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LOCALLY CONVEX LIE GROUPS

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ABSTRACT. We describe some new constructions of infinite dimensional Lie groups based on direct limits in various categories of linear spaces. In each case the limit group takes its topology and analytic structure from the limit Lie algebra. In all of the situations we investigate, the limit of the Lie algebras is a good topological Lie algebra. We prove that the result is essentially the same for limits of Lie groups, but there the situation is much more delicate. For the unrestricted locally convex direct limit, the limit group is a Lie group whose group composition may be only separately continuous.

SECTION 1. PRELIMINARIES.

In this paper we construct analytic structures on various classes of direct limits of finite dimensional Lie groups. In effect, by taking the direct limit of the Lie algebras in various categories of topological vector spaces, and using the exponential map, we carry a topology and an analytic structure to the limit group. We published the case of the usual direct limit, in the category of all Hausdorff topological vector spaces, in [9]. The new results here deal with direct limits in several categories of locally convex topological vector spaces. Some of these results were announced in [10].

Essentially every Lie group G considered in this paper derives its topological and analytic structure in a neighborhood U of the identity from the Lie algebra $\mathfrak{g}$ by means of the exponential map $\exp : \mathfrak{g} \to G$, and then globally on G by means of group translations. Of course this requires each left translation $\ell(x) : \ell(x)^{-1}U \cap U \to U \cap \ell(x)U$ to be analytic, as well as a corresponding condition for the right translations. Thus, as differentiable manifold, G is modelled on the topological vector space $\mathfrak{g}$. Our primary interest in this paper is the case where $\mathfrak{g}$ is a locally convex Hausdorff topological vector space. We will say *locally convex Lie group* for a Lie group modelled on a locally convex Hausdorff topological vector space by means of the exponential map and group translations, as described above. This is formalized in Definition 8.2 below.

Fix a directed set A. Thus A is a partially ordered set, say with order relation $\leqq$, such that if α, $\beta \in A$ then one has $\gamma \in A$ with $\alpha, \beta \leqq \gamma$. We consider directed systems

$$\{G_\alpha,\ \phi_{\beta,\alpha}\ ;\quad V_\alpha,\ \eta_{\beta,\alpha}\ ;\quad \pi_\alpha\}. \tag{1.1}$$

First, by definition, α and β run over A, each G_α is a (finite dimensional real) Lie group, say with real Lie algebra $\mathfrak{g}_\alpha$, and if $\alpha \leqq \beta$ then $\phi_{\beta,\alpha} : G_\alpha \to G_\beta$ is an analytic homomorphism. We require the

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standard $\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \cdot \phi_{\beta,\alpha}$ for $\alpha \leqq \beta \leqq \gamma$ and $\phi_{\alpha,\alpha} = \text{ident}_{G_\alpha}$ for all α. Second, each of the V_α is a finite dimensional complex vector space, and if $\alpha \leqq \beta$ then $\eta_{\beta,\alpha} : V_\alpha \to V_\beta$ is a linear transformation. As above we require the standard $\eta_{\gamma,\alpha} = \eta_{\gamma,\beta} \cdot \eta_{\beta,\alpha}$ for $\alpha \leqq \beta \leqq \gamma$ and $\eta_{\alpha,\alpha} = \text{ident}_{V_\alpha}$ for all α. Third, π_α is a continuous representation of G_α on V_α, and one has the consistency condition that for $\alpha \leqq \beta$ the left hand diagram diagram of

$$\begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ {\scriptstyle \phi_{\beta,\alpha}}\downarrow{\scriptstyle \eta_{\beta,\alpha}} & & \downarrow{\scriptstyle \eta_{\beta,\alpha}} \\ G_\beta \times V_\beta & \xrightarrow{\pi_\beta} & V_\beta \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{d\pi_\alpha} & V_\alpha \\ {\scriptstyle d\phi_{\beta,\alpha}}\downarrow{\scriptstyle \eta_{\beta,\alpha}} & & \downarrow{\scriptstyle \eta_{\beta,\alpha}} \\ \mathfrak{g}_\beta \times V_\beta & \xrightarrow{d\pi_\beta} & V_\beta \end{array} \tag{1.2}$$

is commutative. If $\alpha \leqq \beta$ then $\phi_{\beta,\alpha}$ defines a Lie algebra homomorphism $d\phi_{\beta,\alpha} : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$. The Lie algebra representations $d\pi_\alpha$ satisfy the consistency condition that comes out of the condition for the π_α. So the right hand diagram of (1.2) is commutative.

The *direct limit* or *injective limit* group $G = \varinjlim G_\alpha$ consists of the equivalence classes $[g_\alpha]$ of sets $\{g_\alpha\}_{\alpha \in A}$ where each $g_\alpha \in G_\alpha$ and, for some $\delta \in A$, $\delta \leqq \alpha \leqq \beta$ in A implies $g_\beta = \phi_{\beta,\alpha}(g_\alpha)$. The equivalence relation is such that $[g_\alpha]$ is determined by the eventual behavior of $\{g_\alpha\}$: $\{g_\alpha\} \sim \{'g_\alpha\}$ when, for some $\delta \in A$, if $\delta \leqq \gamma$ then $g_\gamma = {}'g_\gamma$. G is a group with the operations $[g_\alpha] \cdot ['g_\alpha] = [h_\alpha]$ where each $h_\alpha = g_\alpha \cdot {}'g_\alpha$ and $[g_\alpha]^{-1} = [g_\alpha^{-1}]$. We have *canonical homomorphisms*

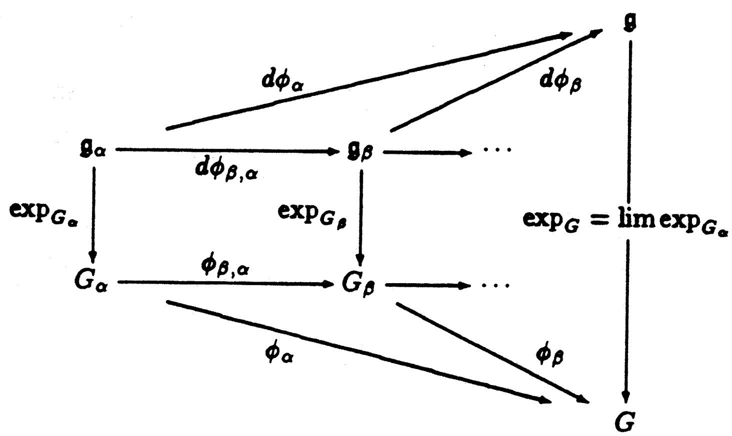
$$\phi_\beta : G_\beta \to G \text{ by } \phi_\beta(x) = [g_\gamma] \text{ where } g_\gamma = \phi_{\gamma,\beta}(x) \text{ for all } \beta \leqq \gamma. \tag{1.3}$$

Those homomorphisms define the naïve *direct limit* (*DL*) topology on G: A subset $U \subset G$ is *DL*–open in G if and only if $\phi_\beta^{-1}(U)$ is open in G_β for every $\beta \in A$. $G_{DL} = \varinjlim_{DL} G_\alpha$ consists of G with this *DL* topology. G_{DL} is a (Hausdorff) topological group. See the Appendix. To be more precise (if more pedantic) one can write $(G, \{\phi_\alpha\}_{\alpha \in A}) = \varinjlim \{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ since the direct limit is in fact dependent on the homomorphisms $\phi_{\beta,\alpha}$, and not on the groups G_α alone. This dependence is illustrated by Examples 4 and 5 in Section 10. It occurs also in the direct limits of vector spaces and of Lie algebras, as described below. However, we will seldom need to use this heavier notation.

Similarly we have the the *direct limit* Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ and Lie algebra homomorphisms $d\phi_\beta : \mathfrak{g}_\beta \to \mathfrak{g}$ defining a naïve direct limit (*DL*) topology on $\mathfrak{g}$. So $\mathfrak{g}_{DL} = \varinjlim_{DL} \mathfrak{g}_\alpha$ consists of $\mathfrak{g}$ with this *DL* topology; it is a topological Lie algebra, and the exponential map

$$\exp_G : \mathfrak{g}_{DL} \to G_{DL} \text{ defined by } \exp_G([\xi_\alpha]) = [\exp_{G_\alpha}(\xi_\alpha)] \tag{1.4}$$

is well defined and continuous. In the terminology of Definition (A.4) of the Appendix, $\exp_G$ is the unique continuous map $\varinjlim \exp_{G_\alpha}$ which makes the following diagram commutative.



The *direct limit* vector space $V = \varinjlim V_\alpha$ is defined as above. One has linear transformations $\eta_\beta : V_\beta \to V$ as in (1.3), they define a naïve direct limit (*DL*) topology on V, and V together with that topology is a topological vector space $V_{DL} = \varinjlim_{DL} V_\alpha$.

The *direct limit* representation $\pi = \varinjlim \pi_\alpha$ is the representation of G on V given by $\pi([g_\alpha])([v_\alpha]) = [\pi_\alpha(g_\alpha)(v_\alpha)]$. It is well defined and is a continuous representation of G_{DL} on V_{DL}. Similarly $d\pi([\xi_\alpha])([v_\alpha]) = [d\pi_\alpha(\xi_\alpha)(v_\alpha)]$ is a continuous representation of $\mathfrak{g}_{DL}$ on V_{DL}, the usual exponential series $\exp(d\pi([\xi_\alpha]))([v_\alpha])$ converges in V_{DL}, and $\pi(\exp_G([\xi_\alpha]))([v_\alpha]) = \exp(d\pi([\xi_\alpha]))([v_\alpha])$. The diagrams

$$\begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ \phi_\alpha \big\downarrow \eta_\alpha & & \big\downarrow \eta_\alpha \\ G_{DL} \times V_{DL} & \xrightarrow{\pi} & V_{DL} \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{d\pi_\alpha} & V_\alpha \\ d\phi_\alpha \big\downarrow \eta_\alpha & & \big\downarrow \eta_\alpha \\ \mathfrak{g}_{DL} \times V_{DL} & \xrightarrow{d\pi} & V_{DL} \end{array} \tag{1.5}$$

are commutative.

Example 1. Direct limits of classical algebras. Let $\mathfrak{g}_n \subset \mathfrak{gl}(V_n)$, where V_n is a finite-dimensional vector space. For instance, $\mathfrak{g}_n$ could be one of the classical algebras $\mathfrak{u}(n), \mathfrak{o}(n)$, or $\mathfrak{u}(q,n)$. Here $d\phi_{n+k,n} : \mathfrak{g}_n \to \mathfrak{g}_{n+k}$ is given by $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}$, obtained by adding $\dim V_{n+k} - \dim V_n$ zero columns and rows to ξ.

$\mathfrak{g}_{DL} = \varinjlim_{DL} \mathfrak{g}_n$ is a locally convex Lie algebra since (see [8], [12] or [15]) for countable direct limits of topological vector spaces the direct limit and the locally convex direct limits coincide.

It is convenient to work in the case where the maps of the directed system are injective. This implies no loss of generality. In fact, we proved [9, Proposition 3.1] that there is a unique quotient directed system with the same limits and with all maps $\phi_{\beta,\alpha}$, $\eta_{\beta,\alpha}$, ϕ_α, and η_α injective. In [9] we denoted this injective quotient system with overlines, but here we just pass to the injective quotient system and drop the extra overline notation.

The spectral growth condition of [9] is

$$\iota(\xi) = \sup\{|\mathrm{Im}\,\lambda| \mid \lambda \text{ is an eigenvalue of } d\pi(\xi)\} < \infty \text{ for all } \xi \in \mathfrak{g}\,. \tag{1.6}$$

This condition (1.6) has strong consequences. It forces

$$\mathcal{O} = \{\xi \in \mathfrak{g} \mid \iota(\xi) < \pi\} \tag{1.7}$$

to contain an open neighborhood $\mathcal{O}_1$ of 0 in $\mathfrak{g}_{DL}$ [9, Proposition 5.5]. If $d\pi : \mathfrak{g} \to \mathrm{End}(V)$ is injective, it says that $\exp_G : \mathcal{O}_1 \to G$ is injective, and that

$$U = U_{DL} = \exp_G(\mathcal{O}_1) \text{ is an open subset of } G_{DL}; \tag{1.8}$$

$$\exp_G : \mathcal{O}_1 \to U \text{ is a homeomorphism; and} \tag{1.9}$$

$$\exp_{G_\alpha} : d\phi_\alpha^{-1}(\mathcal{O}_1) \to \phi_\alpha^{-1}(U) \text{ is a diffeomorphism for all } \alpha \in A. \tag{1.10}$$

Write $C^\omega(\cdot)$ for the sheaf of germs of real analytic functions on a finite dimensional real analytic manifold. The direct limit sheaf is $C^\omega(G_{DL}) = \varinjlim_{DL} C^\omega(G_\alpha)$. If S is open in G_{DL}, then $f : S \to \mathbb{C}$ is a section of $C^\omega(G_{DL})$ over S just when each $f \cdot \phi_\alpha : \phi_\alpha^{-1}(S) \to \mathbb{C}$ is C^ω. That specifies the presheaf and thus specifies $C^\omega(G_{DL})$. Since the ingredients of this construction are invariant under group operations, the sheaf $C^\omega(G_{DL})$ is invariant under left and right group translations and group inversion.

The sheaf $C^\omega(\mathfrak{g}_{DL})$ is defined similarly. It also has a standard definition in the context of linear spaces. If B is open in $\mathfrak{g}_{DL}$, then $f : B \to \mathbb{C}$ is analytic just when $f \cdot \ell$ is analytic in the usual sense for every affine $\ell : \mathbb{R} \to \mathfrak{g}$. The two definitions are equivalent because ℓ must have image in one of the finite dimensional spaces $\mathfrak{g}_\alpha$.

The main result of [9] says that, given the spectral growth condition, G_{DL} has a C^ω Lie group structure modelled on the topological vector space $\mathfrak{g}_{DL}$, such that

$$\text{the } (\exp_G|_{\mathcal{O}_1})^{-1} \cdot \ell(g^{-1}) : gU \to \mathcal{O}_1 \hookrightarrow \mathfrak{g}_{DL} \text{ form a local coordinate cover on } G_{DL} \tag{1.11}$$

and

$$C^\omega(G_{DL}) \text{ is the sheaf of germs of } C^\omega \text{ functions on } G_{DL}\,. \tag{1.12}$$

Furthermore, $\exp_G : \mathcal{O}_1 \to U$ is an analytic diffeomorphism and the $\phi_\alpha : G_\alpha \to G_{DL}$ are analytic.

We are going to consider several sorts of locally convex direct limits of directed systems of Lie algebras. First we study the usual locally convex direct limit; it is the direct limit in the category of locally convex topological vector spaces. Second, we study the norm direct limit, direct limit in the category of normed linear spaces, and its completion, the Banach direct limit. Third, we study a direct limit based on seminorms that respect the Lie algebra structure, the multiplicative direct limit. These concepts are defined below. We carry the resulting structure on the limit algebras to a topology and an analytic structure on the limit group, as indicated above. Thus we obtain results similar to the above–described results of [9] for each of these types of locally convex direct limits.

For each $\alpha \in A$, let ν_α be the operator norm defined on $\mathfrak{g}_\alpha$ by the representation $d\pi_\alpha$ on V_α. The family $\{\nu_\alpha\}_{\alpha \in A}$ is said to be of *bounded growth* if for each $\xi = [\xi_\alpha] \in \mathfrak{g}$, there exists an index $\alpha \in A$ such that $\limsup\{\nu_\beta(\phi_{\beta,\alpha}\,\xi_\alpha) \mid \alpha \leqq \beta\} < \infty$.

In order to construct the groups associated to our various types of limit Lie algebras, we need to assume that the operator norms ν_α are of bounded growth in the sense just specified. This condition implies the spectral growth condition (1.6). It is a condition which is automatically satisfied by the examples that occur most naturally.

In the case of the norm direct limit, we also need to assume that the norm we are dealing with be "not too different" from an operator norm of bounded growth. This will be made precise as condition (5.2) below.

SECTION 2. DIRECT LIMIT TOPOLOGIES.

We focus on direct limits of finite dimensional Lie algebras, but several of the results are valid more generally for direct limits of vector spaces.

Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a directed system of topological vector spaces and continuous linear transformations. The directed system is *strict* if $V_\alpha \cong \psi_{\beta,\alpha}(V_\beta)$, topological isomorphism, whenever $\alpha \leqq \beta$. Note that the index set A may be uncountable.

Let $V = \varinjlim_{\alpha \in A} V_\alpha$, algebraic direct limit. We will deal with several topologies on V:

1. The (naïve) *direct limit* (*DL*) topology, described in §1. It is the strongest topology on V for which the inclusion maps $\psi_\alpha : V_\alpha \to V$ are continuous for all $\alpha \in A$. A linear transformation

$F : V_{DL} \to W$ into any topological vector space W, is continuous if and only if the transformations $F \cdot \psi_\alpha : V_\alpha \to W$ are continuous for all $\alpha \in A$. Such an F is said to be *DL-continuous.*

2. Assume that each V_α is locally convex. The *locally convex direct limit* (*LCDL*) topology is defined to be the strongest locally convex topology on V such that all the $\psi_\alpha : V_\alpha \to V$ are continuous. A *convex* subset U of V is *LCDL* open if and only if $\psi_\alpha^{-1}(U)$ is open for all $\alpha \in A$. A linear transformation $F : V_{LCDL} \to W$ into a locally convex vector space W, is continuous if and only if all the $F \cdot \psi_\alpha : V_\alpha \to W$ are all continuous. We will then say that F is *LCDL-continuous.*

In the theory of linear spaces, the *LCDL* topology is used more than the *DL* topology, and is often simply called the direct limit topology.

3. One can of course define a topology using any set of continuous seminorms on V that separates points. This will be especially useful in the case where those seminorms are consistent in a way with a Lie algebra composition. When the V_α are Lie algebras and the $\psi_{\beta,\alpha}$ are Lie algebra homomorphisms, then V inherits a Lie algebra structure. In this case we say that a seminorm ν on V is *multiplicative* if there is a constant $M > 0$ such that

$$\nu([\xi,\eta]) \leqq M\nu(\xi)\nu(\eta) \text{ for all } \xi,\eta \in V \tag{2.1}$$

Given a set $\mathcal{S}$ of seminorms on V that separate points and satisfy (2.1) the resulting topological structure is a *multiplicative direct limit* topology, denoted V_{MDL}. If confusion threatens we will write $V_{MDL,\mathcal{S}}$.

4. Assume that the V_α have *compatible* norms $||\cdot||_\alpha$ in the sense that

$$||\psi_{\beta,\alpha}(v)||_\beta = ||v||_\alpha \qquad \text{for } \alpha \leqq \beta \text{ and } \quad v \in V_\alpha\,. \tag{2.2}$$

The *norm direct limit* or *NDL* topology on V is the topology given by the norm $||\cdot|| = \varinjlim ||\cdot||_\alpha$, defined as follows. Let $v = [v_\alpha] \in V$, and take $\beta \in A$ large enough so that $\psi_\beta^{-1}(v)$ is not empty. Then

$$||v|| = ||\psi_\beta^{-1}(v)||_\beta \tag{2.3}$$

It follows from condition (2.2) that this norm on V is well-defined. The normed vector space thus obtained is denoted by V_{NDL}, or by $(V, ||\cdot||)$.

5. The *Banach direct limit* V_{BDL} of $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is defined to be the completion of V with respect to the *NDL* topology.

The topological vector spaces V_{DL}, V_{LCDL}, V_{MDL}, V_{NDL} and V_{BDL} can be conveniently considered as the limits of the same $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in different categories. We discuss this in the Appendix.

Example 2. The spaces ℓ_p. Let A be the set of natural numbers, and let $V_n = \mathbb{R}^n$. Let $\phi_{m,n} : \mathbb{R}^n \to \mathbb{R}^m$ be the natural inclusion map

$$(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 0, \ldots, 0) \in \mathbb{R}^m \qquad (n \leqq m).$$

The direct limit space V, often denoted $\mathbb{R}^\infty$, is the vector space of all sequences, with coefficients in $\mathbb{R}$, which are eventually zero. Now consider each of the $\mathbb{R}^n$ as a Banach space for the norm ℓ_p, and observe that the compatibility condition (2.1) is satisfied, so that the ℓ_p *NDL* makes sense. Note that V is not *NDL*-complete. Its completion V_{BDL} is the usual space ℓ_p.

Example 3. Norm- and Banach direct limits of classical Lie algebras. We can also take a norm limit of the algebras in Example 1, as follows. Let $1 \leqq p \leqq \infty$ and define $|| \, ||_{n,p}$ on $\mathfrak{g}_n$ to be the ℓ_p norm, namely $||\xi||_{n,p} = (Tr(\xi^*\xi)^{\frac{p}{2}})^{\frac{1}{p}}$ for $1 \leqq p < \infty$ and $||\xi||_{n,\infty} = \sup_{||v||=1}(||\xi v||)$. Then, for each p, $\{|| \cdot ||_{n,p}\}_{n\in\mathbb{N}}$ is a compatible family of norms and hence we have a well-defined norm, $|| \, ||_p = \varinjlim || \cdot ||_{n,p}$ on $\mathfrak{g}$. We will refer to this situation as $(\mathfrak{g}, || \, ||_p)$.

Section 3. Properties of the Topologies.

We start with some results that do not use finite dimensionality of the V_α. Then, starting with Proposition 3.8, we require $\dim V_\alpha < \infty$ in our directed system.

Lemma 3.1. *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system of vector spaces and linear transformations. Then the natural inclusion maps $V_{DL} \hookrightarrow V_{LCDL} \hookrightarrow V_{MDL} \hookrightarrow V_{NDL}$ are continuous, for the NDL topology given by any family $\{|| \cdot ||_\alpha\}$ of norms satisfying the compatibility condition (2.2) and any MDL topology defined by a set of seminorms that contains a norm $\geqq$ a multiple of the NDL limit norm.*

We say that the system $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *eventually stabilizes* if there exists an index α_0 such that $\psi_{\beta,\alpha}$ is an isomorphism onto V_β whenever $\beta \geqq \alpha \geqq \alpha_0$. In that case V_{DL} and V_{LCDL} are both isomorphic to V_{α_0}. Arguing as in Treves [12, Remark 13.1] one sees

Lemma 3.2. *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a strict directed system of Hausdorff topological vector spaces. Let T be a vector space topology on $V = \varinjlim V_\alpha$ such that each $\psi_\alpha : V_\alpha \to \psi_\alpha(V_\alpha)$ is a topological isomorphism where we use the subspace topology on $\psi_\alpha(V_\alpha)$. This last condition is automatic if each $\dim V_\alpha < \infty$. If the system never stabilizes, then (V, T) is not of Baire second category.*

Corollary 3.3. *No NDL topology on V is complete, unless the system eventually stabilizes.*

Lemma 3.4. *Let T be any one of the topologies DL, LCDL, MDL, NDL or BDL. If $B \subset A$ is a directed set under the partial ordering "$\leqq$" it inherits from A, then*

(a) $\varinjlim_{T,\beta\in B} V_\beta \hookrightarrow \varinjlim_{T,\alpha\in A} V_\alpha$ *is continuous, and*

(b) *if B is cofinal in A, then* $\varinjlim_{T,\beta\in B} V_\beta = \varinjlim_{T,\alpha\in A} V_\alpha$.

Proof. The topological vector spaces $\varinjlim_{DL,\beta\in B} V_\beta$ and $\varinjlim_{DL,\alpha\in A} V_\alpha$ have, as their underlying sets, $\bigsqcup_{\beta\in B} \psi_\beta(V_\beta)$ and $\bigsqcup_{\alpha\in A} \psi_\alpha(V_\alpha)$, respectively. The first is contained in the second.

If a subset U of $\bigsqcup_{\alpha\in A} \psi_\alpha(V_\alpha)$ is open in $\varinjlim_{DL,\alpha\in A} V_\alpha$, then $\psi_\alpha^{-1}(U)$ is open in V_α, for all $\alpha \in A$, hence for all $\beta \in B$. It follows that $U \cap \left(\bigsqcup_{\beta\in B} \psi_\beta(V_\beta)\right)$ is open in $\varinjlim_{DL,\beta\in B} V_\beta$.

Suppose that B is cofinal in A. Let α be any element of A, and take any $\beta \in B$ such that $\alpha \leqq \beta$. Then $\psi_\alpha(V_\alpha) \subset \psi_\beta(V_\beta)$. Hence $\bigsqcup_{\alpha\in A} \psi_\alpha(V_\alpha) \subset \bigsqcup_{\beta\in B} \psi_\beta(V_\beta)$.

Let U be open in $\varinjlim_{DL,\beta\in B} V_\beta$, and take α, β as above. Then $\psi_\alpha^{-1}(U) = (\psi_\beta \cdot \psi_{\beta,\alpha})^{-1}(U)$ is open in V_α. Hence U is open in $\varinjlim_{DL,\alpha\in A} V_\alpha$. This proves Lemma 3.4 for the case of $T = DL$.

For the case of $T = LCDL$, use the same argument, taking U to be convex. The other cases are even easier. □

Note. If B is as in Lemma 3.4 (*a*), then for $T = MDL$, NDL or BDL, the T-topology of $\varinjlim_{T,\beta\in B} V_\beta$ coincides with the subspace topology it inherits from $\varinjlim_{T,\alpha\in A} V_\alpha$. This is not generally true for $T = DL$ or $LCDL$.

As indicated earlier, the *LCDL* topology behaves nicely with respect to continuous linear maps between locally convex spaces. The projective tensor product $V \otimes_\varpi W$ provides us with a device allowing us to turn questions dealing with continuous bilinear maps into questions about continuous linear transformations. If V and W are locally convex spaces, then (see [12, Chapters 43–46]) the projective tensor product topology is the unique locally convex topology on $V \otimes W$ such that: For every locally convex space Z, the canonical isomorphism of the space of bilinear mappings of $V \times W$ into Z, onto the space of linear mappings of $V \otimes W$ into Z, induces an isomorphism of the space of continuous bilinear mappings of $V \times W$ into Z, onto the space of continuous linear mappings of $V \otimes W$ into Z.

Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha, \chi_{\beta,\alpha}\}_{\alpha,\beta\in \hat{A}}$ be two directed systems of locally convex vector spaces and continuous linear transformations. There is no loss of generality in assuming that $A = \hat{A}$, since we could if necessary consider both the given systems as having the index set $A \times \hat{A}$, the product directed set with lexicographic order.

For each $\beta \geqq \alpha$, the map $\psi_{\beta,\alpha} \times \chi_{\beta,\alpha} : V_\alpha \times W_\alpha \to V_\beta \times W_\beta$ is given by $\psi_{\beta,\alpha} \times \chi_{\beta,\alpha}(v,w) = (\psi_{\beta,\alpha}(v), \chi_{\beta,\alpha}(w))$, for all $(v,w) \in V_\alpha \times W_\alpha$. These are continuous bilinear maps, and they satisfy the condition

$$(\psi_{\gamma,\beta} \times \chi_{\gamma,\beta}) \cdot (\psi_{\beta,\alpha} \times \chi_{\beta,\alpha}) = \psi_{\gamma,\alpha} \times \chi_{\gamma,\alpha} \text{ whenever } \gamma \geqq \beta \geqq \alpha. \tag{3.5}$$

Denoting by $\iota_\alpha : V_\alpha \times W_\alpha \to V_\alpha \otimes_\varpi W_\alpha$ the canonical injection, we obtain continuous linear transformations $\psi_{\beta,\alpha} \otimes \chi_{\beta,\alpha} : V_\alpha \otimes_\varpi W_\alpha \to V_\beta \otimes_\varpi W_\beta$, such that $(\psi_{\beta,\alpha} \otimes \chi_{\beta,\alpha}) \cdot \iota_\alpha = \iota_\beta \cdot (\psi_{\beta,\alpha} \times \chi_{\beta,\alpha})$ whenever $\beta \geqq \alpha$.

It follows from (3.5) that $\{V_\alpha \otimes_\varpi W_\alpha, \psi_{\beta,\alpha} \otimes \chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is a strict directed system of locally convex vector spaces. So its *LCDL* exists.

Both $\varinjlim_{LCDL}(V_\alpha \otimes_\varpi W_\beta)$ and $(\varinjlim_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim_{LCDL} W_\alpha)$ have as underlying vector space $(\varinjlim V_\alpha) \otimes (\varinjlim W_\alpha)$. It is straightforward[1] that the natural inclusion map $\varinjlim_{LCDL}(V_\alpha \otimes_\varpi W_\beta) \hookrightarrow (\varinjlim_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim_{LCDL} W_\alpha)$ is continuous. Next we show that this map is a topological isomorphism when both directed systems are strict. That extends a result [3, Proposition 6.2 and its Corollary] of Grothendieck, and it is the projective tensor product analog of a result [14, Theorem A 2.2.5] known for inductive tensor products.

Proposition 3.6. *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha, \chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be two strict directed systems of locally convex vector spaces and continuous linear maps. Then*

$$\varinjlim{}_{LCDL}(V_\alpha \otimes_\varpi W_\beta) = (\varinjlim{}_{LCDL} V_\alpha) \otimes_\varpi (\varinjlim{}_{LCDL} W_\beta).$$

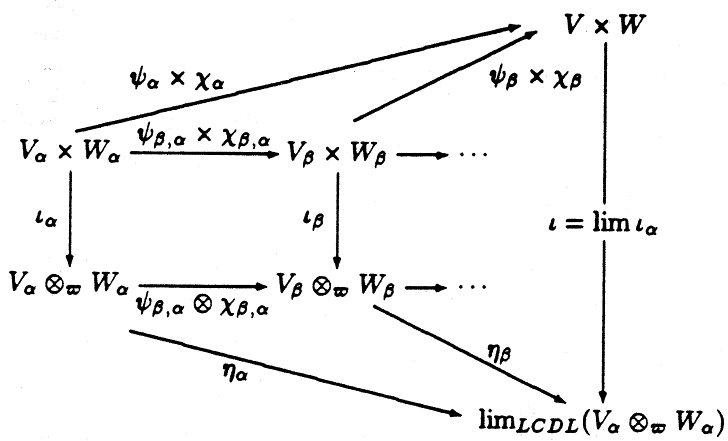
More precisely, if $(V_{LCDL}, \{\psi_\alpha\}_{\alpha\in A})$ and $(W_{LCDL}, \{\chi_\alpha\}_{\alpha\in A})$ are the LCDL limits of $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha, \chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$, respectively, then $\{V_\alpha \otimes_\varpi W_\alpha, \psi_{\beta,\alpha} \otimes \chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is a strict directed system, which has $(V_{LCDL} \otimes_\varpi W_{LCDL}, \{\psi_\alpha \otimes \chi_\alpha\}_{\alpha\in A})$ as its LCDL limit.

[1] The compatible family $\{\psi_\alpha \times \chi_\alpha : V_\alpha \times W_\alpha \to V_{LCDL} \times W_{LCDL}\}_{\alpha\in A}$ of continuous bilinear maps, gives rise to a compatible family $\{\psi_\alpha \otimes \chi_\alpha : V_\alpha \otimes_\varpi W_\alpha \to V_{LCDL} \otimes_\varpi W_{LCDL}\}_{\alpha\in A}$ of continuous linear transformations. Hence $\varinjlim(\psi_\alpha \otimes \chi_\alpha) : \varinjlim(V_\alpha \otimes_\varpi W_\alpha) \to V_{LCDL} \otimes_\varpi W_{LCDL}$ is linear and continuous. By simply checking its action on any point of its domain, we see that this map coincides with the natural inclusion (identity) map $\varinjlim(V_\alpha \otimes_\varpi W_\alpha) \hookrightarrow V_{LCDL} \otimes_\varpi W_{LCDL}$, which is thus shown to be continuous.

Proof. Since $\{\alpha \times \alpha \mid \alpha \in A\}$ is cofinal in the product directed set $A \times A$, it follows from Lemma 3.4 (*b*) that $\varinjlim_{LCDL, \alpha\times\beta \in A\times A}(V_\alpha \otimes_\varpi W_\beta) = \varinjlim_{LCDL, \alpha\in A}(V_\alpha \otimes_\varpi W_\alpha)$.

It remains to be proved that the inclusion map $V_{LCDL} \otimes_\varpi W_{LCDL} \hookrightarrow \varinjlim(V_\alpha \otimes_\varpi W_\alpha)$ is also continuous. To prove this, we use the criterion (see [12, Comment following Definition 43.2]): A convex set N, with $0 \in N \subset V_{LCDL} \otimes W_{LCDL}$ is a neighborhood of 0 for the projective topology, iff it contains a set of the form $C \otimes D := \{c \otimes d \mid c \in C \text{ and } d \in D\}$, where C and D are convex neighborhoods of 0 in V_{LCDL} and W_{LCDL}, respectively.

We now take N to be any convex, open neighborhood of 0 in $\varinjlim_{LCDL}(V_\alpha \otimes_\varpi W_\alpha)$ and show that it meets this criterion, using the diagram



where η_α and η_β are the canonical inclusions and $\iota = \varinjlim \iota_\alpha$.

Then $(\psi_\alpha \times \chi_\alpha)^{-1}(\iota^{-1}(N)) = (\eta_\alpha \cdot \iota_\alpha)^{-1}(N)$, and hence that $\iota^{-1}(N)$ is *DL*-open in $\varinjlim(V_\alpha \times W_\alpha)$. But by Proposition A.10(c) and Corollary A.11(b), $\varinjlim_{DL}(V_\alpha \times W_\alpha) = V_{DL} \times W_{DL}$. Thus there exist open neighborhoods A and B of 0 in V_{DL} and W_{DL}, respectively, such that $A \times B \subset \iota^{-1}(N)$. This means that $A \otimes B \subset N$.

We now remember that N is convex, and so we get that the convex hull, $\operatorname{convh}(A \otimes B) \subset N$. But $(\operatorname{convh} A) \otimes (\operatorname{convh} B) \subset \operatorname{convh}(A \otimes B)$. (Indeed, if $x \otimes b \in (\operatorname{convh} A) \otimes B$, then x can be written as a finite linear combination $x = \sum_i t_i a_i$ of elements $a_i \in A$ with each $t_i \geqq 0$ and $\sum_i t_i = 1$. Then $x \otimes b = (\sum_i t_i a_i) \otimes b = \sum_i t_i (a_i \otimes b) \in \operatorname{convh}(A \otimes B)$. This shows that $(\operatorname{convh} A) \otimes B \subset \operatorname{convh}(A \otimes B)$. We now play the same trick on the other side.)

This shows that $(\operatorname{convh} A) \otimes (\operatorname{convh} B) \subset N$. Since a convex subset of the limit vector space is *LCDL*-open iff it is *DL*-open, we may conclude that convh A and convh B are *LCDL*-open. Therefore, N is open in $V_{LCDL} \otimes_\varpi W_{LCDL}$. □

If $\{\rho_\alpha : V_\alpha \to \mathbb{R}\}_{\alpha \in A}$ is a compatible family of continuous seminorms, then $\rho := \varinjlim \rho_\alpha$ is a continuous seminorm on V_{LCDL}. For the semidisk $D_\rho = \{v \in V \mid \rho(v) < 1\}$ is *LCDL*-open because (i) it is a convex set and (ii) if $\alpha \in A$ then $\phi_\alpha^{-1}(D_\rho) = \{v_\alpha \in V_\alpha \mid \rho_\alpha(v_\alpha) < 1\}$ is open in V_α. Conversely, every continuous seminorm ρ on V_{LCDL} has the form $\rho = \varinjlim \rho_\alpha$ for some compatible family $\{\rho_\alpha\}_{\alpha \in A}$, for example with $\rho_\alpha = \rho \cdot \phi_\alpha$.

Let $\mathcal{R}$ and $\mathcal{S}$ be the respective families of all continuous seminorms on V_{LCDL} and W_{LCDL}. Then the family $\mathcal{R}\otimes\mathcal{S} = \{\rho\otimes\sigma \mid \rho\in\mathcal{R},\ \sigma\in\mathcal{S}\}$ defines the topology of $V_{LCDL}\otimes_{\varpi} W_{LCDL}$. On the other hand, if ρ_α and σ_α are continuous seminorms on V_α and W_α, respectively, then $\rho_\alpha\otimes\sigma_\alpha$ is a continuous seminorm on $V_\alpha\otimes_{\varpi} W_\alpha$. So any pair of compatible families $\{\rho_\alpha : V_\alpha\to\mathbb{R}\}_{\alpha\in A}$ and $\{\sigma_\alpha : W_\alpha\to\mathbb{R}\}_{\alpha\in A}$ gives rise to a continuous seminorm $\varinjlim(\rho_\alpha\otimes\sigma_\alpha)$ on $\varinjlim_{LCDL}(V_\alpha\otimes_{\varpi} W_\alpha)$. It is not difficult to show that $\varinjlim(\rho_\alpha\otimes\sigma_\alpha) = \varinjlim\rho_\alpha\otimes\varinjlim\sigma_\alpha$.

This suggests another proof of Proposition 3.6. There, the difficulty lies in proving there are no "extra" seminorms, i.e. that the family $\mathcal{R}\otimes\mathcal{S}$ defines the *LCDL* topology of $\varinjlim(V_\alpha\otimes_{\varpi} W_\alpha)$. This probably requires that $\{V_\alpha,\psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha,\psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be strict.

Corollary 3.7. *Let $\{V_\alpha,\psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{W_\alpha,\chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be two strict directed systems of locally convex vector spaces. Let Z be any locally convex vector space and $\{b_\alpha : V_\alpha\times W_\alpha\to Z\}_{\alpha\in A}$ a family of jointly continuous bilinear maps such that $b_\beta\cdot(\psi_{\beta,\alpha}\times\chi_{\beta,\alpha}) = b_\alpha$ whenever $\beta\geqq\alpha$ in A. Then the unique map $b : V_{LCDL}\times W_{LCDL}\to Z$ such that $b\cdot(\psi_\alpha\times\chi_\alpha) = b_\alpha$ for all α, is bilinear and jointly continuous. In this sense $(V_{LCDL}\times W_{LCDL},\{\psi_\alpha\times\chi_\alpha\}_{\alpha\in A})$ is the LCDL limit of the system $\{V_\alpha\times W_\alpha\ ,\psi_{\beta,\alpha}\times\chi_{\beta,\alpha}\}_{\alpha,\beta\in A}$.*

Several useful properties hold for the case when the index set $A=\mathbb{N}$ and the directed system is strict. For example, then V_{LCDL} coincides with V_{DL} and is complete and Hausdorff, and V_α is isomorphic to its image $\psi_\alpha(V_\alpha)$ where the latter has the topology induced by V_{LCDL}. See [15]. However, these properties do not hold in general for uncountable directed systems; see [8]. A result of Komura [7] shows that, even if every V_α is Hausdorff, V_{LCDL} can have the property that no two points are separated.

Of course, if the index set A is uncountable, but has a cofinal subsequence, then it follows from Lemma 3.4 that V_{LCDL} has all of the desirable properties known to hold for countable locally convex direct limits.

Unless we state otherwise, we do not assume that the directed system is countable. On the other hand, for the remainder of this section and most of the rest of this paper, we assume that each of the spaces V_α is finite-dimensional. In many situations, it turns out that this finite dimensionality makes up for uncountability of the index set.

Recall our assumption that each of the $\psi_{\beta,\alpha}$ is injective. Together with the finite-dimensionality of the V_α, it will follow that the direct limit is strict. See Corollary 3.12 below.

We will find compatible families of norms as a special case of

Proposition 3.8. *V is unitarizable in the sense that there exist Hilbert space structures on each V_α such that the $\psi_{\beta,\alpha} : V_\alpha\to V_\beta$ are unitary injections. $V=\varinjlim V_\alpha$ has pre–Hilbert space structure defined by: $\langle u,v\rangle_V = \langle\psi_\alpha^{-1}(u),\psi_\alpha^{-1}(v)\rangle_{V_\alpha}$ whenever $u,v\in\psi_\alpha(V_\alpha)$.*

Proof. Let $\mathcal{A}$ consist of all pairs (S,h_S) where $S\subset A$ and h_S assigns a unitary structure to each V_α , $\alpha\in S$, in such a way that if $\alpha\leqq\beta$ with both in S then $\psi_{\beta,\alpha} : V_\alpha\to V_\beta$ is a unitary injection. Thus the subspace $V_S=\sum_{\alpha\in S}\psi_\alpha(V_\alpha)$ of V has a well defined pre–Hilbert space structure into which the ψ_α are unitary injections.

$\mathcal{A}$ is a partially ordered set with the relation: $(S,h_S)\preceq(T,h_T)$ if (1) $S\subset T$ and (2) if $\alpha\in S$ then h_S and h_T assign the same inner product to V_α. Thus V_S is a subspace of V_T and $V_S\hookrightarrow V_T$ is a unitary injection.

Let $\mathcal{S}$ be a linearly ordered subset of $\mathcal{A}$, say $\mathcal{S}=\{(S_n,h_{S_n})\mid n\in N\}$ where N is a linarly ordered subset of the index set for $\mathcal{A}$. So $(S_m, h_{S_m}) \preceq (S_n, h_{S_n})$ for $m \leqq n$ with $m, n \in N$. Set $S = \bigcup_{n \in N} S_n$. If $m \leqq n$ with $m, n \in N$ and $\alpha \in S_m$ then h_{S_m} and h_{S_n} assign the same Hilbert space structure to V_α, and by definition this is the one assigned by h_S. Thus $(S, h_S) \in \mathcal{A}$ and is a least upper bound for $\mathcal{S}$.

Zorn's Lemma now says that $\mathcal{A}$ has a maximal element (M, h_M). If $\beta \in A \setminus M$ then $V_\beta = V'_\beta \oplus V''_\beta$ where $V'_\beta = V_\beta \cap \psi_\beta^{-1}(V_M)$ and V''_β is any vector space complement. Let $P = M \cup \{\beta\}$, let h_P agree with h_M for $\alpha \in M$, and define h_P for β as follows. The inner product on V'_β comes from that of V_M, the inner product on V''_β is arbitrary, and $V'_\beta \perp V''_\beta$. Then $(M, h_M) \prec (P, h_P)$, contradicting maximality. Now $M = A$. That proves Proposition 3.8 □

Corollary 3.9. *There exists a compatible system of norms on $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$, and hence an NDL topology on V.*

By definition, V_{NDL} and V_{MDL} are Hausdorff. From Corollary 3.9, the *LCDL* topology is stronger than a Hausdorff topology. Thus

Corollary 3.10. *The LCDL topology on V is Hausdorff.*

Corollary 3.11 below, follows from the Hausdorff property and the fact for *LCDL* that each ψ_α^{-1}–image of a barrel is a barrel. Its importance stems from these facts: For a barrelled vector space, the conclusion of the Banach-Steinhaus Theorem (Principle of Uniform Boundedness) holds. For a bornological vector space, a linear map is bounded (maps bounded sets into bounded sets) if and only if it is continuous.

Corollary 3.11. *V_{LCDL} is bornological and barrelled.*

When the topology on V is Hausdorff, the subspace topology on each $\psi_\alpha(V_\alpha)$ is Hausdorff, thus is the unique Hausdorff topology compatible with the vector space structure on that finite dimensional space. Thus, using Corollary 3.10,

Corollary 3.12. *Let T denote any of LCDL, MDL, or NDL. Then each V_α is topologically isomorphic to its image $\psi_\alpha(V_\alpha) \subset V_T$ where $\psi_\alpha(V_\alpha)$ carries the subspace topology.*

Section 4. Direct Limits of Finite Dimensional Lie Algebras

We now consider a directed system $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ of finite-dimensional Lie algebras $\mathfrak{g}_\alpha$ and Lie algebra homomorphisms $\psi_{\beta,\alpha}$. Let $\mathfrak{g} = \varinjlim_{\alpha \in A} \mathfrak{g}_\alpha$, as described in §1. Since $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ is in particular a directed system of locally convex vector spaces and continuous linear maps, $\mathfrak{g}_{DL}$, $\mathfrak{g}_{LCDL}$, $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ all make sense. Each of them has a topological vector space structure, and also a Lie algebra structure. We next investigate the question of when these two structures are compatible.

Proposition 4.1. (*a*) *If $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ is a strict directed system of Lie algebras, then the Lie bracket is a continuous map $\mathfrak{g}_{DL} \times \mathfrak{g}_{DL} \to \mathfrak{g}_{DL}$.*

(*b*) *Let $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a strict directed system of locally convex Lie algebras. Then the Lie bracket $\mathfrak{g}_{LCDL} \times \mathfrak{g}_{LCDL} \to \mathfrak{g}_{LCDL}$ is continuous.*

(c) If $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *is a strict directed system of Lie algebras, and if* S *is a set of continuous multiplicative* (2.1) *seminorms on* $\mathfrak{g}_{DL}$ *that separates points, then the Lie bracket* $\mathfrak{g}_{MDL} \times \mathfrak{g}_{MDL} \to \mathfrak{g}_{MDL}$ *is continuous.*

(d) If $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *is a strict directed system of Lie algebras with a compatible family of norms that satisfies* (4.3) *below, then the Lie bracket* $\mathfrak{g}_{NDL} \times \mathfrak{g}_{NDL} \to \mathfrak{g}_{NDL}$ *is continuous.*

To prove statement (a) we need some results on direct limits in the category of topological spaces and continuous maps. These are given in the Appendix. Statement (b) is a consequence of Corollary 3.7. Statements (c) and (d) are immediate from (2.1) and (2.2), respectively.

Consider the *NDL* case. Given a compatible family $\{||\cdot||_\alpha\}$ of norms for $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$, and given $\xi_\alpha \in \mathfrak{g}_\alpha$, for each $\beta \geqq \alpha$ we have an operator norm for the adjoint action of $\psi_{\beta,\alpha}(\xi_\alpha)$ on $\mathfrak{g}_\beta$

$$||\mathrm{ad}\,(\psi_{\beta,\alpha}(\xi_\alpha))||_{\beta,\infty} := \sup\left\{ \frac{||\psi_{\beta,\alpha}(\xi_\alpha)(\xi_\beta)\;||_\beta}{||\xi_\beta||_\beta} \;\middle|\; \xi_\beta \in \mathfrak{g}_\beta \right\}.$$

For any $\alpha \leqq \beta \leqq \gamma$ we then have $||\mathrm{ad}\,(\psi_{\beta,\alpha}(\xi_\alpha))||_{\beta,\infty} \leqq ||\mathrm{ad}\,(\psi_{\gamma,\alpha}(\xi_\alpha))||_{\gamma,\infty}$.

Proposition 4.2. *Let* $\{\mathfrak{g}_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *be a directed system of Lie algebras, with a compatible family of norms* $\{||\cdot||_\alpha\}$. *Suppose that*

$$\text{if } \xi = [\xi_\alpha] \in \mathfrak{g} \text{ and } \alpha \in A \text{ then } \limsup_{\beta \geqq \alpha}\{||ad\,(\xi_\alpha)||_{\beta,\infty}\} < \infty. \tag{4.3}$$

Then there exists $M > 0$ *such that* $||\;[\xi,\zeta]\;|| \leqq M \cdot ||\xi|| \cdot ||\zeta||$ *for all* $\xi,\zeta \in \mathfrak{g}$ *where* $||\cdot|| = \varinjlim ||\cdot||_\alpha$. *Hence the corresponding* $\mathfrak{g}_{NDL}$ *and* $\mathfrak{g}_{BDL}$ *are normed Lie algebras.*

When we speak of the limit Lie algebras $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ we shall always assume that the directed system in question satisfies the hypothesis (4.3) of Proposition 4.2.

Section 5. The Basic Local Coordinate System.

Let T be one of the topologies *DL*, *LCDL*, *MDL*, *NDL* or *BDL*. In this section we discuss the question of finding sufficient conditions for the existence of a "good" neighborhood $\mathcal{O}_1$ of 0 in $\mathfrak{g}_T$, i.e. of a T–open neighborhood $\mathcal{O}_1$ of 0, such that the restriction of the exponential map to $\mathcal{O}_1$ is one-to-one. This neighborhood is essential for construction of analytic manifold structure on G_T in §§4 and 5. The set $U = \exp(\mathcal{O}_1)$ will turn out to be a chart of G_T at 1, and conditions (1.8) to (1.10) of §1 will hold with T in place of *DL*.

We already know that the restriction of $\exp_G$ to the set $\mathcal{O}$, given by (1.7), is injective. So we need only show that $\mathcal{O}$ contains a T–open neighborhood $\mathcal{O}_1$ of 0.

The NDL topology for the operator norm.

Assume that $\{d\pi_\alpha, V_\alpha\}_{\alpha\in A}$ is a compatible family of representations, and that ν_α is the corresponding operator norm on $\mathfrak{g}_\alpha$. Assume that $\limsup\{\nu_a(\xi_\alpha)\} < \infty$ for each $[\xi_\alpha] \in \mathfrak{g}$. The spectral growth condition follows. We have an operator norm $\nu = \limsup \nu_\alpha$ on $\mathfrak{g}$. Let ι and $\mathcal{O}$ be as defined in (1.6) and (1.7), using this system $\{d\pi_\alpha\}$. We'll now show that $\mathcal{O}$ is a neighborhood of 0 in the *NDL* Lie algebra $(\mathfrak{g}, \nu)$, where $\nu = \limsup \nu_\alpha$.

Let $\iota_\alpha(\xi) = \sup\{|\mathrm{Im}\lambda| \mid \lambda \text{ is an eigenvalue of } d\pi_\alpha(\xi)\}$ and $\mathcal{O}_\alpha = \{\xi \in \mathfrak{g}_\alpha \mid \iota_\alpha(\xi) < \pi\}$ for $\alpha \in A$. Then $\iota_\alpha(\xi) \leqq \nu_\alpha(\xi)$. Thus $\mathcal{O}_\alpha$ contains the open neighborhood $\mathcal{N}_\alpha = \{\xi \in \mathfrak{g}_\alpha \mid \nu_\alpha(\xi) < \pi\}$ and hence it is itself a neighborhood of 0. Since this holds for each index α, and $\mathcal{O}_\alpha = \psi_\alpha^{-1}(\mathcal{O})$, it follows that $\mathcal{O}$ is a neighborhood of 0 in $(\mathfrak{g}, \nu)$. We now take as our $\mathcal{O}_1 = \mathcal{O}_{1,\nu}$ any ν-open neighborhood of 0 which is contained in $\mathcal{O}$. We could for example take

$$\mathcal{O}_1 = \mathcal{O}_{1,\nu} := \{\xi \in \mathfrak{g} \mid \nu(\xi) < \pi\}. \tag{5.1}$$

The NDL topology for the general case.

Let $\{||\cdot||_\alpha\}$ be a compatible family of norms for the directed system and let $||\cdot|| = \varinjlim ||\cdot||_\alpha$. We prove that the *NDL* Lie algebra $(\mathfrak{g}, ||\cdot||)$ has a neighborhood on which $\exp_G$ is injective if the norm is "not too different" from an operator norm.

More precisely, let $\{\nu_\alpha\}$ be a family of operator norms that come from some representation with bounded spectral growth, as in §1. For each α, let $c_\alpha(\nu)$ be a positive number such that $\nu_\alpha(\xi) \leqq c_\alpha(\nu)||\xi||_\alpha$ for all $\xi \in \mathfrak{g}_\alpha$. Assume that $c_\alpha(\nu)$ is minimal for this condition. We prove that if

$$\text{the operator norm } \nu = \varinjlim \nu_\alpha \text{ satisfies } \limsup c_\alpha(\nu) < \infty, \tag{5.2}$$

then $\mathcal{O}_{1,\nu}$ is $||\cdot||$-open. In effect, (5.2) implies $\nu(\xi) \leqq \limsup c_\alpha(\nu)||\xi||$ for all $\xi \in \mathfrak{g}$, so the natural inclusion $(\mathfrak{g}, ||\cdot||) \hookrightarrow (\mathfrak{g}, \nu)$ is continuous.

The DL, LCDL and MDL topologies.

Let $\mathcal{O}_1$ be as in (5.1). Let ν be an operator norm, as above. Fix a set $\mathcal{S}$ of continuous multiplicative (as in (2.1)) seminorms on $\mathfrak{g}$ that separates points, in fact contains a norm $\nu' \geqq c\nu$ for some $c > 0$. Since the natural inclusion maps $\mathfrak{g}_{DL} \hookrightarrow \mathfrak{g}_{LCDL} \hookrightarrow \mathfrak{g}_{MDL} \hookrightarrow (\mathfrak{g}, \nu)$ are continuous, $\mathcal{O}_1$ is an open neighborhood of 0 for the *DL*, the *LCDL*, and the *MDL* topologies.

Section 6. The Topologies on the Limit Groups.

In this Section we indicate how the topologized Lie algebras $\mathfrak{g}_{DL}$, $\mathfrak{g}_{LCDL}$, $\mathfrak{g}_{MDL}$, $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{BDL}$ define topological structures G_{DL}, G_{LCDL}, G_{MDL}, G_{NDL} and G_{BDL} on the limit group $G = \varinjlim G_\alpha$. Specifically, we indicate how each G_T carries the structure of a topological manifold modelled on the corresponding $\mathfrak{g}_T$, where T is *DL*, *LCDL*, *MDL*, *NDL* or *BDL*. The C^ω differentiable manifold structures are described in §7.

For reasons of exposition we assume in §§6, 7, 8 that G_{DL} is connected. The results are formulated so that they are correct as stated even when G_{DL} is disconnected. We remove all connectivity requirements in §9.

The topologies of G_{DL}, G_{MDL}, G_{NDL} and G_{BDL} have the property that the group operations are continuous. So each of them is a topological group in the usual sense. For G_{LCDL}, we only obtain a group with a topology that makes $x \mapsto x^{-1}$ continuous and $(x, y) \mapsto xy$ separately continuous.

The key to this process is the following result in [2, Chapter II, §II]: A family $\mathcal{V}$ of subsets of a group G is a fundamental system of neighborhoods of 1 in G, for some Hausdorff topology under which the group operations of G are continuous, if and only if $\mathcal{V}$ satisfies the five conditions:

1. If U_1, $U_2 \in \mathcal{V}$, then there exists a set $U_3 \in \mathcal{V}$ such that $U_3 \subset U_1 \cap U_2$,
2. The intersection of all sets of $\mathcal{V}$ is $\{1\}$,
3. If $U \in \mathcal{V}$, there exists a set $U_1 \in \mathcal{V}$ such that $U_1^{-1} \subset U$,

4. If $U \in \mathcal{V}$ and $g \in G$, there exists a set $U_1 \in \mathcal{V}$ such that $gU_1g^{-1} \subset U$,

5. If $U \in \mathcal{V}$, there exists a set $U_1 \in \mathcal{V}$ such that $U_1U_1 \subset U$.

Let $\mathcal{O}$ be as an (1.7), and let $\mathcal{O}_1 \subset \mathcal{O}$ be an open neighborhood of 0 in $\mathfrak{g}_T$. Set

$$\tilde{\mathcal{V}} = \{\tilde{U} \subset \mathcal{O}_1 \mid \tilde{U} \text{ is } T\text{–open}\} \text{ and } \mathcal{V} = \{U \in G \mid U = \exp(\tilde{U}) \text{ for some } \tilde{U} \in \tilde{\mathcal{V}}\}. \qquad (6.1)$$

The desired properties of $\mathcal{V}$ follow from a list of analogous properties of $\tilde{\mathcal{V}}$. We denote by H the Campbell–Hausdorff series in $\mathfrak{g}$. Consider the five conditions:

$\tilde{1}$. If $\tilde{U}_1, \tilde{U}_2 \in \tilde{\mathcal{V}}$, then there exists a set $\tilde{U}_3 \in \tilde{\mathcal{V}}$ such that $\tilde{U}_3 \subset \tilde{U}_1 \cap \tilde{U}_2$,

$\tilde{2}$. The intersection of all sets of $\tilde{\mathcal{V}}$ is $\{0\}$,

$\tilde{3}$. If $\tilde{U} \in \tilde{\mathcal{V}}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that $-\tilde{U}_1 \subset \tilde{U}$,

$\tilde{4}$. If $\tilde{U} \in \tilde{\mathcal{V}}$ and $\xi \in \mathfrak{g}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that $H(\xi, H(\tilde{U}_1, -\xi)) \subset \tilde{U}$,

$\tilde{5}$. If $\tilde{U} \in \tilde{\mathcal{V}}$, there exists a set $\tilde{U}_1 \in \tilde{\mathcal{V}}$ such that $H(\tilde{U}_1, \tilde{U}_1) \subset \tilde{U}$.

In the following proposition we show that conditions $\tilde{1}$ through $\tilde{5}$ all hold for $\mathfrak{g}_{MDL}$ and $\mathfrak{g}_{NDL}$, consequently that conditions 1 through 5 all hold for G_{MDL} and G_{NDL}. We have conditions $\tilde{1}$ through $\tilde{4}$ for $\mathfrak{g}_{LCDL}$, and therefore conditions 1 through 4 for G_{LCDL}; but in the *LCDL* case condition $\tilde{5}$ does not generally hold.

Proposition 6.2. *(a) Let T = NDL, MDL or LCDL. Properties $\tilde{1}$ through $\tilde{4}$ hold for $\mathfrak{g}_T$. Properties 1 through 4 hold for the corresponding group G_T.*

(b) Property $\tilde{5}$ holds for $\mathfrak{g}_{NDL}$ and $\mathfrak{g}_{MDL}$, hence 5 holds for G_{NDL} and G_{MDL}.

Each of the Properties 1 through 4 follows from the corresponding Property $\tilde{1}$ through $\tilde{4}$. To obtain 4 from $\tilde{4}$ we need to assume that G is *DL*-connected.

Proof (a). $\tilde{1}$ and $\tilde{2}$ are easy. $\tilde{3}$ follows from the continuity of the map $x \mapsto -x$.

We now see why $\tilde{4}$ is true. We have the identity (see [13, Chapter 2, §13, 2.13.7, 2.13.8])

$$H(\xi, H(\eta, -\xi)) = (\exp\, \mathrm{ad}\, \xi)(\eta) \qquad (6.3)$$

and so $\tilde{4}$ follows if $\eta \mapsto (\exp \mathrm{ad}\, \xi)(\eta)$ is continuous on $\mathfrak{g}_T$. Since this is a linear operator on $\mathfrak{g}_T$, it follows that for the *LCDL* case $\eta \mapsto (\exp \mathrm{ad}\, \xi)(\eta)$ is continuous precisely when its restriction to each $\mathfrak{g}_\alpha$ is continuous. But this is clear, since there exists a β such that $(d\phi_\beta)^{-1}(\xi) \neq \emptyset$ and $\eta \mapsto (\exp \mathrm{ad}\, \xi)(\eta)$ is continuous on the image of each finite-dimensional $\mathfrak{g}_\alpha$.

Now consider the *MDL* case. First, $\eta \mapsto \mathrm{ad}\, \xi(\eta)$ is continuous on $\mathfrak{g}_{MDL}$ with $\rho(\mathrm{ad}\, \xi(\eta)) \leqq M_\rho\, \rho(\xi)\, \rho(\eta)$ for every multiplicative seminorm $\rho \in S$ and for every $\eta \in \mathfrak{g}_{MDL}$. Since $(\exp \mathrm{ad}\, \xi)(\eta) = \sum_k \frac{(\mathrm{ad}\, \xi)^k}{k!}(\eta)$,

$$\rho((\exp \mathrm{ad}\, \xi)(\eta)) \leqq \sum_k \tfrac{1}{k!}\rho((\mathrm{ad}\, \xi)^k(\eta)) \leqq \sum_k \tfrac{1}{k!}(M_\rho)^k(\rho(\xi))^k\rho(\eta) = \rho(\eta)\exp(M_\rho\rho(\xi)).$$

The *NDL* situation is the special case of *MDL* where there is just one norm.

Now we turn to the proofs of Properties 1 through 4 for the group. Properties 1 and 2 are easy consequences of $\tilde{1}$ and $\tilde{2}$. Property 3 follows from $\tilde{3}$ and the simple fact that $\exp(-\xi) = \exp(\xi)^{-1}$.

Property 4 is less obvious. Since G_{DL} is a connected topological group, and exp maps a neighborhood of $0 \in \mathfrak{g}_{DL}$ onto a DL neighborhood of $1 \in G_{DL}$, any element $g \in G$ can be written as a finite product $g = \exp(\xi_n) \cdot \exp(\xi_{n-1}) \cdots \exp(\xi_1)$ with $\xi_i \in \mathfrak{g}$. We will use

$$\exp\, H(\xi, H(\eta, -\xi)) = \exp(\xi) \cdot \exp(\eta) \cdot \exp(-\xi).$$

We choose the desired neighborhood U_1 recursively. Use Property $\tilde{4}$, to choose $\tilde{U}_n$ such that $H(\xi_n, H(\tilde{U}_n, -\xi_n)) \subset \tilde{U}$. Set $U_n = \exp(\tilde{U}_n)$ so that $\exp(\xi_n) \cdot U_n \cdot \exp(-\xi_n) \subset U$. Next choose $U_{n-1} = \exp(\tilde{U}_{n-1})$ such that $\exp(\xi_{n-1}) \cdot U_{n-1} \cdot \exp(-\xi_{n-1}) \subset U_n$. Finally, $U_1 = \exp(\tilde{U}_1)$ will satisfy $gU_1g^{-1} \subset U$.

Proof of (b). For the *MDL* case, $\tilde{U} \in \tilde{\mathcal{V}}$. Choose ϵ and $\rho \in \mathcal{S}$ so that $B_\epsilon = \{\xi \,|\, \rho(\xi) < \epsilon\} \subset \tilde{U}$. Then (b) follows from

Lemma 6.4. *Let $0 < r < 1$. Let $\rho \in \mathcal{S}$. Let $M = M_\rho$ be a constant such that $\rho([\xi, \eta]) \leqq M_\rho \rho(\xi)\, \rho(\eta)$ for all $\xi, \eta \in \mathfrak{g}_{MDL}$. If ξ, η satisfy $\rho(\xi)$, $\rho(\eta) < \frac{r}{2M}$ then $\rho(H(\xi, \eta)) < \frac{r}{M(1-r)}$.*

For the continuity of exposition we will prove Lemma 6.4 later in this section.

Now choose r so that $r > 0$ and $r < \frac{\epsilon M}{1+\epsilon M}$. Let $\tilde{U}_1 = \{\xi \,|\, \rho(\xi) < \frac{r}{2M}\}$. By Lemma 6.4, $\rho(H(\xi, \eta)) < \frac{r}{M(1-r)} < \epsilon$ for $\xi, \eta \in \tilde{U}_1$, giving us the required conclusion.

NDL follows as a special case.

For 5, let $U = \exp(\tilde{U})$ and $U_1 = \exp(\tilde{U}_1)$, where $H(\tilde{U}_1, \tilde{U}_1) \subset \tilde{U}$. Then $U \supset \exp(H(\tilde{U}_1, \tilde{U}_1)) = \exp(\tilde{U}_1) \cdot \exp(\tilde{U}_1) = U_1 U_1$. □

Definition 6.5. *Let $T =$ NDL, MDL or LCDL. Define $G_{NDL} = \varinjlim_{NDL} G_\alpha$ to be the abstract group $G = \varinjlim G_\alpha$ with the topology defined by the family $\mathcal{V}$ of (6.1) for the NDL topology. Define $G_{LCDL} = \varinjlim_{LCDL} G_\alpha$ (resp. $G_{MDL} = \varinjlim_{MDL} G_\alpha$) to be the abstract group $G = \varinjlim G_\alpha$ with the topology defined by the family $\mathcal{V}$ of (6.1) for the LCDL (resp. MDL) topology. In other words, a subset $V \subset G_{NDL}$(resp. G_{LCDL} or G_{MDL}) is open if and only if for each $g \in V$ there exists $U \in \mathcal{V}$ such that $gU \subset V$.*

The topological group G_{BDL} is defined as the completion of G_{NDL} with respect to its two-sided group uniformity.

There are (at least) two natural ways in which to consider completing the group G_{NDL}: one might use the two-sided group uniformity (see [7, Chapter 6, Problem Q(b)] and the following Note), or, on the other hand, one might first complete $\mathfrak{g}_{NDL}$ to $\mathfrak{g}_{BDL}$ and then use a process such as the one used above to construct G_T. Proposition 6.6 below says that both $\exp : \mathcal{O}_1 \to U_1$ and its inverse $\log : U_1 \to \mathcal{O}_1$ are uniformly continuous in the *NDL* topology, so both paths lead to the same complete topological group G_{BDL}.

A net $\{g_i\}_{i\in I} \subset G_{NDL}$ is *(two-sided) Cauchy* if for every neighborhood U of 1 in G_{NDL} there exists an $i_0 \in I$ such that both $g_i g_j^{-1}$ and $g_i^{-1} g_j$ are in U, whenever $i, j \geqq i_0$.

Proposition 6.6. *Let $(\mathfrak{g}, ||\cdot||) = \mathfrak{g}_{NDL} = \varinjlim_{NDL} \mathfrak{g}_\alpha$ and let M be a constant such that $||\,[\xi, \eta]\,|| \leqq M||\xi|| \cdot ||\eta||$ for all $\xi, \eta \in \mathfrak{g}$. Let $0 < r < 1$ and fix a net $\{\xi_i\} \subset \mathcal{O}_1$ with each $||\xi_i|| \leqq r/2M$. Then $\{\xi_i\}$ is $||\cdot||$-Cauchy in $\mathfrak{g}$ if and only if $\{\exp(\xi_i)\}$ is Cauchy in G_{NDL}.*

The proofs of Proposition 6.6 and Lemma 6.4 depend on the following Lemma, which gives various useful bounds involving the Campbell-Hausdorff series. We introduce some notation. It is standard (see [13, Chapter 2, §15]) that the Campbell-Hausdorff series can be expanded as

$$H(\xi,\eta)=\sum_{n=1}^{n=\infty}c_n(\xi,\eta)$$

with $c_1(\xi,\eta)=\xi+\eta$, $c_2(\xi,\eta)=\frac{1}{2}[\xi,\eta]$ and in general $c_n(\xi,\eta)$ given recursively by the formula in [13, Chapter 2, Lemma 2.15.3]. We let $\tilde{H}(\xi,\eta)=\sum_{n\geqq 2}c_n(\xi,\eta)$.

Lemma 6.7. *Let* $\|[\xi,\eta]\|\leqq M\|\xi\|\,\|\eta\|$ *and* $\|\xi\|,\|\eta\|\leqq\frac{r}{2M}$ *where* $0<r<1$. *Then*

(a) $\|H(\xi,\eta)\|\leqq\frac{\|\xi\|}{1-r}$

(b) $\|\tilde{H}(\xi,\eta)\|\leqq\frac{r\|\xi\|}{1-r}$

(c) $\|H(\xi,\eta)\|\leqq\frac{\|\xi+\eta\|}{1-r}$.

Proof. Compute $\|c_n(\xi,\eta)\|\leqq\frac{1}{n}M^{n-1}2^{n-1}\|\xi\|\max\{\|\xi\|,\|\eta\|\}^{n-1}$. Hence

$$\|H(\xi,\eta)\|\leqq\sum_{n=1}^{n=\infty}\frac{1}{n}M^{n-1}2^{n-1}\|\xi\|\max(\|\xi\|,\|\eta\|)^{n-1}\leqq\sum_n\frac{r^{n-1}}{n}\|\xi\|\leqq\frac{\|\xi\|}{1-r}.$$

Similarly,

$$\|\tilde{H}(\xi,\eta)\|\leqq\sum_{n=2}^{n=\infty}\frac{1}{n}M^{n-1}2^{n-1}\|\xi\|\max(\|\xi\|,\|\eta\|)^{n-1}\leqq\sum_{n\geqq 2}\frac{r^{n-1}}{n}\|\xi\|\leqq\frac{r\|\xi\|}{1-r}.$$

In order to prove (c), let $\psi(\xi_m,\xi_{m-1},\cdots,\xi_1)=[\xi_m[\xi_{m-1}[\cdots[\xi_2,\xi_1]]]]$. By [13, Chapter 2, Exercise 44(d)] we have

$$H(\xi,\eta)=\sum_{n=1}^{n=\infty}c_n(\xi,\eta)=\sum_{r,s}C_{r,s}(\xi,\eta)\text{ where}$$

$$(r+s)C_{r,s}(\xi,\eta)=\sum_{m\geqq 1}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_i+q_i\geqq 1,\\ \sum p_i=r,\\ \sum q_i=s}}\frac{\psi(\overbrace{\xi,\xi,\cdots,\xi}^{p_1},\overbrace{\eta,\eta,\cdots,\eta}^{q_1},\cdots,\overbrace{\xi,\xi,\cdots,\xi}^{p_m},\overbrace{\eta,\eta,\cdots,\eta}^{q_m})}{p_1!\,q_1!\cdots p_m!\,q_m!}$$

If $q_m>1$ then the farthest bracket in ψ is $[\eta,\eta]$, whence $\psi=0$. Therefore, nonzero terms occur in the sum only when the farthest interior bracket is $[\xi,\eta]$ or $[\eta,\xi]$. However, in either case, ξ in the farthest interior bracket can be replaced by $\xi+\eta$ since $[\xi+\eta,\eta]=[\xi,\eta]$. Using this observation we get that $\|c_n(\xi,\eta)\|\leqq\frac{1}{n}M^{n-1}2^{n-1}\|\xi+\eta\|\max\{\|\xi\|,\|\eta\|\}^{n-1}$. Now the proof of (c) is essentially the same as the argument for (a). □

Proof of Lemma 6.4. Lemma 6.4 is a special case of Lemma 6.7(b).

Proof of Proposition 6.6. We have $\exp(\xi_i)\cdot\exp(-\xi_j)=\exp H(\xi_i,-\xi_j)$ and $\exp(-\xi_i)\cdot\exp(\xi_j)=\exp H(-\xi_i,\xi_j)$. Hence in order to prove the Proposition, we need to prove that $\|\xi_i-\xi_j\|$ converges to 0 if and only if $\|H(\xi_i,-\xi_j)\|$ and $\|H(-\xi_i,\xi_j)\|$ converges to 0. We will actually prove the stronger statement that $||\xi_i - \xi_j||$ converges to 0 if and only if $||H(\xi_i, -\xi_j)||$ (or $||H(-\xi_i, \xi_j)||$) converge to 0. So assume that $||\xi_i - \xi_j|| \to 0$. Choose ϵ and i_0 such that if $i,j \geqq i_0$ then $||\xi_i - \xi_j|| < (1-r)\epsilon$. Then by Lemma 6.7(c), $||H(\xi_i, -\xi_j)|| \leqq \frac{||\xi_i - \xi_j||}{1-r} < \epsilon$. Similarly, $||H(-\xi_i, \xi_j)|| < \epsilon$.

To prove the converse we need some simple identities involving the Campbell-Hausdorff series, namely $H(-\xi, \xi) = 0$ and $H(\xi, H(\eta, \xi)) = H(H(\xi, \eta), \xi)$. The first is easy and the second follows from the associativity of multiplication on the group. Using these we have, $\xi_i = H(\xi_i, 0) = H(\xi_i, H(-\xi_j, \xi_j)) = H(H(\xi_i, -\xi_j), \xi_j) = H(\xi_i, -\xi_j) + \xi_j + \tilde{H}(H(\xi_i, -\xi_j), \xi_j)$. Thus, $\xi_i - \xi_j = H(\xi_i, -\xi_j) + \tilde{H}(H(\xi_i, -\xi_j), \xi_j)$. We wish to show that if $H(\xi_i, -\xi_j)$ converges to 0, then so does $\xi_i - \xi_j$. Now given ϵ, choose i_0 such that $\forall\, i,j \geqq i_0$, $||H(\xi_i, -\xi_j)|| < \epsilon(1-r)$. Assume that ϵ is small enough so that $\epsilon(1-r) < r/2M$. By Lemma 6.7, we have $||\tilde{H}(H(\xi_i, -\xi_j), \xi_j)|| \leqq \frac{r||H(\xi_i, -\xi_j)||}{1-r} < r\epsilon$. Finally, $||\xi_i - \xi_j|| \leqq ||H(\xi_i, -\xi_j)|| + ||\tilde{H}(H(\xi_i, -\xi_j), \xi_j)|| < (1-r)\epsilon + r\epsilon = \epsilon$. □

In summary,

Theorem 6.8. *The DL, LCDL, MDL and NDL topologies on $\mathfrak{g}$ define topologies on G such that, for the resulting topological spaces G_{DL}, G_{LCDL}, G_{MDL} and G_{NDL}, the map* $\exp : \mathcal{O}_1 \to U_1$ *is a homeomorphism onto an open set. G_{DL}, G_{MDL} and G_{NDL} are topological groups. G_{LCDL} is essentially a topological group, except that group multiplication may be only separately continuous.*

Now consider the *BDL* case. Assume (5.2) and define $\mathcal{O}_1$ by the same inequality (5.1) as in the *NDL* case. Then Proposition 6.6 and Theorem 6.8 give us

Corollary 6.9. *Let $\mathfrak{g}_{BDL}$ be the completion of the norm direct limit algebra $\mathfrak{g}_{NDL}$, let G_{BDL} be the completion of the corresponding norm direct limit group G_{NDL}, with respect to the two-sided group uniformity, and assume (5.2). Then G_{BDL} is a topological group and the restriction to $\mathcal{O}_1$ of the exponential map* $\exp : \mathfrak{g}_{BDL} \to G_{BDL}$ *is a homeomorphism onto an open set.*

SECTION 7. STRUCTURE SHEAVES

We are going to carry the sheaves $C^\omega(G_{DL})$ and $C^\omega(\mathfrak{g}_{DL})$ of germs of analytic functions for the *DL* topology to corresponding sheaves $C^\omega(G_{NDL})$ and $C^\omega(\mathfrak{g}_{NDL})$ for the *NDL* topologies, to $C^\omega(G_{BDL})$ and $C^\omega(\mathfrak{g}_{BDL})$ for the *BDL* topologies, to $C^\omega(G_{MDL})$ and $C^\omega(\mathfrak{g}_{MDL})$ for the *MDL* topologies, and to $C^\omega(G_{LCDL})$ and $C^\omega(\mathfrak{g}_{LCDL})$ for the *LCDL* topology.

We use the results of §6 to carry this out in the case where G_{DL} is connected. Later, in §9, we will eliminate that restriction.

Recall the standard construction [1, page 9] of direct image sheaves. Let X and Y be topological spaces and $\psi : X \to Y$ a continuous map. If $\mathcal{F} \to X$ is a sheaf one has a presheaf over Y, which assigns to an open set $W_Y \subset Y$ the abelian group of all sections of $\mathcal{F}$ over $\psi^{-1}(W_Y)$. This presheaf is complete. That defines the *direct image sheaf* $\psi_*\mathcal{F} \to Y$. The assignment ψ_* is a left exact covariant functor.

The natural maps $G_{DL} \to G_{LCDL} \to G_{MDL} \to G_{NDL}$, by $x \mapsto x$, and $\mathfrak{g}_{DL} \to \mathfrak{g}_{LCDL} \to \mathfrak{g}_{MDL} \to \mathfrak{g}_{NDL}$ by $\xi \mapsto \xi$, are continuous. The sections of the direct image sheaves $C^\omega(G_{NDL}) \to G_{NDL}$, $C^\omega(\mathfrak{g}_{NDL}) \to \mathfrak{g}_{NDL}$, $C^\omega(G_{MDL}) \to G_{MDL}$, $C^\omega(\mathfrak{g}_{MDL}) \to \mathfrak{g}_{MDL}$, $C^\omega(G_{LCDL}) \to G_{LCDL}$, and $C^\omega(\mathfrak{g}_{LCDL}) \to \mathfrak{g}_{LCDL}$ are just those sections of the corresponding *DL* sheaf whose domains are *NDL*–, *MDL*–, or *LCDL*–open. So they are the analytic function germ sheaves where we define

Definition 7.1. *Let W be an open set in G_{NDL}, $\mathfrak{g}_{NDL}$, G_{MDL}, or $\mathfrak{g}_{MDL}$, G_{LCDL}, or $\mathfrak{g}_{LCDL}$,* *respectively. Then a function $f : W \to \mathbb{C}$ is real analytic if (a) f is continuous and (b) f is DL–analytic.*

The natural maps $G_{NDL} \hookrightarrow G_{BDL}$ and $\mathfrak{g}_{NDL} \hookrightarrow \mathfrak{g}_{BDL}$ are continuous. The sections of the direct image sheaves $C^\omega(G_{BDL}) \to G_{BDL}$ and $C^\omega(\mathfrak{g}_{BDL}) \to \mathfrak{g}_{BDL}$ are just those continuous functions whose restrictions to G_{NDL} or $\mathfrak{g}_{NDL}$ are sections of the corresponding NDL sheaf. So they are the analytic function germ sheaves where we define

Definition 7.2. *let W be an open set in G_{BDL} or $\mathfrak{g}_{BDL}$, respectively. Then a function $f : W \to \mathbb{C}$ is real analytic if (a) f is continuous and (b) the restriction of f to the dense subset $W \cap G_{NDL}$, respectively $W \cap \mathfrak{g}_{NDL}$, is NDL–analytic.*

Section 8. Structure of the Limit Groups

We now combine the material of §§6 and 7 to obtain Theorems 8.1 and 8.5 below. We prove those theorems here in §8 when G_{DL} is connected, and we remove the connectivity requirement in §9.

Let T be one of the topologies *DL*, *LCDL*, *MDL*, *NDL* or *BDL*. In the *DL* case, assume the spectral growth condition (1.6). In the *LCDL*, *MDL*, *NDL* and *BDL* cases assume the stronger conditions (4.3) and (5.2). In a group, $\ell(x)$ denotes the left translation $y \mapsto xy$. Our results are summarized in

Theorem 8.1. *The real analytic structures on $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$, for the topologies $T =$ DL, LCDL, MDL and NDL, define structures G_T of C^ω differentiable manifold on $G = \varinjlim G_\alpha$ based respectively on the topogical vector spaces $\mathfrak{g}_T$. A C^ω local coordinate cover on G_T, corresponding to the topology T, is given by the $(\exp|_{\mathcal{O}_1})^{-1} \cdot \ell(g^{-1}) : gU_1 \to \mathcal{O}_1 \hookrightarrow \mathfrak{g}_T$; and $C^\omega(G_T)$ is the sheaf of germs of C^ω functions on G_{DL} whose domains are open in G_T. G_{DL} and G_{NDL} are C^ω Lie groups, as is the completion G_{BDL} of G_{NDL}. G_{LCDL} is essentially a C^ω Lie group, except that group multiplication may be only separately analytic.*

In order to make Theorem 8.1 precise we must formalize a few definitions.

Definition 8.2. *A* Lie group *is a triple $(G, \mathfrak{g}, \exp)$ where* (i) *G is both a group and a C^ω manifold in such a way that the map $G \times G \to G$ by $(x, y) \mapsto xy^{-1}$ is jointly C^ω,* (ii) *$\mathfrak{g}$ is a topological Lie algebra,* (iii) *there is an open neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$ such that $U = \exp(\mathcal{O})$ is open in G and $\exp : \mathcal{O} \to U$ is a C^ω diffeomorphism, and* (iv) exp *restricts to the usual Lie group exponential map from any 1-dimensional subalgebra of $\mathfrak{g}$ onto the corresponding 1-parameter subgroup of G.*

As is usual in the finite dimensional case, we usually write G instead of $(G, \mathfrak{g}, \exp)$.

Definition 8.3. *A* separately continuous *Lie group is as above, except that* (i) *is weakened to* (i′) *G is both a group and a C^ω manifold in such a way that the map $G \times G \to G$ by $(x, y) \mapsto xy^{-1}$ is separately C^ω.*

In Definitions 8.2 and 8.3, the underlying differentiable manifold structure of G is modelled on the vector space $\mathfrak{g}$. With this in mind define

Definition 8.4. *A Lie group or separately continuous Lie group $(G, \mathfrak{g}, \exp)$ is called locally convex if the topological vector space $\mathfrak{g}$ is locally convex.*

These definitions are stricter than some used by other authors, e.g. Pressley and Segal. They exclude the group of diffeomorphisms of the circle; for that group the exponential map does not meet all the requirements of (8.2) or (8.3).

Theorem 8.1 says that G_T is an infinite dimensional Lie group, separately continuous in the $LCDL$ case. Next we see that each of the limit Lie groups described in Theorem 8.1 is the direct limit, in the appropriate category, of the directed system $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$.

Theorem 8.5. *(a) Let H be any Lie group, possibly infinite dimensional. Let $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ be a compatible family of analytic group homomorphisms. Then there exists a unique analytic group homomorphism $f = \varinjlim f_\alpha : G_{DL} \to H$ such that $f \cdot \phi_\alpha = f_\alpha$ for all $\alpha \in A$.*

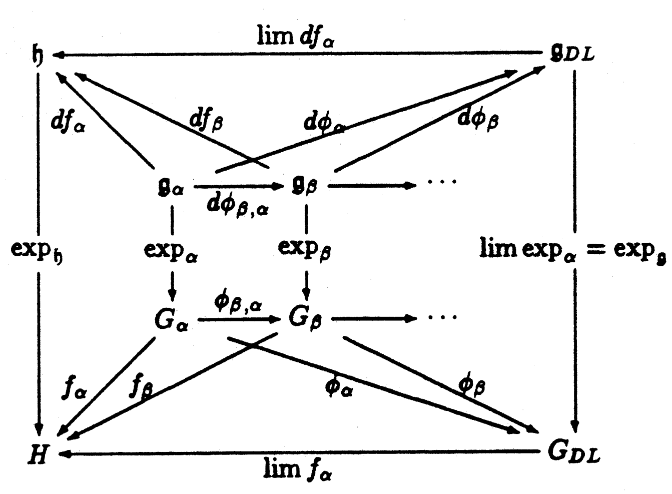
(b) If H is any Lie group modelled on a locally convex space and $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ is a compatible family of analytic group homomorphisms, then there exists a unique analytic group homomorphism $f = \varinjlim f_\alpha : G_{LCDL} \to H$ such that $f \cdot \phi_\alpha = f_\alpha$ for all $\alpha \in A$.

(c) If H is any Lie group modelled on a locally multiplicatively convex Lie algebra, and $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ is a compatible family of analytic group homomorphisms, then the function $f = \varinjlim f_\alpha$ always exists and is a homomorphism of abstract groups. This homomorphism $f : G_{MDL} \to H$ is analytic if and only if it is continuous.

(d) The analogous statement to (c) holds for the NDL and BDL cases. Here we must assume that the Lie group H is modelled on a normed vector space or, respectively, on a Banach space.

Proof of 8.5(a). Consider the function $f = \varinjlim f_\alpha$. This is a group homomorphism, since each f_α is such. It is also continuous, since each f_α is continuous. We'll now show that it is analytic.

Let $\mathfrak{h}$ be the Lie algebra of H, and $df_\alpha : \mathfrak{g}_\alpha \to \mathfrak{h}$ the differential of $f_\alpha : G_\alpha \to H$ ($\alpha \in A$). Then $\{df_\alpha\}_{\alpha\in A}$ is a compatible family of analytic Lie algebra homomorphisms. Hence there exists a unique analytic Lie algebra homomorphism $\varinjlim(df_\alpha) : \mathfrak{g} \to \mathfrak{h}$, such that $\varinjlim(df_\alpha) \cdot \phi_\alpha = df_\alpha$ for each $\alpha \in A$. Consider the diagram



(8.6)

Chase an arbitrary element around (8.6) to see that $\exp_\mathfrak{h} \cdot \varinjlim(df_\alpha) = \varinjlim f_\alpha \cdot \exp_\mathfrak{g}$.

It follows from the definition of the C^ω structure of G_{DL}, that $\varinjlim f_\alpha$ is analytic at 1, and hence everywhere. This diagram chase also shows that

$$\varinjlim(df_\alpha) = d(\varinjlim f_\alpha) \tag{8.7}$$

whenever both $\varinjlim(df_\alpha)$ and $\varinjlim f_\alpha$ exist.

Proof of 8.5(b) when G_{DL} is connected. The argument we use here is similar to the one we used to prove statement (a), above. Use the same diagram, replacing *DL* by *LCDL*.

In this case, we don't *a priori* know that $\varinjlim f_\alpha$ is continuous. We prove continuity at 1 from the results of §6 and the fact that $\varinjlim(df_\alpha)$ is continuous. We've shown that if G is *DL* connected, then the *LCDL* topology makes G into a separately continuous topological group, and $\varinjlim f_\alpha$ is continuous on G_{LCDL}. We have the *LCDL* version of (8.6). It implies the *LCDL* version of (8.7). Combine that with the results of §7 to see that $\varinjlim f_\alpha$ is C^ω.

Proof of 8.5(c) when G_{DL} is connected. Let $\{f_\alpha : G_\alpha \to H\}_{\alpha \in A}$ be as in the statement of Theorem 8.2. Then $\{f_\alpha\}$ is in particular a compatible family of homomorphisms of abstract groups. Hence $\varinjlim f_\alpha$ exists, as a group homomorphism. Using the *MDL* version of (8.6), it is easy to see that $\varinjlim(df_\alpha)$ exists and is continuous precisely when $\varinjlim f_\alpha$ is *MDL*-continuous. In that case, it follows from the *MDL* version of (8.4) that $d(\varinjlim f_\alpha)$ exists, and hence that $\varinjlim f_\alpha$ is C^ω.

Proof of 8.5(d) when G_{DL} is connected. There is an exact *NDL* (resp. *BDL*) analogue of the argument used to prove statement (c), above. This completes the proofs of Theorems 8.1 and 8.5 when G_{DL} is connected. The proofs in general will follow from the results of §9.

Corollary 8.6. *$G_{DL} \hookrightarrow G_{LCDL} \hookrightarrow G_{MDL} \hookrightarrow G_{NDL}$ are analytic group homomorphisms.*

Section 9. Removing the Connectedness Requirement

In §§6, 7 and 8, we needed the hypothesis that G be *DL*-connected. We now show that the results of those three sections remain valid as stated when the connectivity hypothesis is removed. To remove that hypothesis we need Lemma 9.1 below, but the proof of Lemma 9.1 requires the *DL*-connected version of Theorems 8.1 and 8.5.

Lemma 9.1. *Let $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a strict directed system of finite dimensional Lie groups, and let $G = \varinjlim G_\alpha$. Denote by T any one of the topologies DL, LCDL, MDL, NDL or BDL on G. If $g \in G$ then $Ad\, g : G \to G$ is T-continuous.*

Proof. Let $\widetilde{G}$ denote the subgroup of G generated by $\exp(\mathfrak{g})$. The exponential map sends some *DL*-neighborhood of 0 in $\mathfrak{g}$ *onto* some *DL*-neighborhood of 1 in G. It follows that $\widetilde{G}$ is the *DL*-connected component of 1 in G. In particular $\widetilde{G}$ is normal in G.

We must now prove that $\mathrm{Ad}\, g|_{\widetilde{G}}$ is T–continuous.

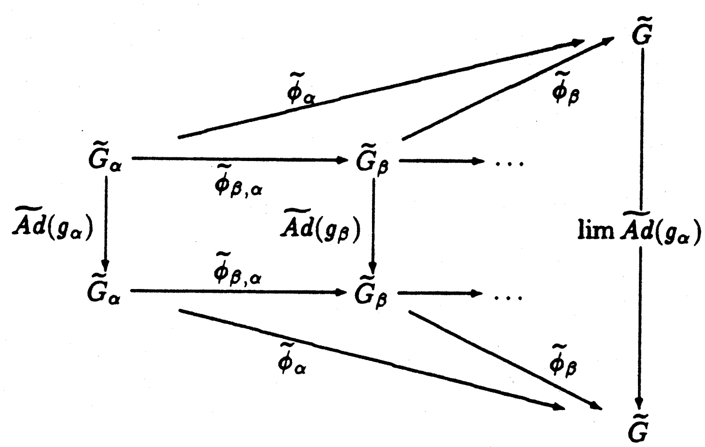
For each index α, set $\widetilde{G}_\alpha = \phi_\alpha^{-1}(\widetilde{G})$, where $\phi_\alpha : G_\alpha \to G$ is the canonical inclusion. We have the maps $\widetilde{\phi}_{\beta,\alpha} := \phi_{\beta,\alpha}|_{\widetilde{G}_\alpha} : \widetilde{G}_\alpha \to \widetilde{G}_\beta$, and $\widetilde{\phi}_\alpha : \widetilde{G}_\alpha \to \widetilde{G}$, and so we obtain a directed system $\{\widetilde{G}_\alpha, \widetilde{\phi}_{\beta,\alpha}\}_{\alpha,\beta \in A}$, which has direct limit $(\widetilde{G}, \{\widetilde{\phi}_\alpha\}_{\alpha \in A})$.

We already proved the results of §§6 and 7, as well as Theorem 8.1, when the limit group is *DL*-connected. We now apply those results to the directed system $\{\widetilde{G}_\alpha, \widetilde{\phi}_{\beta,\alpha}\}_{\alpha,\beta\in A}$, obtaining our assertions for the infinite dimensional Lie group $\widetilde{G}_T$. Conditions (4.3) and (5.2) hold for the original directed system $\{\mathfrak{g}_\alpha, d\phi_{\beta,\alpha}\}$ of Lie algebras. They follow for the new system $\{\widetilde{\mathfrak{g}}_\alpha, d\widetilde{\phi}_{\beta,\alpha}\}$.

Let $g = [g_\alpha] \in G$, say $g = \phi_\alpha(g_\alpha)$ for each $\alpha \geqq \delta$. When $\alpha \geqq \delta$ let $\mathrm{Ad}_\alpha(g_\alpha) : G_\alpha \to G_\alpha$ be the adjoint action of g_α on G_α. The subgroup $\widetilde{G}_\alpha$ is normal in G_α because it is the ϕ_α^{-1}-image of the normal subgroup $\widetilde{G}$ of G. So $\mathrm{Ad}_\alpha(g_\alpha)|_{\widetilde{G}_\alpha}$ is a well defined automorphism of $\widetilde{G}_\alpha$; call it $\widetilde{\mathrm{Ad}}(g_\alpha)$.

Since $\widetilde{\phi}_{\beta,\alpha} \cdot \widetilde{\mathrm{Ad}}(g_\alpha) = \widetilde{\mathrm{Ad}}(g_\beta) \cdot \widetilde{\phi}_{\beta,\alpha}$ for $\delta \leqq \alpha \leqq \beta$ the family $\{\widetilde{\mathrm{Ad}}(g_\alpha)\}_{\alpha\in A,\, \delta\leqq\alpha}$ is compatible.

We proved Theorem 8.5 for the *DL*-connected case, so by (A.4) there is an analytic Lie group homomorphism, $\varinjlim \widetilde{\mathrm{Ad}}(g_\alpha) : \widetilde{G} \to \widetilde{G}$ which completes the commutative diagram



A standard diagram chase shows that $\varinjlim \widetilde{\mathrm{Ad}}(g_\alpha) = \mathrm{Ad}\, g|_{\widetilde{G}}$. It follows that $\mathrm{Ad}\, g : G \to G$ is T-continuous at 1, and hence everywhere. □

The argument of Lemma 9.1 shows that the identity component of G_T coincides with the identity component of G_{DL} for the cases *LCDL*, *MDL* and *NDL* of T.

Corollary 9.2. *Let G and T be as in Lemma 9.1. Take any $g \in G$ and any T-open neighborhood U of 1 in G. Then there exists a T-open neighborhood U_1 of 1 in G such that $gU_1g^{-1} \subset U$.*

Going back to §6, observe that the conclusion of Corollary 9.2 gives us precisely the one missing piece in the proof of Proposition 6.2. This means we now have a proof of that Proposition without using any hypothesis of *DL*-connectedness.

In all the results and constructions of §§6, 7 and 8 we needed hypothesis of *DL*-connectedness, but only because we used it to establish Proposition 6.2. Therefore this hypothesis can now be discarded for all of these results.

SECTION 10. EXAMPLES

Example 4. Direct limits of classical Lie groups. Let $\mathfrak{g}_n$ be as in Example 1, and let G_n be their respective Lie groups. For the instances mentioned explicitly in Example 1, the Lie groups are

$G_n = U(n)$, $O(n)$ and $U(q,n)$ respectively. The embedding $\phi_{n+k,n}$ is given by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix}$ where I is the identity matrix of size $\dim V_{n+k} - \dim V_n$. The group $G_{DL} = G_{LCDL} = \varinjlim_{LCDL} G_n$ is now seen to be a locally convex Lie group modelled on its Lie algebra $\varinjlim_{LCDL} \mathfrak{g}_n$. These groups are often denoted by $U(\infty)$, $O(\infty)$, $U(q,\infty)$, etc.

Example 5. Consider the classical groups $O(n)$ and the homomorphisms $\theta_{n+1,n} : O(n) \to O(n+1)$ given by $\theta_{n+1,n}(g) = \begin{pmatrix} g & 0 \\ 0 & \det g \end{pmatrix}$. This embedding is portrayed in several of M. C. Escher's prints, in particular *Swans*.

Setting $\theta_{n,n} = \mathrm{id}$ and $\theta_{n+k,n} = \theta_{n+k,n+k-1} \cdots \theta_{n+1,n}$ whenever $k > 1$, we obtain a directed system.

The point is that $\varinjlim\{O(n), \theta_{m,n}\}_{n,m\in\mathbb{N}} = SO(\infty)$, which is connected, while in Example 4, $\varinjlim\{O(n), \phi_{m,n}\}_{n,m\in\mathbb{N}} = O(\infty)$, which has two connected components.

Example 6. Norm- and Banach direct limits of classical Lie groups. Using Example 3, each sequence of classical Lie groups in Example 4, and each p with $1 \leqq p \leqq \infty$, gives group $G = \varinjlim G_n$. That group G has the structure of normed Lie group G_{NDL} modelled on the norm direct limit Lie algebra $(\mathfrak{g}, \|\cdot\|_p)$. The compatibility condition (2.1) is satisfied in these cases, so each of the normed Lie groups G_{NDL} can be completed to a Banach Lie group G_{BDL} modelled on $\mathfrak{g}_{BDL}$.

The groups of Examples 4 and 6 have been studied as topological groups by Kolomytsev, Semoilenko, Olshansk'ii and others. For example see [5], [6] and [11]. Here [6] contains some bibliography on the subject.

Example 7. C^∞ functions.

Let Ω be a separable C^∞ manifold, e.g. an open subset of $\mathbb{R}^n$, M a finite-dimensional Lie group with Lie algebra $\mathfrak{m}$. Then $\mathfrak{g} = C^\infty(\Omega, \mathfrak{m})$ and $G = C^\infty(\Omega, M)$ are a topological Lie algebra and group respectively, with the topology of uniform convergence of the functions and their derivatives on compact sets. Here the algebra and group operations are specified pointwise. It is standard that $C^\infty(\Omega, \mathfrak{m})$ is complete with respect to this topology.

For $K \subset \Omega$, K compact, define $C_K^\infty(\Omega, \mathfrak{m}) = \{f \in C^\infty(\Omega, \mathfrak{m}) \mid \operatorname{supp}(f) \subset K\}$ where $\operatorname{supp}(f)$ is the support of the function f. Define $C_c^\infty(\Omega, \mathfrak{m}) = \bigcup_K C_K^\infty(\Omega, \mathfrak{m})$. As Ω is locally compact with a countable basis for open sets, we have a sequence $B_1 \subset B_2 \subset \cdots$ of open sets with $\Omega = \bigcup B_n$ and each $K_i =$ closure B_i compact. Thus, $C_c^\infty(\Omega, \mathfrak{m}) = \bigcup_{K_j} C_{K_j}^\infty(\Omega, \mathfrak{m})$. Give $C_{K_j}^\infty(\Omega, \mathfrak{m})$ the subspace topology inherited from $C^\infty(\Omega, \mathfrak{m})$. Then the inclusion map $C_{K_j}^\infty(\Omega, \mathfrak{m}) \to C_{K_{j+1}}^\infty(\Omega, \mathfrak{m})$ is an isomorphism onto its image. Thus, we can view $C_c^\infty(\Omega, \mathfrak{m})$ as the strict countable direct limit, thus the locally convex direct limit, of the $C_{K_j}^\infty(\Omega, \mathfrak{m})$. The sheaf of analytic functions on $C_c^\infty(\Omega, \mathfrak{m})$ is the direct limit sheaf. Note that this topology on $C_c^\infty(\Omega, \mathfrak{m})$ is in general strictly finer than the topology of uniform convergence on compact sets.

Let $C_{K_j}^\infty(\Omega, M)$ be C^∞ functions on Ω with values in M, whose support lies in K_j. Then, $C_{K_j}^\infty(\Omega, M)$ is a Lie group with Lie algebra $C_{K_j}^\infty(K_j, \mathfrak{m})$. We can now define a differentiable structure on the direct limit group, $C_c^\infty(\Omega, M) = \bigcup C_{K_j}^\infty(\Omega, M)$, modelled on $C_c^\infty(\Omega, \mathfrak{m})$. Let O and U be neighborhoods of 0 and 1 respectively in $\mathfrak{m}$ and M, on which the exponential map is a diffeomorphism.

Then, the exponential map from

$$\tilde{O} = \{f \in C_c^\infty(\Omega, \mathfrak{m}) \mid f \text{ has compact support and } f(\Omega) \subset O\}$$

to

$$\tilde{U} = \{f \in C_c^\infty(\Omega, M) \mid f \text{ has compact support and } f(\Omega) \subset U\}$$

is a homeomorphism. The sheaf of analytic functions on $C_c^\infty(\Omega, M)$ is defined as the direct limit sheaf. The existence of local sections is ensured by the fact that the topology on $C_c^\infty(\Omega, M)$ is given locally by the Lie algebra $C_c^\infty(\Omega, \mathfrak{m})$

Example 8. Lie groups and Lie algebras of operators on a Hilbert space. Let $\mathcal{H}$ be a Hilbert space, not necessarily separable. Let $\{e_i\}_{i\in I}$ be a complete orthonormal set in $\mathcal{H}$. Our indexing set in this example will be the directed set A of all finite subsets of I, with the partial order $\alpha \leqq \beta \iff \alpha \subseteq \beta$.

For each $\alpha \in A$, denote by $\mathcal{H}_\alpha$ the finite-dimensional subspace of $\mathcal{H}$ with basis $\{e_i\}_{i\in\alpha}$.

Let $GL(\mathcal{H}_\alpha)$ denote the group of invertible linear operators on $\mathcal{H}_\alpha$. If $\alpha \leqq \beta$ in A, then $\phi_{\beta,\alpha} : GL(\mathcal{H}_\alpha) \to GL(\mathcal{H}_\beta)$ is the natural inclusion map which identifies $GL(\mathcal{H}_\alpha)$ with the subgroup $\{g \in GL(\mathcal{H}_\beta) \mid g(e_i) = e_i$ whenever $i \in \beta$ but $i \notin \alpha\}$. Thus the directed system $\{GL(\mathcal{H}_\alpha), \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is strict.

The inclusion map $\phi_\alpha : GL(\mathcal{H}_\alpha) \to G(\mathcal{H}) := \varinjlim GL(\mathcal{H}_\alpha)$ is an isomorphism of $GL(\mathcal{H}_\alpha)$ onto its image, which is the subgroup $\{g \in GL(\mathcal{H}) \mid g(e_i) = e_i \text{ for } i \notin \alpha\}$. The limit $G(\mathcal{H})$ is a subgroup of the group $GL(\mathcal{H})$ of all bounded invertible linear operators on $\mathcal{H}$. It consists of those operators which have the form $1 +$ (finite rank).

The choice of embedding makes it clear that the spectral growth condition is satisfied. Hence $G(\mathcal{H})$ with the direct limit topology becomes a Lie group, $G_{DL}(\mathcal{H})$ with Lie algebra $\mathfrak{g}_{DL}(\mathcal{H}) := \varinjlim_{DL} \mathfrak{gl}(\mathcal{H}_\alpha)$. We can also retopologize $G(\mathcal{H})$ using the locally convex Lie algebra $\mathfrak{g}_{LCDL}(\mathcal{H})$. Then $G_{LCDL}(\mathcal{H})$ is a separately continuous locally convex Lie group modelled on its Lie algebra $\mathfrak{g}_{LCDL}(\mathcal{H})$.

For $1 \leqq p \leqq \infty$, we have algebras $(\mathfrak{g}(\mathcal{H}), \|\cdot\|_p)$ as in Examples 2 and 6. Each of them gives us a normed Lie group $(G_{NDL}(\mathcal{H}), \|\cdot\|_p)$. Their completions give us Banach Lie groups $(G_{BDL}(\mathcal{H}), \|\cdot\|_p)$ with Banach Lie algebras $\mathfrak{g}_{BDL}(\mathcal{H})$.

We have as a special case using the uniform operator norm on $\mathfrak{g}(\mathcal{H})$ that the completion is the algebra of compact operators and the completed group is the group of bounded operators of the form $1 +$ (compact).

Example 9. Subgroups and subalgebras of spaces of bounded linear operators. Retain the notation of the previous example. We look at certain subalgebras $\mathfrak{k}_\alpha \subset \mathfrak{gl}(\mathcal{H}_\alpha)$. For example consider $\mathfrak{k}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the Lie algebra of skew-Hermitian operators on $\mathcal{H}_\alpha$. The corresponding Lie group is $K_\alpha = U(\mathcal{H}_\alpha)$, the group of unitary operators on $\mathcal{H}_\alpha$. If the $\{\mathfrak{k}_\alpha, d\phi_{\beta,\alpha}|_{\mathfrak{k}_\alpha}\}$ and $\{K_\alpha, \phi_{\beta,\alpha}|_{K_\alpha}\}$ form directed systems, then the direct limit group K is a Lie group with Lie algebra, $\mathfrak{k}$, the direct limit Lie algebra in the direct limit and various norm topologies. Again with the locally convex topology, K has a differentiable structure modeled on $\mathfrak{k}$, but the group operations are only separately continuous. The normed Lie algebras can be completed to yield Banach Lie algebras and Banach Lie groups. In general, $(G_{BDL}(\mathcal{H}), \|\cdot\|_p) = \{g \in GL(\mathcal{H}) \mid g \text{ preserves a non-degenerate form, and } \|g - 1\|_p < \infty\}$.

For the special case that $\mathfrak{k}_\alpha = \mathfrak{u}(\mathcal{H}_\alpha)$, the limit $\mathfrak{k}$ is the Lie algebra of skew-Hermitian finite rank operators on $\mathcal{H}$ and K is the group of unitary operators of the form $1 +$ (finite rank). The completions of the above with respect to the norm topology yield, respectively, $\mathfrak{k}_{BDL}$, the Lie algebra of skew-Hermitian compact operators and K_{BDL}, the Lie group of unitary operators of the form $1 + (\text{compact})$.

Example 10. The separable case. If $\mathcal{H}$ is separable, the limit Lie algebras and Lie groups of Examples 7 and 8 can be considered as countable direct limits. Indeed, let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal system for $\mathcal{H}$ and define for each $n \in \mathbb{N}$, $\alpha_n = \{i \in \mathbb{N} \mid i \leqq n\}$. Thus the directed set A has a cofinal subsequence $\{\alpha_n\}_{n\in\mathbb{N}}$ and it follows from Lemma 2.6 (*b*) that the direct limit over $\{\alpha_n\}_{n\in\mathbb{N}}$ equals the direct limit over A in each of the direct limit topologies considered here. The classical Banach-Lie algebras and groups of operators studied by de la Harpe [4] all fit into this scheme.

Appendix: Direct Limits and Category Theory

For most results on direct limits of topological vector spaces, topological Lie algebras, or Lie groups, the simplest proofs are obtained by viewing the direct limit as a universal, as in Definition A.3, below. We take that approach in the main body of this paper. At the same time, we try to minimize use of categorical language. In this Appendix, we touch lightly on the subject of direct limits in an arbitrary category, giving those definitions and facts that we need for our proofs. We then try to put our direct limits in proper perspective by proving certain consistency results on direct limits of topological groups, topological vector spaces, and topological Lie algebras.

The concept of "naïve direct limit" (our *DL* limit) of a directed system of topological vector spaces corresponds, in terms of Definition A.3, with the direct limit in the category of topological vector spaces and continuous linear transformations.

The usual concept of "locally convex direct limit" (our *LCDL* limit) corresponds to the direct limit in the category of locally convex vector spaces and continuous linear transformations.

Definition A.1. *Let C be a category and A a directed set. A* directed system $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ *in C indexed by A consists of assignments of* (i) *an object S_α of C for every $\alpha \in A$ and* (ii) *a morphism (arrow) $\phi_{\beta,\alpha}$ of C from S_α to S_β for every $\alpha, \beta \in A$ with $\alpha \leqq \beta$, such that* (a) *each $\phi_{\alpha,\alpha}$ is the identity morphism and* (b) $\phi_{\gamma,\beta} \cdot \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$ *whenever $\alpha \leqq \beta \leqq \gamma$ in A.*

Definition A.2. *Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system in C. Let T be an object of C. Fix an index $\delta \in A$ and a family $\{f_\alpha\}_{\alpha\in A,\, \delta\leqq\alpha}$, where f_α is a morphism in C, from S_α to T, for each $\alpha \geqq \delta$. Then the family $\{f_\alpha\}_{\alpha\in A,}$ is called* compatible *if $f_\beta \cdot \phi_{\beta,\alpha} = f_\alpha$ whenever $\delta \leqq \alpha \leqq \beta$ in A.*

Definition A.3. *Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system in C. The* direct limit *(filtered colimit, filtered right root) of $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in C is a pair $(S, \{\phi_\alpha\}_{\alpha\in A})$ where S is an object of C and ϕ_α is a morphism in C from S_α to S such that*

A.3 (i) *$\phi_\beta \cdot \phi_{\beta,\alpha} = \phi_\alpha$ whenever $\alpha \leqq \beta$ in A, and*

A.3(ii) *for every object T of C and every compatible family of morphisms f_α in C from S_α to T, there is a unique morphism f from S to T in C such that $f \cdot \phi_\alpha = f_\alpha$ for all $\alpha \in A$.*

The direct limit of $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in C is usually denoted $(S, \{\phi_\alpha\}_{\alpha\in A}) = \varinjlim\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ or simply $S = \varinjlim S_\alpha$. The morphism f of A.3(ii) *is called* the direct limit *of $\{f_\alpha\}_{\alpha\in A}$, denoted $f = \varinjlim f_\alpha$.*

It is usual to refer to S as the direct limit of $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and to write $S = \varinjlim S_\alpha$. We'll follow this standard abuse of notation, but it is important to remember that the limit depends on the $\phi_{\beta,\alpha}$ as well as the S_α. See Examples 4 and 5 in Section 10.

Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{T_\alpha, \psi_{\beta,\alpha}\}_{\beta,\alpha\in A}$ be two directed systems in $\mathcal{C}$, both indexed by the same directed set A. Let $(S, \{\phi_\alpha\}_{\alpha\in A}) = \varinjlim\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $(T, \{\psi_\alpha\}_{\alpha\in A}) = \varinjlim\{T_\alpha, \psi_{\beta,\alpha}\}_{\beta,\alpha\in A}$. If $\{g_\alpha : S_\alpha \to T_\alpha\}_{\alpha\in A,\, \delta\leqq\alpha}$ is a family of morphisms in $\mathcal{C}$ which is *compatible* in the sense that $\psi_{\beta,\alpha} \cdot g_\alpha = g_\beta \cdot \phi_{\beta,\alpha}$ whenever $\delta \leqq \alpha \leqq \beta \in A$, then the family $\{\psi_\alpha \cdot g_\alpha : S_\alpha \to T\}_{\alpha\in A,\, \delta\leqq\alpha}$ of morphisms in $\mathcal{C}$, is compatible in the sense of Definition A.2. So its limit morphism exists. We denote it

$$\varinjlim g_\alpha := \varinjlim(\psi_\alpha \cdot g_\alpha). \tag{A.4}$$

Direct limits do not always exist. But when a directed system has a direct limit in a given category, that limit is unique.

In certain categories it is easy to prove that every directed system has a direct limit. For example, this is the case in the category **Set** of all sets (in a given universe) and all functions between sets. Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a directed system in **Set**. Its direct limit $(S, \{\phi_\alpha\}_{\alpha\in A}) = \varinjlim\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is constructed as follows:

A *string* is a set $\{s_\alpha\}$ consisting of an element $s_\alpha \in S_\alpha$ for each $\alpha \geqq \delta$, for some fixed $\delta \in A$ that may depend on $\{s_\alpha\}$, such that $\phi_{\beta,\alpha}(s_\alpha) = s_\beta$ whenever $\delta \leqq \alpha \leqq \beta$ in A. An element $[s_\alpha]$ of S is an equivalence class of strings, two strings $\{s_\alpha\}$ and $\{t_\alpha\}$ being equivalent if they eventually coincide, i.e. if there exists an index γ such that $s_\alpha = t_\alpha$ whenever $\gamma \leqq \alpha$ in A.

For each index $\epsilon \in A$, the canonical map $\phi_\epsilon : S_\epsilon \to S$ is given by $\phi_\epsilon(s_\epsilon) = [s_\alpha]$ where $s_\alpha = \phi_{\alpha,\epsilon}(s_\epsilon)$ whenever $\epsilon \leqq \alpha$ in A. Thus $S = \bigcup_{\alpha\in A} \phi_\alpha(S_\alpha)$.

In the category **Grp** of groups and group homomorphisms, the direct limit $(G, \{\phi_\alpha\}_{\alpha\in A})$ of a directed system $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is given as follows:

For each α in A, denote by S_α the underlying set of G_α, and let $(S, \{\phi_\alpha\}_{\alpha\in A})$ be the direct limit of $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in **Set**. View the group operation $\mu_\alpha : G_\alpha \times G_\alpha \to G_\alpha$ a map from $S_\alpha \times S_\alpha$ to S_α. This defines a compatible family of maps $\{\mu_\alpha\}_{\alpha\in A}$. Hence we obtain a map $\mu = \varinjlim \mu_\alpha : \varinjlim(S_\alpha \times S_\alpha) \to \varinjlim S_\alpha$. But for **Set** it is easy to show that taking direct limit commutes with taking Cartesian product. So $\mu = \varinjlim \mu_\alpha : S \times S \to S$.

Let $\iota_\alpha : G_\alpha \to G_\alpha$ be the map sending each element to its inverse. Again, the ι_α form a compatible family and define a limit map $\iota = \varinjlim \iota_\alpha : S \to S$.

The group G has S as its underlying set, μ as its group operation and ι as the map sending each element to its inverse. This makes group homomorphisms of the maps ϕ_α.

Thus, taking direct limits commutes with taking the forgetful functor from **Grp** to **Set**.

Similarly, one obtains the direct limits of vector spaces, Lie algebras, etc.

In the category **Top** of topological spaces and continuous maps, the direct limit $(T, \{\phi_\alpha\}_{\alpha\in A})$ of a directed system $\{T_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ is constructed as follows:

The underlying set of T and the ϕ_α are obtained as in **Grp**. The topology of T is defined by: $U \subset T$ is open iff $\phi_\alpha^{-1}(U)$ is open in T_α for all α. Again it is clear that taking direct limits commutes with taking the forgetful functor from **Top** to **Set**.

The question becomes more delicate when one considers a category whose objects have an algebraic structure and a topological structure (and perhaps also an analytic one), and these structures are required to be compatible.

For example, given a directed system $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in the category **TopGrp** of all (Hausdorff) topological groups, one has the direct limit $(G, \{\phi_\alpha\}_{\alpha\in A})$ in **Grp**, and also the direct limit $(T, \{\psi_\alpha\}_{\alpha\in A})$ in **Top**. Then G and T have the same underlying set, and each ϕ_α coincides with the corresponding ψ_α when they are both considered as morphisms in **Set**. One must prove that μ and ι are continuous for the direct limit topology.

A directed system $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in a category $\mathcal{C}$, whose objects are defined by a topology and possibly also an algebraic structure, is called *strict* if $\phi_{\beta,\alpha}$ is a homeomorphism of S_α onto $\phi_{\beta,\alpha}(S_\alpha)$ whenever $\alpha \leqq \beta$ in A. Here $\phi_{\beta,\alpha}(S_\alpha)$ carries the subspace topology from S_β.

The existence of a direct limit for every strict direct system in **TopGrp** is proved here by Corollaries A.11 and A.12. Similarly, strict direct limits exist in the category of (Hausdorff) topological Lie algebras and continuous Lie algebra homomorphisms.

For the category of locally convex Lie algebras and continuous Lie algebra homomorphisms, the existence of a direct limit for every strict directed system is given by Proposition 4.1.

Theorem 8.5 (a) says that any strict directed system of finite dimensional Lie groups has a direct limit in the category of (possibly) infinite dimensional Lie groups, and analytic group homomorphisms. Here the concept of infinite dimensional Lie group must be understood in the sense of Definition 8.2.

Theorem 8.5 (b) says that any strict directed system of finite dimensional Lie groups has a direct limit in the category of separately continuous locally convex Lie groups, and analytic group homomorphisms.

We now turn to the proofs of those results on direct limits in **Top** and **TopGrp** which we need in this paper.

Lemma A.5. *Let $(S, \{\phi_\alpha\}) = \varinjlim S_\alpha$ be the direct limit of a strict directed system $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ of topological spaces and continuous maps. Then for every $\alpha \in A$, ϕ_α is a homeomorphism of S_α onto $\phi_\alpha(S_\alpha)$ where the latter carries the subspace topology from S.*

Proof. Each ϕ_α is injective because the $\phi_{\beta,\alpha}$ are injective and because $\phi_\alpha(x) = \phi_\alpha(y)$ implies $\phi_{\beta,\alpha}(x) = \phi_{\beta,\alpha}(y)$ for all $\beta \geqq \alpha$.

ϕ_α is continuous by construction, so we need only prove it open. In other words, given $\alpha \in A$ and an open subset $U_\alpha \subset S_\alpha$, we must find an open subset $U \subset S$ such that

$$\phi_\alpha(U_\alpha) = \phi_\alpha(S_\alpha) \cap U. \tag{A.6}$$

If $\gamma \geqq \alpha$ in A, then let U_γ be the largest open subset W of S_γ such that

$$\phi_{\gamma,\alpha}(U_\alpha) = \phi_{\gamma,\alpha}(S_\alpha) \cap W. \tag{A.7}$$

Since $\phi_{\gamma,\alpha}(U_\alpha)$ is relatively open in S_γ, there exists an open set W satisfying (A.7), and U_γ is the union of all such W.

One can prove $\phi_{\gamma,\beta}^{-1}(U_\gamma) = U_\beta$ for $\alpha \leqq \beta \leqq \gamma$. But we only need the weaker

$$\phi_{\gamma,\beta}(U_\beta) \subset U_\gamma \text{ whenever } \alpha \leqq \beta \leqq \gamma. \tag{A.8}$$

To prove (A.8) we take any open subset Z of S_γ such that $\phi_{\gamma,\beta}(U_\beta) = \phi_{\gamma,\beta}(S_\beta) \cap Z$, and we prove (A.7) with Z in place of W.

$\phi_{\beta,\alpha}(U_\alpha) \subset U_\beta$ implies $\phi_{\gamma,\beta}(\phi_{\beta,\alpha}(U_\alpha)) \subset \phi_{\gamma,\beta}(U_\beta) \subset Z$, so $\phi_{\gamma,\alpha}(U_\alpha) \subset \phi_{\gamma,\alpha}(S_\alpha) \cap Z$.

Assume that $z \in \phi_{\gamma,\alpha}(S_\alpha) \cap Z$. Then $z \in Z$ and $z = \phi_{\gamma,\alpha}(x)$ for some $x \in S_\alpha$. Hence $z = \phi_{\gamma,\beta}(y)$ for $y = \phi_{\beta,\alpha}(x) \in S_\beta$. By the hypothesis on Z this $y \in U_\beta$ and hence $x \in U_\alpha$. This shows that $\phi_{\gamma,\alpha}(S_\alpha) \cap Z \subset \phi_{\gamma,\alpha}(U_\alpha)$, so $\phi_{\gamma,\alpha}(S_\alpha) \cap Z = \phi_{\gamma,\alpha}(U_\alpha)$. Therefore $\phi_{\gamma,\beta}(U_\beta) \subset Z \subset U_\gamma$, and (A.8) is proved.

Let $U = \bigsqcup_{\beta \geqq \alpha} \phi_\beta(U_\beta) \subset S$. Using (A.8) now U is an open set which satisfies (A.6). □

Lemma A.9. *Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a strict directed system of topological spaces and continuous maps. Let $(S, \{\phi_\alpha\}_{\alpha\in A}) = \varinjlim\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$. Let Y be a topological space, and $h : Y \to S$ a map that we do not require to be continuous. Then there exists a strict directed system $\{Y_\alpha, \eta_{\beta,\alpha}\}_{\alpha,\beta\in A}$ of topological spaces and continuous maps such that $Y = \varinjlim Y_\alpha$ and $h = \varinjlim h_\alpha$ where $h_\alpha : Y_\alpha \to S_\alpha$. The map h is continuous if and only if all the h_α are continuous.*

Proof. For each $\alpha \in A$, set $Y_\alpha = h^{-1}(\phi_\alpha(S_\alpha))$, with the subspace topology it inherits from Y. Define $\eta_{\beta,\alpha} : Y_\alpha \to Y_\beta$ and $\eta_\alpha : Y_\alpha \to Y$ to be the natural inclusion maps. Then $(Y, \{\eta_\alpha\}_{\alpha\in A}) = \varinjlim\{Y_\alpha, \eta_{\beta,\alpha}\}_{\alpha,\beta\in A}$. Define the h_α by $h_\alpha(y) = h \cdot \eta_\alpha(y)$. □

Proposition A.10. *Let $\{S_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ and $\{T_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be two directed systems of topological spaces and continuous maps. Then*

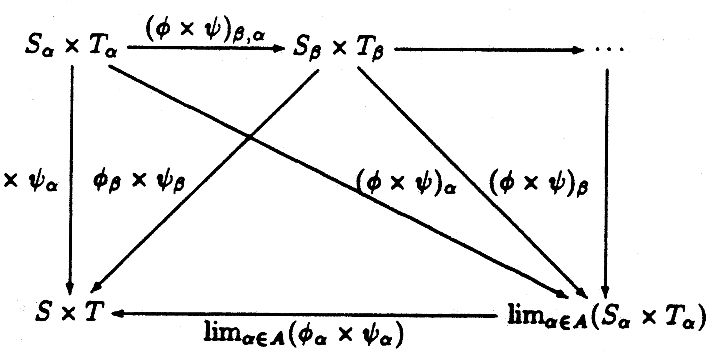
(a) $\varinjlim_{(\alpha,\beta)\in A\times A} (S_\alpha \times T_\beta) = \varinjlim_{\alpha\in A} (S_\alpha \times T_\alpha)$.

(b) *The inclusion $\iota : \varinjlim_{\alpha\in A} (S_\alpha \times T_\alpha) \to (\varinjlim_{\alpha\in A} S_\alpha) \times (\varinjlim_{\alpha\in A} T_\alpha)$ is continuous.*

(c) *If the directed systems $\{S_\alpha, \phi_{\beta,\alpha}\}$ and $\{T_\alpha, \psi_{\beta,\alpha}\}$ are both strict, then ι is a homeomorphism, so $\varinjlim_{\alpha\in A} (S_\alpha \times T_\alpha) = (\varinjlim_{\alpha\in A} S_\alpha) \times (\varinjlim_{\alpha\in A} T_\alpha)$ as topological spaces.*

Proof. The first statement follows from the fact that the set $\{(\alpha,\alpha) : \alpha \in A\}$ is cofinal in the product directed set $A \times B$.

The two spaces of the second statement have the same underlying set. Let $S = \varinjlim S_\alpha$ and $T = \varinjlim T_\alpha$, and consider the commutative diagram.

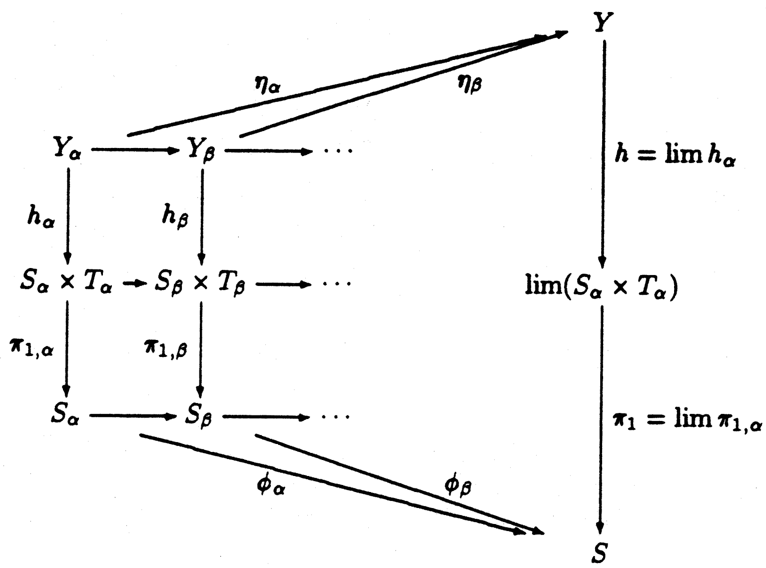


Since the $\phi_\alpha \times \psi_\alpha$ form a compatible family of continuous maps, there is a unique continuous map $\varinjlim(\phi_\alpha \times \psi_\alpha)$. This map is obviously the natural inclusion, which is the identity.

To prove (c) we use the characterization of the product topology as a universal. The product topology on $S \times T$ is the unique topology satisfying: whenever Y is a topological space and $h : Y \to$

$S \times T$ is a map, then h is continuous if and only if both the maps $\pi_1 \cdot h : Y \to S$ and $\pi_2 \cdot h : Y \to T$ are continuous. Here π_1 and π_2 are the natural projections from $S \times T$ to its factors.

To show that $\varinjlim (S_\alpha \times T_\alpha)$ has this property, consider any topological space Y and any map $h : Y \to \varinjlim (S_\alpha \times T_\alpha)$. Construct the strict directed system $\{Y_\alpha, \eta_{\beta,\alpha}\}_{\alpha,\beta \in A}$ and the compatible family of maps $\{h_\alpha : Y_\alpha \to S_\alpha \times T_\alpha\}_{\alpha \in A}$ such that $h = \varinjlim h_\alpha$, as in Lemma A.9. Let $\pi_{1,\alpha} : S_\alpha \times T_\alpha \to S_\alpha$ and $\pi_{2,\alpha} : S_\alpha \times T_\alpha \to T_\alpha$ denote the natural projections. For $i = 1, 2$ and $\alpha \in A$, set $f_{i,\alpha} = \phi_\alpha \cdot \pi_{i,\alpha} \cdot h_\alpha$, obtaining a compatible family $\{f_{i,\alpha}\}_{\alpha \in A}$ whose limit is $\varinjlim f_{i,\alpha} = \pi_i \cdot h$.



The maps $\pi_i \cdot h$ are continuous for $i = 1,\ 2$, exactly when the $f_{i,\alpha}$ are continuous for each i and each α. Since the ϕ_α are continuous, and $S_\alpha \times T_\alpha$ has the product topology, the $f_{i,\alpha}$ are continuous if and only if the h_α are continuous. But $h = \varinjlim h_\alpha$, so h is continuous if and only if the h_α are all continuous.

We conclude that h is continuous if and only if $\pi_i \cdot h$ is continuous for $i = 1,\ 2$. Therefore the direct limit topology $\varinjlim (S_\alpha \times T_\alpha)$ coincides with the product topology of $S \times T$. □

The content of the following Corollary is that direct limit commutes with the forgetful functor in certain categories. We need that basic fact in dealing with our categories of locally convex Lie algebras and Lie groups.

Corollary A.11.

(a) *Let $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a strict directed system of topological groups and continuous group homomorphisms. Then this system has a direct limit $(G, \{\phi_\alpha\}_{\alpha \in A})$ in the category of topological groups and continuous group homomorphisms. Let $\widetilde{G}$ denote the underlying topological space of G. Then $(\widetilde{G}, \{\phi_\alpha\})$ is the topological space direct limit of the system.*

(b) *Let $\{V_\alpha, \psi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a strict directed system of topological vector spaces and continuous linear maps. Then its topological vector space direct limit $(V, \{\phi_\alpha\})$ always exists. The pair $(\widetilde{V}, \{\phi_\alpha\})$, where $\widetilde{V}$ is the topological space underlying V, coincides with the topological space direct limit of the given system.*

(c) *Let $\{\mathfrak{g}_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a strict directed system of topological Lie algebras and continuous*

Lie algebra homomorphisms. Then the topological Lie algebra direct limit $(\mathfrak{g}, \{\phi_\alpha\})$ always exists. The pair $(\widetilde{\mathfrak{g}}, \{\phi_\alpha\})$, where $\widetilde{\mathfrak{g}}$ is the topological space underlying $\mathfrak{g}$, coincides with the topological space direct limit of the given system.

Proof. (a) Let S be the direct limit of $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ in the category of topological spaces and continuous maps. Then S and G have the same underlying set and the identity map $S \to G$ is continuous. It remains to be proved that the topology of S is compatible with the limit group structure, in other words, that the group operations of G are continuous when they are considered as maps $S \times S \to S$.

For each $\alpha \in A$, let $\sigma_\alpha : G_\alpha \times G_\alpha \to S$ be the continuous map $\sigma_\alpha(x, y) = \phi_\alpha(x \cdot y^{-1})$. The family $\{\sigma_\alpha\}_{\alpha\in A}$ is compatible so there is a unique continuous map $\sigma = \varinjlim \sigma_\alpha$ such that $\sigma \cdot \phi_\alpha = \sigma_\alpha$ for all α. The domain of σ is the topological space direct limit of the $G_\alpha \times G_\alpha$, which is the same as $S \times S$ by Proposition A.10(c). Thus $(x, y) \mapsto x \cdot y^{-1}$ is a continuous map $S \times S \to S$.

(b) Addition is a continuous map $S \times S \to S$, by (a). The base field F can be considered as the direct limit $\varinjlim_{\alpha\in A}(F_\alpha, \{\iota_{\beta,\alpha}\})$ where each F_α is a copy of F and each $\iota_{\beta,\alpha}$ is the identity. This permits us to apply Proposition A.10(c) to the topological space direct limit $\varinjlim(F_\alpha \times V_\alpha) = F \times S$, and so to prove that scalar multiplication is a continuous map $F \times S \to S$ by an argument similar to the one used for the group operation.

(c) The argument of (b) is essentially the same in any category of topological spaces with a continuous binary operation, whose morphisms are all continuous maps that respect the binary operation. So it proves (c) as well. □

Ordinarily the definition of "topological group" includes the Hausdorff condition. We finish by checking that the Hausdorff condition survives the topological group strict direct limit.

Corollary A.12. *Let $\{G_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta\in A}$ be a strict directed system of (Hausdorff) topological groups and continuous group homomorphisms. Then $G = \varinjlim G_\alpha$ is Hausdorff. In particular, the strict direct limit of Hausdorff topological vector spaces is a Hausdorff topological vector space, and similarly for topological Lie algebras.*

Proof. Let $0 \neq x \in G$. Let α be any index such that $\phi_\alpha^{-1}(x)$ is not empty. Since ϕ_α is injective, there exists a neighborhood U_α of 0 in G_α such that $\phi_\alpha^{-1}(x) \notin U_\alpha$. Let U be the open subset of G which is constructed from U_α as in the proof of Lemma A.5. Then U is a neighborhood of 0 in G. Since $\phi_\alpha^{-1}(U) = U_\alpha$ we have that $x \notin U$. That is the T_0 separation condition – points are closed. But T_0 implies Hausdorff for topological groups. □

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