, \beta \in A} \) be two strict directed systems of finite-dimensional vector spaces and continuous linear maps. Then

\[
\lim_{\text{LCDO}} (V_\alpha \otimes \omega W_\beta) = (\lim_{\text{LCDO}} V_\alpha) \otimes (\lim_{\text{LCDO}} W_\beta).
\]

**Direct limits of finite-dimensional Lie algebras.** We now consider a directed system \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) of finite-dimensional Lie algebras \( a_\alpha \) and Lie algebra homomorphisms \( \psi_{\beta, \alpha} \). Let \( g = \lim_{\alpha \in A} a_\alpha \), as described in §1. Since \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) is in particular a directed system of locally convex vector spaces and continuous linear maps, \( g_{\text{DL}}, g_{\text{LCDO}}, g_{\text{NDL}}, \) and \( g_{\text{BDL}} \) all make sense. Each of them has a topological vector space structure, and also a Lie algebra structure. We next investigate the question of when these two structures are compatible.

**Proposition 2.12.** (a) If \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) is a strict directed system of Lie algebras, then the Lie bracket is a continuous map \( g_{\text{DL}} \times g_{\text{BDL}} \to g_{\text{BDL}} \).

(b) Let \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) be a strict directed system of locally convex Lie algebras. If the index set \( A \) is countable, or each \( a_\alpha \) is finite-dimensional, then the Lie bracket is a continuous map \( a_{\text{LCDO}} \times a_{\text{LCDO}} \to a_{\text{LCDO}} \).

To prove statement (a) we need some results on direct limits in the category of topological spaces and continuous maps. These are given in [8]. Statement (b) is a consequence of Proposition 2.11.

Consider the NDL case. Given a compatible family \( \{|| \cdot ||_\alpha\} \) of norms for \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \), and given \( \xi_\alpha \in a_\alpha \), for each index \( \beta \geq \alpha \) we have an operator norm for the adjoint action of \( \psi_{\beta, \alpha} \xi_\alpha \) on \( g_\beta \).

\[
||\text{ad}(\psi_{\beta, \alpha} \xi_\alpha)||_{\beta, \infty} = \sup \left\{ \frac{||[\psi_{\beta, \alpha} \xi_\alpha, \xi_\beta]||_{\beta}}{||\xi_\beta||_{\beta}} \right\}.
\]

For any \( \alpha \leq \beta \leq \gamma \) we then have \( ||\text{ad}(\psi_{\beta, \alpha} \xi_\alpha)||_{\beta, \infty} \leq ||\text{ad}(\psi_{\beta, \gamma} \xi_\gamma)||_{\gamma, \infty} \).

**Proposition 2.13.** Let \( \{a_\alpha, \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) be a directed system of Lie algebras, with a compatible family of norms \( \{|| \cdot ||_\alpha\} \). Suppose that

\[
(2.14) \quad \text{if } \xi = [\xi_\alpha] \in g \text{ and } \alpha \in A \text{ then } \lim_{\beta \geq \alpha} ||\text{ad}(\xi_\alpha)||_{\beta, \infty} < \infty.
\]

Then there exists \( M > 0 \) such that \( ||[\xi, \zeta]|| \leq M \cdot ||\xi|| \cdot ||\zeta|| \) for all \( \xi, \zeta \in g \) where \( || \cdot || = \lim_{\alpha \in A} || \cdot ||_\alpha \). Hence the corresponding \( g_{\text{NDL}} \) and \( g_{\text{BDL}} \) are normed Lie algebras.

When we speak of the limit Lie algebras \( g_{\text{NDL}} \) and \( g_{\text{BDL}} \) we shall always assume that the directed system in question satisfies the hypothesis (2.14) of Proposition 2.13.

3. The basic local coordinate system

Let \( T \) be one of the topologies DL, LCDO, NDL, or BDL. In this section we discuss the question of finding sufficient conditions for the existence of a "good" neighborhood \( \mathcal{O}_I \) of 0 in \( g_T \), i.e., of a T-open neighborhood \( \mathcal{O}_I \) of 0, such that the restriction of the exponential map to \( \mathcal{O}_I \) is one-to-one. This neighborhood is essential for construction of analytic manifold structure on \( G_T \) in §§4 and 5. The set \( U = \exp(\mathcal{O}_I) \) will turn out to be a chart of \( G_T \) at 1, and conditions (1.8) to (1.10) of §1 will hold with \( T \) in place of DL.

We already know that the restriction of \( \exp_G \) to the set \( \mathcal{O}_I \), given by (1.7), is injective. So we need only show that \( \mathcal{O}_I \) contains a T-open neighborhood \( \mathcal{O}_I \) of 0.

The NDL topology for the operator norm. Assume that \( \{d_\alpha \psi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) is a compatible family of representations, and that \( \nu \) is the corresponding operator norm on \( g_\alpha \). Assume that

\[
\limsup \nu(\xi_\alpha) < \infty
\]

for each \( \xi_\alpha \in g_\alpha \). The spectral growth condition follows. We have an operator norm \( \nu = \lim \sup \nu_\alpha \) on \( g_\alpha \). Let \( r \) and \( \mathcal{O}_I \) be as defined in (1.6) and (1.7), using this system \( \{d_\alpha \psi_{\beta, \alpha}\} \). We will now show that \( \mathcal{O}_I \) is a neighborhood of 0 in the NDL Lie algebra \( (g, \nu) \), where \( \nu = \lim \sup \nu_\alpha \).

For each \( \alpha \in A \) we set \( \tau_\alpha(\xi) = \sup \{|\lambda| : \lambda \text{ is an eigenvalue of } d_\alpha \psi_{\beta, \alpha}(\xi)\} \), and \( \mathcal{O}_I = \{\xi \in g_\alpha : \tau_\alpha(\xi) < \pi\} \). Thus \( \mathcal{O}_I \) contains the open neighborhood \( \mathcal{N}_\alpha = \{\xi_\alpha \in g_\alpha : \nu_\alpha(\xi_\alpha) < \pi\} \) and hence it is itself a neighborhood of 0. Since this holds for each index \( \alpha \), and \( \mathcal{O}_I = \psi^{-1}_\alpha(\mathcal{O}_I) \), it follows that \( \mathcal{O}_I \) is a neighborhood of 0 in \( (g, \nu) \). We now take as our \( \mathcal{O}_I = \mathcal{O}_{1, \nu} \) any \( \nu \)-open neighborhood of 0 which is contained in \( \mathcal{O}_I \). We could for example take

\[
(3.1) \quad \mathcal{O}_I = \mathcal{O}_{1, \nu} := \{\xi \in g : \nu(\xi) < \pi\}.
\]

The DL and LCDO topologies. Let \( \mathcal{O}_I \) be as in (3.1), and \( \nu \) be an operator norm, as above. Since the natural inclusion maps \( g_{\text{DL}} \hookrightarrow g_{\text{LCDO}} \hookrightarrow (g, \nu) \) are continuous, \( \mathcal{O}_I \) is an open neighborhood of 0 for both the LCDO and the DL topologies.

The NDL topology for the general case. Let \( \{|| \cdot ||_\alpha\} \) be a compatible family of norms for the directed system and let \( || \cdot || = \lim_{\alpha \in A} || \cdot ||_\alpha \). We prove that the NDL Lie algebra \( (g, || \cdot ||) \) has a neighborhood on which \( \exp_G \) is injective if the norm is "not too different" from an operator norm.

More precisely, let \( \{\nu_\alpha\} \) be a family of operator norms that come from some representation with bounded spectral growth, as in §1. For each \( \alpha \),
let $c_{a}(\nu)$ be a positive number such that $\nu(\xi) \leq c_{a}(\nu) ||x||_{\nu}$ for all $\xi \in \mathfrak{g}$. Assume that $c_{a}(\nu)$ is minimal for this condition. We prove that if

$$\text{(3.2)} \quad \text{the operator norm } \nu = \lim_{\nu_{n}} \text{ satisfies } \lim \sup_{\nu_{n}} c_{a}(\nu) < \infty,$$

then $O_{1,\nu}$ is $||\cdot||$-open. In effect, (3.2) implies $\nu(\xi) \leq \lim \sup_{\nu_{n}} c_{a}(\nu) ||x||$ for all $\xi \in \mathfrak{g}$, so the natural inclusion $(g, ||\cdot||) \mapsto (g, \nu)$ is continuous.

4. The topologies on the limited groups

In this section we indicate how the topologized Lie algebras $\mathfrak{g}_{\text{DL}}, \mathfrak{g}_{\text{LCDL}}, \mathfrak{g}_{\text{NDL}},$ and $\mathfrak{g}_{\text{BDL}}$ define topological structures $\mathfrak{g}_{\text{DL}}, \mathfrak{g}_{\text{LCDL}}, \mathfrak{g}_{\text{NDL}}$, and $\mathfrak{g}_{\text{BDL}}$ on the limited group $G = \lim G_{n}$. Specifically, we indicate how each $G_{n}$ carries the structure of a topological manifold modelled on the corresponding $\mathfrak{g}_{n}$, where $T$ is DL, LCDL, NDL, or BDL. The $C^{\infty}$-differentiable manifold structures are described in §5.

The topologies of $\mathfrak{g}_{\text{DL}}, \mathfrak{g}_{\text{NDL}},$ and $\mathfrak{g}_{\text{BDL}}$ have the property that the group operations are continuous. So each of them is a topological group in the usual sense. For $\mathfrak{g}_{\text{LCDL}}$, we only obtain a group with a topology that makes $x \mapsto x^{-1}$ continuous and $(x, y) \mapsto xy$ separately continuous.

The key to this process is the following result in [2, Chapter II, §II]: A family $\mathcal{V}$ of subsets of a group $G$ is a fundamental system of neighborhoods of $1$ in $G$, for some Hausdorff topology under which the group operations of $G$ are continuous, and only if $\mathcal{V}$ satisfies the five conditions:

1. If $U_{1}, U_{2} \in \mathcal{V}$, then there exists a set $U_{3} \in \mathcal{V}$ such that $U_{1} \subseteq U_{3} \cap U_{2}$.
2. The intersection of all sets of $\mathcal{V}$ is $\{1\}$.
3. If $U \in \mathcal{V}$, there exists a set $U_{1} \in \mathcal{V}$ such that $U_{1}^{-1} \subseteq U$
4. If $U \in \mathcal{V}$ and $g \in G$, there exists a set $U_{1} \in \mathcal{V}$ such that $gU_{1}g^{-1} \subseteq U$
5. If $U \in \mathcal{V}$, there exists a set $U_{1} \in \mathcal{V}$ such that $U_{1}U_{1} \subseteq U$.

Let $O_{1} \subseteq \mathcal{O}$ be an open neighborhood (for topology to be specified) of 0 and set

$$\mathcal{V} = \{ U \in \mathcal{O} \mid U \text{ is T-open} \} \text{ and } \mathcal{V} = \{ U \in G \mid U = \exp(U) \text{ for some } U \in \mathcal{V} \}.$$

The desired properties of $\mathcal{V}$ follow from a list of analogous properties of $\mathcal{V}$. We denote by $H$ the Campbell-Hausdorff-Dynkin series in $\mathfrak{g}$. Consider the five conditions:

1. If $U_{1}, U_{2} \in \mathcal{V}$, then there exists a set $U_{3} \in \mathcal{V}$ such that $U_{3} \subseteq U_{1} \cap U_{2}$.
2. The intersection of all sets of $\mathcal{V}$ is $\{0\}$.
3. If $U \in \mathcal{V}$, there exists a set $U_{1} \in \mathcal{V}$ such that $-U_{1} \subseteq U$
4. If $U \in \mathcal{V}$ and $\xi \in \mathfrak{g}$, there exists a set $U_{1} \in \mathcal{V}$ such that

$$H(\xi, H(U_{1}, -\xi)) \subseteq U.$$
assume (3.2). Then $G_{BDL}$ is a topological Lie group and the restriction to $O_1$ of the exponential map $\exp : \mathcal{G}_{BDL} \to G_{BDL}$ is a homeomorphism onto an open set.

5. Structure sheaves

In this section we carry the sheaves $C^\omega(G_{DL})$ and $C^\omega(\mathcal{G}_{DL})$ of germs of analytic functions for the DL topology to corresponding sheaves $C^\omega(G_{NDL})$ and $C^\omega(\mathcal{G}_{NDL})$ for the NDL topologies, to $C^\omega(G_{BDL})$ and $C^\omega(\mathcal{G}_{BDL})$ for the BDL topologies, and to $C^\omega(G_{LCLD})$ and $C^\omega(\mathcal{G}_{LCLD})$ for the LCLD topology.

Recall the standard construction [1, p. 9] of direct image sheaves. Let $X$ and $Y$ be topological spaces and $\psi : X \to Y$ a continuous map. If $F \to X$ is a sheaf one has a presheaf over $Y$, which assigns to an open set $W \subset Y$ the abelian group of all sections of $F$ over $\psi^{-1}(W)$. This presheaf is complete. That defines the direct image sheaf $\psi_* F \to Y$. The assignment $\psi_*$ is a left exact covariant functor.

The natural maps $G_{DL} \to G_{LCLD} \to G_{NDL}$, by $x \mapsto x$, and $\mathcal{G}_{DL} \to \mathcal{G}_{LCLD} \to \mathcal{G}_{NDL}$ by $\xi \mapsto \xi$, are continuous. The sections of the direct image sheaves $\mathcal{G}_{NDL} \to \mathcal{G}_{LCLD} \to \mathcal{G}_{NDL}$, $C^\omega(G_{NDL}) \to C^\omega(G_{LCLD}) \to C^\omega(G_{NDL})$, and $C^\omega(\mathcal{G}_{LCLD}) \to \mathcal{G}_{LCLD}$ are just those sections of the corresponding DL sheaf whose domains are NDL- or LCLD-open. So they are the analytic function germ sheaves where we define

DEFINITION 5.1. Let $\mathcal{W}$ be an open set in $G_{NDL}$, $\mathcal{G}_{NDL}$, $G_{LCLD}$, or $\mathcal{G}_{LCLD}$, respectively. Then a function $f : \mathcal{W} \to C$ is real analytic if (a) $f$ is continuous and (b) $f$ is DL-analytic.

The natural maps $G_{NDL} \to G_{BDL}$ and $\mathcal{G}_{NDL} \to \mathcal{G}_{BDL}$ are continuous. The sections of the direct image sheaves $C^\omega(G_{BDL}) \to G_{BDL}$ and $C^\omega(\mathcal{G}_{BDL}) \to \mathcal{G}_{BDL}$ are just those continuous functions whose restrictions to $G_{NDL}$ or $\mathcal{G}_{NDL}$ are sections of the corresponding NDL sheaf. So they are the analytic function germ sheaves where we define

DEFINITION 5.2. Let $\mathcal{W}$ be an open set in $G_{BDL}$ or $\mathcal{G}_{BDL}$, respectively. Then a function $f : \mathcal{W} \to C$ is real analytic if (a) $f$ is continuous and (b) the restriction of $f$ to the dense subset $\mathcal{W} \cap G_{NDL}$, respectively $\mathcal{W} \cap \mathcal{G}_{NDL}$, is NDL-analytic.

6. Conclusion and examples

We now combine the material of §§4 and 5. Let $T$ be one of the topologies DL, LCLD, NDL or BDL. In the DL case, assume the spectral growth condition (1.6). In the LCLD, NDL and BDL cases assume the stronger conditions (2.14) and (3.2). In a group, $L_x$ denotes the left translation $y \mapsto xy$. Our results are summarized in

THEOREM 6.1. The real analytic structures on $G_{\lim} \mathcal{G}_{\lim}$, corresponding to the topologies $T = DL$, LCLD or NDL, respectively, define structures $G_T$ of $C^\omega$ differentiable manifold on $G = \lim G_n$ based on the topological vector space $g_T$. A $C^\omega$ local coordinate cover on $G_T$, corresponding to the topology $T$, is given by the $(\exp|_\xi)_{\xi}^{-1} \cdot L_{\xi} : g U_1 \to O_1 \subset g_T$; and $C^\omega(G_T)$ is the sheaf of germs of $C^\omega$ functions on $G_{\lim}$ whose domains are open in $G_T$. $G_{DL}$ and $G_{NL}$ are $C^\omega$ Lie groups, as is the completion $G_{BDL}$ of $G_{BDL}$. $G_{LCLD}$ is essentially a $C^\omega$ Lie group, except that group multiplication may be only separately analytic.

EXAMPLE 4. DIRECT LIMITS OF CLASSICAL LIE GROUPS. Let $G_n$ be their respective Lie groups. For the instances mentioned explicitly in Example 1, the Lie groups are $G_n = U(n)$, $O(n)$ and $U(q, n)$ respectively. The embedding $\phi_{n \to \infty} \phi_n$ is given by $\xi \mapsto (\xi, 0)$ where $I$ is the identity matrix of size $\dim V_n \times \dim V_{n}$. The group $\mathcal{G}_{DL} = \mathcal{G}_{LCLD} = \lim \mathcal{G}_{NDL} G_n$ is now seen to be a locally convex Lie group modelled on its Lie algebra $\lim \mathcal{L}_{NDL} G_n$.

EXAMPLE 5. NORM- AND BANACH DIRECT LIMITS OF CLASSICAL LIE GROUPS. Using Example 3, each sequences of classical Lie groups in Example 4, and for each $p$ with $1 \leq p \leq \infty$, gives group $G = \lim G_n$. That group $G$ has the structure of normed Lie group $\mathcal{G}_{NDL}$ modelled on the norm direct limit Lie algebra $(g, \| \cdot \|_\nu)$. The compatibility condition (2.1) is satisfied in these cases, so each of the normed Lie groups $\mathcal{G}_{NDL}$ can be completed to a Banach Lie group $\mathcal{G}_{BDL}$ modelled on $\mathcal{G}_{BDL}$.

The groups of Examples 4 and 5 have been studied as topological groups by Kolomietsev, Semenko, Ofshanski and others. For example see [4, 5, and 9]. Here [5] contains some bibliography on the subject.

EXAMPLE 6. $C^\omega$ FUNCTIONS. Let $\Omega$ be a separable $C^\omega$ manifold, e.g. an open subset of $R^n$, $G$ a finite-dimensional Lie group with Lie algebra $g$. Then $C^\omega(\Omega, g)$ and $C^\omega(\Omega, G)$ are a topological Lie algebra and group respectively, with the topology of uniform convergence of the functions and their derivatives on compact sets. Here the algebra and group operations are specified pointwise. It is standard that $C^\omega(\Omega, g)$ is complete with respect to this topology.

For $K \subset \Omega$, $K$ compact, define $C^\omega_K(\Omega, g) = \{ f \in C^\omega(\Omega, g) \mid \text{supp}(f) \subset K \}$ where $\text{supp}(f)$ is the support of the function $f$. Define $C^\omega_{\Omega}(\Omega, g) = \bigcup K \text{ compact } C^\omega_K(\Omega, g)$. As $\Omega$ is locally compact with a countable basis for open sets, we have a sequence $B_1 \subset B_2 \subset \cdots$ of open sets with $\Omega = \bigcup B_i$ and each $K_i$ is closure $B_i$ compact. Thus, $C^\omega_{\Omega}(\Omega, g) = \bigcup K_i C^\omega_{K_i}(\Omega, g)$. Give $C^\omega_{\Omega}(\Omega, g)$ the subspace topology inherited from $C^\omega(\Omega, g)$. Then the inclusion map $C^\omega_{\Omega}(\Omega, g) \to C^\omega_{\Omega}(\Omega, g)$ is an isomorphism onto its image. Thus, we can view $C^\omega_{\Omega}(\Omega, g)$ as the strict countable direct limit, thus the locally convex direct limit, of the $C^\omega_{\Omega}(\Omega, g)$. The sheaf of analytic functions on $C^\omega_{\Omega}(\Omega, g)$ is the direct limit sheaf. Note that this topology on $C^\omega_{\Omega}(\Omega, g)$
is in general strictly finer than the topology of uniform convergence on compact sets.

Let $C^\infty_c(\Omega, G)$ be $C^\infty$ functions on $\Omega$ with values in $G$, whose support lies in $K$. Then, $C^\infty_c(\Omega, G)$ is a Lie group with Lie algebra $C^\infty_c(K, g)$. We can now define a differentiable structure on the direct limit group, $C^\infty_c(\Omega, G) = \bigcup C^\infty_c(\Omega, G)$, modelled on $C^\infty_c(\Omega, G)$. Let $O$ and $U$ be neighborhoods of 0 and 1 respectively in $g$ and $G$, on which the exponential map is a diffeomorphism. Then, the exponential map from $
abla = \{ f \in C^\infty_c(\Omega, G) \mid f \text{ has compact support and } f(\Omega) \subset O \}$

to

$
\nabla = \{ f \in C^\infty_c(\Omega, G) \mid f \text{ has compact support and } f(\Omega) \subset U \}
$

is a homeomorphism. The sheaf of analytic functions on $C^\infty_c(\Omega, G)$ is defined as the direct limit sheaf. The existence of local sections is ensured by the fact that the topology on $C^\infty_c(\Omega, G)$ is given locally by the Lie algebra $C^\infty_c(\Omega, g)$.

**Example 7. Lie Groups and Lie Algebras of Operators on a Hilbert Space.** Let $\mathcal{H}$ be a Hilbert space, not necessarily separable. Let $\{e_i\}_{i \in I}$ be a complete orthonormal set in $\mathcal{H}$. Our indexing set in this example will be the directed set $A$ of all finite subsets of $I$, with the partial order $\alpha \leq \beta \iff \alpha \subseteq \beta$.

For each $\alpha \in A$, denote by $\mathcal{H}_\alpha$ the finite-dimensional subspace of $\mathcal{H}$ with basis $\{e_i\}_{i \in \alpha}$.

Let $GL(\mathcal{H}_\alpha)$ be the group of invertible linear operators on $\mathcal{H}_\alpha$. If $\alpha \leq \beta$ in $A$, then $\phi_{\beta, \alpha} : GL(\mathcal{H}_\beta) \to GL(\mathcal{H}_\alpha)$ is the natural inclusion map which identifies $GL(\mathcal{H}_\beta)$ with the subgroup $\{g \in GL(\mathcal{H}_\beta) \mid g(e_i) = e_i \text{ whenever } i \notin \alpha \}$. Thus the directed system $\{GL(\mathcal{H}_\alpha), \phi_{\beta, \alpha}\}_{\alpha \in A}$ is strict.

The inclusion map $\phi_{\alpha} : GL(\mathcal{H}_\alpha) \to GL(\mathcal{H}) := \lim_{\alpha} GL(\mathcal{H}_\alpha)$ is an isomorphism of $GL(\mathcal{H}_\alpha)$ onto its image, which is the subgroup $\{g \in GL(\mathcal{H}) \mid g(e_i) = e_i \text{ for } i \notin \alpha \}$. The limit $\mathcal{G}(\mathcal{H})$ is a subgroup of the group $GL(\mathcal{H})$ of all bounded invertible linear operators $\mathcal{H}$. It consists of those operators which have the form $1+\mathcal{H}$.

The choice of embedding makes it clear that the spectral growth condition is satisfied. Hence $\mathcal{G}(\mathcal{H})$ with the direct limit topology becomes a Lie group, $\mathcal{G}_{DL}(\mathcal{H})$ with Lie algebra $\mathfrak{g}_{DL}(\mathcal{H}) := \lim_{DL} g(\mathcal{H}_\alpha)$. We can also repoliterate $\mathcal{G}(\mathcal{H})$ using the locally convex Lie algebra $\mathfrak{g}_{LCDL}(\mathcal{H})$. Then $\mathcal{G}_{LCDL}(\mathcal{H})$ is a separately continuous locally convex Lie group modelled on its Lie algebra $\mathfrak{g}_{LCDL}(\mathcal{H})$.

For $1 \leq p \leq \infty$, we have algebras $(\mathfrak{g}(\mathcal{H}), \| \cdot \|_p)$ as in Examples 2 and 5. Each of them gives us a normed Lie group $(\mathcal{G}_{BDL}(\mathcal{H}), \| \cdot \|_p)$. Their completions give us Banach Lie groups $(\mathcal{G}_{BDL}(\mathcal{H}), \| \cdot \|_p)$ with Banach Lie algebras $\mathfrak{g}_{BDL}(\mathcal{H})$.

We have as a special case using the uniform operator norm on $\mathfrak{g}(\mathcal{H})$ that the completion is the algebra of compact operators and the completed group is the group of bounded operators of the form $1+(\mathcal{H})$.

**Example 8. Subgroups and Subalgebras of Spaces of Bounded Linear Operators.** We retain the notation of the previous example. We look at certain subalgebras $\mathfrak{t}_\alpha \subset \mathfrak{g}(\mathcal{H}_\alpha)$. For example consider $\mathfrak{t}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the Lie algebra of skew-Hermitian operators on $\mathcal{H}_\alpha$. The corresponding Lie group is $K_\alpha = GL(\mathcal{H}_\alpha)$, the group of unitary operators on $\mathcal{H}_\alpha$. If the $\{t_\alpha, d\phi_{\beta, \alpha} \}$ and $\{K_\alpha, \phi_{\beta, \alpha} \}$ form directed systems, then the direct limit group $K$ is a Lie group with algebra $t$, the direct limit Lie algebra in the direct limit and various norm topologies. Again with the locally convex topology, $K$ has a differentiable structure modeled on $t$, but the group operations are only separately continuous. The normed Lie algebras can be completed to yield Banach Lie algebras and Banach Lie groups. In general, $(\mathcal{G}_{BDL}(\mathcal{H}), \| \cdot \|_p) = \{g \in GL(\mathcal{H}) \mid g \text{ preserves a nondegenerate form, and } \|g-I\|_p < \infty\}$.

For the special case that $\mathfrak{t}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the limit $t$ is the Lie algebra of skew-Hermitian finite rank operators on $\mathcal{H}$ and $K$ is the group of unitary operators of the form $1+(\text{finite rank})$. The completions of the above with respect to the norm topology yield, respectively, $\mathcal{G}_{BDL}(\mathcal{H})$, the Lie algebra of skew-Hermitian compact operators and $\mathcal{K}_{BDL}$, the Lie group of unitary operators of the form $1+(\text{compact})$.

**Example 9. The Separable Case.** If $\mathcal{H}$ is separable, the limit Lie algebras and Lie groups of Examples 7 and 8 can be considered as countable direct limits. Indeed, let $\{e_i\}_{i \in \mathbb{N}}$ be a complete orthonormal system for $\mathcal{H}$ and define for each $n \in \mathbb{N}$, $\alpha_n = \{i \in \mathbb{N} \mid i \leq n \}$. Thus the directed set $A$ has a cofinal subsequence $\{\alpha_n\}_{n \in \mathbb{N}}$ and it follows from Lemma 2.6 (b) that the direct limit over $\{\alpha_n\}_{n \in \mathbb{N}}$ equals the direct limit over $A$ in each of the direct limit topologies considered here. The classical Banach-Lie algebras and groups of operators studied by de la Harpe [3] all fit into this scheme.

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