Nova Journal of Algebra and Geometry
Vol. 1, No. 4, pp. 383-396, 1992

ISSN 1060-9881

THE UNCERTAINTY PRINCIPLE FOR GELFAND PAIRS

JOSEPH A. WOLF

Department of Mathematics, University of California, Berkeley California 94720; jawolf@math.berkeley.edu

ABSTRACT. We extend the classical Uncertainty Principle to the context of Gelfand pairs. The Gelfand pair setting includes riemannian symmetric spaces, compact topological groups, and locally compact abelian groups. If the locally compact abelian group is $\mathbb{R}^n$ we recover a sharp form of the classical Heisenberg uncertainty principle.

SECTION 1. INTRODUCTION.

The classical uncertainty principle says that a function and its Fourier transform cannot both be mostly concentrated on short intervals: if $f(t)$ has has most of its support in an interval of length ℓ, and its Fourier tranform $\hat{f}(\tau)$ has has most of its support in an interval of length $\hat{\ell}$ then $\ell \cdot \hat{\ell} \geqq 1 - \eta$ where η is specified by the precise meaning of "most of its support". In quantum mechanics this says that the position and momentum of a particle cannot be determined simultaneously [9]. In signal processing it says that instantaneous frequency cannot be measured precisely [8]. And of course the Heisenberg Lie algebra and its commutation relations are the basic building block in harmonic analysis on nilpotent Lie groups [10], and thus in the associated theories of Cauchy–Riemann spaces, Heisenberg manifolds, and hypoellilptic operators ([6], [1], [7]).

A few years ago, D. L. Donoho and P. B. Stark proved [4] a sharp classical extension of the uncertainty principle: $|T| \cdot |W| \geqq 1 - \eta$ whenever $f(t)$ has most of its support in a *set* (not necessarily an interval) T of measure $|T|$ and $\hat{f}(\tau)$ has most of its support in a *set* (not necessarily an interval) W of measure $|W|$.

Let T be a measurable set, 1_T its indicator (= characteristic) function, and $||\cdot|| = ||\cdot||_p$, an L_p norm. We say that

$$f \in L_p \text{ is } \epsilon-\textbf{concentrated} \text{ on } T \text{ if } ||f - 1_T f|| \leqq \epsilon ||f||. \tag{1.1}$$

The precise form of Donoho and Stark's stronger L_2 version mentioned above is [4]

$$\begin{aligned}&\text{Let } f, \hat{f} \in L_2(\mathbb{R}), \text{ let } \epsilon, \delta \geqq 0, \text{ and let } T, W \subset \mathbb{R} \text{ be measurable sets.}\\ &\text{Suppose that } f \text{ is } \epsilon\text{-concentrated on } T \text{ and } \hat{f} \text{ is } \delta\text{-concentrated on } W.\\ &\text{Then Lebesgue measures satisfy } |T| \cdot |W| \geqq (1 - \epsilon - \delta)^2.\end{aligned} \tag{1.2}$$

The key point in the Donoho–Stark L_2 argument is the operator norm inequality

$$||QP||^2 \leqq |T| \cdot |W| \text{ where } Pf = 1_T f \text{ and } Qf = (1_W \hat{f})^\vee \tag{1.3}$$

Research partially supported by N.S.F. Grant DMS 91 00578.

where $\hat{}$ denotes Fourier transform as usual and $^\vee$ is its inverse.

K. T. Smith [11] extended the uncertainty principle (1.2) to locally compact abelian groups by extending the inequality (1.3). In this paper we modify Smith's arguments so that they apply more generally to Gelfand pairs, thus to riemannian symmetric spaces and compact topological groups as well as locally compact abelian groups.

In the notation to be specified just below, we have two apparently different generalizations to Smith's extension of the uncertainty principle. Fix a Gelfand pair (G, K). Theorem 4.3 extends the uncertainty principle to functions on $K\backslash G/K$. It depends on the spherical transform for the Gelfand pair (G, K) and the resulting decomposition of $L_2(K\backslash G/K)$ by positive definite zonal spherical functions. Theorem 6.7 extends the uncertainty principle to functions on G/K; it depends on the vector valued transform corresponding to a direct integral decomposition of $L_2(G/K)$. These two extensions are in fact equivalent because the underlying L_2 decompositions use the same Plancherel measure.

In Section 2 we summarize function theory on $K\backslash G/K$ for a Gelfand pair (G, K). We start with the basic definitions, describe the spherical transform, and discuss the consequences for $L_2(K\backslash G/K)$. Then in Section 3 we prove analogs of (1.3) for $K\backslash G/K$. The corresponding uncertainty principle is in Section 4.

In Section 5 we describe the direct integral decomposition of $L_2(G/K)$ and the corresponding vector valued transform on G/K. Then in Section 6 we verify the analogs of the operator norm inequalities of §3 and derive the uncertainty principle for G/K.

The reader will notice that we spend more space reviewing Gelfand pairs than we spend proving our uncertainty principles. That is the nature of the situation: we are showing that certain classical considerations are valid without essential change in a much broader context. Of course the reader familiar with Gelfand pairs need only glance at Sections 2 and 5 to catch the notation.

SECTION 2. COMMUTATIVE FUNCTION THEORY OF GELFAND PAIRS.

The theory of Gelfand pairs includes many of the general aspects of hamonic analysis on locally compact abelian groups and on symmetric spaces. There are some excellent treatments available, for example [2] and [5]. But we only need the basic structural facts. In this Section we summarize those aspects of Gelfand pair theory that correspond most closely to the Pontrjagin–Plancherel theory for locally compact abelian groups and to the analysis of central functions on compact groups. For brevity, we defer to [2] and [5] for references to original sources.

A. Some Basic Definitions.

Let G be a locally compact topological group and K a compact subgroup. Let m_G be Haar measure on G, subject to the usual convention: counting measure if G is infinite and discrete, total mass 1 if G is compact. Let m_K be Haar measure of total mass 1 on K.

Convolution on G is the action of functions under the left regular representation:

$$f_1 * f_2(y) = \int_G f_1(x) f_2(x^{-1}y)\, dm_G(x). \tag{2.1}$$

As in the case $G = \mathbb{R}$, Young's Inequality

$$\|f * h\|_p \leqq \|f\|_1 \|h\|_p \text{ for } f \in L_1(G) \text{ and } h \in L_p(G) \tag{2.2}$$

is immediate.

A function f on G is called K–bi–invariant if $f(k_1xk_2) = f(x)$ for all $x \in G$ and $k_i \in K$. We say that a function $f : G \to \mathbb{C}$ vanishes at ∞ if, given $\epsilon > 0$, there is a compact subset $C_\epsilon \subset G$ such that $|f(x)| < \epsilon$ for $x \notin C_\epsilon$. Each of the spaces

$C_0(G)$:	continuous functions $G \to \mathbb{C}$ with compact support
$C_\infty(G)$:	continuous functions $G \to \mathbb{C}$ vanishing at ∞
$L_p(G)$:	standard Lebesgue space of measurable functions $G \to \mathbb{C}$, $1 \leqq p \leqq \infty$

projects onto its subspace

$$C_0(K\backslash G/K), \qquad C_\infty(K\backslash G/K), \qquad L_p(K\backslash G/K), \quad 1 \leqq p \leqq \infty$$

of K–bi–invariant functions, by $f \mapsto f^\natural$ where $f^\natural(x) = \int_K \int_K f(k_1xk_2)dm_K(k_1)dm_K(k_2)$.

$C_0(G)$ is an associative algebra under convolution. Note that $C_0(K\backslash G/K)$ is a subalgebra. Similarly, this time using Young's Inequality, $L_1(K\backslash G/K)$ is a subalgebra of the convolution algebra $L_1(G)$. If $x \in G$ then G carries a unique K–bi–invariant probability measure m_x, defined by

$$\int_G f(y)dm_x(y) = \int_K \int_K f(k_1xk_2)dm_K(k_1)dm_K(k_2) \tag{2.3}$$

with support KxK.

2.4. Lemma. *The following conditions are equivalent:*

(1) *The convolution algebra $C_0(K\backslash G/K)$ is commutative.*
(2) *The convolution algebra $L_1(K\backslash G/K)$ is commutative.*
(3) *If $x, y \in G$ then $KxK \cdot KyK = KyK \cdot KxK$.*
(4) *If $x, y \in G$ then $m_x * m_y = m_y * m_x$.*

If those conditions hold, then G is unimodular.

2.5. Definition. (G, K) *is* **commutative**, *i.e., is a* **Gelfand pair** *if the conditions of Lemma* 2.4 *obtain.*

If G is a locally compact abelian group, and we set $K = \{1\}$, then of course (G, K) is commutative. But there are other interesting examples, riemannian symmetric spaces and compact groups, as follows.

The name "Gelfand pair" comes from I. Gelfand's result

2.6. Theorem. *Let G be a locally compact group and θ an involutive automorphism of G such that $G = SK$ where*

(1) *K is a compact subgroup of G,*
(2) *if $k \in K$ then $\theta(k) = k$, and*
(3) *if $s \in S$ then $\theta(s) = s^{-1}$.*

Then G is unimodular and (G, K) is commutative.

2.7. Corollary. *Let M be a connected riemannian symmetric space, let G be any group of isometries of M that contains the identity component of the group of all isometries, and let K be the isotropy subgroup of G at some point of M. Then (G, K) is commutative.*

2.8. Corollary. *Let M be a compact topological group. Let $G = M \times M$, so G acts on M by $(x,y) : m \mapsto xmy^{-1}$. Let K be the stablizer of the identity element of M, so $K = \{(x,x^{-1}) \mid x \in M\} = \Delta M$, the diagonal M in G. Then (G,K) is commutative.*

The special case of Corollary 2.7 is due to É. Cartan. In modern language, Cartan proved that the commuting algebra for the left regular representation of G on $L_2(G/K)$ is commutative, i.e. that this left regular representation is multiplicity–free. Corollary 2.7 includes the Lie group case.

B. The Spherical Transform.

In the special case of a locally compact abelian group G, the spherical transform is the map $f \to \hat{f}$,

$$\hat{f}(\omega) = \int_G f(x)\omega(x^{-1})dm_G(x), \quad \omega \in \widehat{G} \tag{2.9}$$

from $L_p(G)$ to $L_{p'}(\widehat{G})$ with the usual $\frac{1}{p} + \frac{1}{p'} = 1$. This generalizes the classical Fourier transform. Here $\widehat{G}$ is the Pontrjagin dual of G. By definition, $\widehat{G}$ consists of all unitary characters on G, i.e., all continuous homomorphisms $\omega : G \to \{z \in \mathbb{C} \mid |z| = 1\}$ with group composition $(\omega_1\omega_2)(x) = \omega_1(x)\omega_2(x)$. In a certain topology, the case $K = \{1\}$ of the one described in (2.18) – (2.22) below, $\widehat{G}$ is a locally compact abelian group.

Zonal spherical functions are the analog of continuous homomorphisms

$$G \to \{z \in \mathbb{C} \mid z \neq 0\},$$

and positive zonal spherical functions are the analog of unitary characters.

Now fix a Gelfand pair (G,K).

2.10. Definition. *A nonzero Radon measure μ on G is* **spherical** *if*

(1) *it is K–bi–invariant: $\mu(k_1Ek_2^{-1}) = \mu(E)$ for measurable $E \subset G$, and*
(2) *$\mu : C_0(K\backslash G/K) \to \mathbb{C}$ is an algebra homomorphism, i.e., $\mu(f * h) = \mu(f)\mu(h)$, where $\mu(f)$ means $\int_G f(x)d\mu(x)$.*

2.11. Definition. *A continuous function $\omega : G \to \mathbb{C}$ is a* **zonal spherical function** *or* **zsf** *if $d\mu(x) = \omega(x^{-1})dm_G(x)$ defines a spherical measure.*

All continuous homomorphisms $\omega : G \to \{z \in \mathbb{C} \mid z \neq 0\}$ (often called *quasi-characters*) are zsf. Every spherical measure μ is absolutely continuous with respect to Haar measure m_G, so it has form $d\mu(x) = \omega(x^{-1})dm_G(x)$ where ω is K–bi–invariant. Here we may choose ω continuous, hence zsf, and check $\omega(1) = 1$. So the notions of spherical measure and zsf are essentially equivalent. They correspond to quasi–characters on locally compact abelian groups. The following shows that they enjoy properties close to those of quasi–characters.

2.12. Proposition. *These are equivalent for a function $\omega : G \to \mathbb{C}$:*

(1) *ω is a zonal spherical function on G.*
(2) *ω (i) is continuous, (ii) is K–bi–invariant, (iii) satisfies $\omega(1) = 1$, and (iv) if $f \in C_0(K\backslash G/K)$ there is a constant $\lambda_f \in \mathbb{C}$ such that $f * \omega = \lambda_f\omega$.*
(3) *ω is not identically zero, and if $x, y \in G$ then $\omega(x)\omega(y) = \int_K \omega(xky)dm_K(k)$.*

The Fourier transform, whether on $\mathbb{R}$ or an arbitrary locally compact abelian group, only uses unitary characters. So we recall

2.13. Definition. *A function* $\phi : G \to \mathbf{C}$ *is* **positive definite** *if* $\sum \overline{c_i} c_j \phi(x_i^{-1}x_j) \geqq 0$ *whenever* $n \geqq 1$, $\{c_1, \ldots, c_n\} \subset \mathbf{C}$, *and* $\{x_1, \ldots, x_n\} \subset G$.

Note that a positive definite function ϕ satisfies $\phi(1) \geqq |\phi(x)|$ and $\phi(x^{-1}) = \overline{\phi(x)}$ for all $x \in G$.

2.14. Theorem. *Let ϕ be a continuous positive definite function on G with $\phi(1) = 1$. Then there is a unitary representation π of G, and a cyclic unit vector u in the representation space $\mathcal{H}_\pi$, such that $\phi(x) = \langle u, \pi(x)u \rangle$ for all $x \in G$. The pair (π, u) is unique up to unitary equivalence.*

The connection with Gelfand pairs is

2.15. Theorem. *Let ϕ be a positive definite zonal spherical function and (π, u) the corresponding cyclic unitary representation, $\phi(x) = \langle u, \pi(x)u \rangle$. Then π is irreducible and u spans the space $\mathcal{H}_\pi^K$ of $\pi(K)$–fixed vectors. Conversely, if π is an irreducible unitary representation of G and $\mathcal{H}_\pi^K$ is spanned by a unit vector u then $\phi(x) = \langle u, \pi(x)u \rangle$ is a positive definite zonal spherical function.*

Write $S = S(G, K)$ for the set of all zonal spherical functions $f : G \to \mathbf{C}$, and write $P = P(G, K)$ for the set of all positive definite zonal spherical functions.

2.16. Definition. *The* **spherical transform** *is the map $f \mapsto \hat{f}$, from K–bi–invariant functions on G to functions on $S = S(G, K)$, given by*

$$\hat{f}(\omega) = \mu_\omega(f) = \int_G f(x)\omega(x^{-1})dm_G(x). \tag{2.17}$$

If $f \in L_1(K\backslash G/K)$, in particular if $f \in C_0(K\backslash G/K)$, then the integral is absolutely convergent. Map

$$S \to \prod \mathbf{C}_f \text{ by } \omega \mapsto (\hat{f}(\omega)) \text{ as } f \text{ ranges over } C_0(K\backslash G/K). \tag{2.18}$$

This is injective, and we now view S as a subspace of the topological product space $\prod \mathbf{C}_f$. The subspace topology on S is the weak topology for the functions $\hat{f}$ with $f \in C_0(K\backslash G/K)$. Since positive definite functions are bounded by their value at the identity element $1 \in G$,

$$P = P(G, K) \subset \prod D_f \text{ where } D_f = \{z_f \in \mathbf{C}_f \mid |z_f| \leqq \|f\|_1\}. \tag{2.19}$$

If $\omega \in S$, a calculation shows that

$$\omega \in P \text{ if and only if } \mu_\omega(f * f^*) \geqq 0 \text{ for all } f \in C_0(K\backslash G/K). \tag{2.20}$$

Here $f^*(x) = \overline{f(x^{-1})}$ because G is unimodular. Combining (2.19) and (2.20), one proves that

$$\begin{aligned} &P \text{ has closure } \mathrm{cl}(P) \text{ consisting of all } z \in \textstyle\prod D_f \text{ such that} \\ &\quad a)\ \ f \to z_f \text{ is a linear functional on } C_0(K\backslash G/K), \\ &\quad b)\ \ z_{f_1 * f_2} = z_{f_1} z_{f_2} \text{ for all } f_1, f_2 \in C_0(K\backslash G/K), \text{ and} \\ &\quad c)\ \ z_{f * f^*} \geqq 0 \text{ for all } f \in C_0(K\backslash G/K). \end{aligned} \tag{2.21}$$

From this

2.22. Proposition. *P is locally compact. Either P is compact and equal to its closure in $\prod \mathbf{C}_f$, or $\mathrm{cl}P = P \cup 0$ is the 1-point compactification of P.*

Proposition 2.22 gives us an analog of the Riemann-Lebesgue Lemma:

2.23. Corollary. *If $f \in L_1(K\backslash G/K)$ then $\hat{f} \in C_\infty(P)$, i.e., f is a continuous function on P vanishing at ∞.*

C. The Godement–Plancherel Theorem for $K\backslash G/K$.

Godement's extension of the classical Pontrjagin–Plancherel Theorem from locally compact abelian groups to Gelfand pairs is

2.24. Theorem. *Let (G, K) be commutative. Then there is a unique positive Radon measure ν on P, concentrated on a certain subset M, such that*

$$\text{if } f \in C_0(K\backslash G/K) \text{ then } \hat{f} \in L_2(P,\nu) \text{ and } ||\hat{f}||_{L_2(P,\nu)} = ||f||_{L_2(G)} \,. \tag{2.25}$$

Moreover $f \to \hat{f}$ extends by continuity to a Hilbert space isomorphism of $L_2(K\backslash G/K)$ onto $L_2(P,\nu)$ which intertwines the (left) convolution representation

$$\ell: \quad C_0(K\backslash G/K) \text{ on } L_2(K\backslash G/K) \text{ by } \ell(f)h = f * h \tag{2.26a}$$

with the (left) multiplication representation

$$\hat{\ell}: \quad C_0(K\backslash G/K) \text{ on } L_2(P,\nu) \text{ by } \hat{\ell}(f)q = \hat{f}q \,. \tag{2.26b}$$

ν is called **Plancherel measure** for (G, K). The uniform closure of $\ell(C_0(K\backslash G/K))$ in the algebra of bounded linear operators on $L_2(K\backslash G/K)$ is a commutative C^* algebra, and M is its maximal ideal space. If $f \in C_0(K\backslash G/K)$ then $\ell(f)$ has dual $\ell(f)^\frown$, function on M given by $\ell(f)^\frown(m) = \ell(f)$ modulo m. If $m \in M$ there is a unique zonal spherical function ω_m such that

$$\ell(f)^\frown(m) = \int_G f(x)\omega_m(x^{-1})dm_G(x) \,. \tag{2.27}$$

Furthermore ω_m is positive definite, so M is identified with a subset of P, and $M \hookrightarrow P$ extends to a homeomorphism of $M \cup \{0\}$ onto a closed subset of $P \cup \{0\}$. Also,

$$\text{if } f \in C_0(K\backslash G/K) \text{ then } \ell(f)^\frown(m) = \hat{f}(\omega_m), \text{ i.e., } \ell(f)^\frown = \hat{f}|_M \,. \tag{2.28}$$

With these identifications, the construction of Plancherel measure proceeds along the same lines as the standard construction of Haar measure. Along the way one gets the analog of Bochner's Theorem

2.29. Corollary. $f(x) = \int_M \omega_m(x)\hat{f}(m)d\nu(m)$.

and the analog of the Fourier inversion formula

2.30. Corollary. $\hat{f} \in L_1(P,\nu)$ and $f(x) = \int_P \hat{f}(\omega)\omega(x)d\nu(\omega)$.

for f in the dense subspace

$$\text{finite linear combinations } \sum a_{ij} f_i * f_j^* \text{ where the } f_k \in C_0(K\backslash G/K)$$

of $L_2(K\backslash G/K)$.

For comparison it is worth recalling the Pontrjagin–Plancherel Theorem for a locally compact abelian group G. There is a unique normalization of Haar measure $m_{\widehat{G}}$ on $\widehat{G}$ such that $f \mapsto \hat{f}$ defines[1] an isometry of $L_2(G)$ onto $L_2(\widehat{G})$. We always use that normalization for $m_{\widehat{G}}$. The inverse Fourier transform is given by

$$\check{\phi}(x) = \int_{\widehat{G}} \phi(\omega)\omega(x)dm_{\widehat{G}}(\omega). \tag{2.31}$$

This is essentially the same as the Fourier transform on $\widehat{G}$. So here the Titchmarch Inequality for G (compare [3, §51]),

$$\|\hat{f}\|_{p'} \leqq \|f\|_p \text{ whenever } 1 \leqq p \leqq 2 \text{ and } f \in L_p(G) \tag{2.32}$$

and the corresponding inequality for $\widehat{G}$, combine for $p = 2$ to give the isometry of the Plancherel Theorem.

Section 3. Operator Norm Inequalities.

In this section we prove certain operator norm estimates on Gelfand pairs. Just as K. T. Smith used an extension [11] to locally compact abelian groups of the Donoho–Stark estimate (1.3) to prove his uncertainty principle, the estimates we prove will be used in §4 to extend that uncertainty principle to spaces $K\backslash G/K$ where (G, K) is a Gelfand pair.

Fix a Gelfand pair (G, K). To adapt (1.3) we use the spherical transform (2.17) and its inverse (2.30),

$$\hat{f}(\omega) = \int_G f(x)\omega(x^{-1})dm_G(x) \text{ and } h^\vee(x) = \int_P h(\omega)\omega(x)d\nu(\omega). \tag{3.1}$$

As in the classsical case, straightforward computation gives

$$\|\hat{f}\|_\infty \leqq \|f\|_1 \text{ for } f \in L_1(K\backslash G/K) \text{ and } \|h^\vee\|_\infty \leqq \|h\|_1 \text{ for } h \in L_1(P, \nu),$$

so Riesz-Thorin interpolation gives us the analog for $K\backslash G/K$ of the Titchmarch Inequality

$$\begin{aligned} \|\hat{f}\|_{p'} &\leqq \|f\|_p \quad \text{for} \quad f \in L_p(K\backslash G/K), \quad 1 \leqq p \leqq 2, \\ \|h^\vee\|_{p'} &\leqq \|h\|_p \quad \text{for} \quad h \in L_p(P, \nu), \qquad 1 \leqq p \leqq 2, \end{aligned} \tag{3.2}$$

with the usual $\frac{1}{p} + \frac{1}{p'} = 1$. Compare [3, §51].

Fix subsets $T = KTK \subset G$ and $U \subset P$ of finite measure. Let 1_T and 1_U denote their respective indicator functions. Define operators $P = P_T$ and $Q = Q_U$ by

$$Pf = 1_T f \quad \text{and} \quad Qf = (1_U \hat{f})^\vee. \tag{3.3}$$

3.4. Proposition. *If $1 \leqq p \leqq 2$, $q \geqq 1$ and $f \in L_p(K\backslash G/K)$ then*

$$\|PQf\|_q \leqq m_G(T)^{1/q}\nu(U)^{1/p}\|f\|_p .$$

[1] In fact map $f \mapsto \hat{f}$ is initially defined only for $f \in L_1(G) \cap L_2(G)$ and then is extended by continuity as in Theorem 2.24.

Case $p = q$: the operator norm on $L_p(K\backslash G/K)$ satisfies $\|PQ\|_p \leqq m_G(T)^{1/p}\nu(U)^{1/p}$.

Proof. We compute

$$\begin{aligned} PQf(x) &= 1_T(x)Qf(x) \\ &= 1_T(x)(1_U\hat{f})^{\vee}(x) \\ &= 1_T(x)\int_P (1_U\hat{f})(\omega)\omega(x)d\nu(\omega) \\ &= 1_T(x)\int_P 1_U(\omega)\left\{\int_G f(y)\omega(y^{-1})dm_G(y)\right\}\omega(x)d\nu(\omega) \\ &= 1_T(x)\int_G f(y)\left\{\int_P 1_U(\omega)\omega(y^{-1})\omega(x)d\nu(\omega)\right\}dm_G(y). \end{aligned}$$

As $\omega(y^{-1})\omega(x) = \int_K \omega(y^{-1}kx)dm_K(k)$ and f is K-bi-invariant, now

$$PQf(x) = 1_T(x)\int_G f(y)\left\{\int_P 1_U(\omega)\omega(y^{-1}x)d\nu(\omega)\right\}dm_G(y) = \langle f, \overline{k_x}\rangle$$

where $k_x(y) = 1_T(x)\int_P 1_U(\omega)\omega(y^{-1}x)d\nu(\omega) = 1_T(x)1_U^{\vee}(y^{-1}x)$. Using Hölder,

$$|PQf(x)| = |\langle f, \overline{k_x}\rangle| \leqq \|f\|_p\|k_x\|_{p'} = \|f\|_p\|1_U^{\vee}\|_{p'}|1_T(x)|.$$

Integration $\int_G |PQf(x)|^q dm_G(x)$ and the inequality (3.2) give

$$\|PQf\|_q \leqq \|f\|_p\,\|1_U^{\vee}\|_{p'}\,m_G(T)^{1/q} \leqq \|f\|_p\,\|1_U\|_p\,m_G(T)^{1/q} = m_G(T)^{1/q}\,\nu(U)^{1/p}\,\|f\|_p\,.$$

That completes the proof of Proposition 3.4. □

Proposition 3.4 is sufficient for the uncertainty principle of Theorem 4.3 below. But there are some other useful operator norm estimates for Gelfand pairs.

The estimate analogous to that of Proposition 3.4, but in the other order, is

3.5. Proposition. *If $1 \leqq p \leqq 2$, $q \geqq p'$, and $f \in L_p(K\backslash G/K)$, then*[2]

$$\|QPf\|_q \leqq m_G(T)^{1/q}\nu(U)^{1/p}\|f\|_p \text{ and } \|QPf\|_q \leqq m_G(T)^{1/p}\nu(U)^{1/p}\|Pf\|_q\,.$$

Proof. As in Proposition 3.4, compute

$$QPf(x) = \int_P 1_U(\omega)\left\{\int_G 1_T(y)f(y)\omega(y^{-1})dm_G(y)\right\}\omega(x)d\nu(\omega) = \langle f, \overline{j_x}\rangle$$

where $j_x(y) = 1_T(y)1_U^{\vee}(y^{-1}x)$. So $|QPf(x)| \leqq \|f\|_p\|j_x\|_{p'}$ and

$$\|QPf\|_q \leqq \|f\|_p\left(\int_G\left\{\left(\int_G 1_T(y)|1_U^{\vee}(y^{-1}x)|^{p'}dm_G(y)\right)^{1/p'}\right\}^q dm_G(x)\right)^{1/q}.$$

[2]There is a typographical error (a misplaced prime) in the published statement of [11, Theorem 3.2]. The correction is implicit in the argument below. The argument for the second inequality of Proposition 3.5 follows [12].

Now

$$\begin{aligned}
\|QPf\|_q &\leqq \|f\|_p \left(\int_G \left\{(1_T \underset{r}{*} |1_U^\vee|^{p'})(x)\right\}^{q/p'} dm_G(x)\right)^{1/q} \\
&= \|f\|_p(\|(1_T * |1_U^\vee|^{p'})\|_{q/p'})^{1/p'} \\
&\leqq \|f\|_p \left(\|1_T\|_{q/p'}\|(|1_U^\vee|^{p'})\|_1\right)^{1/p'} \quad \text{by Young's Inequality, using } q \geqq p' \\
&= \|f\|_p m_G(T)^{1/q}(\|(|1_U^\vee|^{p'})\|_1)^{1/p'} \\
&= \|f\|_p m_G(T)^{1/q}\|1_U^\vee\|_{p'} \\
&\leqq \|f\|_p m_G(T)^{1/q}\|1_U\|_p \text{ by Titchmarch (3.2), using } 1 \leqq p \leqq 2 \\
&= m_G(T)^{1/q}\nu(U)^{1/p}\|f\|_p \,.
\end{aligned}$$

That proves the first inequality of Proposition 3.5. For the second inequality replace f by Pf in the first inequality and use $P^2 = P$:

$$\|QPf\|_q \leqq m_G(T)^{1/q}\nu(U)^{1/p}\|Pf\|_p \,. \tag{i}$$

If $r = \frac{q}{p}$ and $\frac{1}{r} + \frac{1}{r'} = 1$ then

$$\begin{aligned}
\|Pf\|_p^p &= \langle |Pf|^p, 1_T\rangle \\
&\leqq \|(|Pf|)^p\|_r \; \|1_T\|_{r'} \quad \text{by the Hölder inequality} \\
&= (\|Pf\|_q)^p \; \|1_T\|_{r'} \quad \text{since } rp = q \text{ and } \frac{1}{r} = \frac{p}{q}\,.
\end{aligned} \tag{ii}$$

Use $\frac{1}{pr'} = \frac{1}{p} - \frac{1}{q}$ to combine (i) and (ii) as

$$\|QPf\|_q \leqq m_G(T)^{1/q}\nu(U)^{1/p}\|Pf\|_q m_G(T)^{1/pr'} = m_G(T)^{1/p}\nu(U)^{1/p}\|Pf\|_q \,.$$

That completes the proof. □

If $K\backslash G/K$ has finite measure, then G is compact, and one has a stronger form of part of (3.2), analog of the Hausdorff-Young Inequality (again compare [3, §51]):

$$\|\hat{f}\|_q \leqq \|f\|_p \text{ for } f \in L_p(K\backslash G/K) \text{ with } 1 \leqq p \leqq \infty \text{ and } \frac{1}{q} \leqq \min(\frac{1}{p'}, \frac{1}{2})\,. \tag{3.6}$$

Similarly, if P has finite measure, for example if P is compact, then

$$\|h^\vee\|_q \leqq \|h\|_p \text{ for } h \in L_p(P, \nu) \text{ with } 1 \leqq p \leqq \infty \text{ and } \frac{1}{q} \leqq \min(\frac{1}{p'}, \frac{1}{2})\,. \tag{3.7}$$

These lead to slightly stronger forms of Propositions 3.4 and 3.5.

Section 4. The Uncertainty Principle for $K\backslash G/K$.

As in §3 fix a Gelfand pair (G, K) and subsets $T = KTK \subset G$ and $U \subset P$ of finite measure. Define operators $P = P_T$ and $Q = Q_U$ as in (3.3). Given $\epsilon \geqq 0$ we say that

$$f \in L_p(K\backslash G/K) \text{ is } L_p\ \epsilon-\textbf{concentrated} \text{ on } T \text{ if } \|f - 1_T f\|_p \leqq \epsilon\|f\|_p \tag{4.1a}$$

and similarly, given $\delta \geqq 0$,

$$h \in L_{p'}(P, \nu) \text{ is } L_{p'}\ \delta-\textbf{concentrated} \text{ on } U \text{ if } \|h - 1_U h\|_{p'} \leqq \delta\|h\|_{p'} \tag{4.1b}$$

Somewhat analogously, we say (compare [4] and [11]) that

$$\begin{aligned}
&f \in L_p(K\backslash G/K) \text{ is } L_p\ \delta-\textbf{bandlimited} \text{ to } U \text{ if there exists} \\
&\quad f_U \in L_p(K\backslash G/K) \text{ with } \widehat{f_U} \text{ supported in } U \text{ and } \|f - f_U\|_p \leqq \delta\|f\|_p\,.
\end{aligned} \tag{4.2}$$

4.3. Theorem. *Suppose that $0 \neq f \in L_p(K\backslash G/K)$ with $1 \leqq p \leqq 2$ and $\epsilon, \delta \geqq 0$ such that f is ϵ–concentrated on T and δ–bandlimited to U. Then*

$$m_G(T)^{1/p}\nu(U)^{1/p} \geqq ||PQ||_p \geqq \frac{1-\epsilon-\delta}{1+\delta}.$$

And if $p = 2$ then, further, $||PQ||_2 \geqq 1 - \epsilon - \delta$.

Proof. The first inequality is the case $p = q$ of Proposition 3.4. Note $f_U = Qf_U$ because $\widehat{f_U}$ is supported in U, and of course $||P|| \leqq 1$, to compute

$$\begin{aligned} ||f||_p - ||PQf||_p &\leqq ||f - PQf||_p \\ &\leqq ||f - Pf||_p + ||Pf - Pf_U||_p + ||PQf_U - PQf||_p \\ &\leqq \epsilon||f||_p + \delta||f||_p + ||PQ||_p\delta||f||_p \\ &= (\epsilon + \delta + \delta||PQ||_p)||f||_p\ , \end{aligned}$$

so $||PQf||_p \geqq (1 - \epsilon - \delta - \delta||PQ||_p)||f||_p$. Now $||PQ||_p \geqq 1 - \epsilon - \delta - \delta||PQ||_p$ so

$$(1+\delta)||PQ||_p \geqq 1 - \epsilon - \delta\ .$$

That proves the general assertion. If $p = 2$ we can take $\widehat{f_U} = Qf$ so that $Qf = Qf_U$ and the $||PQf_U - PQf||_p$ term does not occur. □

Section 5. Noncommutative Function Theory of Gelfand Pairs.

In this Section we describe the vector-valued transform that leads to an analysis of $L_2(G/K)$ where (G, K) is a Gelfand pair. As before, we defer to [2] and [5] for reference to original sources.

Recall Theorem 2.15. Every positive definite zonal spherical function $\omega \in P$ determines an essentially unique irreducible unitary representation π_ω and unit vector $u_\omega \in \mathcal{H}_\omega = \mathcal{H}_{\pi_\omega}$ such that $\omega(x) = \langle u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega}$ for all $x \in G$. It is immediate that

$$\text{if } f \in L_1(G/K) \text{ and } x \in G \text{ then } (f * \omega)(x) = \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega}\ . \tag{5.1}$$

On the other hand, a calculation shows that

$$\text{if } f \in C(G/K) \cap L_1(G/K) \text{ and } x \in G \text{ then } f(x) = \int_P (f * \omega)(x)d\nu(\omega)\ . \tag{5.2}$$

So $f * \omega \in C(G/K) \cap L_1(G/K)$ corresponds to $\pi_\omega(f)u_\omega \in \mathcal{H}_\omega$. In fact (5.1) and (5.2) combine to prove

5.3. Lemma. *If $f \in C_0(G/K)$ and $x \in G$ then*

$$f(x) = \int_P \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \text{ and } ||f||^2_{L_2(G/K)} = \int_P ||\pi_\omega(f)u_\omega||^2_{\mathcal{H}_\omega} d\nu(\omega)\ .$$

We need the details of the notion of direct integral. Let (Y, τ) be a measure space. For each $y \in Y$ let $\mathcal{H}_y$ be a Hilbert space. Let $\{s_\alpha\}$ be a countable family of maps $Y \to \bigcup_{y\in Y} \mathcal{H}_y$ such that

$s_\alpha(y) \in \mathcal{H}_y$ and $\{s_\alpha(y)\}$ spans $\mathcal{H}_y$ for all $y \in Y$, and the functions $y \mapsto \langle s_\alpha(y), s_\alpha(y)\rangle_{\mathcal{H}_y}$ belong to $L_1(Y,\tau)$.

Then the **direct integral** $\mathcal{H} = \int_Y \mathcal{H}_y d\tau(y)$ **modeled on the** $\{s_\alpha\}$ is the linear span of all the maps $s : Y \to \bigcup_{y\in Y} \mathcal{H}_y$ such that

$s(y) \in \mathcal{H}_y$ a.e. (Y,τ) and the functions $y \mapsto \langle s(y), s_\alpha(y)\rangle_{\mathcal{H}_y}$ belong to $L_1(Y,\tau)$.

$\mathcal{H}$ is a separable Hilbert space with inner product $\langle s,s'\rangle_{\mathcal{H}} = \int_Y \langle s(y), s'(y)\rangle_{\mathcal{H}_y} d\tau(y)$.

More generally, fix $1 \leqq p \leqq \infty$. Then the L_p **direct integral** $\mathcal{H}_p$ is the linear span of the maps $s : Y \to \bigcup_{y\in Y} \mathcal{H}_y$ such that

$s(y) \in \mathcal{H}_y$ a.e. (Y,τ) and the functions $y \mapsto \langle s(y), s(y)\rangle_{\mathcal{H}_y}^{1/2}$ belong to $L_p(Y,\tau)$.

$\mathcal{H}_p$ is a Banach space with norm $||s||_p = \left(\int_P \langle s(y), s(y)\rangle_{\mathcal{H}_y}^{p/2}\right)^{1/p}$ for $1 \leqq p < \infty$, norm $||s||_\infty =$ ess $\sup_{(P,\nu)} \langle s(y), s(y)\rangle_{\mathcal{H}_y}^{1/2}$. Of course $\mathcal{H}_2$ is the Banach structure underlying the Hilbert space structure of $\mathcal{H}$.

Let $BL(\mathcal{H}_y)$ denote the algebra of bounded linear operators on the Hilbert space $\mathcal{H}_y$. Let $T : Y \to \bigcup_{y\in Y} BL(\mathcal{H}_y)$ such that

$T(y) \in BL(\mathcal{H}_y)$ a.e. (Y,τ), and if $s,s' \in \mathcal{H}$ then $y \mapsto \langle T(y)s(y), s'(y)\rangle$ belongs to $L_1(Y,\tau)$.

Then $s \mapsto Ts \in \mathcal{H}$, $Ts(y) = T(y)s(y)$, defines an element $T \in BL(\mathcal{H})$. This element is denoted $T = \int_Y T(y)d\nu(y)$ and is called the **direct integral** of the $T(y)$.

In our case, (P,ν) is the measure space; for $\omega \in P$ we have the Hilbert space $\mathcal{H}_\omega$; for $f \in C_0(G/K)$ we have $s_f(\omega) = \pi_\omega(f)u_\omega$, and $\{s_\alpha\} = \{s_{f_\alpha}\}$ where $\{f_\alpha\}$ is a countable dense subset of $C_0(G/K)$.

5.4. Definition. *The* **Fourier transform** *on G/K is the map*

$$\mathcal{F} : L_1(G/K) \to \int_P \mathcal{H}_\omega d\nu(\omega)$$

given by $\mathcal{F}(f)(\omega) = s_f(\omega) = \pi_\omega(f)u_\omega$.

Now Lemma 5.3 gives us the Godement–Plancherel Theorem for G/K:

5.5. Theorem. *Let (G,K) be commutative and let ν be its Plancherel measure. Then the Fourier transform*

$$\mathcal{F} : C_0(G/K) \to \mathcal{H}\ , \qquad \textit{where}\ \mathcal{H} = \int_P \mathcal{H}_\omega d\nu(\omega)\ , \tag{5.7}$$

satisfies

$$f(x) = \int_P \langle \mathcal{F}(f), \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \text{ and } ||\mathcal{F}(f)||_{\mathcal{H}} = ||f||_{L_2(G/K)}\,. \tag{5.8}$$

Moreover $f \mapsto \mathcal{F}(f)$ extends by continuity to a Hilbert space isomorphism of $L_2(G/K)$ onto $\mathcal{H}$ which intertwines the (left) regular representation

$$\lambda :\quad L_1(G) \text{ on } L_2(G/K) \text{ by } \lambda(\psi)f = \psi * f \tag{5.9a}$$

with the direct integral representation

$$\begin{aligned} &\pi = \int_P \pi_\omega\, d\nu(\omega) : \\ &L_1(G) \text{ on } \mathcal{H} = \int_P \mathcal{H}_\omega d\nu(\omega) \quad \text{by} \quad \pi(\psi) = \int_P \pi_\omega(\psi) d\nu(\omega)\,. \end{aligned} \tag{5.9b}$$

The standard Plancherel-type formula is of the form

$$f(x) = \int_{\widehat{G}} \text{trace } \pi(\ell(x^{-1})f)\, d\mu(\pi), \tag{5.10}$$

where $\widehat{G}$ is the unitary dual of G and $\ell(y)f(g) = f(y^{-1}g)$. The connection with Theorem 5.5 is given by

$$\text{trace } \pi_\omega(\ell(x^{-1})f) = \langle \pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega \rangle. \tag{5.11}$$

Formula (5.11) depends on the fact that if $v_\omega \perp u_\omega$ in $\mathcal{H}_\omega$ and $f \in L_1(G/K)$ then $\pi_\omega(f)v_\omega = 0$. The same calculation shows that if $\pi \in \widehat{G}$ has no K–fixed vector and $f \in L_1(G/K)$ then $\pi(f) = 0$. So for functions on G/K, only the π_ω occur in the expression (5.10).

Section 6. The Uncertainty Principle for G/K

We now modify the considerations of §§3 and 4 to carry them from $K\backslash G/K$ to G/K. We start with the formulae for the Fourier transform (5.4) and its inverse (5.8),

$$\mathcal{F}(f)(\omega) = \pi_\omega(f)u_\omega \text{ and } \mathcal{F}^{-1}(v)(x) = \int_P \langle v_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega). \tag{6.1}$$

Again as in the classical case,

$$\begin{aligned} ||\mathcal{F}(f)||_\infty = \ & \text{ess sup}_{(P,\omega)}\ ||\mathcal{F}(f)(\omega)||_{\mathcal{H}_\omega} = \text{ ess sup}_{(P,\omega)}\ ||\pi_\omega(f)u_\omega||_{\mathcal{H}_\omega} \\ \leqq \ & \text{ess sup}_{(P,\omega)}\ ||f||_1 = ||f||_1 \end{aligned}$$

for $f \in L_1(G/K)$, and

$$||\mathcal{F}^{-1}(v)||_\infty = \text{ ess sup} \int_P \langle v(\omega), \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \leqq \int_P ||v(\omega)||_{\mathcal{H}_\omega} d\nu(\omega) = ||v||_1$$

for $v \in \mathcal{H}_1$. So as before, Riesz-Thorin interpolation results in a G/K analog of the Titchmarch Inequality

$$\begin{aligned} ||\mathcal{F}f||_{p'} \quad &\leqq ||f||_p \quad \text{for} \quad f \in L_p(G/K), \quad 1 \leqq p \leqq 2, \\ ||\mathcal{F}^{-1}(v)||_{p'} &\leqq ||v||_p \quad \text{for} \quad v \in \mathcal{H}_p\,, \qquad 1 \leqq p \leqq 2, \end{aligned} \tag{6.2}$$

with the usual $\frac{1}{p} + \frac{1}{p'} = 1$.

Fix sets $T = TK \subset G$ and $U \subset P$ of finite measure. As in (3.3) define

$$Pf = 1_T f \qquad \text{and} \qquad Qf = \mathcal{F}^{-1}(1_U \mathcal{F}(f))\,. \tag{6.3}$$

6.4. Proposition. *If $1 \leqq p \leqq 2, q \geqq 1$ and $f \in L_p(G/K)$ then*

$$||PQf||_q \leqq m_G(T)^{1/q}\nu(U)^{1/p}||f||_p\,.$$

Case $p = q$: the operator norm on $L_p(G/K)$ satisfies $||PQ||_p \leqq m_G(T)^{1/p}\nu(U)^{1/p}$.

Proof. We compute essentially as in the proof of Proposition 3.4:

$$\begin{aligned} PQf(x) &= 1_T(x)\mathcal{F}^{-1}(1_U\mathcal{F}(f))(x) \\ &= 1_T(x) \int_P \langle 1_U(\omega)\pi_\omega(f)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} \\ &= 1_T(x) \int_P 1_U(\omega) \left\{ \int_G f(y)\langle \pi_\omega(y)u_\omega, \pi_\omega(x)u_\omega\rangle_{\mathcal{H}_\omega} dm_G(y) \right\} d\nu(\omega) \\ &= 1_T(x) \int_G f(y) \left\{ \int_P 1_U(\omega)\langle u_\omega, \pi_\omega(y^{-1}x)u_\omega\rangle_{\mathcal{H}_\omega} d\nu(\omega) \right\} dm_G(y) \\ &= \langle f, \overline{k_x}\rangle_{L_2(G/K)} \end{aligned}$$

where

$$k_x(y) = 1_T(x) \int_P 1_U(\omega) \langle u_\omega, \pi_\omega(y^{-1}x) u_\omega \rangle_{\mathcal{H}_\omega} d\nu(\omega)$$
$$= 1_T(x) \mathcal{F}^{-1}(\omega \mapsto 1_U(\omega) u_\omega)(y^{-1}x) .$$

Using Hölder, integration $\int_G |PQf(x)|^q dm_G(x)$, and the inequality (6.2), we see, as before, that

$$||PQf||_q \leqq ||f||_p \, ||\mathcal{F}^{-1}(\omega \mapsto 1_U(\omega) u_\omega)||_{p'} \, m_G(T)^{1/q}$$
$$\leqq ||f||_p \, ||1_U||_p \, m_G(T)^{1/q} = m_G(T)^{1/q} \nu(U)^{1/p} \, ||f||_p .$$

That completes the proof of Proposition 6.4. □

One can also prove operator norm inequalities corresponding to (3.5), (3.6) and (3.7), but we leave their formulation and proof to the reader.

We now proceed as for $K\backslash G/K$ in §4. Given $\epsilon \geqq 0$ we say that

$$f \in L_p(G/K) \text{ is } L_p \ \epsilon - \text{concentrated on } T \text{ if } ||f - 1_T f||_p \leqq \epsilon ||f||_p \tag{6.5a}$$

and similarly, given $\delta \geqq 0$,

$$v \in \mathcal{H}_{p'} \text{ is } L_{p'} \ \delta - \text{concentrated on } U \text{ if } ||v - 1_U v||_{p'} \leqq \delta ||v||_{p'} \tag{6.5b}$$

Analogously,

$$\begin{aligned} &f \in L_p(G/K) \text{ is } L_p \ \delta - \text{bandlimited to } U \text{ if there exists} \\ &\quad f_U \in L_p(G/K) \text{ with } \mathcal{F}(f_U) \text{ supported in } U \text{ and } ||f - f_U||_p \leqq \delta ||f||_p . \end{aligned} \tag{6.6}$$

We proved Theorem 4.3, the uncertainty principle for $K\backslash G/K$, as a formal conequence of Proposition 3.4. Now exactly the same argument proves the uncertainty principle for G/K as a formal consequence of Proposition 6.4:

6.7. Theorem. *Suppose that $0 \neq f \in L_p(G/K)$ with $1 \leqq p \leqq 2$ and $\epsilon, \delta \geqq 0$ such that f is ϵ–concentrated on T and δ–bandlimited to U. Then*

$$m_G(T)^{1/p} \nu(U)^{1/p} \geqq ||PQ||_p \geqq \frac{1 - \epsilon - \delta}{1 + \delta}.$$

And if $p = 2$ then, further, $||PQ||_2 \geqq 1 - \epsilon - \delta$.

There is an equivalent uncertainty principle for the operator–valued transform $f \mapsto \mathbf{F}(f)$ where $\mathbf{F}(f)(\omega) = \pi_\omega(f)$ for $f \in L_p(G/K)$ and $\omega \in P$. See the discussion surrounding (5.10) and (5.11).

References

1. R. Beals and P. Greiner, *Calculus on Heisenberg Manifolds*, Annals of Math. Studies **119**, Princeton University Press, 1988.
2. J. Dieudonné, *Éléments d'Analyse, Tome VI, Chapitre XXII: Analyse Harmonique*, Gauthier–Villars, 1976.
3. W. F. Donoghue Jr., *Distributions and Fourier Transforms*, Academic Press, 1969.
4. D. L. Donoho & P. B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Applied Math. **49** (1989), 906–931.
5. J. Faraut, *Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques*, Analyse Harmonique, Les Cours de CIMPA (1982), 315–446.
6. C. L. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.
7. G. B. Folland, *Harmonic Analysis in Phase Space*, Annals of Math. Studies **122**, Princeton University Press, 1989.
8. D. Gabor, *Theory of Communication*, J. Inst. Electrical Engineering **93** (1946), 429–457.
9. W. Heisenberg, *The Physical Principles of Quantum Theory*, University of Chicago Press, 1930.
10. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.
11. K. T. Smith, *The uncertainty principle on groups*, SIAM J. Appl. Math. **50** (1990), 876–882.
12. K. T. Smith, *private communication* (February 15, 1992).

Received March 9, 1992