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Admissible Representations and Geometry of Flag Manifolds

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ABSTRACT. We describe geometric realizations for various classes of admissible representations of reductive Lie groups. The representations occur on partially holomorphic cohomology spaces corresponding to partially holomorphic homogeneous vector bundles over real group orbits in complex flag manifolds. The representations in question include standard tempered and limits of standard tempered representations, and representations induced from finite dimensional representations of real parabolic subgroups.

Section 1. Introduction.

Harmonic analysis on Lie groups and their homogeneous spaces has been guided and influenced by various geometric constructions of unitary representations. Those unitary representations are the building blocks for the extensions of classical Fourier analysis relevant to the analytic problems in question. Here I'll try to indicate some aspects of the background, concentrating on the interplay between geometry and analysis, I'll indicate some extensions that now seem worth writing down, and I'll mention some interesting open problems.

The best known geometric realization of group representations is the Bott–Borel–Weil Theorem from the 1950's ([**2**], [**19**]). If G is a compact connected Lie group and T is a maximal torus, then a choice $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of positive root system defines a G-invariant complex manifold structure on G/T by: $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ represents the holomorphic tangent space. Now fix that structure and let $\lambda \in i\mathfrak{t}_0^*$ be integral, that is, e^λ is a well defined character of T. View e^λ as a representation of T on a 1-dimensional vector space E_λ and let $\mathbb{E}_\lambda \to G/T$ denote the associated homogeneous holomorphic hermitian line bundle. We write

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$\mathcal{O}(\mathbb{E}_\lambda) \to G/T$ for the sheaf of germs of holomorphic sections of $\mathbb{E}_\lambda \to G/T$. The group G acts on everything here, including the cohomologies $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda))$. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. The Bott–Borel–Weil Theorem says

THEOREM. *If $\lambda + \rho$ is singular then every $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda)) = 0$. Now suppose that $\lambda + \rho$ is regular, let w denote the unique Weyl group element such that $\langle w(\lambda + \rho), \alpha \rangle > 0$ for all $\alpha \in \Phi^+$, and let $\ell(w)$ denote its length as a word in the simple root reflections. Then* (i) *$H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda)) = 0$ for $q \neq \ell(w)$, and* (ii) *G acts irreducibly on $H^{\ell(w)}(G/T; \mathcal{O}(\mathbb{E}_\lambda))$ by the representation with highest weight $w(\lambda + \rho) - \rho$.*

In the Bott–Borel–Weil Theorem, $\ell(w)$ can be described as the number of positive roots that w carries to negative roots, the representation of G with highest weight $w(\lambda + \rho) - \rho$ can be described as the discrete series representation with Harish–Chandra parameter $w(\lambda + \rho)$, and, by Kodaira–Hodge Theory, $H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda))$ is naturally G-isomorphic to the space of harmonic differential forms of bidegree $(0, q)$ on G/T with values in $\mathbb{E}_\lambda$.

In 1965 Kostant and Langlands independently conjectured an analog of the Bott–Borel–Weil Theorem for connected noncompact semisimple Lie groups with finite center. The conjecture was proved in the 1970's in two stages by Schmid ([**23**], [**25**]), and I extended the result (also in the 70's) to general semisimple Lie groups [**31**]. The representations in question there are the *discrete series* representations of G. They are the fundamental building blocks for the *tempered representations* of G, which in turn are the representations that enter into the Plancherel formula for G. My structure theory for the geometry of real group orbits $G(z) \subset Z \cong G_{\mathbb{C}}/P$ on complex flag manifolds from the late 1960's [**30**] also led [**31**] to corresponding geometric realizations for all standard tempered representations of general semisimple Lie groups G. This followed a line of attack that in retrospect was modelled on the Kostant–Kirillov–Souriau theory of geometric quantization.

In the context of semisimple Lie groups, the theory of geometric quantization seemed to founder on several seemingly intractable technical problems. Typically these involved questions of closed range or of vanishing for cohomology except in a particular degree, especially in regard to representations whose infinitesimal character was singular or even just not very nonsingular.

By the middle 1970's many mathematicians began to look for alternatives to or variations on standard geometric quantization. Methods involving varying polarizations or structure group liftings had specialized success, and the derived functor modules of Vogan and Zuckerman [**29**] took a central position in the representation theory of semisimple Lie Groups G. Those derived functor modules are simultaneously modules for the Lie algebra $\mathfrak{g}_0$ of G and for the maximal compact subgroup K of G, but are not G–modules. The passage from $(\mathfrak{g}_0, K)$–modules to G–modules, called *globalization*, was understood by Schmid [**26**] in the middle 1980's in a form that turned out to be suitable for geometric quantization. Schmid and I [**27**] used exactness of the maximal globalization functor of [**26**] to make a change of polarization argument, starting with the tempered case, which had become the case of a maximally real polarization. In the setting of real group orbits on the flag manifold $X \cong G_{\mathbb{C}}/B$ of Borel subalgebras of $\mathfrak{g}$, this gave us the connection between hyperfunction quantization and the derived functor modules of Vogan and Zuckerman. That settled the technical problems, mentioned above, for geometric quantization on $X \cong G_{\mathbb{C}}/B$, and at the same time identified the resulting representations.

Of course much of the geometric interest in this requires more general flag manifolds than the flag of Borel subalgebras of $\mathfrak{g}$. There is some work on pushing the hyperfunction quantization method down from the flag $X \cong G_{\mathbb{C}}/B$ to a more general flag manifold $W \cong G_{\mathbb{C}}/P$. This was done in [**31**] for standard tempered representations and certain well behaved (measurable integrable – defined in §2) G–orbits, in [**33**] for the realizations of discrete series representations that are "closest" to the realization of holomorphic discrete series as spaces of holomorphic sections of vector bundles, and in [**34**] for finite rank bundles over measurable open orbits. In this paper we show how those results all fit into a common framework.

I am not going to discuss localization methods here, but rather just indicate some of the work in that area. It starts, of course with the seminal work [**1**] of Beilinson and Bernstein. There are many unpublished results of Bernstein and Miličić, or at least I have this impression from Miličić. There is some work ([**9**],[**10**]) of Hecht, Miličić, Schmid and myself in which we draw the connection between $\mathcal{D}$–module realizations of representations and realizations by Zuckerman derived functor modules, and draw consequences for completeness, vanishing and irreducibility. There are papers of Hecht and Taylor ([**11**], [**12**], [**13**]) where a minimal–globalization form of localization is developed and applied to $\mathfrak{n}$–homology and an elegant geometric character formula. Finally, there is work of Kashiwara, Schmid, Vilonen and many others which would take too much space to catalog.

In Section 2 we specify our class of real Lie groups, and we recall the basic facts [**30**] concerning real group orbits on complex flag manifolds. We concentrate on the type of orbit that comes into the geometric constructions of representations. Those orbits are the measurable open orbits for the Dolbeault cohomology realization of representations such as those of the discrete series, measurable integrable orbits for the partially holomorphic cohomology realization of representations such as those of the various tempered series.

In Section 3 we recall the solution ([**25**] for connected semisimple groups of finite center, [**31**] for more general groups) to the Kostant–Langlands Conjecture. This realizes relative discrete series representations on spaces of square integrable harmonic bundle–valued forms.

In Section 4 we recall the corresponding result [**31**] for the various series of standard tempered representations. They are realized on spaces of square integrable partially harmonic forms with values in a partially holomorphic vector bundle. Those representations are the ones that enter into the Plancherel formula.

In Section 5 we show how one obtains (partial) Dolbeault coholomogy realizations of standard tempered representations on partially holomorphic negative vector bundles. This material was essentially known ([**22**], [**31**], [**27**]) and we just put it together.

In Section 6 we combine methods of [**56**], [**64**] and [**65**] to describe holomorphic cohomology realizations over measurable open orbits for several classes of representations, not necessarily tempered. In Section 7 we apply the results of §6 to the holomorphic arc components of a measurable integrable orbit, obtaining corresponding partially holomorphic cohomology realizations. Finally, in Section 8, we list some important open questions for this circle of ideas.

Section 2. Real Group Orbits on Complex Flag Manifolds.

We recall some of the main points of [**30**] that we need to describe our geometric constructions of representations, specifically the constructions in [**31**] and some new constructions. If G is a real Lie group then $\mathfrak{g}_0$ denotes its real Lie algebra, $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$, G^0 is the topological component of the identity of G, and if $A, B \subset G$ are subgroups then $Z_A(B)$ denotes the centralizer of B in A and Z_B denotes the center $Z_B(B)$ of B. We say that G is *reductive* if $\mathfrak{g}_0$ is direct sum of a semisimple ideal and a commutative ideal. If G is reductive then $Int(\mathfrak{g})$ denotes the group of inner automorphisms of $\mathfrak{g}$, group generated by the $\exp(ad(\xi))$ for $\xi \in \mathfrak{g}$. It is the complexification of the adjoint group $Ad(G^0)$.

Here we work with the class of *general semisimple Lie groups*, consisting of all real reductive Lie groups G such that

$$\begin{aligned} &\text{if } g \in G \text{ then } Ad(g) \text{ is an inner automorphism of } \mathfrak{g} \\ &G \text{ has a closed normal abelian subgroup } Z \text{ such} \\ &\quad \text{that } Z \subset Z_G(G^0) \text{ and } |G/ZG^0| < \infty. \end{aligned} \tag{2.1}$$

The *Harish–Chandra class* of reductive Lie groups is the case where G/G^0 is finite and $[G^0, G^0]$ has finite center. As in the case of Harish–Chandra class groups, the first part of (2.1) says that irreducible admissible representations of G have well defined infinitesimal characters. The second part says that irreducible admissible representations of G more or less have central characters – that if π is any such representation then the restriction of π to the commutative group ZZ_{G^0} is a sum that involves at most $|G/ZG^0| < \infty$ distinct quasicharacters.

The first condition of (2.1) also says that G has a well defined natural action on all complex flag manifolds for $\mathfrak{g}$. Specifically, if $\mathfrak{q}$ is any parabolic subalgebra of $\mathfrak{g}$ and if $g \in G$ it says that $Ad(g)\mathfrak{q}$ is $Int(\mathfrak{g})$–conjugate to $\mathfrak{q}$. Now G acts on the flag manifold W consisting of all $Int(\mathfrak{g})$–conjugates of $\mathfrak{q}$ by $g : Ad(g_1)\mathfrak{q} \mapsto Ad(gg_1)\mathfrak{q}$. We identify W with the compact complex manifold $Int(\mathfrak{g})/Q$ where Q is the parabolic subgroup of $Int(\mathfrak{g})$ that is the normalizer of $\mathfrak{q}$.

The subgroup $Z_G(G^0) \subset G$ acts trivially on the complex flag manifold W, so for purposes of G–orbit structure we may replace G by the group $\overline{G} = G/Z_G(G^0)$ of Harish–Chandra class. In the remainder of §2 we make that replacement, but we will have to make the distinction in §§3 and 4.

With the replacement just described, $G \subset G_{\mathbb{C}} = Int(\mathfrak{g})$ and $W = G_{\mathbb{C}}/Q$ with G acting as a subgroup of $G_{\mathbb{C}}$. If $w = gQ \in G_{\mathbb{C}}/Q = W$ then we will write Q_w and $\mathfrak{q}_w$ for the isotropy subgroup gQg^{-1} of $G_{\mathbb{C}}$ at w and the isotropy subalgebra of $\mathfrak{g}$ there. We write τ for complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$.

The intersection of any two parabolic subalgebras of $\mathfrak{g}$ contains a Cartan subalgebra. From this,

$$(2.2) \qquad \mathfrak{q}_w \cap \tau\mathfrak{q}_w \text{ contains a } \tau\text{–stable Cartan subalgebra } \mathfrak{h} \text{ of } \mathfrak{g}.$$

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ denote the corresponding root system. There exist a positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ and a subset $\Psi \subset \Pi$ of the corresponding system of simple roots such that

$$(2.3) \quad \begin{aligned} &\mathfrak{q}_w = \mathfrak{q}_\Psi = \mathfrak{q}_\Psi^n + \mathfrak{q}_\Psi^r, \text{ nilradical plus reductive complement, where} \\ &\mathfrak{q}_\Psi^n = \sum_{\Psi^n} \mathfrak{g}_\alpha \text{ and } \mathfrak{q}_\Psi^r = \mathfrak{h} + \sum_{\Psi^r} \mathfrak{g}_\alpha \text{ with} \\ &\Psi^r = \{\alpha \in \Phi \mid \alpha \in \mathrm{span}(\Psi)\} \text{ and } \Psi^n = \{\alpha \in \Phi \mid \alpha \in -\Phi^+ \text{ but } \alpha \notin \Psi^r\} \end{aligned}$$

Since $\mathfrak{g}_0$ has only finitely many G–conjugacy classes of Cartan subalgebras $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$, and since $\Phi(\mathfrak{g}, \mathfrak{h})$ admits only finitely many subsystems of positive roots, it follows that there are only finitely many G–orbits on W. In particular there are open orbits, and the union of the open orbits is dense. It also follows that there is just one closed orbit, necessarily the lowest dimensional orbit, and that the closed orbit is in the closure of every orbit. The complexification of the isotropy subalgebra of $\mathfrak{g}_0$ at $w \in W$ is $(\mathfrak{g}_0 \cap \mathfrak{q}_w)_{\mathbb{C}} = \mathfrak{q}_w \cap \tau\mathfrak{q}_w$, sum of

$$(2.4) \quad \begin{aligned} &\text{nilradical: } (\mathfrak{q}_\Psi^n \cap \tau\mathfrak{q}_\Psi^r) + (\mathfrak{q}_\Psi^r \cap \tau\mathfrak{q}_\Psi^n) + (\mathfrak{q}_\Psi^n \cap \tau\mathfrak{q}_\Psi^n) = \\ &\qquad \left(\sum_{\Psi^n \cap \tau\Psi^r} + \sum_{\Psi^r \cap \tau\Psi^n} + \sum_{\Psi^n \cap \tau\Psi^n} \right) \mathfrak{g}_\alpha \\ &\text{reductive: } (\mathfrak{q}_\Psi^r \cap \tau\mathfrak{q}_\Psi^r) = \mathfrak{h} + \sum_{\Psi^r \cap \tau\Psi^r} \mathfrak{g}_\alpha. \end{aligned}$$

In particular $|\Psi^n \cap \tau\Psi^n|$ is the real codimension of $G(w)$ in W, and $G(w)$ is open in W just when $\Psi^n \cap \tau\Psi^n$ is empty.

One can be more specific. The real Cartan subalgebra $\mathfrak{h}_0$ has a unique decomposition $\mathfrak{h}_0 = \mathfrak{h}_T + \mathfrak{h}_A$ where the roots are pure imaginary on $\mathfrak{h}_T$ and real on $\mathfrak{h}_A$. Those are the ± 1 eigenspaces on $\mathfrak{h}_0$ for a Cartan involution θ of $\mathfrak{g}_0$ that stabilizes $\mathfrak{h}_0$. Let $\mathfrak{k}_0$ denote the fixed point set of θ on $\mathfrak{g}_0$, Lie algebra of the maximal compact subgroup $K = G^\theta$ of G. These are equivalent: (i) $\mathfrak{h}_T$ is a Cartan subalgebra of $\mathfrak{k}_0$, (ii) $\mathfrak{h}_T$ contains a regular element of $\mathfrak{g}_0$, (iii) there is a system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ of positive roots such that $\tau\Phi^+ = -\Phi^+$. The orbit $G(w)$ is open in W precisely when some Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0 \cap \mathfrak{q}_w$, (which necessarily maximizes $\dim \mathfrak{h}_T$), has a positive root system Φ^+ such that (2.3) holds and $\tau\Phi^+ = -\Phi^+$. Since any two maximally compact Cartan subalgebras of $\mathfrak{g}_0$ are G–conjugate, now the open orbits are enumerated by a double coset space $W_K \backslash W_{\mathfrak{g}} / W_{\mathfrak{q}^r}$ of Weyl groups. Note that $W_K = W_G$ catches the action of the various components of G.

The open orbits that seem to enter most strongly into representation theory are the *measurable open orbits.* They are the open orbits $G(w) \subset W$ that carry a G–invariant volume element. If that is the case, then the invariant volume element is the volume element of a G–invariant, possibly indefinite, kaehler metric on the orbit, and the isotropy subgroup $G \cap P_w$ is the centralizer in G of a (compact) torus subgroup of G. In terms of Lie algebras, measurable open orbits are characterized by the following equivalent conditions: (i) $\mathfrak{q}_w \cap \tau\mathfrak{q}_w$ is reductive, i.e. $\mathfrak{q}_w \cap \tau\mathfrak{q}_w = \mathfrak{q}_w^r \cap \tau\mathfrak{q}_w^r$, (ii) $\mathfrak{q}_w \cap \tau\mathfrak{q}_w = \mathfrak{q}_w^r$, (iii) $\tau\Psi^r = \Psi^r$, and $\tau\Psi^n = -\Psi^n$. We know just when this happens: if one open G–orbit on W is measurable, then they all are measurable; and the open G–orbits on W are measurable if and only if $\tau\mathfrak{q}$ is conjugate to the parabolic subalgebra opposite to $\mathfrak{q}$. So in particular the open orbits are measurable in several important situations: the case $\operatorname{rank} K = \operatorname{rank} G$ and the case where $\mathfrak{q}$ is a Borel subalgebra (that is, Ψ is empty) of $\mathfrak{g}$.

There are other useful conditions, mostly automatic for measurable open orbits. For example, an orbit $G(w) \subset W$ is *integrable* if $\mathfrak{q}_w + \tau\mathfrak{q}_w$ is a subalgebra of $\mathfrak{g}$. Let $\mathfrak{u} = \mathfrak{q}_w^n \cap \tau\mathfrak{q}_w^n$ and let $\mathfrak{v}$ denote the normalizer of $\mathfrak{u}$ in $\mathfrak{g}$. Then the following conditions are equivalent to integrality of $G(w)$: (i) $\mathfrak{q}_w + \tau\mathfrak{q}_w = \mathfrak{v}$, (ii) $\mathfrak{q}_w \subset \mathfrak{v}$, (iii) $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{v}$, and (iv) $\mathfrak{q}_w + \tau\mathfrak{q}_w$ is an algebra and $\mathfrak{u}$ is its nilpotent radical. See [**30**, Theorem 7.10] for a complete analysis of integrable orbits.

A *holomorphic arc* in the orbit $G(w) \subset W$ is a holomorphic map from the unit disk in $\mathbb{C}$ to W with image in $G(w)$. A *chain of holomorphic arcs* in $G(w)$ means a sequence $\{f_1, \cdots, f_k\}$ of holomorphic arcs in $G(w)$ such that the image of f_i meets the image of f_{i+1} for $1 \leqq i < k$. The *holomorphic arc components* of $G(w)$ are the equivalence classes of elements of $G(w)$ under the relation: $w_1 \sim w_2$ if there is a chain $\{f_1, \cdots, f_k\}$ of holomorphic arcs in $G(w)$ such that w_1 is in the image of f_1 and w_2 is in the image of f_k. Any connected complex submanifold of W contained in $G(w)$ is contained in a holomorphic arc component. If S is a holomorphic arc component of $G(w)$ and $g \in G$ such that $g(S)$ meets S then $g(S) = S$. It follows that the G–normalizer $N_G(S) = \{g \in G \mid g(S) = S\}$ is a Lie subgroup of G that is transitive on S. In particular S is an embedded C^ω submanifold of W. This notion is nicely set up for combining holomorphic and real induction of group representations, but the catch is that holomorphic arc components might not be complex submanifolds.

Fix an orbit $G(w) \subset W$ and let S_w denote the holomorphic arc component of w. If $g \in G$ then the holomorphic arc component of $g(w)$ is $S_{g(w)} = gS_w$, and $N_G(S_{g(w)}) = gN_G(S_w)g^{-1}$. Write $\mathfrak{n}_G(S_w)_0$ for the real Lie algebra of $N_G(S_w)$, $\mathfrak{n}_G(S_w)$ for its complexification, and $N_G(S_w)_{\mathbb{C}}$ for the corresponding complex analytic subgroup of $G_{\mathbb{C}}$. Now we need a certain τ–stable subspace $\mathfrak{m}_G(S_w)$ of $\mathfrak{g}$. The linear form $\delta_w = \sum_{\Psi^n \cap \tau\Psi^n} \alpha : \mathfrak{h} \to \mathbb{C}$ defines a τ–stable parabolic subalgebra $\mathfrak{s}_G(S_w) = \mathfrak{s}_G(S_w)^n + \mathfrak{s}_G(S_w)^r$, where $\mathfrak{s}_G(S_w)^n = \sum_{\langle \alpha, \delta_w \rangle > 0} \mathfrak{g}_\alpha$ and $\mathfrak{s}_G(S_w)^r = \mathfrak{h} + \sum_{\alpha \perp \delta_w} \mathfrak{g}_\alpha$. Now $\mathfrak{m}_G(S_w) = \mathfrak{s}_G(S_w) + \sum_\Gamma \mathfrak{g}_\alpha$ where $\Gamma = \{\alpha \in \Phi \mid -\alpha \notin \Psi^n \cap \tau\Psi^n, \langle \alpha, \delta_w \rangle < 0, \alpha + \tau\alpha \notin \Phi\}$. Then [**30**, Theorem 8.9] the following are equivalent: (i) the holomorphic arc components of $G(w)$ are complex submanifolds of W, (ii) $\mathfrak{n}_G(S_w) \subset \mathfrak{q}_w + \tau\mathfrak{q}_w$, (iii) $\mathfrak{n}_G(S_w) = \mathfrak{m}_G(S_w)$, and (iv) $\mathfrak{m}_G(S_w)$ is a subalgebra of $\mathfrak{g}$. When those conditions hold, we say that the orbit $G(w)$ is *partially complex*. In particular, if Γ is empty, then $\mathfrak{n}_G(S_w) = \mathfrak{s}_G(S_w) = \mathfrak{m}_G(S_w)$ and $G(w)$ is partially complex.

We will say that the orbit $G(w) \subset W$ is of *flag type* if $N_G(S_{w'})_{\mathbb{C}}(w')$ is a complex flag manifold for $w' \in G(w)$. We say that $G(w)$ is *measurable* if $S_{w'}$ carries an $N_G(S_{w'})$–invariant positive Radon measure for $w' \in G(w)$. We say that $G(w)$ is *polarized* if the $\mathfrak{q}_{w'}$ have τ–stable reductive parts for $w' \in G(w)$. Set $\mathfrak{t}_w = \sum_{\Psi^n \cap -\tau\Psi^n} \mathfrak{g}_\alpha + \sum_{-\Psi^n \cap \tau\Psi^n} \mathfrak{g}_\alpha$. Then [**30**, Theorem 9.2] $G(w)$ is measurable if and only if $\mathfrak{n}_G(S_w) = (\mathfrak{q}_w \cap \tau\mathfrak{q}_w) + \mathfrak{t}_w$, in other words just when $\mathfrak{n}_G(S_w)$ has nilpotent radical $(\mathfrak{q}_w \cap \tau\mathfrak{q}_w)^n$ and reductive part $(\mathfrak{q}_w \cap \tau\mathfrak{q}_w)^r + \mathfrak{t}_w$. It follows that if $G(w)$ is measurable then (i) $G(w)$ is partially complex, (ii) $G(w)$ is of flag type, (iii) $G(w)$ is polarized if and only if it is integrable, and (iv) the $N_G(S_w)$–invariant positive Radon measure on S_w is the volume element of an $N_G(S_w)$–invariant, possibly indefinite, kaehler metric. Furthermore [**30**, Theorem 9.9] if $G(w)$ is polarized then the following are equivalent: (i) $G(w)$ is measurable, (ii) $G(w)$ is integrable, and (iii) $G(w)$ is partially complex and of flag type. Under those conditions, $\mathfrak{n}_G(S_w) = (\mathfrak{q}_w \cap \tau\mathfrak{q}_w) + \mathfrak{t}_w = \mathfrak{s}_G(S_w)$.

Section 3. Harmonic Form Realizations of Relative Discrete Series Representations.

In the setting of the Bott–Borel–Weil Theorem, described in §1 above, the classical theorems of Dolbeault, Hodge and Kodaira tell us that every cohomology class $[c] \in H^q(G/T; \mathcal{O}(\mathbb{E}_\lambda))$ is represented by exactly one harmonic $(0,q)$–form on G/T with values in $\mathbb{E}_\lambda$. In this section we suppose that G is a general semisimple group as in (2.1). We first describe the realization of relative discrete series representations by square integrable harmonic differential forms. Then, using real group orbit results mentioned in §2, we describe the realization of standard tempered representations by square integrable partially harmonic forms.

If $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, then by definition the corresponding Cartan subgroup of G is given by $H = \{g \in G \mid Ad(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}_0\}$. Note that $H = Z_G(G^0)(H \cap G^0)$ and that $H \cap G^0$ is the Cartan subgroup of G^0 with Lie algebra $\mathfrak{h}_0$. Let $Z \subset Z_G(G^0)$ as in (2.1). We may (and do) replace Z by ZZ_{G^0}, which still satisfies the requirements of (2.1), but which also satisfies: $Z \cap G^0 = Z_{G^0}$. An irreducible unitary representation $\pi \in \widehat{G}$ belongs to the *relative discrete series* if its coefficients $f_{u,v} : G \to \mathbb{C}$, given by $f_{u,v}(g) = \langle u, \pi(g)v \rangle_{\mathcal{H}_\pi}$, are square integrable on G modulo Z.

Suppose that G has a relatively compact Cartan subgroup, that is, has a Cartan subgroup T such that T/Z is compact. That is the condition ([**4**], [**5**], [**31**]) for the existence of relative discrete series representations of G. Fix a relatively compact Cartan subgroup $T \subset K$ of G. Then $T \cap G^0 = T^0$, in particular the Cartan subgroup $T \cap G^0$ of G^0 is commutative. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system, let $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ a choice of positive root system, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$, half the trace of $ad_{\mathfrak{g}}|_{\mathfrak{t}}$ on $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

If π is a relative discrete series representation of G and Θ_π is its distribution character, then the equivalence class of π is determined by the restriction of Θ_π to $T \cap G'$. So we can parameterize the relative discrete series of G by parameterizing those restrictions. Here we follow [**5**], [**6**] and [**31**].

Let $G^\dagger$ denote the finite index subgroup $TG^0 = Z_G(G^0)G^0$ of G. The Weyl group $W^\dagger = W(G^\dagger, T)$ coincides with $W^0 = W(G^0, T^0)$ and is a normal subgroup of $W = W(G, T)$. Let $\chi \in \widehat{T}$. It follows from (2.1) that the irreducible unitary representation χ is finite dimensional. Since T^0 is commutative, χ has differential $d\chi(\xi) = \lambda(\xi)I$ where $\lambda \in i\mathfrak{t}_0^*$ and where I is the identity on the representation space of χ. Suppose that $\lambda + \rho$ is regular, i.e., that $\langle \lambda + \rho, \alpha\rangle \neq 0$ for all $\alpha \in \Phi$. Then there are unique relative discrete series representations π_χ^0 of G^0 and $\pi_\chi^\dagger$ of $G^\dagger$ whose distribution characters satisfy

$$(3.1) \qquad \Theta_{\pi_\chi^0}(x) = \pm\frac{\sum_{w\in W^0} sign(w) e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \text{ and } \Theta_{\pi_\chi^\dagger}(zx) = \chi(z)\Theta_{\pi_\chi^0}(x)$$

for $z \in Z_G(G^0)$ and $x \in T^0 \cap G'$. Here note that $\pi_\chi^\dagger = \chi|_{Z_G(G^0)} \otimes \pi_\chi^0$. The same datum χ specifies a relative discrete series representation $\pi_\chi = \mathrm{Ind}_{G^\dagger}^G(\pi_\chi^\dagger)$ of G. π_χ is characterized by the fact that its distribution character is supported in $G^\dagger$, where

$$(3.2) \qquad \Theta_{\pi_\chi} = \sum_{1\leqq i\leqq r} \Theta_{\pi_\chi^\dagger} \cdot \gamma_i^{-1}$$

with $\gamma_i = Ad(g_i)|_{G^\dagger}$ where $\{g_1, \dots, g_r\}$ is any system of coset representatives of G modulo $G^\dagger$. To combine these into a single formula one chooses the g_i so that they normalize T, i.e. chooses the γ_i to be a system of coset representatives of W modulo $W^\dagger$.

Every relative discrete series representation of G is equivalent to a representation π_χ as just described. Relative discrete series representations π_χ and $\pi_{\chi'}$ are equivalent if and only if $\chi' = \chi \cdot w^{-1}$ for some $w \in W$.

A choice $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of positive root system defines a G-invariant complex manifold structure on G/T such that $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ represents the holomorphic tangent space. In effect, a choice of Φ^+ is a choice of Borel subalgebra $\mathfrak{b} = \mathfrak{t} + \sum_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha} \subset \mathfrak{g}$. Let X denote the flag variety of Borel subalgebras of $\mathfrak{g}$ and let $x \in X$ stand for the just–described Borel subalgebra $\mathfrak{b}$. Then $gT \mapsto g(x)$ defines a G–equivariant holomorphic diffeomorphism of G/T onto the open real group orbit $G(x) \subset X$.

More generally let W be a complex flag manifold of $\mathfrak{g}$, let $w \in W$, set $Y = G(w)$, and suppose

$$
\begin{aligned}
&Y \text{ is open in } W \text{ and}\\
&\overline{G} = G/Z_G(G^0) \text{ has compact isotropy subgroup at } w.
\end{aligned}
\tag{3.3}
$$

Let $\mathfrak{q} \subset \mathfrak{g}$ denote the parabolic subalgebra of $\mathfrak{g}$ corresponding to w, so that W consists of all $Int(\mathfrak{g})$–conjugates of $\mathfrak{q}$. As in §2 we may view $\overline{G} = G/Z_G(G^0)$ inside $\overline{G}_{\mathbb{C}} = Int(\mathfrak{g})$. Let $\overline{L}$ denote the isotropy subgroup of $\overline{G}$ at w and let L denote its inverse image in G under the projection $G \to G/Z_G(G^0) = \overline{G}$ So $\overline{L}$ is compact and L is the isotropy subgroup of G at w. Passing to a conjugate, equivalently moving w within Y, we may suppose $T \subset L$.

Let $\chi \in \widehat{L}$, let E_χ denote the representation space, and let $\mathbb{E}_\chi \to Y \cong G/L$ denote the associated holomorphic homogeneous vector bundle. Using the Mackey machine, Cartan's highest weight theory, and the methods of [**31**, §2.4, 3.4, 3.5], we see that χ is finite dimensional and is constructed as follows. First, $L \cap G^0 = L^0$ and there is an irreducible representation χ^0 of L^0 with highest weight λ. Second, there is a representation $\psi \in \widehat{Z_G(G^0)}$ that agrees with χ^0 on $Z_{G^0} = Z_G(G^0) \cap L^0$. So we have the irreducible unitary representation $\chi^\dagger = \psi \otimes \chi^0$ of $L^\dagger = Z_G(G^0)L^0$. Third, $\lambda + \rho_{\mathfrak{l}}$ is $\Phi(\mathfrak{l}, \mathfrak{t})$–regular and this implies $\chi = Ind_{L^\dagger}^{L}(\chi^\dagger)$. We will call λ the *highest weight* of χ.

Note that $\lambda + \rho_{\mathfrak{l}}$ is the Harish–Chandra parameter for the infinitesimal character of χ. This of course is a special case of the relative discrete series picture. We will simply refer to $\lambda + \rho_{\mathfrak{l}}$ as the infinitesimal character of χ.

Since χ is unitary, the bundle $\mathbb{E}_\chi \to Y$ has a G–invariant hermitian metric. Let $\square$ denote the Kodaira–Hodge–Laplace operator $\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $\mathbb{E}_\chi$. Then we have Hilbert spaces

$$
\mathcal{H}^q(Y; \mathbb{E}_\chi): \text{ harmonic } L_2\ \mathbb{E}_\chi\text{-valued } (0,q)\text{-forms on } Y \tag{3.4}
$$

on which G acts naturally, and the natural actions of G on those spaces are unitary representations.

As remarked before for the flag manifold of Borel subalgebras of $\mathfrak{g}$, if G is compact (and in fact if $\overline{G}$ is compact) then the space $\mathcal{H}^q(Y; \mathbb{E}_\chi)$ of L_2 harmonic forms is naturally identified with the sheaf cohomology $H^q(Y; \mathcal{O}(\mathbb{E}_\chi))$.

The root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ decomposes as the disjoint union of the *compact roots* $\Phi_K = \Phi(\mathfrak{k}, \mathfrak{t}) = \{\alpha \in \Phi : \mathfrak{g}_\alpha \subset \mathfrak{k}\}$ and the *noncompact roots* $\Phi_{G/K} = \Phi \setminus \Phi_K$. Write Φ_K^+ for $\Phi^+ \cap \Phi_K$ and $\Phi_{G/K}^+$ for $\Phi^+ \cap \Phi_{G/K}$.

My proof [**31**, Theorem 7.2.3] of Theorem 3.5 below only applied to the case where $\lambda + \rho$ is "sufficiently" nonsingular, because it relied on Schmid's proof [**23**] for connected linear Lie groups. Later [**25**] Schmid was able to drop the condition of sufficient nonsingularity. With this in mind, the proof of [**31**, Theorem 7.2.3] now yields

3.5. THEOREM. *Let $\chi \in \widehat{L}$ with highest weight λ. Express $\chi = Ind_{L^\dagger}^{L}(\psi \otimes \chi^0)$. If $\lambda + \rho$ is $\Phi(\mathfrak{g}, \mathfrak{t})$-singular then every $\mathcal{H}^q(Y; \mathbb{E}_\chi) = 0$. Now suppose that $\lambda + \rho$ is $\Phi(\mathfrak{g}, \mathfrak{t})$-regular and define*

$$q(\lambda + \rho) = |\{\alpha \in \Phi_K^+ : \langle \lambda + \rho, \alpha \rangle < 0\}| + |\{\beta \in \Phi_{G/K}^+ : \langle \lambda + \rho, \beta \rangle > 0\}|.$$

Then $\mathcal{H}^q(Y; \mathbb{E}_\chi) = 0$ for $q \neq q(\lambda + \rho)$, and G acts irreducibly on $\mathcal{H}^{q(\lambda+\rho)}(Y; \mathbb{E}_\chi)$ by the relative discrete series representation $\pi_{\psi \otimes e^\lambda}$ of infinitesimal character $\lambda + \rho$.

An interesting variation on this result realizes the relative discrete series on spaces of L_2 bundle-valued harmonic spinors. See [**20**], [**24**] and [**32**].

Section 4. Harmonic Form Realizations of Tempered Representations.

The representations of G that enter into its Plancherel formula are the *tempered representations.* They are constructed from a certain class of real parabolic subgroups of G, the *cuspidal parabolic subgroups*, combining the relative discrete series construction for the reductive part of cuspidal parabolic with unitary induction from the parabolic up to G. We start by recalling the definitions.

Let H be a Cartan subgroup of our general semisimple (2.1) Lie group G. Fix a Cartan involution θ of G such that $\theta(H) = H$. Its fixed point set $K = G^\theta$ is a maximal compactly embedded (compact modulo $Z_G(G^0)$) subgroup of G. We decompose

$$\begin{aligned} &\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ and } H = T \times A \\ &\text{where } T = H \cap K,\ \theta(\xi) = -\xi \text{ on } \mathfrak{a}, \text{ and } A = \exp_G(\mathfrak{a}_0). \end{aligned} \tag{4.1}$$

Then the centralizer $Z_G(A)$ of A in G has form $M \times A$ where $\theta(M) = M$. Now [**31**] M is a reductive Lie group in the same class (2.1), and T is a compactly embedded Cartan subgroup of M, so M has relative discrete series representations.

Suppose that the positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ is defined by positive root systems $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. This means that

$$\Phi^+(\mathfrak{m}, \mathfrak{t}) = \{\alpha|_\mathfrak{t} : \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h}),\ \alpha|_\mathfrak{a} = 0\} \tag{4.2a}$$

and

$$\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\beta|_{\mathfrak{a}_0} : \beta \in \Phi^+(\mathfrak{g}, \mathfrak{h}),\ \beta|_\mathfrak{a} \neq 0\}. \tag{4.2b}$$

In other words, given $\mathfrak{h}$, the Borel subalgebra

$$\mathfrak{b} = \mathfrak{b}^n + \mathfrak{b}^r \text{ where } \mathfrak{b}^r = \mathfrak{h} \text{ and } \mathfrak{b}^n = [\mathfrak{b}, \mathfrak{b}] = \textstyle\sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{-\alpha} \tag{4.2c}$$

is chosen to maximize $\mathfrak{b}^n \cap \overline{\mathfrak{b}^n}$. Note that $\mathfrak{b}^n \cap \overline{\mathfrak{b}^n}$ has real form $(\mathfrak{b}^n \cap \overline{\mathfrak{b}^n})_0 = \mathfrak{b}^n \cap \mathfrak{g}_0$, which is the sum $\sum_{\gamma \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_\gamma$ of the positive restricted root spaces.

A subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ is a *(real) parabolic subalgebra* if its complexification $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, in other words if $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ for some τ–stable parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. A subgroup $P \subset G$ is a *parabolic subgroup* of G if it is the G–normalizer of a parabolic subalgebra of $\mathfrak{g}_0$. A parabolic subgroup $P \subset G$ is called *cuspidal* if the Levi component (reductive part) has a relatively compact Cartan subgroup. We now have the cuspidal parabolic subgroup $P = MAN$ of G, where M and A are as above, where $MA = M \times A$ is the Levi component of P, and where $N = exp_G((\mathfrak{b}^n \cap \overline{\mathfrak{b}^n})_0)$.

Let $\chi \in \widehat{H}$ and consider the basic datum $(H, \mathfrak{b}, \chi)$. The representation of $\mathfrak{b}$ is determined because χ represents H irreducibly: $\chi(\mathfrak{b}^n) = 0$ and $\chi|_{\mathfrak{h}}$ is the differential of the representation of H. Decompose $\chi = \psi \otimes e^\nu \otimes e^{i\sigma}$, $\psi \in \widehat{Z_G(G^0)}$, $e^\nu \in \widehat{T}$, $\sigma \in \mathfrak{a}_0^*$. Suppose that $\nu + \rho_\mathfrak{m}$ is $\Phi(\mathfrak{m}, \mathfrak{t})$–regular. Then $\psi \otimes e^\nu$ specifies a relative discrete series representation $\eta_{\psi\otimes e^\nu}$ of M. The Levi component $M \times A$ of P acts irreducibly and unitarily on $H_{\eta_{\psi\otimes e^\nu}}$ by $\eta_{\psi\otimes e^\nu} \otimes e^{i\sigma}$. That extends uniquely to a representation (which we still denote $\eta_{\psi\otimes e^\nu} \otimes e^{i\sigma}$) of P on $H_{\eta_{\psi\otimes e^\nu}}$ whose kernel contains N. Now we have the *standard tempered representation*

$$\pi_\chi = \pi_{\psi,\nu,\sigma} \ = \ \mathrm{Ind}_{P\uparrow G}(\eta_{\psi\otimes e^\nu} \otimes e^{i\sigma}) \tag{4.3}$$

of G. One can compute the character of π_χ and see that it is independent of the choice of positive root system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ that is defined by choices of $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. With H fixed up to conjugacy, and as ψ and σ vary, we have the *H-series* of tempered representations of G.

The various tempered series exhaust enough of $\widehat{G}$ for a decomposition of $L_2(G)$ essentially as

$$\sum\nolimits_{H\in Car(G)} \sum\nolimits_{\psi\otimes e^\nu \in \widehat{T}} \int_{\widehat{A}} H_{\pi_{\psi,\nu,\sigma}} \otimes H^*_{\pi_{\psi,\nu,\sigma}} m(H : \psi : \nu : \sigma) d\sigma.$$

Here $m(H : \psi : \nu : \sigma)d\sigma$ is the *Plancherel measure* on $\widehat{G}$. This was worked out by Harish–Chandra ([**6**], [**7**], [**8**]) for groups of Harish–Chandra class, and somewhat more generally by Herb and myself ([**31**]; [**14**], [**15**]; [**17**], [**18**]). Harish–Chandra's approach is based on an analysis of the structure of the Schwartz space, while Herb and I use explicit character formulae (compare [**3**], [**21**], [**14**], [**16**]).

Fix a θ-stable Cartan subgroup $H \subset G$ and a positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ defined by positive root systems $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ as in (4.2) above. Then we have the associated cuspidal parabolic subgroup $P = MAN \subset G$.

We now need a complex flag manifold $W \cong \overline{G}_\mathbb{C}/Q$ consisting of the $Int(\mathfrak{g})$–conjugates of a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$, and a real group orbit $Y = G(w) \subset W$, such that

(4.4a) Y is measurable, hence partially complex and of flag type, and

(4.4b) the normalizer $N_G(S_w)$ of the holomorphic arc component S_w has Lie algebra $\mathfrak{p}_0$.

Then S_w will be a topological component of the open M–orbit $M(w)$ in the subflag $\overline{M}_{\mathbb{C}}(w) \subset W$. Thus AN will act trivially on S_w and the isotropy subgroup of G at w will be of the form UAN where $U \subset M$ is of the M–centralizer of a subtorus of T. Suppose in addition that

$$U/Z_G(G^0) = \{m \in M \mid m(w) = w\}/Z_G(G^0) \text{ is compact.} \tag{4.4c}$$

Then $M(w) \cong M/U$ will be a measurable integrable open orbit in $\overline{M}_{\mathbb{C}}(w)$, $U = Z_M(M^0)U^0$ with $U \cap M^0 = U^0$, $UM^0 = M^\dagger$, and $M/M^\dagger$ will enumerate the topological components of $M(w)$.

It is straightforward to construct all pairs (W, w) that satisfy (4.4) and such that $G(w)$ is integrable as well as measurable [**31**, §6.7]. They are given by:

(4.5a) $\mathfrak{u}_0 \subset \mathfrak{m}_0$ is the $\mathfrak{m}_0$–centralizer of a subspace of $\mathfrak{t}$,

(4.5b) the corresponding analytic subgroup $U \subset M$ has compact image in $M/Z_G(G^0)$,

(4.5c) $\mathfrak{r} \subset \mathfrak{m}$ is a parabolic subalgebra with reductive part $\mathfrak{r}^r = \mathfrak{u}$, and

(4.5d) $\mathfrak{q}_w$ is the $\mathfrak{g}$–normalizer of $\mathfrak{r}^n + \mathfrak{n}$.

Then the considerations of [**31**, §6.7] show that

$$\mathfrak{q}_w^n = \mathfrak{r}^n + \mathfrak{n} \text{ and } \mathfrak{q}_w^r = \mathfrak{u} + \mathfrak{a}. \tag{4.6}$$

In the case $U = T$ this constructs $|W_{\mathfrak{m}}|$ pairs (W, w) that satisfy (4.4), though of course there will be some identifications under the Weyl group of G. In particular there are many pairs (W, w) that satisfy (4.4).

Now assume that the situation (4.4) is given. Then the holomorphic arc components $gS_w \cong M^\dagger/U$ of $Y = G(w) \cong G/UAN$ are topological components of the fibres of

(4.7a) $Y \to G/P$, G–equivariant fibration with structure group M and typical fibre M/U

given by $gUAM \mapsto gMAN$. To say it in a slightly different way, $M(w)$ has finitely many topological components $m_iM^\dagger(w) = m_iS_w$, as $m_iM^\dagger$ ranges over $M/M^\dagger$. The holomorphic arc components of Y are the fibres of

(4.7b) $Y \to G/P^\dagger$, G–equivariant fibration with structure group $M^\dagger$ and typical fibre $M^\dagger/U$

where $P^\dagger = M^\dagger AN$. The complex structure on the holomorphic arc component S_w, as complex submanifold of W, is the M-invariant complex structure on M/U for which $\sum_{\alpha \in \Phi^+(\mathfrak{m},\mathfrak{t}) \setminus \Phi^+(\mathfrak{m},\mathfrak{u})} \mathfrak{m}_\alpha$ is the holomorphic tangent space.

Consider irreducible unitary representations

$$\mu \in \widehat{U}, \text{ with representation space } E_\mu, \text{ and } e^{i\sigma} \in \widehat{A} \text{ where } \sigma \in \mathfrak{a}_0^*. \tag{4.8a}$$

Let $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)}(\dim \mathfrak{g}_\phi)\phi$ as usual. This is the quasicharacter on UAN that must be inserted for ordinary induction to become unitary induction from UAN to G. Now UAN acts on E_μ by

$$\gamma_{\mu,\sigma}(uan) = e^{\rho_{\mathfrak{a}}+i\sigma}(a)\mu(u). \tag{4.8b}$$

That specifies the associated G–homogeneous vector bundle

$$p : \mathbb{E}_{\mu,\sigma} \to Y \cong G/UAN. \tag{4.8c}$$

This bundle has a natural CR–structure and is holomorphic over every holomorphic arc component of Y. Furthermore K is transitive on Y so the bundle has a natural K–invariant hermitian metric based on the unitary structure of its typical fibre E_μ.

Restrict the holomorphic tangent bundle of W to Y and let $\mathbb{T} \to Y$ denote the sub–bundle whose fibre at $w' \in Y$ is the holomorphic tangent space to $S_{w'}$ at w'. The space of partially smooth (p,q)–forms with values in $\mathbb{E}_{\mu,\sigma}$ is

$$\begin{aligned} A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) &: \text{measurable sections of } \mathbb{E}_{\mu,\sigma}\otimes \wedge^p\mathbb{T}^*\otimes\wedge^q\overline{\mathbb{T}}^* \\ &\qquad \text{that are } C^\infty \text{ on each holomorphic arc component.}\end{aligned} \tag{4.9a}$$

The subspace of square integrable partially smooth forms is defined just as one might guess. Let # denote the Kodaira–Hodge orthocomplementation mapping $A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \to A^{n-p,n-q}(Y;\mathbb{E}^*_{\mu,\sigma})$ on each holomorphic arc component, and let $\bar\wedge$ denote exterior product followed by contraction, so pointwise $\omega\bar\wedge\#\omega$ is $||\omega||^2$ times the volume element of the holomorphic arc component. Now we have

$$\begin{aligned} A_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) &: \text{all } \omega \in A^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) \text{ such that} \\ &\int_{S_{kw}} \omega\bar\wedge\#\omega < \infty \ a.e.\ k \in K \text{ and} \\ &\int_{K/U}\left(\int_{S_{kw}} \omega\bar\wedge\#\omega\right) d(kU) < \infty. \end{aligned} \tag{4.9b}$$

The Kodaira–Hodge–Laplace operators on the restrictions of $\mathbb{E}_{\mu,\sigma}$ to the holomorphic arc components fit together to give us essentially self adjoint operators $\square$ on the Hilbert space completions of the $A_2^{p,q}(Y;\mathbb{E}_{\mu,\sigma})$. Their kernels are the spaces

$$\begin{aligned} \mathcal{H}^{p,q}(Y;\mathbb{E}_{\mu,\sigma}) &: \text{ square integrable partially harmonic} \\ &\qquad \mathbb{E}_{\mu,\sigma}\text{–valued}(p,q)\text{–forms on } Y.\end{aligned} \tag{4.9c}$$

We'll only use the $\mathcal{H}^q(Y;\mathbb{E}_{\mu,\sigma}) = \mathcal{H}^{0,q}(Y;\mathbb{E}_{\mu,\sigma})$.

The natural action of G on $\mathcal{H}^q(Y;\mathbb{E}_{\mu,\sigma})$ is a unitary representation. It is unitarily equivalent to the representation of G on the Hilbert space of L_2 sections of a certain homogeneous vector bundle

$$\begin{aligned} &\mathbb{H}^q(M/U;\mathbb{E}_\mu|_{M/U}) \to G/P, \\ &\quad \text{fibre } \mathcal{H}^q(M/U;\mathbb{E}_\mu|_{M/U}), \text{ structure group } P = MAN.\end{aligned} \tag{4.10}$$

Here, as in the discussion of the realization of the relative discrete series, $\mathcal{H}^q(M/U;\mathbb{E}_\mu|_{M/U})$ is the Hilbert space of $L_2(M/U)$ harmonic, $\mathbb{E}_\mu|_{M/U}$–valued, $(0,q)$–forms on M/U. Let η denote the (necessarily unitary) representation of M on $\mathcal{H}^q(M/U;\mathbb{E}_\mu|_{M/U})$. Then MAN acts on $\mathcal{H}^q(M/U;\mathbb{E}_\mu|_{M/U})$ by $\eta \otimes e^{\rho_\mathfrak{a}+i\sigma}(man) = e^{\rho_\mathfrak{a}+i\sigma}(a)\eta(m)$ and $\mathbb{H}^q(M/U;\mathbb{E}_\mu|_{M/U}) \to G/P$ is the associated vector bundle over G/P. Again the $\rho_\mathfrak{a}$ means that the natural action of G on L_2 sections is unitary for $\langle f,f'\rangle = \int_K \langle f(k), f'(k)\rangle dk$.

Theorem 3.5 combines with the considerations above to give the realization of standard tempered representations described in Theorem 4.9 below. Originally I proved Theorem 4.11 only for the case where $\nu+\rho_\mathfrak{m}$ is "sufficiently" nonsingular [**31**, Theorem 8.3.4], so that the realization for the corresponding relative discrete series representation of M would be available. As described just before Theorem 3.5, we can now drop that condition.

The representation $\mu \in \widehat{U}$ is of the form $Ind_{U^\dagger}^U(\mu^\dagger)$ where $U^\dagger = Z_M(M^0)U^0$, where $U^0 = U \cap M^0$, and where $\mu^\dagger = \psi \otimes \mu^0$ in such a way that $\psi \in \widehat{Z_M(M^0)}$ agrees with $\mu^0 \in \widehat{U^0}$ on $Z_{M^0} = Z_M(M^0) \cap U^0$. Here μ^0 has some highest weight, say ν, relative to $\Phi^+(\mathfrak{u},\mathfrak{t})$, and $\nu + \rho_\mathfrak{u}$ is $\Phi(\mathfrak{u},\mathfrak{t})$–nonsingular. In general we will just refer to ν as the highest weight of μ.

4.11. THEOREM. *Let $\mu \in \widehat{U}$ with highest weight ν. If $\nu + \rho_\mathfrak{m}$ is $\Phi(\mathfrak{m},\mathfrak{t})$–singular then every $\mathcal{H}^q(Y;\mathbb{E}_{\mu,\sigma}) = 0$. Let*

$$q(\nu+\rho_\mathfrak{m}) = |\{\alpha \in \Phi^+_{K\cap M} : \langle \nu+\rho_\mathfrak{m},\alpha\rangle < 0\}| + |\{\beta \in \Phi^+_{M/K\cap M} : \langle \nu+\rho_\mathfrak{m},\beta\rangle > 0|.$$

Then $\mathcal{H}^q(Y;\mathbb{E}_{\mu,\sigma}) = 0$ for $q \neq q(\nu+\rho_\mathfrak{m})$, and G acts on $\mathcal{H}^q(Y;\mathbb{E}_{\mu,\sigma})$ by the standard H–series representation $\pi_{\psi,\nu,\sigma} = Ind_P^G(\eta_{\psi\otimes e^\nu}\otimes e^{i\sigma})$ of G of infinitesimal character $\nu + \rho_\mathfrak{m} + i\sigma$.

A variation on this theorem realizes the tempered series on spaces of L_2 bundle-valued partially harmonic spinors. See [**32**].

Whenever $\sigma \in \mathfrak{a}_0^*$ is $\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$–regular, the standard H–representation $\pi_{\psi,\nu,\sigma}$ is irreducible. Plancherel measure for G thus is carried by the irreducible representations among the H–series representations realized above, as H varies over the conjugacy classes of Cartan subgroups of G.

Section 5. Sheaf Cohomology Realizations of Tempered Representations.

An important variation on the Kostant–Langlands Conjecture result — which in fact preceded its solution — is Schmid's Dolbeault cohomology realization [**22**] of discrete series representations of connected semisimple Lie groups with finite center. Suppose that G has a compactly embedded Cartan subgroup T and that we are in the situation of (3.3). Then the open orbit $Y = G(w) \cong G/L$ in W contains $K(w) \cong K/L$ as a maximal compact complex submanifold. We denote

$s = \dim_{\mathbb{C}} K(w)$. Whenever

$$\text{(5.1)}\qquad \begin{aligned} \lambda+\rho \text{ is } Y\text{–antidominant: } & \langle \lambda+\rho, \gamma\rangle < 0 \text{ for all } \beta \in \Phi^+(\mathfrak{g},\mathfrak{t}) \setminus \Phi^+(\mathfrak{l},\mathfrak{t}), \\ & \langle \lambda, \alpha\rangle \geqq 0 \text{ for all } \alpha \in \Phi^+(\mathfrak{l},\mathfrak{t}) \end{aligned}$$

we have $s = q(\lambda+\rho)$. This is the case where the associated holomorphic vector bundles $\mathbb{E}_\chi \to Y$, $\chi \in \widehat{L}$ with highest weight λ, are negative vector bundles.

If π is a relative discrete series representation of G, we can choose the positive root system $\Phi^+(\mathfrak{g},\mathfrak{t})$ so that $\pi = \pi_\chi$ where the highest weight λ of χ satisfies (5.1). This is because $\mathfrak{l}$ is the reductive part of a parabolic subalgebra of $\mathfrak{g}$. Thus there is no restriction on $\pi_{\psi\otimes e^\lambda}$ in

5.2. THEOREM. *Let $\chi \in \widehat{L}$ with highest weight λ. Suppose that the infinitesimal character $\lambda+\rho$ is Y-antidominant* (5.1). *Then $H^q(Y;\mathcal{O}(\mathbb{E}_\chi)) = 0$ for $q \neq s$, $H^s(Y;\mathcal{O}(\mathbb{E}_\chi))$ has a natural structure of infinite dimensional Fréchet space, and the natural action of G on $H^s(Y;\mathcal{O}(\mathbb{E}_\chi))$ is a continuous representation of infinitesimal character $\lambda+\rho$. Express $\chi = Ind_{L^\dagger}^{L}(\psi \otimes \chi^0)$. The representation of G on $H^s(Y;\mathcal{O}(\mathbb{E}_\chi))$ is infinitesimally equivalent[1] to the relative discrete series representation $\pi_{\psi\otimes e^{\lambda+\rho}}$.*

Theorem 5.2 was proved by Schmid [**22**] in the case where G is connected with finite center, $L = T$, and λ is sufficiently nonsingular. It follows by now–standard techniques [**31**, §3] for general semisimple groups G with $L = T$, and λ is sufficiently nonsingular. The analogous statement for Zuckerman's derived functor modules now holds without the requirement of sufficient nonsingularity, because of their analytic continuation properties [**29**]. In view of [**27**, §9] the same holds for our cohomology modules. That proves Theorem 5.2 completely for the case $L = T$. The result as stated now is more or less immediate from the Leray spectral sequence for the holomorphic fibration $G/T \to G/L$. Compare [**33**].

In Theorem 5.2, the infinitesimal equivalence is

$$\text{(5.3)}\qquad \mathcal{H}^s(Y;\mathbb{E}_\chi)_{(K)} \to H^s(Y;\mathcal{O}(\mathbb{E}_\chi))_{(K)}$$

as a map on spaces of K-finite vectors. That is the map that sends an L_2 harmonic form to (the sheaf cohomology class that corresponds to) its Dolbeault class.

We now look at the tempered case. Fix a Cartan subgroup $H = T \times A$ as in (4.1), a positive root system $\Phi^+(\mathfrak{g},\mathfrak{h})$ as in (4.2), and a measurable open orbit $Y = G(w) \subset W$ on a complex flag manifold $W \cong \overline{G}_{\mathbb{C}}/Q$ as in (4.4). We are going to replace $\mathcal{H}^q(M/U;\mathbb{E}_\mu|_{M/U})$ by the partial Dolbeault cohomology space that realizes the relative discrete series representations $\eta_{\psi\otimes e^\nu}$ of M and $\eta_{\psi\otimes e^\nu} \otimes e^{i\sigma}$

[1]Let π and ϕ be continuous representations of G on complete locally convex topological vector spaces V_π and V_ψ. Let $(V_\pi)_{(K)}$ and $(V_\psi)_{(K)}$ denote the respective subspaces of K-finite vectors. They are modules for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. An *infinitesimal equivalence* of V_π with V_ψ means a $\mathcal{U}(\mathfrak{g})$-isomorphism of $(V_\pi)_{(K)}$ onto $(V_\psi)_{(K)}$.

of MA and $P = MAN$. The space $(K \cap M)(w) \cong (K \cap M)/U$ is a maximal compact complex submanifold of $Y \cong G/UAN$. Let $s = \dim_{\mathbb{C}} (K \cap M)/U$. Whenever $\nu + \rho_{\mathfrak{m}}$ is S_w–antidominant (5.1), that is

$$\text{(5.4)} \qquad \begin{aligned} &\langle \nu + \rho_{\mathfrak{m}}, \gamma \rangle < 0 \text{ for all } \beta \in \Phi^+(\mathfrak{m}, \mathfrak{t}) \setminus \Phi^+(\mathfrak{u}, \mathfrak{t}) \text{ and} \\ &\langle \nu, \alpha \rangle \geqq 0 \text{ for all } \alpha \in \Phi^+(\mathfrak{u}, \mathfrak{t}), \end{aligned}$$

we have $s = q(\nu + \rho_{\mathfrak{m}})$. As before, this is the case where the bundle $\mathbb{E}_\mu|_{M/U} \to M/U$ is negative.

Let $\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\mu) \to Y$ denote the sheaf of germs of C^∞ sections of $\mathbb{E}_\mu \to G/Y$ that are holomorphic along the holomorphic arc components of Y, i.e. holomorphic along the fibres of $Y \to G/P$. By irreducibility of μ, these are the sections annihilated by the right action of $\mathfrak{q}^n$. See (5.8) below for the condition when μ may be reducible.

Given a relative discrete series representation η of M, we need a positive root system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ such that:

(1) $\Phi^+(\mathfrak{m}, \mathfrak{t})$ satisfies (5.4),
(2) $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ is arbitrary, and
(3) $\Phi^+(\mathfrak{g}, \mathfrak{h})$ is defined by $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$.

Given a choice of $\Phi^+(\mathfrak{g}, \mathfrak{h})$ as above, we realize the corresponding standard H–series representations of G on partial Dolbeault cohomology as follows. Given $\mathbb{E}_{\mu,\sigma} \to Y$ as above, we have the sheaf

$$\text{(5.5a)} \qquad \begin{aligned} &\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}) \to G/UA: \text{ germs of } C^\infty \text{ sections } f \text{ of } \mathbb{E}_{\mu,\sigma} \to G/UA \\ &\qquad\qquad \text{such that } f(x;\xi) + \chi(\xi) \cdot f(x) = 0 \text{ for all } x \in G \text{ and } \xi \in \mathfrak{q}. \end{aligned}$$

$\mathbb{E}_{\mu,\sigma} \to G/UA$ pushes down to a bundle $\mathbb{E}_{\mu,\sigma} \to Y = G(\mathfrak{q}) = G(w) \subset W$, so the sheaf $\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}) \to G/UA$ pushes down to a sheaf

$$\text{(5.5b)} \qquad \begin{aligned} &\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}) \to Y: \text{ germs of } C^\infty \text{ sections } f \text{ of } \mathbb{E}_{\mu,\sigma} \to Y \\ &\qquad\qquad \text{such that } f(x;\xi) + \chi(\xi) \cdot f(x) = 0 \text{ for all } x \in G \text{ and } \xi \in \mathfrak{q}. \end{aligned}$$

The germs in $\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}) \to UA$ are equivariant along fibres of $G/UA \to G/UAN \cong Y$, and the collapse of the Leray spectral sequence of $G/UA \to G/UAN \cong Y$ yields natural G–equivariant isomorphisms

$$\text{(5.5c)} \qquad H^q(G/UA; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma})) \cong H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma})).$$

When χ is irreducible, (5.5a) and (5.5b) reduce to: $f(x;\xi) = 0$ for all $x \in G$ and $\xi \in \mathfrak{q}^n$. In the more general case considered here, (5.5) is the appropriate condition; see [**28**]. Now we can state

5.6. THEOREM. *Let $\mu \in \widehat{U}$ and $\sigma \in \mathfrak{a}_0^*$ as in* (4.8a), *let ν be the highest weight of μ, and suppose that the infinitesimal character $\nu + \rho_{\mathfrak{m}}$ is S_w–antidominant* (5.4). *So $s = q(\nu + \rho_{\mathfrak{m}})$. If $q \neq s$ then $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma})) = 0$. Express $\mu = Ind_{U^\dagger}^U(\psi \otimes \mu^0)$. Then $H^s(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}))$ has a natural structure of infinite dimensional Fréchet space, the natural action of G on $H^s(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}))$*

is a continuous representation of infinitesimal character $\nu + \rho_{\mathfrak{m}} + i\sigma$, and this representation of G is infinitesimally equivalent to the standard H–series representation $\pi_{\psi,\nu,\sigma} = Ind_P^G(\eta_{\psi\otimes e^\nu} \otimes e^{i\sigma})$.

The Fréchet space structure on the cohomologies $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\mu,\sigma}))$ is described, in a larger context, in §7 below.

On the flag manifold X of Borel subalgebras of $\mathfrak{g}$, our cohomology construction for standard tempered representations can be formulated as follows (compare [**27**]). Fix a basic datum $(H, \mathfrak{b}, \chi)$: H is a Cartan subgroup of G, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{b}$, and χ is a finite dimensional representation of $(\mathfrak{b}, H)$. We have the associated homogeneous vector bundles $\mathbb{E}_\chi \to G/H$ and the sheaf $\mathcal{O}_{\mathfrak{b}}(\mathbb{E}_\chi) \to G/H$ of germs of sections defined by the right action of $\mathfrak{b}$. Note that $Y \cong G/HN$ here and the partial complex structure (CR structure) induced on $G(\mathfrak{b})$ by X is the one for which the holomorphic tangent space of the typical fibre $M(\mathfrak{b}) \cong M/T$ of $G/HN \to G/P$ is $\sum_{\alpha\in\Phi^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_\alpha$.

Now the case of Theorem 5.6, where W is the flag manifold X of Borel subalgebras of $\mathfrak{g}$, can be reformulated as

5.7. PROPOSITION. *Every standard tempered series representation $\pi_{\psi,\nu,\sigma}$ of G, $\psi\otimes e^\nu\otimes e^{i\sigma} \in \widehat{H}$ and $H \in Car(G)$, is realized up to infinitesimal equivalence as the natural action of G on a partial Dolbeault cohomology space $H^s(Y, \mathcal{O}_{\mathfrak{b}}(\mathbb{E}_{\mu,\sigma}))$, $Y = G(\mathfrak{b}) \subset X$, for a basic datum $(H, \mathfrak{b}, \chi)$ as follows. $\mathfrak{b}$ is maximally real subject to the condition $\mathfrak{h} \subset \mathfrak{b}$; $\mu = Ind_{U^\dagger}^U(\psi \otimes \mu^o) \in \widehat{U}$ where μ^0 has highest weight ν; χ is given on $H = T \times A$ by $\psi \otimes e^\nu \otimes e^{i\sigma+\rho_{\mathfrak{a}}}$, on $\mathfrak{h}$ by the differential, and on $\mathfrak{b}^n$ by 0; and $s = \dim_{\mathbb{C}} (K \cap M)/T$. Here $H^s(Y, \mathcal{O}_{\mathfrak{b}}(\mathbb{E}_{\mu,\sigma}))$ has a natural Fréchet space structure and the action of G is continuous.*

We may assume $\langle \nu + \rho_{\mathfrak{m}}, \gamma\rangle < 0$ for all $\gamma \in \Phi^+(\mathfrak{m}, \mathfrak{t})$. With that assumption, if $q \neq s$ then $H^q(Y, \mathcal{O}_{\mathfrak{b}}(\mathbb{E}_{\mu,\sigma})) = 0$.

Except for the Fréchet space structure on the cohomologies, Proposition 5.7 is the starting point of [**27**] for the construction of standard admissible representations. The Fréchet space structure itself comes out of some variations (see §§6 and 7 below) on the methods of [**27**].

In our more general context, a basic datum corresponding the the setup of Theorem 5.6 is of the form $(UA, \mathfrak{q}, \chi)$ where UA is a Levi component (reductive part) of the isotropy subgroup of G at $w \in W$, where $\mathfrak{q}$ is the parabolic subalgebra of $\mathfrak{g}$ represented by w in the complex flag manifold W, and where χ is a finite dimensional representation of $(\mathfrak{q}, UA)$. Then the appropriate sheaf is given by (5.5), and we have natural G–equivariant isomorphisms $H^q(G/UA; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\chi) \cong H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\chi)$. Then one has the analog for W of Proposition 5.7.

Section 6. Representations for Open Orbits

Fix a complex flag manifold W and a measurable open G–orbit $Y = G(w)$. Let $\mathfrak{q} = \mathfrak{q}_w \subset \mathfrak{g}$ denote the parabolic subalgebra represented by w, and let L

denote the isotropy subgroup of G at w. Then L contains a fundamental Cartan subgroup H of G, $L = Z_G(G^0)L^0$ and $L^0 = L \cap G^0$, by the argument of [**31**, Lemma 7.1.2]. The image $\overline{L}$ of L in $\overline{G} = G/Z_G(G^0) \subset \overline{G}_{\mathbb{C}}$ has Lie algebra $\mathfrak{q}_0^r$, real form of a Levi component $\mathfrak{q}^r$ of $\mathfrak{q}$.

We write $s = s_Y$ in for the complex dimension of the maximal compact subvariety $K(w) \cong K/(K \cap L)$ of Y.

For two classes of representations η of $(\mathfrak{q}, L)$ we'll describe a bundle $\mathbb{E}_\eta \to Y$ and a sheaf $\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\eta) \to Y$, and we'll discuss the representations of G on to $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\eta))$.

First, consider the case where η is finite dimensional. Then the main result is Theorem 6.1 below. It was proved in [**27**] for the case where W is the flag of Borel subalgebras of $\mathfrak{g}$, and in [**34**] the argument of [**27**] was reworked to apply to general W. The argument of [**34**] is for connected linear semisimple groups G, but those restrictions on G can be dropped by the techniques of [**31**] and [**9**, Appendix].

6.1. THEOREM. *Suppose χ is a finite dimensional representation of $(\mathfrak{q}, L)$. Then the $\overline{\partial}$ operator for the Dolbeault complex of $\mathbb{E}_\chi \to Y$ has closed range. $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\chi))$ is an admissible G-module with finite composition series. Its underlying Harish-Chandra module is the Zuckerman derived functor module $A^q(G, L, \mathfrak{q}, \chi)$. If E_χ has infinitesimal character with Harish-Chandra parameter $\lambda + \rho_{\mathfrak{l}}$ (corresponding to highest weight λ) and if $\chi(\mathfrak{q}^n) = 0$ then $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\chi))$ has infinitesimal character $\lambda + \rho$; and then if $\lambda + \rho$ is Y-antidominant then $H^q(Y; \mathcal{O}_{\mathfrak{q}}(\mathbb{E}_\chi)) = 0$ for $q \neq s$.*

Second, we consider the case where η may be infinite dimensional but is constrained to be one of the cohomology space representations described in Theorem 6.1 for an open L-orbit on the flag manifold of Borel subalgebras of $\mathfrak{l} = \mathfrak{q}^r$. As we saw in §5, this includes all fundamental series[2] representations of L, and a moment's thought shows that it includes all standard representations of L whose character has support that meets the elliptic set[3] in G. In particular this class of representations η contains all the representations in the analytic continuation of the fundamental series.

Let X denote the flag manifold of Borel subalgebras of $\mathfrak{g}$ and consider the natural projection

$$p : X \to W \text{ defined by } \mathfrak{b} \subset p(\mathfrak{b}) \text{ for all } \mathfrak{b} \in X. \tag{6.2}$$

The typical fibre of $p : X \to W$ is the complex flag manifold $F \cong Q^r/(B \cap Q^r)$ of all Borel subalgebras $'\mathfrak{b} \subset \mathfrak{q}^r$. The real group L acts on F just as G acts on

[2]The *fundamental series* for G is the tempered series associated to a fundamental Cartan subgroup. It is the relative discrete series in case that fundamental Cartan subgroup is compact modulo $Z_G(G^0)$.

[3]The *elliptic set* in G is the union of the G-conjugates of $T \cap G'$ where $H = T \times A$ is a fundamental Cartan subgroup, $T = H \cap K$, and G' is the regular set. In other words, the elliptic set in G is the set of all G-conjugates of G-regular elements of K.

X. Let $L(x) \subset F$ be an open orbit. Since L contains the fundamental Cartan subgroup H of G, and all fundamental Cartan subgroups of L are L^0–conjugate, we may assume that H is the isotropy subgroup of L at x. Let $\mathfrak{b} = \mathfrak{b}_x \subset \mathfrak{g}$ denote the Borel subalgebra represented by $x \in X$; then $'\mathfrak{b} = {'\mathfrak{b}_x} \subset \mathfrak{q}^r$ is the Borel subalgebra represented by $x \in F$. Note that $\mathfrak{b}$ and $'\mathfrak{b}$ determine each other by: $'\mathfrak{b} = \mathfrak{b} \cap \mathfrak{q}^r$ and $\mathfrak{b} = {'\mathfrak{b}} + \mathfrak{q}^n$.

Let χ be a finite dimensional representation of $(\mathfrak{b}, H)$. Let $'\chi$ be its restriction to a representation of $('\mathfrak{b}, H)$. consider the associated homogeneous holomorphic vector bundles of finite rank

$$(6.3) \qquad \mathbb{E}_\chi \to G(x) \cong G/H \text{ and } {'\mathbb{E}_\chi} \to L(x) \cong L/H; \text{ so } {'\mathbb{E}_\chi} = \mathbb{E}_\chi|_{L(x)}.$$

They define the sheaves of germs of holomorphic sections

$$(6.4) \qquad \mathcal{O}_\mathfrak{b}(\mathbb{E}_\chi) \to G(x) \text{ and } \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi}) \to L(x). \text{ so } \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi}) = \mathcal{O}_\mathfrak{b}(\mathbb{E}_\chi)|_{L(x)}$$

which conversely define the holomorphic structure of the bundles [**28**].

We now suppose that χ, as a representation of H, is $G(x)$–antidominant with infinitesimal character λ. We also suppose that $\chi(\mathfrak{b}^n) = 0$. Denote dimensions of maximal compact subvarieties by

$$(6.5)\ u = \dim_\mathbb{C} K(x),\ t = \dim_\mathbb{C}(K \cap L)(x), \text{ and } s = \dim_\mathbb{C} K(w); \text{ so } u = t + s\ .$$

Then $H^p(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})) = 0$ for $p \neq t$ and $H^q(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_\chi)) = 0$ for $q \neq u$ by Theorem 6.1. Thus the Leray spectral sequence of $G(x) \to G(w)$ collapses at E_2,

$$(6.6) \qquad E_2^{a,b} = H^a(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{H}^b(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})))) \text{ with } d_2 : E_2^{a,b} \to E_2^{a+2,b-1}$$

where $\mathbb{H}^v(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})) \to G(w)$ is the homogeneous vector bundle whose typical fibre is the $(\mathfrak{q}, L)$–module $H^v(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})$. Thus

$$(6.7) \qquad H^q(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_\chi)) = \sum_{a+b=q} H^a(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{H}^b(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})))).$$

Again by the vanishing of Theorem 6.1, the left side vanishes for $q \neq u$ and the right side vanishes for $b \neq t$, so (6.5) forces

$$(6.8) \qquad H^{s+t}(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_\chi)) = H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{H}^t(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi})))).$$

Modulo the Fréchet space results described in §7 below, we have proved

6.9. THEOREM. *Let η denote $H^t(L(x); \mathcal{O}_{'\mathfrak{b}}({'\mathbb{E}_\chi}))$ as a representation of $(\mathfrak{q}, L)$ and let $\mathbb{V}_\eta \to G(w)$ denote the associated homogeneous vector bundle. Then $H^q(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta)) = 0$ for $q \neq s$, $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ has a natural Fréchet space structure, and the natural action of G on $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ is a continuous representation. Further, that action of G on $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ is an admissible representation with finite composition series. It has infinitesimal character $\lambda + \rho$. Its underlying Harish–Chandra module is the Zuckerman derived functor module $A^{s+t}(G, H, \mathfrak{b}, \chi) = A^s(G, L, \mathfrak{q}, \eta)$.*

Section 7. Representations for Measurable Orbits.

Fix a parabolic subgroup $P \subset G$, not necessarily cuspidal. In this Section we study measurable integrable orbits $Y = G(w) \subset W$ in complex flag manifolds. For the appropriate choices of (W, w), which means $\mathfrak{q} = \mathfrak{q}_w$ such that $\mathfrak{p} = \mathfrak{q} + \tau\mathfrak{q}$, we show that the fibres of $Y \to G/P$ are (up to topological components) the holomorphic arc components of Y. With this, we study representations induced from P from the viewpoint of our orbit picture, extending the scope of our geometric construction of the standard tempered representations.

We now look for all complex flag manifolds $W \cong \overline{G}_{\mathbb{C}}$ consisting of the $Int(\mathfrak{g})$–conjugates of a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$, and all real group orbits $Y = G(w) \subset W$, such that

(7.1a) Y is measurable, thus partially complex and of flag type,

(7.1b) Y is integrable, so $\mathfrak{q}_w + \tau\mathfrak{q}_w$ is an algebra, and

(7.1c) the normalizer $N_G(S_w)$ of the holomorphic arc component S_w has Lie algebra $\mathfrak{p}_0$.

Recall the Langlands decomposition $P = MAN$. Let H denote a fundamental Cartan subgroup of a Levi component P^r of P, let $\mathfrak{a}_0$ denote the "split component" of the center of $\mathfrak{p}_0^r$,

$$\mathfrak{a}_0 = \{\xi \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] \mid \alpha(\xi) \in \mathbb{R} \text{ for all } \alpha \in \Phi(\mathfrak{g}, \mathfrak{h}) \text{ and } \alpha(\xi) = 0 \text{ for all } \alpha \in \Phi(\mathfrak{p}^r, \mathfrak{h})\}. \tag{7.2}$$

Then $P^r = M \times A$ and $H = J \times A$ where A is the analytic subgroup of G for $\mathfrak{a}_0$, where $\mathfrak{j}_0 = \mathfrak{a}_0^{\perp} \cap \mathfrak{h}_0$, and where $\mathfrak{m} = \mathfrak{j} + \sum_{\alpha \in \Phi(\mathfrak{p}^r, \mathfrak{h})} \mathfrak{g}_\alpha$. Also, here, J is a fundamental Cartan subgroup of M and N is the analytic subgroup of G for the real form $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$ of $\mathfrak{n} = \mathfrak{p}^n$. The real parabolic P is cuspidal if and only if $J/Z_G(G^0)$ is compact.

Given (7.1), the algebras $\mathfrak{p}$ and $\mathfrak{q}$ are related by

(7.3a) $\mathfrak{u}_0 \subset \mathfrak{m}_0$ is the $\mathfrak{m}_0$–centralizer of a subspace of $\mathfrak{j}$,

(7.3b) $\mathfrak{r} \subset \mathfrak{m}$ is a parabolic subalgebra with reductive part $\mathfrak{r}^r = \mathfrak{u}$, and

(7.3c) $\mathfrak{q} = \mathfrak{q}^r + \mathfrak{q}^n$ where $\mathfrak{q}^r = \mathfrak{u} + \mathfrak{a}$ and $\mathfrak{q}^n = \mathfrak{r}^n + \mathfrak{n}$.

Conversely, given P and its Langlands decomposition, (7.3) gives the construction of all $\mathfrak{q}$ that satisfy (7.1). The proof is essentially the same as the proof [**31**, §6.7] of the case (4.5) where the analytic group U for $\mathfrak{u}_0$ is compact modulo $Z_G(G^0)$.

Now fix a complex flag manifold W and a measurable integrable orbit $Y = G(w)$ as in (7.1). Then W consists of the $Int(\mathfrak{g})$–conjugates of a parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_w \subset \mathfrak{g}$ that satisfies (7.3). In particular G has isotropy subgroup UAN at w, and M has isotropy subgroup U at w, where $U/Z_G(G^0)$ may be noncompact but the conditions immediately following (4.4c) remain valid. The holomorphic arc components of Y are the $gS_w = S_{gw} \cong M^\dagger/U$, and gS_w is the topological component of w in the fibre of $Y \cong G/UAN \to G/MAN = G/P$ over gP.

S_w is a measurable open M–orbit on the sub–flag $\overline{M}_{\mathbb{C}}(w) \subset W$ and AN acts trivially on S_w. Now consider representations

(7.4) β : representation of $(\mathfrak{r}+\mathfrak{a}, UA)$, admissible and of finite length on UA.

Let $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)}(\dim \mathfrak{g}_\phi)\phi$ as before. Then as in (4.8), UAN also acts on the representation space E_β by $\gamma_\beta(uan) = e^{\rho_{\mathfrak{a}}}(a)\beta(ua)$, and that specifies a homogeneous vector bundle $\mathbb{E}_{\gamma_\beta} \to Y \cong G/UAN$. Then γ_β is also a representation of $\mathfrak{q} = \mathfrak{r}+\mathfrak{n}$ because β is defined on $\mathfrak{r}$ and we defined γ_β to annihilate $\mathfrak{n}$. Now γ_β is a representation of $(\mathfrak{q}, UAN)$. Thus we have

(7.5a) $\pi_{\beta,q}$: representation of G on $H^q(Y;\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\gamma_\beta}))$ and

(7.5b) $\eta_{\beta,q}$: representation of MAN on $H^q(M/U;\mathcal{O}_{\mathfrak{r}}(\mathbb{E}_{\gamma_\beta}|_{M/U}))$

such that $\pi_{\beta,q} = Ind_P^G(\eta_{\beta,q})$.

First consider the case where β is finite dimensional. Apply Theorem 6.1 to $M(w) \subset \overline{M}_{\mathbb{C}}(w)$. The $\overline{\partial}$ operator for the Dolbeault complex of $\mathbb{E}_{\gamma_\beta} \to M(w) \cong MA/UA$ has closed range. The cohomologies $H^q(M(w);\mathcal{O}_{\mathfrak{q}}(\mathbb{E}_{\gamma_\beta}))$ are admissible Fréchet MA–modules with finite composition series and underlying Harish–Chandra modules $A^s(MA, UA, \mathfrak{r}+\mathfrak{a}, \gamma_\beta)$. The representations here are the $\eta_{\beta,q}$ of (7.5b).

Given $gP \in G/P$ we have the $\overline{\partial}$ operator for the Dolbeault complex of $\mathbb{E}_{\gamma_\beta}|_{gM(w)} \to gM(w)$. The base space there is the fibre of

$$Y \cong G/UAN \to G/MAN = G/P \text{ over } gP.$$

These Dolbeault operators $\overline{\partial}_{gP}$ for the fibre–restrictions of $\mathbb{E}_{\gamma_\beta}$ fit together to form

(7.6) $\overline{\partial}_Y$: Cauchy–Riemann operator for the partial Dolbeault complex of $\mathbb{E}_{\gamma_\beta} \to Y$.

The partial Dolbeault complex here consists of the spaces $C^{-\omega}(Y;\mathbb{E}_{\gamma_\beta}\otimes\Lambda^\bullet\mathbb{N}_Y^*)$ with the operators $\overline{\partial}_Y$. Here $C^{-\omega}$ denotes hyperfunction sections. $\mathbb{N}_Y \to Y$ is the antiholomorphic tangent bundle, intersection of the complexified tangent bundle of Y with the antiholomorphic tangent bundle of W, so $\mathbb{E}_{\gamma_\beta}\otimes\Lambda^q\mathbb{N}_Y^* \to Y$ consists of the $\mathbb{E}_{\gamma_\beta}$–valued $(0,q)$–forms on Y.

The cohomology of that partial Dolbeault complex is computed from a subcomplex with hyperfunction coefficients that are C^∞ along the fibres of $Y \to G/P$. More precisely the inclusion

$$(7.7) \qquad \{C^{-\omega}_{G/P}(Y;\mathbb{E}_{\gamma_\beta}\otimes\Lambda^\bullet\mathbb{N}_Y^*), \overline{\partial}_Y\} \hookrightarrow \{C^{-\omega}(Y;\mathbb{E}_{\gamma_\beta}\otimes\Lambda^\bullet\mathbb{N}_Y^*), \overline{\partial}_Y\}$$

of that subcomplex in the partial Dolbeault complex of $\mathbb{E}_{\gamma_\beta}$ induces isomorphisms of cohomology. The point is that the subcomplex has a natural Fréchet topology adapted to the fibration $Y \to G/P$, and in that topology the operator $\overline{\partial}_Y$ is continuous. This is the content of [**27**, §7].

7.8. LEMMA. *Each* $\overline{\partial}_Y : C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^q \mathbb{N}^*_Y) \to C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^{q+1} \mathbb{N}^*_Y)$ *has closed range.*

PROOF. For each fibre $gM(w)$ of $Y \to G/P$ and each integer $q \geqq 0$ we write C^q_{gP} for the space $C^\infty(gM(w); (\mathbb{E}_{\gamma_\beta} \otimes \wedge^q \mathbb{N}^*_Y)|_{gM(w)})$ of C^∞ bundle valued forms over that fibre, we write Z^q_{gP} for the kernel of $\overline{\partial}_{gP} : C^q_{gP} \to C^{q+1}_{gP}$, and we write B^q_{gP} for the image of $\overline{\partial}_{gP} : C^{q-1}_{gP} \to C^q_{gP}$. We know from Theorem 6.1 that Z^q_{gP} and B^q_{gP} are closed subspaces of the Fréchet space C^q_{gP}. In particular $\overline{\partial}_{gP}$ induces a Fréchet space isomorphism of C^q_{gP}/Z^q_{gP} onto B^{q+1}_{gP}.

Now write $\mathbb{C}^q \to G/P$, $\mathbb{Z}^q \to G/P$ and $\mathbb{B}^q \to G/P$ for the G–homogeneous Fréchet bundles over G/P with fibre over gP given by C^q_{gP}, Z^q_{gP} and B^q_{gP}, respectively. $C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^q \mathbb{N}^*_Y)$ is the space of $C^{-\omega}$ sections of $\mathbb{C}^q \to G/P$, and there the kernel and image of $\overline{\partial}_Y$ are the respective spaces of $C^{-\omega}$ sections of $\mathbb{Z}^q \to G/P$ and $\mathbb{B}^q \to G/P$.

If $\phi \in C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^q \mathbb{N}^*_Y)$ then $\overline{\partial}_Y(\phi) = 0 \iff$ each $\overline{\partial}_{gP}(\phi|_{gM(w)}) = 0$. It follows that the subbundle $\mathbb{Z}^q \to G/P$ is the kernel of $\overline{\partial}_Y : \mathbb{C}^q \to \mathbb{C}^{q+1}$. Thus $\overline{\partial}_Y$ induces an injective bundle map $\gamma^q : \mathbb{C}^q/\mathbb{Z}^q \to \mathbb{B}^{q+1}$. But γ^q is invertible on each fibre, so it must be invertible. Now γ^q is surjective, so $\overline{\partial}_Y$ maps $\mathbb{C}^q$ onto $\mathbb{B}^{q+1}$. In other words, the range of $\overline{\partial}_Y : C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^q \mathbb{N}^*_Y) \to C^{-\omega}_{G/P}(Y; \mathbb{E}_{\gamma_\beta} \otimes \wedge^{q+1} \mathbb{N}^*_Y)$ is the space of $C^{-\omega}$ sections of the Fréchet subbundle $\mathbb{B}^{q+1} \to G/P$ of $\mathbb{C}^{q+1} \to G/P$. Thus $\overline{\partial}_Y$ has closed range as asserted. □

Now we have a serious extension of Theorem 6.1:

7.9. THEOREM. *Let β be a finite dimensional representation of $(\mathfrak{r} + \mathfrak{a}, UA)$. Then the cohomologies $H^q(Y; \mathcal{O}_\mathfrak{q}(\mathbb{E}_{\gamma_\beta}))$ are admissible Fréchet G–modules with finite composition series. Their underlying Harish–Chandra modules are the Zuckerman derived functor modules $A^q(G, UA, \mathfrak{r} + \mathfrak{a}, \beta)$. If E_β has highest weight λ, then the representation $\pi_{\beta,q}$ of G on $H^q(Y; \mathcal{O}_\mathfrak{q}(\mathbb{E}_{\gamma_\beta}))$ has infinitesimal character $\lambda + \rho_\mathfrak{m}$. If $\lambda + \rho_\mathfrak{m}$ is S_w–antidominant* (5.4), *then $H^q(Y; \mathcal{O}_\mathfrak{q}(\mathbb{E}_{\gamma_\beta})) = 0$ for $q \neq s_{S_w}$.*

Finally we consider the case where β may be infinite dimensional but is constrained to be one of the cohomology representations of Theorem 7.9 for an open M–orbit on the flag manifold of all Borel subalgebras of $\mathfrak{m}$.

Start with the projection $p : X \to W$ from the flag manifold of all Borel subalgebras of $\mathfrak{g}$, as in (6.2). Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_{-\alpha} \subset \mathfrak{q}$ where $\mathfrak{h} = \mathfrak{j} + \mathfrak{a}$ corresponds to a fundamental Cartan subgroup of $MA = P^r$ and where $\tau\Phi^+(\mathfrak{m}, \mathfrak{j}) = -\Phi^+(\mathfrak{m}, \mathfrak{j})$. Now $p(x) = w$ where $\mathfrak{b} = \mathfrak{b}_x$ and $M(x)$ is open in the flag manifold $\overline{M}_\mathbb{C}(x)$ of Borel subalgebras of $\mathfrak{m}$.

Let χ be a finite dimensional representation of $(\mathfrak{b}, H)$. Then we have the bundles and sheaves of (6.3) and (6.4), except that those over $G(x)$ are holomorphic only over the holomorphic arc components. Now suppose that χ, as a representation of H, is $G(x)$–antidominant with infinitesimal character λ. Suppose also that $\chi(\mathfrak{b}^n) = 0$. Essentially as in (6.5) write

$$\begin{aligned} &u = \dim_{\mathbb{C}}(K \cap M)(x),\ t = \dim_{\mathbb{C}}(K \cap U)(x), \text{ and} \\ &s = \dim_{\mathbb{C}}(K \cap M)(w);\ \text{so } u = t + s\ . \end{aligned} \tag{7.10}$$

Then $H^p(U(x); \mathcal{O}_{'\mathfrak{b}}('\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})) = 0$ for $p \neq t$ and $H^q(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})) = 0$ for $q \neq u$ by Theorem 7.9. As in (6.6) and (6.7), the Leray spectral sequence of $G(x) \to G(w)$ collapses at E_2 and

$$H^q(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})) = \sum_{a+b=q} H^a(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{H}^b(U(x); \mathcal{O}_{'\mathfrak{b}}('\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})))). \tag{7.11}$$

Here we use the fact that the isotropy subgroup UAN of G at w has orbit $UAN(x) = U(x)$ because $\mathfrak{a} + \mathfrak{n} \subset \mathfrak{b}$. The vanishing in Theorem 7.9 now tells us that

$$H^{s+t}(G(x); \mathcal{O}_\mathfrak{b}(\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})) = H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{H}^t(U(x); \mathcal{O}_{'\mathfrak{b}}('\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}})))). \tag{7.12}$$

It also follows from Theorem 7.9 that the space (7.12) has a natural Fréchet space structure for which the action of G is a continuous representation. Now we have the extension of Theorem 6.9 from measurable open orbits to measurable integrable orbits:

7.13. THEOREM. *Let η denote $H^t(U(x); \mathcal{O}_{'\mathfrak{b}}('\mathbb{E}_{\chi\otimes e^{\rho_\mathfrak{a}}}))$ as a representation of $(\mathfrak{q}, UAN)$ and let $\mathbb{V}_\eta \to G(w)$ denote the associated homogeneous vector bundle. Then $H^q(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta)) = 0$ for $q \neq s$, $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ has a natural Fréchet space structure and the natural action of G on $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ is a continuous representation. Further, that action of G on $H^s(G(w); \mathcal{O}_\mathfrak{q}(\mathbb{V}_\eta))$ is an admissible representation with finite composition series. It has infinitesimal character $\lambda + \rho_\mathfrak{m}$. Its underlying Harish–Chandra module is the Zuckerman derived functor module $A^{s+t}(G, H, \mathfrak{b}, \chi) = A^s(G, UA, \mathfrak{q}, \eta)$.*

Section 8. Open Problems.

The first obvious open problem is to remove the requirement of finite dimensionality from the representations χ of Theorem 6.1. This is done, but only in special cases, in Theorem 6.9. The problem divides into several parts: a clean functorial definition of the topology of the Dolbeault complex, the closed range problem for the $\overline{\partial}$–operator, keeping track of the infinitesimal character, and a vanishing theorem for the antidominant case. It would be especially interesting here to understand whether the infinitesimal character and the vanishing theorem really need $\chi(\mathfrak{q}^n) = 0$, even in the finite dimensional case. Most of this was done by H.-W. Wong [**34**] for finite dimensional χ, but it is not at all clear how to proceed in the infinite dimensional case.

The second obvious open problem is to remove the requirement of finite dimensionality from the representations β of Theorem 7.9. This is done, again only in special cases, in Theorem 7.13, respectively. The problems include those of the measurable open orbit case, but here one must first find a good subcomplex of the partial Dolbeault complex that computes the cohomologies, and then one must have the solution to the first problem for the holomorphic arc components.

Third, it would be good to use this geometric setting to obtain character formulas. This is done in another setting by Hecht and Taylor [**13**].

Fourth, one needs a better connection between representations constructed as in this paper from the G–orbit structure of a complex flag manifold W and those constructed by localization methods from the $\overline{K}_{\mathbb{C}}$–orbit structure. This is fine up on the flag of Borel subalgebras of $\mathfrak{g}$ ([**9**], [**10**]), but not yet satisfactory in general.

Finally, of course, one can ask how much of this goes over to the quantum group setting.

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