Annals of Mathematics, **136** (1992), 541–555

The Stein condition for cycle spaces of open orbits on complex flag manifolds

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Introduction and statement of results

Let G be a connected, reductive, real Lie group, $\mathfrak{g}_0$ be its real Lie algebra and $\mathfrak{g} = \mathfrak{g}_0 \otimes_R \mathbb{C}$ be its complexified Lie algebra. As usual, $\mathrm{Int}(\mathfrak{g})$ denotes the complex, connected, semisimple Lie group of all inner automorphisms of $\mathfrak{g}$, consisting of the automorphisms $\mathrm{Ad}(g)$ as g runs over any connected Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. Given

$$(0.1) \qquad \mathfrak{p}: \text{ any parabolic subalgebra of } \mathfrak{g},$$

we have the complex flag manifold

$$(0.2) \qquad X = G_{\mathbb{C}}/P: \text{ all } \mathrm{Int}(\mathfrak{g})\text{-conjugates of } \mathfrak{p},$$

where P is the parabolic subgroup of $G_{\mathbb{C}}$ that is the analytic subgroup for $\mathfrak{p}$.

Here G acts on X through its adjoint action on $\mathfrak{g}$. Since we will only be interested in the G orbits and their structure, we may, and do, assume that

$$(0.3) \qquad G_{\mathbb{C}} \text{ is simply connected and semisimple, and that } G \subset G_{\mathbb{C}}.$$

The G-orbit structure of X is well understood (see [13]). There are only finitely many orbits, in particular there are open orbits. If $x \in X$, let $\mathfrak{p}_x$ be the corresponding parabolic subalgebra of $\mathfrak{g}$; that is, if $x = gP$, then $\mathfrak{p}_x = \mathrm{Ad}(g)\mathfrak{p}$. Let $\xi \mapsto \overline{\xi}$ denote the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}$. Then $\mathfrak{p}_x \cap \overline{\mathfrak{p}_x}$ contains a Cartan subalgebra of $\mathfrak{g}$ of the form $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ denote the root system. Fix

$$(0.4a) \qquad \Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h}): \text{ positive root system}$$

*Research partially supported by N.S.F. Grant DMS 88-05816.

such that the corresponding[1] Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha} \text{ is contained in } \mathfrak{p}_x. \tag{0.4b}$$

Then there is a set Φ of simple roots such that

$$\mathfrak{p}_x = \mathfrak{p}_x^r + \mathfrak{p}_x^{-n}, \quad \mathfrak{p}_x^r = \mathfrak{h} + \sum_{\alpha \in \Phi^r} \mathfrak{g}_\alpha, \quad \mathfrak{p}_x^{-n} = \sum_{\beta \in \Phi^n} \mathfrak{g}_{-\beta}, \tag{0.5a}$$

where

$$\begin{array}{l} \Phi^r \text{ consists of all roots that are linear combinations from } \Phi, \\ \Phi^n \text{ consists of all positive roots that are not contained in } \Phi^r. \end{array} \tag{0.5b}$$

Here $\mathfrak{p}_x^{-n}$ is the nilradical of $\mathfrak{p}_x$ and $\mathfrak{p}_x^r$ is a reductive complement. Given $\mathfrak{h}$ and $\Delta^+(\mathfrak{g}, \mathfrak{h})$, every parabolic subalgebra of $\mathfrak{g}$ is $\mathrm{Int}(\mathfrak{g})$-conjugate to one of the forms (0.5) for a unique set Φ of simple roots.

In the context of equations (0.4), one knows from [13], Thm. 4.5, that

$$\begin{array}{c} G(x) \text{ is open in } X \text{ if and only if } \mathfrak{h} \text{ and } \Delta^+ \\ \text{can be chosen such that } \overline{\Delta^+} = -\Delta^+. \end{array} \tag{0.6}$$

Here note that $\overline{\Delta^+} = -\Delta^+$ implies that $\mathfrak{h}_0$ contains a regular elliptic element, so that $\mathfrak{h}_0$ is a fundamental (as compact as possible) Cartan subalgebra of $\mathfrak{g}_0$.

We now fix

$$\begin{cases} D = G(x) \subset X: \text{ open real group orbit on the complex} \\ \qquad \text{flag manifold } X, \\ \mathfrak{h},\ \Delta^+: \text{ Cartan subalgebra and positive root system,} \\ \qquad \text{as in (0.6), and} \\ K: \text{ maximal compact subgroup of } G \text{ such that } \mathfrak{k} \cap \mathfrak{h} \\ \qquad \text{is a Cartan subalgebra of } \mathfrak{k}, \end{cases} \tag{0.7}$$

where $\mathfrak{k}_0$ and $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C}$ are the real and complexified Lie algebras of K. Then the isotropy subgroup of G at x is

$$V = G \cap P_x; \text{ then } D \cong G/V \text{ and } \mathfrak{v} = \mathfrak{p}_x \cap \overline{\mathfrak{p}_x}, \tag{0.8}$$

where, of course, $\mathfrak{v}_0$ and $\mathfrak{v} = \mathfrak{v}_0 \otimes \mathbb{C}$ are the real and complexified Lie algebras of V.

[1] Note that our parabolic subalgebras—including Borel subalgebras—have nilradicals that are sums of negative root spaces. This is so that holomorphic tangent spaces will be spanned by positive root spaces so that, in turn, positive linear functionals will correspond to positive vector bundles.

The most interesting case for current applications is the case (cf. [13], §6) of a *measurable* open orbit—the case where D carries a G-invariant measure. If D is measurable, then, in fact, the measure is induced by the volume form of a G-invariant, indefinite-Kähler metric. The following conditions are equivalent, and D is measurable if and only if they hold ([13], Thm. 6.3):

$$(0.9)\qquad \begin{cases} V \text{ is the centralizer of a torus subgroup } Z \text{ of } K \cap V, \\ \mathfrak{p}_x \cap \overline{\mathfrak{p}_x} \text{ is reductive}, \\ \mathfrak{p}_x \cap \overline{\mathfrak{p}_x} = \mathfrak{p}_x^r, \\ \overline{\mathfrak{p}_x^n} = \mathfrak{p}_x^{-n}. \end{cases}$$

Here $\mathfrak{p}_x^n = \sum_{\beta\in\Phi^n} \mathfrak{g}_\beta$. Note that, in general, $D = G(x)$ is open in X if and only if $\overline{\mathfrak{p}_x^n} \subset \mathfrak{p}_x$, which is implied by the last equality of conditions (0.9). For $\mathfrak{p}_x^n$ represents the holomorphic tangent space to X at x, thus to D at x in the case of an open orbit; so in that case, $\overline{\mathfrak{p}_x^n}$ represents the antiholomorphic tangent space.

Conditions (0.9) are automatic if K contains a Cartan subgroup of G, that is, if $\operatorname{rank} K = \operatorname{rank} G$, in particular if V is compact. They are also automatic if P is a Borel subgroup of G. More generally they are equivalent (cf. [13], Thm. 6.7) to the condition that $\overline{\mathfrak{p}}$ be $\mathrm{Int}(\mathfrak{g})$-conjugate to the parabolic subalgebra of $\mathfrak{g}$ that is opposite to $\mathfrak{p}$.

Whether D is measurable or not, $\mathfrak{k} \cap \mathfrak{p}_x$ is a parabolic subalgebra of $\mathfrak{k}$, for Δ^+ consists of all roots whose value on some element $\xi \in \mathfrak{k}_0 \cap \mathfrak{h}$ has a positive imaginary part. It follows that

$$(0.10)\qquad \begin{gathered} Y = K(x) \cong K/(K \cap V) \cong K_{\mathbb{C}}/(K_{\mathbb{C}} \cap P_x) \\ \text{is a complex submanifold of } D. \end{gathered}$$

Furthermore Y is not contained in any compact complex submanifold of D of greater dimension. So Y is a maximal compact subvariety of D. We will refer to

$$(0.11)\qquad M_D = \{gY \mid g \in G_{\mathbb{C}} \text{ and } gY \subset D\}$$

as the *linear cycle space* or the *space of maximal compact-linear subvarieties* of D. Since Y is compact and D is open in X, then M_D is open in

$$(0.12a)\qquad M_X = \{gY \mid g \in G_{\mathbb{C}}\} \cong G_{\mathbb{C}}/L,$$

where

$$(0.12b)\qquad L = \{g \in G_{\mathbb{C}} \mid gY = Y\}, \text{ a closed complex subgroup of } G_{\mathbb{C}}.$$

Thus M_D has a natural structure of a complex manifold. The point of this paper is to prove the following theorem:

THEOREM 0.13. *Let D be a measurable open G orbit on a complex flag manifold $X = G_{\mathbb{C}}/P$. Then the linear cycle space M_D is a Stein manifold.*

Some time ago, R.O. Wells, Jr. and I gave an argument for Theorem 0.13 in the case where V is compact (see [12], Thm. 2.5.6). That argument made essential use of an exhaustion function of W. Schmid [9] and techniques from integral geometry. Recently D.N. Akhiezer and S.G. Gindikin found some combinatorial problems[2] with the proof, and I found a problem[3] in the use of Schmid's exhaustion function.

The new elements in the present proof of Theorem 0.13 are the idea behind the reorganization that settles the combinatorial problems in [12]; the somewhat more serious use of semisimple structure theory and the structure of bounded symmetric domains; an exhaustion function [10] for D, whose Levi form has the appropriate number of negative eigenvalues; and a variation on classical methods (see [6] and [2]; or [5], §2.6) for constructing certain sorts of strictly plurisubharmonic exhaustion functions.

In the situation we consider in Section 3, the circle of ideas considered by Docquier and Grauert in [3] and by Andreotti and Narasimhan in [1] suggests the path from a certain plurisubharmonic function ϕ_M on M_D to a strictly plurisubharmonic exhaustion function and, thus, to the Stein condition on M_D. I wish to thank Alan Huckleberry for steering me to the work of Docquier and Grauert [3] and Andreotti and Narasimhan [1].

1. Structure of the complex isotropy subgroup

In this section we work out the structure of the $G_{\mathbb{C}}$-stabilizer L of the maximal compact-linear subvariety Y in our open orbit $D = G(x) \cong G/V$. At this point we do not need to assume measurability of D. The starting point is the following lemma, which is obvious.

LEMMA 1.1. *The kernel of the action of $L = \{g \in G_{\mathbb{C}} \mid gY = Y\}$ on Y is*

$$E = \bigcap_{k \in K} kP_xk^{-1} = \bigcap_{k \in K_{\mathbb{C}}} kP_xk^{-1} \tag{1.2}$$

and $K_{\mathbb{C}}E \subset L \subset K_{\mathbb{C}}P_x$.

In general, $G_{\mathbb{C}}, P, X, D, K$ and Y break up as direct products according to any decomposition of $\mathfrak{g}_0$ as a direct sum of ideals or, equivalently, any

[2] These problems are settled as a consequence of the material in §§1 and 2 below.

[3] The Levi form of that function has $n - s = \dim_{\mathbb{C}} D - \dim_{\mathbb{C}} S$ negative eigenvalues, but in [12] it was also assumed to have s positive eigenvalues. This problem is not amenable to a simple patch, but it is avoided in the arguments of §3 below.

decomposition of G as a direct product. Here we are taking advantage of assumption (0.3). So for purposes of determining the group L specified in equations (0.12) and just above, we may, and do, assume that G and $\mathfrak{g}_0$ are noncompact and simple. This is equivalent to the assumption that G/K is an irreducible Riemannian-symmetric space of noncompact type.

We will say that G is of *hermitian type* if the irreducible Riemannian-symmetric space G/K carries the structure of a hermitian-symmetric space.

Let θ be the Cartan involution of G with the fixed point set K. We also write θ for its holomorphic extension to $G_{\mathbb{C}}$ and its differential on $\mathfrak{g}_0$ and $\mathfrak{g}$. Now $\mathfrak{g}$ decomposes as $\mathfrak{k}+\mathfrak{s}$ into ± 1-eigenspaces of θ. Our irreducibility assumption says, exactly, that the adjoint action of K on $\mathfrak{s}_0 = \mathfrak{g}_0 \cap \mathfrak{s}$ is irreducible. Thus G is of hermitian type if and only if this action fails to be absolutely irreducible. Then there is a positive root system $'\Delta^+ = {}'\Delta^+(\mathfrak{g}, \mathfrak{h})$ with the following property: $\mathfrak{s} = \mathfrak{s}_+ + \mathfrak{s}_-$, where $\mathfrak{s}_+$ is a sum of $'\Delta^+$-positive root spaces and represents the holomorphic tangent space of G/K, and $\mathfrak{s}_- = \overline{\mathfrak{s}_+}$ is a sum of $'\Delta^+$-negative root spaces and represents the antiholomorphic tangent space. Write $S_\pm = \exp(\mathfrak{s}_\pm) \subset G_{\mathbb{C}}$. Then G/K is an open G orbit on $G_{\mathbb{C}}/K_{\mathbb{C}}S_-$.

The complex Lie algebra $\mathfrak{g}$ has the compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{s}_0$. In other words, the corresponding real-analytic subgroup G_u of $G_{\mathbb{C}}$ is the compact real form of $G_{\mathbb{C}}$ and, thus, is a maximal compact subgroup. The maximal compact subgroup of G satisfies $K = G \cap G_u$. Clearly K is its own normalizer in G, but its normalizer $N_{G_u}(K)$ in G_u can have several components.

PROPOSITION 1.3. *Either G is of hermitian type and $L = K_{\mathbb{C}}E = K_{\mathbb{C}}S_\pm$, connected, or*[4] *$L = K_{\mathbb{C}}N_{G_u}(K)$ with the identity component $L^0 = K_{\mathbb{C}}$. In either case, $G \cap L = K$. In general, if V is compact, then $L = K_{\mathbb{C}}E$ and L is connected.*

Proof. We first run through certain structural possibilities for G and $\mathfrak{p}_x$. The group $V = G \cap P_x$ is compact in Cases 1 and 2 below, is noncompact in Cases 3 and 4, and can be either compact or noncompact in Cases 5 and 6.

Case 1. G is of hermitian type with $P_x \subset K_{\mathbb{C}}S_-$. Then $S_+ \cap P_x = \{1\}$. Since $\mathfrak{p}_x$ is a parabolic subalgebra of $\mathfrak{g}$, it contains one or both root spaces $\mathfrak{g}_{\pm\alpha}$ for every root α. Now $S_- \subset P_x$. As K normalizes S_-, it follows that $S_- \subset E$. So $K_{\mathbb{C}}S_- \subset K_{\mathbb{C}}E \subset L \subsetneqq G_{\mathbb{C}}$. But $K_{\mathbb{C}}S_-$ is a maximal parabolic subgroup of $G_{\mathbb{C}}$ and, thus, is a maximal subgroup. We conclude that $L = K_{\mathbb{C}}E = K_{\mathbb{C}}S_-$ and $G \cap L = K$.

Case 2. G is of hermitian type with $P_x \subset K_{\mathbb{C}}S_+$. Arguing as in Case 1, we conclude that $L = K_{\mathbb{C}}E = K_{\mathbb{C}}S_+$ and $G \cap L = K$.

[4] This latter situation occurs both for G of hermitian type and for G not of hermitian type.

Case 3. G is of hermitian type with $P_x \not\subset K_{\mathbb{C}}S_-$, $P_x \not\subset K_{\mathbb{C}}S_+$ and $S_- \subset P_x$. Since $\mathfrak{s}_+$ and $\mathfrak{s}_-$ together generate $\mathfrak{g}$, and $\mathfrak{s}_- \subset \mathfrak{p}_x$, the intersection $\mathfrak{p}_x \cap \mathfrak{s}_+ \subsetneqq \mathfrak{s}_+$. But K acts irreducibly on $\mathfrak{s}_+$ by conjugation. It follows from equation (1.2) that $\mathfrak{e} \cap \mathfrak{s}_+ = 0$, where $\mathfrak{e}$ is the Lie algebra of the complex group E. Now the complex group L has the Lie algebra $\mathfrak{l} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{s}_-$. As in Case 1, now L must be equal to the corresponding maximal parabolic subgroup of $G_{\mathbb{C}}$. We conclude that $L = K_{\mathbb{C}}E = K_{\mathbb{C}}S_-$ and $G \cap L = K$.

Case 4. G is of hermitian type with $P_x \not\subset K_{\mathbb{C}}S_-$, $P_x \not\subset K_{\mathbb{C}}S_+$ and $S_+ \subset P_x$. Arguing as in Case 3, we conclude that $L = K_{\mathbb{C}}E = K_{\mathbb{C}}S_+$ and $G \cap L = K$.

Case 5. G is of hermitian type with $P_x \not\subset K_{\mathbb{C}}S_-$, $P_x \not\subset K_{\mathbb{C}}S_+$, $S_- \not\subset P_x$ and $S_+ \not\subset P_x$. Then $\mathfrak{e} \cap \mathfrak{s}_+ = 0$, as in Case 3, and $\mathfrak{e} \cap \mathfrak{s}_- = 0$ similarly. Thus $\mathfrak{l} = \mathfrak{k}_{\mathbb{C}}$; that is, $L^0 = K_{\mathbb{C}}$.

If the isotropy subgroup V of G at $x \in D$ is compact, then $L = K_{\mathbb{C}}$, for every component of L meets its maximal compact subgroup, which is $L \cap G_u$. If $\ell \in L \cap G_u$, we choose $k \in K \subset G_u$ such that ℓk leaves x fixed. Now $\ell k \in G_u \cap P_x$. But $X \cong G_{\mathbb{C}}/P_x \cong G_u/(G_u \cap P_x)$ is simply connected and so $G_u \cap P_x$ is connected. However $G_u \cap P_x = V \subset K$ whenever V is compact. So $\ell \in K \subset L^0$, proving $L = K_{\mathbb{C}}$ when V is compact.

Whether V is compact or not, $G \cap L = K$, because K is a maximal subgroup of G.

Case 6. G is not of hermitian type. Then $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$; so the identity component $L^0 = K_{\mathbb{C}}$. As in Case 5, it follows that $G \cap L = K$ in general, and $L = K_{\mathbb{C}}$ when V is compact.

Now the proof of Proposition 1.3 is almost complete. In fact, it is complete if V is compact or if we are in the situation of Cases $1, 2, 3$ or 4 above. To complete the proof in all cases we must prove that

(1.4) $\quad N_{G_u}(K) \cap P_x \subset L$ and meets every topological component of L.

Let $g \in N_{G_u}(K) \cap P_x$. In order to show that $g \in L$ we must prove the following: if $k_1 \in K$, then there exists $k_2 \in K$ such that $gk_1(x) = k_2(x)$. In other words, we must show that if $k_1 \in K$, then there exists $k_2 \in K$ such that $\mathrm{Ad}(g)\mathfrak{p}_{k_1(x)} = \mathfrak{p}_{k_2(x)}$. By assumption on g we can use $k_2 = gk_1g^{-1}$.

Let $\ell \in L$. We must show that its topological component ℓL^0 meets $N_{G_u}(K) \cap P_x$. Arguing as in Case 5 above, we may assume that $\ell \in G_u \cap P_x$. There is nothing to prove except in Cases 5 and 6 above, where $\mathfrak{l} = \mathfrak{k}$, so we may suppose that ℓ normalizes $\mathfrak{k}$. Now ℓ normalizes $\mathfrak{g}_u \cap \mathfrak{k} = \mathfrak{k}_0$ and, hence, normalizes K. Thus $\ell \in N_{G_u}(K) \cap P_x$, as required.

We have proved condition (1.4), completing the proof of Proposition 1.3. $\square$

COROLLARY 1.5. *Either L is a parabolic subgroup $K_{\mathbb{C}}S_{\pm}$ of $G_{\mathbb{C}}$ and $M_X = G_{\mathbb{C}}/L$ is a projective algebraic variety, or L is a reductive subgroup of $G_{\mathbb{C}}$ with the identity component $K_{\mathbb{C}}$ and $M_X = G_{\mathbb{C}}/L$ is an affine algebraic variety.*

2. The case where M_X is projective

We now consider the first of the two cases of Corollary 1.5.

THEOREM 2.1. *Suppose that M_X is a projective algebraic variety. Then the open orbit $D \subset X$ is measurable and M_D is a bounded symmetric domain. In particular M_D is a Stein manifold.*

Note that G is of hermitian type and $L = K_{\mathbb{C}}S_{\pm}$, a maximal parabolic subgroup. We can replace $'\Delta^+$ by its negative if necessary and assume that $L = K_{\mathbb{C}}S_-$. Thus we are either in Case 1 (when V is compact) or in Case 3 (when V is noncompact) of the proof of Proposition 1.3. Also $M_X = G_{\mathbb{C}}/L$ is the standard complex realization of the compact hermitian-symmetric space G_u/K. Denote

$$G_{\mathbb{C}}\{D\} = \{g \in G_{\mathbb{C}} \mid gY \subset D\}. \tag{2.2}$$

It is an open subset of $G_{\mathbb{C}}$, and $M_D \subset M_X \cong G_{\mathbb{C}}/L$ consists of the cosets gL with $g \in G_{\mathbb{C}}\{D\}$. Evidently M_D is stable under the action of G. Thus

$$G_{\mathbb{C}}\{D\} \text{ is a union of double cosets } GgL \text{ with } g \in G_{\mathbb{C}}. \tag{2.3}$$

The proof of Theorem 2.1 will consist of showing that only the identity double coset occurs in $G_{\mathbb{C}}$.

The case where V is compact is so much easier than the general case, here, that we indicate the argument separately. With V compact we have $P_x \subset L$ and thus have the holomorphic fibration

$$\pi : X \to M_X \text{ given by } gP_x \mapsto gL. \tag{2.4}$$

Here $\pi(D)$ is the bounded symmetric domain $\{gL \mid g \in G\}$ and the gY, $g \in G_{\mathbb{C}}$, are the fibers of (2.4). Thus M_D is the bounded symmetric domain $\{gL \mid g \in G\}$, as asserted.

We return to the general case, where V may be noncompact. The double cosets GgL of statement (2.3) are in one-to-one correspondence with the G orbits on M_X. Those orbits are given as follows ([13], Ch. III, or see [14], §7):

Roots $\alpha, \beta \in \Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ are called *strongly orthogonal* if neither of $\alpha \pm \beta$ belongs to Δ. A root $\alpha \in \Delta$ is called *compact* if the root space $\mathfrak{g}_\alpha \subset \mathfrak{k}$ and *noncompact* if $\mathfrak{g}_\alpha \subset \mathfrak{s}$. Kostant's cascade construction for

$$\Psi = \{\psi_1, \dots, \psi_\ell\} : \text{ maximal set of strongly orthogonal, noncompact, positive roots} \tag{2.5}$$

is: ψ_1 is the maximal root; ψ_{r+1} is a root maximal among the noncompact positive roots that are orthogonal to ψ_j for $1 \leqq j \leqq r$.

Each $\psi \in \Psi$ leads to a 3-dimensional simple subalgebra $\mathfrak{g}[\psi] = \mathfrak{g}_{-\psi} + [\mathfrak{g}_\psi, \mathfrak{g}_{-\psi}] + \mathfrak{g}_\psi \subset \mathfrak{g}$, the corresponding complex group $G_{\mathbb{C}}[\psi] \cong \mathrm{SL}(2;\mathbb{C})$, and the Riemann sphere $Z[\psi] = G_{\mathbb{C}}[\psi](z)$, where z is the base point $1 \cdot L \in G_{\mathbb{C}}/L = M_X$. The real group $G[\psi] = G_{\mathbb{C}}[\psi] \cap G$ has three orbits on $Z[\psi]$: the lower hemisphere $D[\psi] = G[\psi](z)$, the equator $G[\psi](c_\psi z)$, and the upper hemisphere $G[\psi](c_\psi^2 z)$, where $c_\psi \in G_{\mathbb{C}}[\psi]$ is the Cayley transform for $Z[\psi]$.

If $\psi, \psi' \in \Psi$ with $\psi \neq \psi'$, then strong orthogonality says that $[\mathfrak{g}[\psi], \mathfrak{g}[\psi']] = 0$. If $\Gamma \subset \Psi$, we then have the direct sum $\mathfrak{g}[\Gamma] = \sum_{\gamma\in\Gamma} \mathfrak{g}[\gamma]$, the corresponding complex group $G_{\mathbb{C}}[\Gamma] = \prod_{\gamma\in\Gamma} G_{\mathbb{C}}[\gamma]$, the product $Z[\Gamma] = G_{\mathbb{C}}[\Gamma](z) = \prod_{\gamma\in\Gamma} Z[\gamma]$ of Riemann spheres, and the Cayley transform $c_\Gamma = \prod_{\gamma\in\Gamma} c_\gamma$.

In [13], Theorem 10.6, I proved the following statement:

Proposition 2.6. *The G orbits on M_X are the $G(c_\Gamma c_\Sigma^2 z)$, where Γ and Σ are disjoint subsets of Ψ. Two such orbits $G(c_\Gamma c_\Sigma^2 z) = G(c_{\Gamma'} c_{\Sigma'}^2 z)$ if and only if the cardinalities $|\Gamma| = |\Gamma'|$ and $|\Sigma| = |\Sigma'|$. The open orbits are the $G(c_\Sigma^2 z)$. An orbit $G(c_{\Gamma'} c_{\Sigma'}^2 z)$ is in the closure of $G(c_\Gamma c_\Sigma^2 z)$ if and only if $|\Sigma'| \leqq |\Sigma|$ and $|\Sigma \cup \Gamma| \leqq |\Sigma' \cup \Gamma'|$.*

Note that $G_{\mathbb{C}}\{D\}$ is open in $G_{\mathbb{C}}$ and the map $G_{\mathbb{C}} \to G_{\mathbb{C}}/L = M_X$ is open. So $G_{\mathbb{C}}\{D\}(z)$ is open in M_X. Thus, if $c_{\Gamma'} c_{\Sigma'}^2 \in G_{\mathbb{C}}\{D\}$ and if $G(c_{\Gamma'} c_{\Sigma'}^2 z)$ is in the closure of $G(c_\Gamma c_\Sigma^2 z)$, then $c_\Gamma c_\Sigma^2 \in G_{\mathbb{C}}\{D\}$. Now statement (2.3) and Proposition 2.6 combine to give us the following corollary:

Corollary 2.7. *There is a (necessarily finite) set $\mathcal{C}$ of transforms $c_\Gamma c_\Sigma^2$, where Γ and Σ are disjoint subsets of Ψ such that*

(i) *if $c_\Gamma c_\Sigma^2, c_{\Gamma'} c_{\Sigma'}^2 \in \mathcal{C}$ with $|\Gamma| = |\Gamma'|$ and $|\Sigma| = |\Sigma'|$, then $\Gamma = \Gamma'$ and $\Sigma = \Sigma'$, and*

(ii) *$G_{\mathbb{C}}\{D\} = \bigcup_{c\in\mathcal{C}} GcL$.*

So, if $c_\Gamma c_\Sigma^2 \in \mathcal{C}$, then $c_{\Gamma\cup\Sigma'}^2 \in \mathcal{C}$ for every subset $\Sigma' \subset \Sigma$. In particular, if $c_\Sigma^2 \notin \mathcal{C}$ whenever $\emptyset \neq \Sigma \subset \Psi$, then $\mathcal{C} = \{1\}$ and $G_{\mathbb{C}}\{D\} = GL$.

Now proving Theorem 2.1 is reduced to the proof that $c_\Sigma^2 \notin \mathcal{C}$ for all nonempty subsets $\Sigma \subset \Psi$.

Since $S_- \subset L \cap P_x$, $Q = L \cap P_x$ is a parabolic subgroup of $G_{\mathbb{C}}$ and $W = G_{\mathbb{C}}/Q$ is a complex flag manifold. Consider the holomorphic projections

$$(2.8) \qquad \begin{cases} \pi' \colon W \to X \text{ by } gQ \mapsto gP_x, \\ \qquad \text{fiber } F' = {\pi'}^{-1}(x) = V_{\mathbb{C}}(w) \cong V_{\mathbb{C}}/V_{\mathbb{C}} \cap L, \\ \pi'' \colon W \to M_X \text{ by } gQ \mapsto gL, \\ \qquad \text{fiber } F'' = {\pi''}^{-1}(z) = K_{\mathbb{C}}(w) \cong K_{\mathbb{C}}/K_{\mathbb{C}} \cap P_x, \end{cases}$$

where g runs over $G_{\mathbb{C}}$ and w is the base point $1 \cdot Q$ in W. The restriction of π' to $\tilde{D} = G(w)$ is

$$\begin{aligned} &\pi' \colon \tilde{D} \to D \text{ by } g(w) \mapsto g(x), \\ &\text{fiber } \tilde{D} \cap F' = V(w) \cong V/K \cap V, \text{ open in } F'. \end{aligned} \tag{2.9}$$

Since F' is a complex flag manifold of $V_{\mathbb{C}}$, since $V(w)$ is open in F' and since K is a maximal compact subgroup of V, here $V(w)$ is a bounded symmetric domain and F' is its compact dual.

We have the usual positive definite hermitian inner product on $\mathfrak{g}$, given by

$$\langle \xi, \eta \rangle = -b(\xi, \tau\theta\eta) \text{ for } \xi, \eta \in \mathfrak{g}, \tag{2.10}$$

where b is the Killing form. Use the associated length function to define

$$\|\xi\|_{\mathfrak{g}} \colon \text{ operator norm of } \mathrm{Ad}(\xi) \colon \mathfrak{g} \to \mathfrak{g} \text{ for } \xi \in \mathfrak{g}, \tag{2.11}$$

where $\mathfrak{v} = \mathfrak{p}_x^r$, the complexified Lie algebra of V. A convexity theorem of R. Hermann (see [14], p. 286) says that the bounded symmetric domain

$$G(z) = \pi''(\tilde{D}) = \{\exp(\zeta)(z) \mid \zeta \in \mathfrak{s}_+ \text{ with } \|\xi\|_{\mathfrak{g}} < 1\}.$$

We will need that result in the following form:

LEMMA 2.12. *Every $g \in G$ has the expression $g = \exp(\zeta_1 + \zeta_2) \cdot k \cdot \exp(\eta)$, where $\eta \in \mathfrak{s}_-$, $k \in K_{\mathbb{C}}$, $\zeta_2 \in \mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+)$, and where $\zeta_1 \in \mathfrak{s}_+$ is orthogonal to $\mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+)$. There is a number $a = a_G > 0$ such that, in this expression, $\|\zeta_1\|_{\mathfrak{g}} < a_G$.*

Proof. The inclusion $G \subset \exp(\mathfrak{s}_+) K_{\mathbb{C}} \exp(\mathfrak{s}_-)$, say, $g = \exp(\zeta) \cdot k \cdot \exp(\eta)$, is standard. It is one of the main steps in the proof of the Harish–Chandra embedding $gK \mapsto \zeta \in \mathfrak{s}_+$ of $G(z) = G/K$ as a bounded symmetric domain in the complex euclidean space $\mathfrak{s}_+$.

Let $\|\xi\|$ (no subscript) denote the norm on $\mathfrak{g}$ associated to the positive definite hermitian inner product (2.10). Let $a = a_G > 0$ be large enough that $\|\zeta\|_{\mathfrak{g}} \leqq a_G \|\zeta\|$ for all $\zeta \in \mathfrak{s}_+$. Then, since we know $\|\zeta_1 + \zeta_2\|_{\mathfrak{g}} < 1$, the triangle inequality for the euclidean norm $\|\cdot\|$ gives us $\|\zeta_1\|_{\mathfrak{g}} \leqq a_G \|\zeta_1\| \leqq a_G \|\zeta\| < a_G$, as asserted. □

We now pull back our operator-norm information from $G(z)$ to D.

LEMMA 2.13. *Decompose $g \in G$ as above, $g = \exp(\zeta_1 + \zeta_2) \cdot k \cdot \exp(\eta)$, where $\eta \in \mathfrak{s}_-$, $k \in K_{\mathbb{C}}$, $\zeta_2 \in \mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+)$, and where $\zeta_1 \in \mathfrak{s}_+$ is orthogonal to $\mathrm{Ad}(k)(\mathfrak{v} \cap \mathfrak{s}_+)$. Define $\tilde{f} \colon G \to \mathbb{R}$ by $\tilde{f}(g) = \|\zeta_1\|_{\mathfrak{g}}$. Then $f(gx) = \tilde{f}(g)$ is a well-defined function $f \colon D \to \mathbb{R}$. If $gx \in D$, then $0 \leqq f(gx) < a_G$, where a_G is given by Lemma 2.12.*

Proof. Suppose that two elements of G carry x to the same point of D, say,

$$\exp(\zeta_1+\zeta_2)\cdot k\cdot \exp(\eta)(x)=\exp(\zeta_1'+\zeta_2')\cdot k'\cdot \exp(\eta')(x), \tag{2.14}$$

as in the statement of Lemma 2.13. We must prove that $\|\zeta_1\|_{\mathfrak{g}}=\|\zeta_1'\|_{\mathfrak{g}}$.

In the situation (2.14) we have $v\in V$ such that

$$\exp(\zeta_1+\zeta_2)\cdot k\cdot \exp(\eta)\cdot v=\exp(\zeta_1'+\zeta_2')\cdot k'\cdot \exp(\eta'). \tag{2.15}$$

Since $\mathfrak{s}_-\subset \mathfrak{p}_x$, we can decompose $\exp(\eta)\in S_-=\exp(\mathfrak{s}_-)$ as $v''p''$, where v'' belongs to the reductive part $V_{\mathbb{C}}=P_x^r$ of P_x and p'' belongs to the unipotent radical $P_x^{-n}=\exp(\mathfrak{p}_x^{-n})$. Of course, $\mathrm{Ad}(v^{-1})$ keeps v'' in $V_{\mathbb{C}}$ and keeps p'' in P_x^{-n}. We may assume that the parabolic subalgebras $\mathfrak{p}_x$ and $\mathfrak{k}+\mathfrak{s}_-$ are defined by the same positive root system, so that $P_x^{-n}\subset K_{\mathbb{C}}S_-$; now $\mathrm{Ad}(v^{-1})p''\in K_{\mathbb{C}}S_-$. Consequently equation (2.15) becomes

$$\exp(\zeta_1+\zeta_2)\cdot k\cdot \exp(\eta)\cdot v=\exp(\zeta_1+\zeta_2)\cdot k\cdot v'''\cdot k'''\cdot s''' \tag{2.16}$$

with $v'''\in V_{\mathbb{C}}$, $k'''\in P_x^{-n}\cap K_{\mathbb{C}}$ and $s'''\in P_x^{-n}\cap S_-$.

Suppose for the moment that

$$v'''\in \exp(\mathfrak{v}\cap\mathfrak{s}_+)(V_{\mathbb{C}}\cap K_{\mathbb{C}})\exp(\mathfrak{v}\cap\mathfrak{s}_-)\subset S_+K_{\mathbb{C}}S_-, \tag{2.17}$$

say, $v'''=\exp(\zeta_3)k_3\exp(\eta_3)$. Then equation (2.16) becomes

$$\begin{aligned}&\exp(\zeta_1+\zeta_2)\cdot k\cdot \exp(\eta)\cdot v\\ &\quad=\exp\big(\zeta_1+\zeta_2+\mathrm{Ad}(k)\zeta_3\big)\cdot kk_3k'''\cdot \mathrm{Ad}(k''')^{-1}(\exp(\eta_3))s'''.\end{aligned} \tag{2.18}$$

The right-hand side of (2.18) is decomposed as in the statement of Lemma 2.13. The reason is that $\mathrm{Ad}(k''')^{-1}(\exp(\eta_3))s'''\in S_-$, $kk_3k'''\in K_{\mathbb{C}}$, and $\zeta_1+\zeta_2+\mathrm{Ad}(k)\zeta_3\in\mathfrak{s}_+$. Evidently $k_3\in V_{\mathbb{C}}\cap K_{\mathbb{C}}$ normalizes $\mathfrak{v}\cap\mathfrak{s}_+$. Similarly $k'''\in P_x^{-n}\cap K_{\mathbb{C}}$ normalizes $\mathfrak{p}_x\cap\mathfrak{s}_+$. But, since $\mathfrak{s}_-\subset\mathfrak{p}_x$ and $\mathfrak{v}=\mathfrak{p}_x^r$, we have $\mathfrak{p}_x\cap\mathfrak{s}_+=\mathfrak{v}\cap\mathfrak{s}_+$. Now

$$\zeta_2+\mathrm{Ad}(k)\zeta_3\in\mathrm{Ad}(k)(\mathfrak{v}\cap\mathfrak{s}_+)=\mathrm{Ad}(kk_3k''')(\mathfrak{v}\cap\mathfrak{s}_+)$$

and

$$\zeta_1\in\mathfrak{s}_+ \text{ is orthogonal to } \mathrm{Ad}(k)(\mathfrak{v}\cap\mathfrak{s}_+)=\mathrm{Ad}(kk_3k''')(\mathfrak{v}\cap\mathfrak{s}_+),$$

as asserted.

Given equation (2.14) and assumption (2.17), we have proved that $\zeta_1=\zeta_1'$ and, in particular, that $\|\zeta_1\|_{\mathfrak{g}}=\|\zeta_1'\|_{\mathfrak{g}}$. But our construction is such that $v'''=1$ when $v=1$, and v''' varies continuously with $v\in V$. This shows that $\|\zeta_1\|_{\mathfrak{g}}=\|\zeta_1'\|_{\mathfrak{g}}$ for v''' in the dense open subset $\exp(\mathfrak{v}\cap\mathfrak{s}_+)(V_{\mathbb{C}}\cap K_{\mathbb{C}})\exp(\mathfrak{v}\cap\mathfrak{s}_-)$ of $V_{\mathbb{C}}$. Now it follows for all $v\in V$ that equation (2.15) implies $\|\zeta_1\|_{\mathfrak{g}}=\|\zeta_1'\|_{\mathfrak{g}}$. We have proved that $f\colon D\to\mathbb{R}$ is well defined.

Obviously $0 \leqq \|\zeta_1\|_{\mathfrak{g}} = \tilde{f}(g) = f(gx)$. Lemma 2.12 says that $f(gx) = \|\zeta_1\|_{\mathfrak{g}} < a_G$, where the number $a_G > 0$ is specified in the lemma. This completes the proof of Lemma 2.13. □

Proof of Theorem 2.1. Let $\emptyset \neq \Sigma \subset \Psi$ with $c^2_\Sigma \in \mathcal{C}$ in the notation of Corollary 2.7. Conjugation by an element of K amounts to changing our base point from x to another point on the maximal compact subvariety $Y \subset D$. In the notation (2.5), an appropriate such conjugation carries Σ to $\{\psi_1, \dots, \psi_m\} \subset \Psi$ with $1 \leqq m \leqq \ell$. According to Corollary 2.7, $G_{\mathbb{C}}\{D\}$ contains the diagonal subgroup $G^d_{\mathbb{C}}[\Sigma] \cong \mathrm{SL}(2;\mathbb{C})$ in $G_{\mathbb{C}}[\Sigma]$. Since ψ_1 is not a root of $\mathfrak{p}^r_x = \mathfrak{v}$, the orbit $X^d[\Sigma] = G^d_{\mathbb{C}}[\Sigma](x)$ is a Riemann sphere contained in D.

Let Σ' consist of all roots in Σ that are not roots of $\mathfrak{v}$. It is nonempty because it contains ψ_1. Now the diagonal subgroup $G^d_{\mathbb{C}}[\Sigma'] \cong \mathrm{SL}(2;\mathbb{C})$ in $G_{\mathbb{C}}[\Sigma']$ has these properties:

$$(2.19)\qquad \begin{cases} G^d_{\mathbb{C}}[\Sigma'](x) = X^d[\Sigma'] \text{ is the same Riemann sphere } X^d[\Sigma], \\ \mathfrak{g}^d[\Sigma'] \cap \mathfrak{s}_+ \text{ is orthogonal to } \mathfrak{v} \cap \mathfrak{s}_+, \text{ and} \\ G^d_{\mathbb{C}}[\Sigma'] \cap K_{\mathbb{C}} \text{ is contained in the Cartan subgroup } H \\ \qquad \text{with Lie algebra } \mathfrak{h}. \end{cases}$$

Now look at the corresponding orbits in the hermitian-symmetric flag variety $M_X = G_{\mathbb{C}}/K_{\mathbb{C}}S_-$. The orbit $G^d_{\mathbb{C}}[\Sigma'](z)$ again is a Riemann sphere; this time it is the diagonal $Z^d[\Sigma']$ in $Z[\Sigma']$, and its intersection with the bounded symmetric domain $G(z)$ is the hemisphere $G^d[\Sigma'](z)$, where $G^d[\Sigma'] = G \cap G^d_{\mathbb{C}}[\Sigma']$. Let f^* denote the restriction of the function $\tilde{f}$, of Lemma 2.13, from G to $G^d[\Sigma']$. Then f^* is real analytic and has a unique real-analytic extension $f^\dagger$ to $G^d_{\mathbb{C}}[\Sigma'] \cap \exp(\mathfrak{s}_+)K_{\mathbb{C}}\exp(\mathfrak{s}_-)$, because $\|\cdot\|$ and $\|\cdot\|_{\mathfrak{g}}$ are proportional on $\mathfrak{g}^d[\Sigma'] \cap \mathfrak{s}_+$. Evidently $f^\dagger$ is unbounded.

We now come back to X. The function f, of Lemma 2.13, is real analytic on the lower hemisphere of the Riemann sphere $X^d[\Sigma] = X^d[\Sigma']$, and its restriction to that hemisphere has a unique real-analytic extension h to the complement $X^d[\Sigma] \setminus c^2_\Sigma(x)$ of the pole opposite to x. That extension is defined by $f^\dagger$ just as f is defined by $\tilde{f}$. In view of properties (2.19), h is just the restriction of f from D to $X^d[\Sigma] \setminus c^2_\Sigma(x)$. Since $f^\dagger$ is unbounded, it follows that f is unbounded. This contradicts Lemma 2.13.

We conclude that $\mathcal{C}$ cannot contain any c^2_Σ with $\emptyset \neq \Sigma \subset \Psi$. According to Corollary 2.7, that completes the proof of Theorem 2.1. □

3. The case where M_X is affine

The second case of Corollary 1.5 is settled in the following fashion:

THEOREM 3.1. *Suppose that the open orbit $D \subset X$ is measurable and M_X is an affine algebraic variety. Then M_D is an open Stein subdomain of the Stein manifold M_X.*

The first step in the proof is to bring in the exhaustion function on D. Schmid and I proved[5] that D has a real analytic exhaustion function ϕ, whose Levi form

$$\mathcal{L}(\phi) = \sqrt{-1}\partial\bar{\partial}\phi \tag{3.2}$$

has at least $n - s$ positive eigenvalues at every point of D (see [10]). Here $n = \dim_{\mathbb{C}} D$ and $s = \dim_{\mathbb{C}} Y$. Since ϕ is an exhaustion function, the subdomains

$$D_c = \{z \in D \mid \phi(z) < c\} \tag{3.3}$$

are relatively compact in D.

The next step is to transfer ϕ to M_D. Define $\phi_M\colon\ M_D \to \mathbb{R}^+$ by

$$\phi_M(gY) = \sup_{y \in Y}\ \phi\big(g(y)\big) = \sup_{k \in K}\phi\big(gk(x)\big). \tag{3.4}$$

LEMMA 3.5. *ϕ_M is a real-analytic plurisubharmonic[6] function on M_D. If Y_∞ is a point on the boundary of M_D in M_X, and $\{Y_i\}$ is a sequence in M_D that tends to Y_∞, then* $\lim_{Y_i \to Y_\infty}\ \phi_M(Y_i) = \infty$.

Proof. Let $W = G_{\mathbb{C}}\{D\} = \{g \in G_{\mathbb{C}} \mid gY \subset D\}$. It is an open subset of $G_{\mathbb{C}}$. Define $\psi\colon\ W \times K \to \mathbb{R}^+$ by $\psi(g,k) = \phi(gk(x))$. Since W is a C^ω manifold and ψ is a C^ω function on it, the set defined by the vanishing of the differential in the K-variable,

$$\tilde{Z} = \{(g,k) \in W \times K \mid d_K\psi(g,k) = 0\},$$

is a C^ω subvariety of $W \times K$. Observe that $\tilde{Z}$ is a union of subvarieties, one of which is

$$Z = \big\{(g,k) \in W \times K \mid \psi(g,k) = \sup_{k' \in K}\phi\big(gk(x)\big)\big\}.$$

We have a well-defined C^ω map $f\colon\ Z \to M_D$ given by $f(g,k) = gY$. If $(g,k) \in Z$, then $\phi(gk(x)) = \psi(g,k)\ \phi_M(gY)$. Since $f\colon\ Z \to M_D$ is C^ω and surjective, and since $\psi|_Z$ is C^ω, now ϕ_M is C^ω.

---

[5] The result unfortunately is stated in [10] for arbitrary open orbits, but it is obvious that the proof there is for the measurable case. It is not clear whether the result holds in the nonmeasurable case. We constructed this exhaustion function in order to show that D is $(s+1)$-complete, in the sense of Andreotti and Grauert, so that cohomologies $\mathrm{H}^q(D;\mathcal{F}) = 0$ whenever $q > s$ and $\mathcal{F} \to D$ is a coherent analytic sheaf. The measurable case was sufficient for our representation-theoretic applications [11], where the parabolic subgroup P is a Borel subgroup of G.

[6] A C^2 function f on a complex manifold is called *plurisubharmonic* if the hermitian form $\mathcal{L}(f)$ is positive semidefinite at every point, and *strictly plurisubharmonic* if $\mathcal{L}(f)$ is positive definite everywhere. See [6], [2] or the exposition in [5], §2.6.

By construction, $\psi(g,k)$ is constant in the second variable $k \in K$. The Levi form $\mathcal{L}(\phi)$ has its positive eigenvalues in directions transversal to the compact subvarieties $gY = gK(x)$. So the Levi form $\mathcal{L}(\phi_M)$ on M_D is positive semidefinite. In other words, the function ϕ_M is plurisubharmonic.

Since D is the increasing union of the open sets D_c defined in (3.3), M_D is the increasing union of its subsets

$$M_c = \{gY \in M_D \mid \phi_M(gY) < c\}. \tag{3.6}$$

The definition (3.4) of ϕ_M shows: if $c < c'$, then $c \leqq \phi_M(gY) < c'$ whenever $gY \in M_{c'} \setminus M_c$. Now let Y_∞ be a point on the boundary of M_D in M_X and $\{Y_i\}$ be a sequence in M_D that tends to Y_∞. Passing to a subsequence, we have positive numbers $\{c_i\} \uparrow \infty$ such that $Y_{c_{i+1}} \in M_{c_{i+1}} \setminus M_{c_i}$ for all i. Now $\lim_{Y_i \to Y_\infty} \phi_M(Y_i) = \infty$. This completes the proof of Lemma 3.5. □

The next step in proving our theorem is to modify ϕ_M to obtain a strictly plurisubharmonic exhaustion function on M_D. As mentioned at the end of the Introduction, the idea behind this modification is suggested by results of Docquier and Grauert ([3], or see [1]).

LEMMA 3.7. *Let M be an open submanifold of a Stein manifold $\tilde{M}$. Suppose that M carries a C^r plurisubharmonic function $\xi, r \in \{2,3,\ldots,\infty,\omega\}$, that blows up on the boundary of M in $\tilde{M}$ in this sense: if $y_\infty \in \operatorname{bd} M$, and $\{y_i\} \subset M$ tends to y_∞, then $\lim_{i\to\infty} \xi(y_i) = \infty$. Then M carries a C^r strictly plurisubharmonic exhaustion function.*

Proof. Since $\tilde{M}$ is Stein, we have (see [2], [7]; or [5], Thm. 5.3.9)

$$F\colon\ \tilde{M} \to \mathbb{C}^{2m+1} \text{ proper holomorphic embedding as closed analytic submanifold,} \tag{3.8}$$

where $m = \dim_{\mathbb{C}} \tilde{M}$. Now the norm square function

$$N\colon\ \tilde{M} \to \mathbb{R}^+ \text{ defined by } N(m) = \|F(m)\|^2 \tag{3.9}$$

has a positive definite Levi form. Moreover the sets $\{m \in \tilde{M} \mid N(m) < c\}$ are relatively compact. Let ξ be the given plurisubharmonic function on M and define

$$\zeta\colon\ M \to \mathbb{R}^+ \text{ by } \zeta(m) = \xi(m) + N(m). \tag{3.10}$$

Now we have the Levi form $\mathcal{L}(\zeta) = \mathcal{L}(\xi) + \mathcal{L}(N)$. It is positive definite, because $\mathcal{L}(\xi)$ is positive semidefinite by hypothesis and $\mathcal{L}(N)$ is positive definite by construction. Now ζ is strictly plurisubharmonic. Since N is differentiable of class C^ω, the function ζ is at least as differentiable as ξ.

The function ζ tends to ∞ at every boundary point of M in $\tilde{M}$, because ξ has that property by hypothesis and N has values $\geqq 0$. So every set

$$M_{\zeta,c} = \{m \in M \mid \zeta(m) < c\} \tag{3.11}$$

has a closure contained in $\tilde{M}$. But F is a proper embedding of $\tilde{M}$ in $\mathbb{C}^{2m+1}$; hence the sets $M_{\zeta,c}$ of equation (3.11) have a compact closure in $\tilde{M}$. Now each $M_{\zeta,c}$ has a compact closure in M. This completes the proof of Lemma 3.7. □

Proof of Theorem 3.1. Lemma 3.5 tells us that Lemma 3.7 applies directly to $M_D \subset M_X$ and the real-analytic plurisubharmonic function ϕ_M. Thus Lemma 3.7 constructs a C^ω strictly plurisubharmonic exhaustion function there. Recall Grauert's solution to the Levi problem from [4]: A complex manifold is Stein if and only if it has a C^∞ strictly plurisubharmonic exhaustion function[7]. We have just constructed such a function on M_D, thus completing the proof of Theorem 3.1. □

Proof of Theorem 0.13. Theorem 0.13 follows from Theorem 2.1 when M_X is a projective algebraic variety; it follows from Theorem 3.1 when M_X is an affine algebraic variety. By Corollary 1.5, these are the only cases. □

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(Received April 29, 1991)