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Differentiable Structure for Direct Limit Groups

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Abstract. A direct limit $G = \varinjlim G_\alpha$ of (finite-dimensional) Lie groups has Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ and exponential map $\exp_G : \mathfrak{g} \to G$. Both G and $\mathfrak{g}$ carry natural topologies. G is a topological group, and $\mathfrak{g}$ is a topological Lie algebra with a natural structure of real analytic manifold. In this Letter, we show how a special growth condition, natural in certain physical settings and satisfied by the usual direct limits of classical groups, ensures that G carries an analytic group structure such that $\exp_G$ is a diffeomorphism from a certain open neighborhood of $0 \in \mathfrak{g}$ onto an open neighborhood of $1_G \in G$. In the course of the argument, one sees that the structure sheaf for this analytic group structure coincides with the direct limit $\varinjlim \mathscr{C}^\omega(G_\alpha)$ of the sheaves of germs of analytic functions on the G_α.

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1. Introduction

Direct limits of topological groups and topological vector spaces have been studied extensively [7], but little has been done outside the topological category for direct limits of Lie groups. In this Letter, we take a first step in that direction by defining the differentiable structure. Our main result is Theorem 8.2, which lets us consider the direct limit as an 'infinite-dimensional Lie group' in a reasonable sense.

On the most classical level, an infinite particle system can be viewed as a limit of finite particle approximations. Thus, its natural symmetry group is a direct limit of finite-dimensional Lie groups. The case where the approximating finite systems consist of independent particles, as in [2], leads to direct limit groups of the form (see (5.9) below) studied by Ol'shanskii [9] in his work on infinite classical groups. The groups studied in this Letter allow the possibility that the particles in the approximating finite systems are coupled. It is natural to assume that the coupling forces on each particle are bounded, corresponding to a certain spectral growth condition (see (5.3) below). That spectral growth condition is the key to our construction.

Our construction depends on an analysis of direct limits of linear Lie algebras. Those limits include algebras such as the $\mathfrak{gl}_\infty$ whose basic representation plays a key

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role in the boson-fermion correspondence. See the exposition and references in the third edition of V. Kac's book [6, §14].

Ol'shanskii [9] studies (countable strict) direct limits of classical finite dimensional Lie groups and some of their representations. These direct limit objects have only been seen as topological groups. Our construction allows these direct limit groups to be seen as infinite-dimensional Lie groups.

In general, there is no consensus on the definition of infinite dimensional Lie group, except for the case of Banach Lie groups [3]. Milnor's exposition [8] describes the basic ideas and references up to 1984. Prominent examples of infinite-dimensional groups include the loop groups [1, 10] and more general Kac–Moody groups [11]. There are several other constructions in [5].

2. Preliminaries on Direct Limits

In this section, we record some basic facts on direct limits of systems based on Lie groups with finite-dimensional representations. This material cannot be new, but our viewpoint is very specific and we have not found any appropriate references.

Fix a directed set A. Thus A is a partially ordered set, say with order relation $\leqslant$, such that if $\alpha, \beta \in A$ then one has $\gamma \in A$ with $\alpha, \beta \leqslant \gamma$. Now consider a directed system

$$\{G_\alpha, \phi_{\beta,\alpha}; V_\alpha; \eta_{\beta,\alpha}; \pi_\alpha\}. \tag{2.1}$$

First, by definition, α and β run over A, each G_α is a (finite-dimensional real) Lie group, say with real Lie algebra $\mathfrak{g}_\alpha$, and if $\alpha \leqslant \beta$ then $\phi_{\beta,\alpha}\colon G_\alpha \to G_\beta$ is an analytic homomorphism. We require the standard

$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \cdot \phi_{\beta,\alpha} \quad \text{for } \alpha \leqslant \beta \leqslant \gamma \quad \text{and} \quad \phi_{\alpha,\alpha} = \mathrm{ident}_{G_\alpha} \quad \text{for all } \alpha. \tag{2.2}$$

Second, each of the V_α is a finite-dimensional complex vector space, and if $\alpha \leqslant \beta$ then $\eta_{\beta,\alpha}\colon V_\alpha \to V_\beta$ is a linear transformation. As above we require the standard

$$\eta_{\gamma,\alpha} = \eta_{\gamma,\beta} \cdot \eta_{\beta,\alpha} \quad \text{for } \alpha \leqslant \beta \leqslant \gamma \quad \text{and} \quad \eta_{\alpha,\alpha} = \mathrm{ident}_{V_\alpha} \quad \text{for all } \alpha. \tag{2.3}$$

Third, π_α is a continuous representation of G_α on V_α, and one has the consistency condition that for $\alpha \leqslant \beta$ the left hand diagram of

$$\begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ {\scriptstyle \phi_{\beta\alpha}}\big\downarrow {\scriptstyle \eta_{\beta,\alpha}} & & \big\downarrow {\scriptstyle \eta_{\beta,\alpha}} \\ G_\beta \times V_\beta & \xrightarrow{\pi_\beta} & V_\beta \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{\mathrm{d}\pi_\alpha} & V_\alpha \\ {\scriptstyle \mathrm{d}\phi_{\beta,\alpha}}\big\downarrow {\scriptstyle \eta_{\beta,\alpha}} & & \big\downarrow {\scriptstyle \eta_{\beta,\alpha}} \\ \mathfrak{g}_\beta \times V_\beta & \xrightarrow{\mathrm{d}\pi_\beta} & V_\beta \end{array} \tag{2.4}$$

is commutative. If $\alpha \leqslant \beta$ then $\phi_{\beta,\alpha}$ defines a Lie algebra homomorphism $\mathrm{d}\phi_{\beta,\alpha}\colon \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$. The Lie algebra representations $\mathrm{d}\pi_\alpha$ satisfy the consistency condition that comes out of the condition for the π_α. So the right hand diagram of (2.4) is commutative.

The *direct limit* or *injective limit* group $G = \varinjlim G_\alpha$ consists of the equivalence classes $[g_\alpha]$ of sets $\{g_\alpha\}$ where each $g_\alpha \in G_\alpha$ and, for some $\beta \in A$, if $\beta \leqslant \gamma$ then $g_\gamma = \phi_{\gamma,\beta}(g_\beta)$. The equivalence relation is such that $[g_\alpha]$ is determined by the eventual behavior of $\{g_\alpha\}$. Precisely,

$$\{g_\alpha\} \sim \{'g_\alpha\} \quad \text{when, for some } \beta \in A, \quad \text{if } \beta \leqslant \gamma \quad \text{then } g_\gamma = 'g_\gamma. \tag{2.5}$$

G is a group with the operations

$$[g_\alpha] \cdot ['g_\alpha] = [h_\alpha] \quad \text{where each } h_\alpha = g_\alpha \cdot 'g_\alpha \quad \text{and} \quad [g_\alpha]^{-1} = [g_\alpha^{-1}]. \tag{2.6}$$

We have homomorphisms

$$\phi_\beta : G_\beta \to G \text{ by } \phi_\beta(x) = [g_\gamma] \text{ where } g_\gamma = \phi_{\gamma,\beta}(x) \text{ for } \beta \leqslant \gamma, g_\gamma = 1_{G_\gamma} \text{ otherwise.} \tag{2.7}$$

Those homomorphisms define a topology on G:

$$\text{A subset } U \subset G \text{ is open in } G \text{ if and only if } \phi_\beta^{-1}(U) \text{ is open in } G_\beta \text{ for every } \beta \in A. \tag{2.8}$$

G is a (Hausdorff) topological group with the operations (2.6) and the topology (2.8).

Similarly the *direct limit* Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ consists of the equivalence classes $[\xi_\alpha]$ of sets $\{\xi_\alpha\}$ where each $\xi_\alpha \in \mathfrak{g}_\alpha$ and, for some $\beta \in A$, if $\beta \leqslant \gamma$, then $\xi_\gamma = \mathrm{d}\phi_{\gamma,\beta}(\xi_\beta)$. The equivalence relation is the Lie algebra version of (2.5). $\mathfrak{g}$ is a Lie algebra, one has Lie algebra homomorphisms $\mathrm{d}\phi_\beta : \mathfrak{g}_\beta \to \mathfrak{g}$, and these homomorphisms define a topology on $\mathfrak{g}$ as in (2.8). Now $\mathfrak{g}$ is a topological Lie algebra, and the exponential map

$$\exp_G : \mathfrak{g} \to G \text{ defined by } \exp_G([\xi_\alpha]) = [\exp_{G_\alpha}(\xi_\alpha)] \tag{2.9}$$

is well defined and continuous.

The *direct limit* vector space $V = \varinjlim V_\alpha$ is defined as in (2.5). It is a vector space in a manner analogous to (2.6), one has linear transformations $\eta_\beta : V_\beta \to V$ as in (2.7), they define a topology on V as in (2.8), and V is a topological vector space.

The *direct limit* representation $\pi = \varinjlim \pi_\alpha$ is the representation

$$\pi([g_\alpha])([v_\alpha]) = [\pi_\alpha(g_\alpha)(v_\alpha)] \tag{2.10a}$$

of G on V. Here π is well defined and is a continuous topological group representation. Similarly

$$\mathrm{d}\pi([\xi_\alpha])([v_\alpha]) = [\mathrm{d}\pi_\alpha(\xi_\alpha)(v_\alpha)] \tag{2.10b}$$

is a well defined continuous representation of $\mathfrak{g}$ on V. The exponential power series $\exp(\mathrm{d}\pi([\xi_\alpha]))([v_\alpha])$ converges for every $[\xi_\alpha] \in \mathfrak{g}$ and every $[v_\alpha] \in V$, and

$$\pi(\exp_G([\xi_\alpha]))([v_\alpha]) = \exp(\mathrm{d}\pi([\xi_\alpha]))([v_\alpha]). \tag{2.10c}$$

We do not yet have a differentiable structure on G so it does not yet make sense to say that $\mathrm{d}\pi$ is the differential of π. But, when we do construct that differentiable structure, $\mathrm{d}\pi$ will be the differential of π.

In any case, (2.4) now gives us commutative diagrams

$$\begin{array}{ccc} G_\alpha \times V_\alpha & \xrightarrow{\pi_\alpha} & V_\alpha \\ {\scriptstyle \phi_\alpha}\downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle \eta_\alpha} \\ G \times V & \xrightarrow{\pi} & V \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_\alpha \times V_\alpha & \xrightarrow{\mathrm{d}\pi_\alpha} & V_\alpha \\ {\scriptstyle \mathrm{d}\phi_\alpha}\downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle \eta_\alpha} \\ \mathfrak{g} \times V & \xrightarrow{\mathrm{d}\pi} & V \end{array} \tag{2.11}$$

that are useful for our construction of the differentiable structure on G.

3. Injective Quotient System

It will be convenient to work in the case where the maps of the directed system are injective. In this section, we describe the appropriate quotient of (2.1).

3.1. PROPOSITION. *Consider the directed system of Section* 2. *Define*

$$\bar{G}_\alpha = G_\alpha/\mathrm{Ker}\,\phi_\alpha \quad \textit{and} \quad \bar{\phi}_\alpha : \bar{G}_\alpha \hookrightarrow G; \qquad \bar{V}_\alpha = V_\alpha/\mathrm{Ker}\,\eta_\alpha \quad \textit{and} \quad \bar{\eta}_\alpha : \bar{V}_\alpha \hookrightarrow V. \tag{3.2}$$

If $\alpha \leqslant \beta$ *then* $\phi_{\beta,\alpha}$ *induces an injective Lie group homomorphism* $\bar{\phi}_{\beta,\alpha} : \bar{G}_\alpha \to \bar{G}_\beta$ *and* $\eta_{\beta,\alpha}$ *induces an injective linear transformation* $\bar{\eta}_{\beta,\alpha} : \bar{V}_\alpha \to \bar{V}_\beta$. *The projections* $p_\alpha : G_\alpha \to \bar{G}_\alpha$ *and* $q_\alpha : V_\alpha \to \bar{V}_\alpha$ *give transformations*

$$\{G_\alpha, \phi_{\beta,\alpha}\} \mapsto \{\bar{G}_\alpha, \bar{\phi}_{\beta,\alpha}\} \quad \textit{and} \quad \{V_\alpha, \eta_{\beta,\alpha}\} \mapsto \{\bar{V}_\alpha, \bar{\eta}_{\beta,\alpha}\} \tag{3.3}$$

that induce topological isomorphisms

$$p : G \cong \varinjlim \bar{G}_\alpha, \qquad \mathrm{d}p : \mathfrak{g} \cong \varinjlim \mathfrak{g}_\alpha \quad \textit{and} \quad q : V \cong \varinjlim \bar{V}_\alpha \tag{3.4}$$

of the direct limits. Finally, the π_α *induce a well defined continuous representation* $\bar{\pi}_\alpha$ *of* $\bar{G}_\alpha$ *on* $\bar{V}_\alpha$, *and the isomorphisms* (3.4) *induce* $\pi \cong \varinjlim \bar{\pi}_\alpha$ *and* $\mathrm{d}\pi \cong \varinjlim \mathrm{d}\bar{\pi}_\alpha$.

Proof. To see that $\phi_{\beta,\alpha}$ induces a well defined homomorphism $\bar{\phi}_{\beta,\alpha} : \bar{G}_\alpha \to \bar{G}_\beta$ we use

$$\begin{array}{ccccc} G_\alpha & \xrightarrow{\phi_{\beta,\alpha}} & G_\beta & \xrightarrow{\phi_\beta} & G \\ {\scriptstyle p_\alpha}\downarrow & & {\scriptstyle p_\beta}\downarrow & & \| \\ \bar{G}_\alpha & \xrightarrow{\bar{\phi}_{\beta,\alpha}} & \bar{G}_\beta & \xrightarrow{\bar{\phi}_\beta} & G \end{array}$$

with $\phi_\alpha = \phi_\beta \cdot \phi_{\beta,\alpha}$ to see that $\bar{\phi}_{\beta,\alpha}$, as indicated above, is well defined by $\bar{\phi}_\alpha = \bar{\phi}_\beta \cdot \bar{\phi}_{\beta,\alpha}$. Then it is immediate that $\bar{\phi}_{\beta,\alpha}$ fills in the above diagram as indicated and is an injective Lie group homomorphism. The argument for $\eta_{\beta,\alpha}$ and $\bar{\eta}_{\beta,\alpha}$ is similar.

In the diagram above, p_α and p_β are surjective. Now the associated transformation $p: G \to \varinjlim \bar{G}_\alpha$ is an isomorphism. Similarly $q: V \to \varinjlim \bar{V}_\alpha$ is an isomorphism.

Finally, combine (2.11) and (3.4) as above to see that $\pi \cong \varinjlim \bar{\pi}_\alpha$ and $\mathrm{d}\pi \cong \varinjlim \mathrm{d}\bar{\pi}_\alpha$. □

We now have a directed system, quotient of (2.1),

$$\{\bar{G}_\alpha, \bar{\phi}_{\beta,\alpha}; \bar{V}_\alpha, \bar{\eta}_{\beta,\alpha}; \bar{\pi}_\alpha\}. \tag{3.5}$$

with canonical isomorphisms

$$\varinjlim \bar{G}_\alpha \cong G, \qquad \varinjlim \bar{\mathfrak{g}}_\alpha \cong \mathfrak{g} \quad \text{and} \quad \varinjlim \bar{V}_\alpha \cong V \tag{3.6}$$

such that the maps $\bar{\phi}_{\beta,\alpha}, \bar{\phi}_\alpha, \bar{\eta}_{\beta,\alpha}$ and $\bar{\eta}_\alpha$ injective. We will take the canonical isomorphisms (3.6) as identifications. Then pull-backs are reduced to intersections and it is easier to keep track of eigenvalues.

4. Discreteness of the Lie Algebra Spectrum

Our first application of the construction of Section 3 is

4.1. PROPOSITION. *If $\xi \in \mathfrak{g}$ then $\mathrm{d}\pi(\xi)$ has discrete spectrum.*

This proposition is an immediate consequence of the following lemma, which is essentially constructive despite its transfinite appearance.

4.2. LEMMA. *Let $\xi \in \mathfrak{g}$. If $\lambda \in \mathbb{C}$ let $V(\xi : \lambda)$ denote the space of all generalized λ-eigenvectors of $\mathrm{d}\pi(\xi)$, So $v \in V(\xi : \lambda)$ just when there is an integer $n = n(\xi : \lambda)$ such that $(\mathrm{d}\pi(\xi) - \lambda)^n v = 0$. Then $V = \Sigma_{\lambda \in \mathbb{C}} V(\xi : \lambda)$, algebraic direct sum.*

Proof. Let $\mathscr{A}$ consist of all subsets $S \subset A$ such that $V_S = \Sigma_{\alpha \in S} \bar{\eta}_\alpha(\bar{V}_\alpha)$ is of the form $\Sigma_{\lambda \in \mathbb{C}} V_S(\xi : \lambda)$, algebraic direct sum, where $V_S(\xi : \lambda)$ is the space of all generalized λ-eigenvectors of $\mathrm{d}\pi(\xi)$ that are contained in V_S. $\mathscr{A}$ has partial order $\preccurlyeq$ given by inclusion.

Let $\mathscr{S} = \{S_n \mid n = 1, 2, \ldots\}$ be a linearly ordered subset of $\mathscr{A}$. Set $S = \bigcup_n S_n$. Then $V_S = \bigcup_n V_{S_n}$. If $v \in V_S$ then v belongs to some V_{S_n} and $v = \Sigma\, v_\lambda$, finite sum with $v_\lambda \in V_{S_n}(\xi : \lambda)$. So V_S is the algebraic direct sum of subspaces $V_S(\xi : \lambda)$. Thus $S \in \mathscr{A}$ and is a maximal element for $\mathscr{S}$.

Zorn's Lemma now says that $\mathscr{A}$ has a maximal element M. If $\beta \leqslant \gamma$ and $\gamma \in M$ then $\bar{\eta}_\beta(\bar{V}_\beta) \subset \bar{\eta}_\gamma(\bar{V}_\gamma) \subset V_M$ so $\beta \in M$. If $M \neq A$ choose $\beta \in A \setminus M$. As just seen, we may replace β with any index $\geqslant \beta$, so we may assume that we are out along the ordering of A far enough so that $\beta \leqslant \gamma$ implies $\xi_\gamma = \bar{\eta}_{\gamma,\beta}(\xi_\beta)$. Now (2.4b) and (2.11) show that $\mathrm{d}\pi(\xi)$ preserves $\bar{\eta}_\beta(\bar{V}_\beta)$ and acts there just as $\mathrm{d}\bar{\pi}_\beta(\xi_\beta)$ acts on $\bar{V}_\beta$. So $V_{\{\beta\}} = \bar{\eta}_\beta(\bar{V}_\beta)$ is of the form $\Sigma_{\lambda \in \mathbb{C}} V_{\{\beta\}}(\xi : \lambda)$, algebraic direct sum. Compare this with $V_M = \Sigma_{\lambda \in \mathbb{C}} V_M(\xi : \lambda)$ to see that $V_N = \Sigma_{\lambda \in \mathbb{C}} V_N(\xi : \lambda)$, algebraic direct sum, when $N = M \cup \{\beta\}$. Thus $M \prec N$, contradicting maximality. Now $M = A$. Lemma 4.2 follows. □

5. Control of the Imaginary Spectrum

In this section, we discuss a spectral condition that ensures the existence of a certain open neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$; that is the condition (5.3) which, we remarked in the Introduction, corresponds to boundedness of coupling forces on any given particle in an infinite particle system. Later we will see that $\exp_G$ carries the analytic structure from $\mathfrak{g}$ to G in such a way that it is a diffeomorphism from $\mathcal{O}$ to an open set in G.

Let $\xi = [\xi_\alpha] \in \mathfrak{g}$ with each $\xi_\alpha \in \bar{\mathfrak{g}}_\alpha$. Define

$$\iota_\alpha(\xi_\alpha) = \max\{|\mathrm{Im}\,\lambda|\,|\lambda \text{ is an eigenvalue of } \mathrm{d}\bar{\pi}_\alpha(\xi_\alpha)\}. \tag{5.1}$$

If $\alpha \leqslant \beta$ then every eigenvalue of $\mathrm{d}\bar{\pi}_\alpha(\xi_\alpha)$ is an eigenvalue of $\mathrm{d}\bar{\pi}_\beta(\mathrm{d}\bar{\phi}_{\beta,\alpha}(\xi_\alpha))$. It follows that $\iota_\alpha(\xi_\alpha) \leqslant \iota_\beta(\mathrm{d}\bar{\phi}_{\beta,\alpha}(\xi_\alpha))$. Let $\beta \in A$ such that $\xi_\gamma = \mathrm{d}\bar{\phi}_{\gamma,\beta}(\xi_\beta)$ for $\beta \leqslant \gamma$. Define $\iota(\xi) \leqslant \infty$ by

$$\begin{aligned}\iota(\xi) &= \sup\{|\mathrm{Im}|\,|\lambda \text{ is an eigenvalue of } \mathrm{d}\pi(\xi)\}\\ &= \sup_{\beta \leqslant \gamma}\{|\mathrm{Im}\,\lambda|\,|\lambda \text{ is an eigenvalue of } \mathrm{d}\bar{\pi}_\gamma(\xi_\gamma)\}\\ &= \lim_{\alpha \in A}\sup\{||\mathrm{Im}\,\lambda|\,|\lambda \text{ is an eigenvalue of } \mathrm{d}\bar{\pi}_\alpha(\xi_\alpha)\}.\end{aligned} \tag{5.2}$$

As γ increases past the 'stability point' β, the real number $\iota_\gamma(\xi_\gamma)$ increases toward $\iota(\xi)$. We will need a spectral growth condition on π

$$\text{if } \xi \in \mathfrak{g} \quad \text{then } \iota(\xi) < \infty. \tag{5.3}$$

In other words, for each β there exists $c_\beta > 0$ such that if $\xi = [\xi_\alpha] \in \mathfrak{g}$, and if $\beta \leqslant \gamma$ implies $\xi_\gamma = \bar{\phi}_{\gamma,\beta}(\xi_\beta)$, then $\iota(\xi) \leqslant c_\beta \cdot \iota_\beta(\xi_\beta)$. If c_β exists then $1 \leqslant c_\beta < \infty$. The uniform case is

$$\begin{aligned}&\text{there exists } c_G = c(\{G_\alpha, \phi_{\beta,\alpha}; V_\alpha, \eta_{\beta,\alpha}; \pi_\alpha\}) > 0 \text{ such that}\\ &\text{if } \xi = [\xi_\alpha] \in \mathfrak{g}, \text{ and if } \beta \leqslant \gamma \text{ implies } \xi_\gamma = \bar{\phi}_{\gamma,\beta}(\xi_\beta), \text{ then } \iota(\xi) \leqslant c_G \cdot \iota_\beta(\xi_\beta),\end{aligned} \tag{5.4}$$

Again, if c_G exists then $1 \leqslant c_G < \infty$. The point of this is

5.5. PROPOSITION. *If the spectral growth condition* (5.3) *holds, then*

$$\mathcal{O} = \{\xi \in \mathfrak{g} \mid \iota(\xi) < \pi\} \tag{5.6}$$

is an open neighborhood of 0 *in* $\mathfrak{g}$.

Proof. Evidently $0 \in \mathcal{O}$ so we need only prove that $\mathcal{O}$ is open in $\mathfrak{g}$. The functions $\iota_\alpha : \mathfrak{g} \to \mathbb{R}$ of (5.1) are continuous. The second equality of (5.2) tells us that

$$\iota \cdot \mathrm{d}\bar{\phi}_\alpha = \sup_{\alpha \leqslant \gamma} \iota_\gamma \cdot \mathrm{d}\bar{\phi}_{\gamma,\alpha}. \tag{5.7}$$

Our hypothesis (5.3) says that ι takes finite values. Thus (5.7) presents $\iota \cdot \mathrm{d}\bar{\phi}_\alpha$ as a finite supremum of continuous functions. Now $\iota \cdot \mathrm{d}\bar{\phi}_\alpha$ is continuous, so we have

$$\mathcal{O}_\alpha = \{\zeta_\alpha \in \bar{\mathfrak{g}}_\alpha \,|\, \iota(\mathrm{d}\bar{\phi}_\alpha(\zeta_\alpha)) < \pi\} = \mathrm{d}\bar{\phi}_\alpha^{-1}(\mathcal{O}) \tag{5.8}$$

open in $\bar{\mathfrak{g}}_\alpha$. Thus $\mathcal{O}$ is open in $\mathfrak{g}$. □

We mention an important case: direct limits of classical groups as studied by Ol'shanskii. Those are given by sequences of classical matrix groups $\pi_n : G_n \hookrightarrow \mathrm{GL}(V_n)$ of the form

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots \tag{5.9a}$$

where the directed system maps are specified on the matrix level by

$$\phi_{n+k,n} : g_n \mapsto \begin{pmatrix} g_n & 0 \\ 0 & I \end{pmatrix}. \tag{5.9b}$$

Here the identity matrix I has size $\dim V_{n+k} - \dim V_n$. The point is that $d\pi_{n+k}(\phi_{n+k,n}\xi_n))$ and $d\pi_n(\xi_n)$ have the same spectrum, so (5.3) and (5.4) are automatic with $c_G = 1$. A typical example would be the case where G_n is the indefinite unitary group $\mathrm{U}(p, n)$ acting in the usual way on $V_n = \mathbb{C}^{(p,n)}$.

The spectral growth condition (5.3) is automatic when each G_α acts on its space V_α as a unipotent group, or, more generally, as an $\mathbb{R}$-split linear group. That occurs (see below) for the Heisenberg groups.

Each sequence of classical real reductive Lie groups gives us interesting cases that satisfy the spectral growth condition (5.3) but do not fit the pattern (5.9). We illustrate this with the sequence of real symplectic groups. Let V_n denote $\mathbb{R}^{2^n}$ with the antisymmetric bilinear form that has matrix with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

down the diagonal in the basis $\{e_1, \ldots, e_{2^n}\}$. Let $G_n = \mathrm{Sp}(V_n) \cong \mathrm{Sp}(2^{n-1}; \mathbb{R})$. Consider the mapping

$$G_n \to G_{n+1} \quad \text{by} \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

corresponding to the map $\eta_{n+1,n} : e_i \mapsto e_i + e_{i+2^n}$ of V_n into V_{n+1}. Then (5.3) is automatic as $\xi \in \mathfrak{g}_n$ and its image in $\mathfrak{g}_{n+1}$ have the same spectrum.

Now let $E_n \subset V_n$ be totally isotropic (J_n-null) subspaces such that $\eta_{n+1,n}(E_n) \subset E_{n+1}$. Then the groups

$$Q_n = \{g \in G_n \mid g|E_n = \text{identity}\}$$

satisfy $\eta_{n+1,n}(Q_n) \subset Q_{n+1}$. These groups have structure [12, §8]

$$Q_n \cong N_{s_n, 2(2^{n-1} - s_n)} \cdot \mathrm{Sp}(2^{n-1} - s_n; \mathbb{R}),$$

where $N_{s,2t}$ is a generalized Heisenberg group [12, (8.4)]. If $s = 1$ then $N_{s,2t}$ reduces to the ordinary Heisenberg group of dimension $2t + 1$ with $\mathrm{Sp}(t; \mathbb{R})$ acting in the usual way. So the case where each $s_n = 1$ corresponds to a certain particle coupling scheme in classical mechanics. In that case the variations $\widetilde{Q}_n$, in which the reductive parts $\mathrm{Sp}(2^{n-1} - 1; \mathbb{R})$ are replaced by the oscillator group $\widetilde{\mathrm{Sp}}(2^{n-1} - 1; \mathbb{R})$, give the initial setting for quantization.

6. Local Injectivity

Now we can start the proof that $\exp_G$ carries the analytic structure of $\mathcal{O}$ (5.6) to G.

6.1. PROPOSITION. *If the spectral growth condition* (5.3) *holds, and if* $\mathrm{d}\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ *is injective, then* $\exp_G\colon \mathcal{O} \to G$ *is injective.*

We will prove Proposition 6.1 as a consequence of

6.2. LEMMA. *Let S and T be linear transformations of a finite-dimensional complex vector space F. Suppose that every eigenvalue λ of e^T satisfies $|\mathrm{Im}\,\lambda| < \pi$. If S commutes with e^T then S commutes with T.*

Proof. S commutes with every $(e^T - b)^n$, preserving its null space, so S preserves every generalized eigenspace of e^T. If $\lambda_1 \neq \lambda_2$ are eigenvalues of T, then $|\mathrm{Im}\,\lambda_i| < \pi$ ensures that $e^{\lambda_1} \neq e^{\lambda_2}$. Now S preserves every generalized eigenspace of T. The proof is reduced to the case where $T = \lambda I + N$ with N nilpotent. Then $e^T = e^\lambda e^N$, so S commutes with e^N. As N is nilpotent now S commutes with N. So S commutes with $T = \lambda I + N$. □

Proof of Proposition. Let $\xi, \zeta \in \mathcal{O}$ with $\exp_G(\xi) = \exp_G(\zeta)$. Express $\xi = [\xi_\alpha]$ and $\zeta = [\zeta_\alpha]$ with $\xi_\alpha, \zeta_\alpha \in \bar{\mathfrak{g}}_\alpha$. Fix an index β such that $\beta \leqslant \gamma$ implies $\xi_\gamma = \mathrm{d}\bar{\phi}_{\gamma,\beta}(\xi_\beta)$ and $\zeta_\gamma = \mathrm{d}\bar{\phi}_{\gamma,\beta}(\zeta_\beta)$.

Fix $\gamma \geqslant \beta$. Denote $F = \bar{V}_\gamma$, $S = \mathrm{d}\pi_\gamma(\xi_\gamma)$, and $T = \mathrm{d}\bar{\pi}_\gamma(\zeta_\gamma)$. We will show $S = T$. By definition (5.6) of $\mathcal{O}$, every eigenvalue λ of S or T satisfies $|\mathrm{Im}\,\lambda| < \pi$. Since $\exp_G(\xi) = \exp_G(\zeta)$ we have $e^S = e^T$. In particular, e^S commutes with e^T. With two applications of the Lemma we see that S commutes with T. Now F is the direct sum of the $F_{\lambda,\mu}$ where $F_{\lambda,\mu}$ is the intersection of the generalized λ-eigenspace of S with the generalized μ-eigenspace of T. Each $F_{\lambda,\mu}$ is stable under both S and T. So the proof that $S = T$ is reduced to the case where $S = \lambda I + L$ and $T = \mu I + M$ with L and M nilpotent. Comparing eigenvalues of $e^S = e^T$ we see $e^\lambda = e^\mu$. Now $|\mathrm{Im}\,\lambda|$, $|\mathrm{Im}\,\mu| < \pi$ says $\lambda = \mu$. So the proof that $S = T$ is reduced to the case where S and T are commuting nilpotent matrices. Putting them simultaneously in upper triangular form we see that $e^S = e^T$ implies $S = T$.

We have just proved that $\mathrm{d}\bar{\pi}_\gamma(\xi_\gamma) = \mathrm{d}\bar{\pi}_\gamma(\zeta_\gamma)$ for $\beta \leqslant \gamma$. That proves $\mathrm{d}\pi(\xi) = \mathrm{d}\pi(\zeta)$. Since $\mathrm{d}\pi$ is injective now $\xi = \zeta$. □

7. Local Differentiability

In this section, we see when $\exp_G\colon \mathcal{O} \to \exp_G(\mathcal{O})$ is diffeomorphic at each finite stage of the injective direct limit process.

7.1. PROPOSITION. *Assume that the spectral growth condition* (5.3) *holds, and suppose that* $\mathrm{d}\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ *is injective. Then*

(1) $U = \exp_G(\mathcal{O})$ *is an open subset of* G;
(2) $\exp_G\colon \mathcal{O} \to U$ *is a homeomorphism; and*

(3) *for each index* $\alpha \in A$ *the map* $\exp_{G_\alpha}: d\bar{\phi}_\alpha^{-1}(\mathcal{O}) \to \bar{\phi}_\alpha^{-1}(U)$ *is a diffeomorphism.* (*Here recall from Proposition* 5.5 *that* $\mathcal{O}$ *is an open subset of* $\mathfrak{g}$.)

We have two elements to combine: injectivity of $\exp_{G_\alpha}$ on $d\bar{\phi}_\alpha^{-1}(\mathcal{O})$ and nonsingularity of $d\exp_{G_\alpha}$ at every point of $d\bar{\phi}_\alpha^{-1}(\mathcal{O})$. The injectivity relies on Proposition 6.1. The nonsingularity depends on

7.2. LEMMA. *The differential* $d\exp_{\bar{G}_\alpha}: T_{\xi_\alpha}(\bar{\mathfrak{g}}_\alpha) \to T_{x_\alpha}(\bar{G}_\alpha)$, $x_\alpha = \exp_{\bar{G}_\alpha}(\xi_\alpha)$, *is nonsingular unless* $\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)$ *has an eigenvalue that is a nonzero integral multiple of* $2\pi\sqrt{-1}$.

Proof. It is known [4, p. 105] that the differential

$$d\exp_{\bar{G}_\alpha}\Big|_{\xi_\alpha} = dL_{x_\alpha}\Big|_{1_{\bar{G}_\alpha}} \cdot \frac{I - e^{-\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)}}{\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)} \tag{7.3}$$

where L_{x_α} is left translation by x_α on $\bar{G}_\alpha$. Consider this on the complexified tangent spaces. Look at the restriction to the $(dL_{x_\alpha}|_{1_{\bar{G}_\alpha}})$-image of the generalized λ-eigenspace of $\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)$. If $\lambda = 0$ we expand

$$\frac{I - e^{-\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)}}{\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)} = I - \frac{\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)}{2!} + \frac{\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)^2}{3!} \cdots = I + \text{nilpotent},$$

which is nonsingular on that generalized eigenspace. If $\lambda \neq 0$, it is nonsingular on that generalized eigenspace just when $I = e^{-\mathrm{ad}_{\bar{G}_\alpha}(\xi_\alpha)}$ is nonsingular there, i.e. when $e^\lambda \neq 1$, which of course happens exactly when λ is not an integral multiple of $2\pi\sqrt{-1}$. □

Proof of Proposition. Since ad_G is a subrepresentation of $d\pi \otimes d\pi^*$, the eigenvalues λ of $\mathrm{ad}_G(\xi)$ all are of the form $\mu - \nu$ where μ and ν are eigenvalues of $d\pi(\xi)$. If $\xi \in \mathcal{O}$ then $|\mu|, |\nu| < \pi$, so $|\lambda| < 2\pi$. By the Lemma, $\exp_{G_\alpha}$ is nonsingular at every point of $d\bar{\phi}_\alpha^{-1}(\mathcal{O})$.

Denote $\mathcal{O}_\alpha = d\bar{\phi}_\alpha^{-1}(\mathcal{O})$ as in (5.8), $U = \exp_G(\mathcal{O})$, and $U_\alpha = \bar{\phi}_\alpha^{-1}(U)$. In the proof of Proposition 5.5 we saw that $\mathcal{O}_\alpha$ is open in $\bar{\mathfrak{g}}_\alpha$. Proposition 6.1 and the Implicit Function Theorem say that U_α is open in $\bar{G}_\alpha$ and that $\exp_{G_\alpha}$ maps $\mathcal{O}_\alpha$ diffeomorphically onto U_α.

U is open in G because the U_α are open. Since $\exp_{G_\alpha}: \mathcal{O}_\alpha \to U_\alpha$ is a homeomorphism for each α, $\exp_G: \mathcal{O} \to U$ also is a homeomorphism. □

8. The Differentiable Structure

We construct the differentiable structure on G from the components of Proposition 7.1. The point is to do it in a way that shows invariance under the group operations. This uses the direct limit of the structure sheaves of the G_α as structure sheaf of G.

If M is any finite dimensional real analytic manifold we denote the sheaf of germs of real analytic functions by $\mathscr{C}^\omega(M)$. Associated to our directed system (2.1) and its injective quotient (3.5) we have direct limit sheaves

$$\mathscr{C}^{\omega}(G) = \varinjlim \mathscr{C}^{\omega}(\bar{G}_{\alpha}) \quad \text{and} \quad \mathscr{C}^{\omega}(\mathfrak{g}) = \varinjlim \mathscr{C}^{\omega}(\bar{\mathfrak{g}}_{\alpha}). \tag{8.1}$$

By definition, if S is an open subset in G, then a function $f: S \to \mathbb{C}$ is a section of $\mathscr{C}^{\omega}(G)$ over S if and only if, for each index α, $f \cdot \bar{\phi}_{\alpha}: \bar{\phi}_{\alpha}^{-1}(S) \to \mathbb{C}$ is a section of $\mathscr{C}^{\omega}(\bar{G}_{\alpha})$, in other words is a C^{ω} function on $\bar{\phi}_{\alpha}^{-1}(S)$. That specifies the presheaf and thus specifies the sheaf $\mathscr{C}^{\omega}(G)$ over G. The sheaf $\mathscr{C}^{\omega}(\mathfrak{g})$ over $\mathfrak{g}$ is defined similarly.

Recall the usual analytic structure on $\mathfrak{g}$. Let $f: B \to \mathbb{C}$ where B is open in $\mathfrak{g}$. Then f is analytic just when $f \cdot l$ is analytic in the usual sense for every affine $l: \mathbb{R} \to \mathfrak{g}$. It is immediate from (8.1) and finite dimensionality of the $\bar{\mathfrak{g}}_{\alpha}$ that $\mathscr{C}^{\omega}(\mathfrak{g})$ is the sheaf of germs of real analytic functions on $\mathfrak{g}$.

$\mathscr{C}^{\omega}(G)$ is stable under the group operations on G. In other words, $g \mapsto g^{-1}$ induces an automorphism of $\mathscr{C}^{\omega}(G)$, and $(g, h) \mapsto gh$ induces a morphism $\mathscr{C}^{\omega}(G) \to \mathscr{C}^{\omega}(G \times G)$. In particular left and right translations induce automorphisms of $\mathscr{C}^{\omega}(G)$. So G is a "ringed group" in the ringed structure for which $\mathscr{C}^{\omega}(G)$ is the structure sheaf. The corresponding 'ringed algebra' structure on $\mathfrak{g}$, the structure for which $\mathscr{C}^{\omega}(\mathfrak{g})$ is the structure sheaf, is just the usual analytic structure on $\mathfrak{g}$.

By Proposition 7.1, $\exp_G|_{\mathcal{O}}$ induces an isomorphism $\exp_G^*|_{\mathcal{O}}: \mathscr{C}^{\omega}(U) \cong \mathscr{C}^{\omega}(\mathcal{O})$. That defines a real analytic structure on U modeled on the topological vector space $\mathfrak{g}$. The invariance described just above, allows us to translate this analytic structure to an open neighborhood of any point of G, defining a C^{ω} Lie group structure on G. In summary,

8.2. THEOREM. *G has a C^{ω} Lie group structure modeled on the topological vector space $\mathfrak{g}$, such that*

(1) *the $(\exp_G|_{\mathcal{O}})^{-1} \cdot L_{g^{-1}}: gU \to \mathcal{O} \hookrightarrow \mathfrak{g}$ form a local coordinate cover on G; and*
(2) *$\mathscr{C}^{\omega}(G)$ (defined in (8.1)) is the sheaf of germs of C^{ω} functions on G.*

Further $\exp_G; \mathcal{O} \to U$ is an analytic diffeomorphism and the $\phi_{\alpha}: G_{\alpha} \to G$ are analytic.

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