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Observability of Evolution Equations for Invariant Differential Operators*

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Abstract

A strong form of discrete observability is proved for solutions of evolution equations $D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0$ on a compact homogeneous space $X = G/K$, where D is a G-invariant closed normal operator on $L^2(X)$. Some interesting special cases are (i) where D is the Laplace-Beltrami operator, so that $D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0$ is the heat equation, and (ii) where X is a riemannian symmetric space, e.g. a sphere. For example, if X is connected then any solution $f(x:t)$ to the usual heat equation is discretely observable at any time $t_0 \geqq 0$ on any neighborhood of an arbitrary point $x_0 \in X$.

To fix the ideas we start by studying the heat equation on the sphere S^n. Considerations there are reduced to certain aspects of the representation theory of the orthogonal group $SO(n+1)$. This representation theoretic viewpoint allows us to carry the results over to invariant evolution equations $D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0$ on compact homogeneous spaces $X = G/K$. This becomes quite explicit in the case of the heat equation on a normal riemannian homogeneous space. Finally, the technical condition on D is shown to be automatic when X is a riemannian symmetric space.

1 Heat Equation on the Sphere

In this section we study the heat equation on the sphere

$$S^n = SO(n+1)/SO(n)$$

and the observability properties of its solutions. In later sections we will examine more general evolution equations on more general manifolds. But

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here we first fix the ideas by considering a concrete case that is of independent interest.

The heat equation on S^n for initial data $b(x)$ is given by

$$\Delta_x f(x:t)+\frac{\partial}{\partial t}f(x:t)=0 \text{ and } f(x:0)=b(x), \quad x\in S^n \text{ and } t\geqq 0 \quad (1.1)$$

where Δ is the (positive) Laplace-Beltrami operator[1] on the sphere. The function $f(x:t)$ represents temperature distribution at time t on S^n evolving from temperature distribution $b(x)$ at time 0.

Observability is the study of just which types of data on (samples of) the values $f(x:t)$ allow us to reconstruct the function $b(x)$ accurately. We will study a restricted version of this question. Suppose that we have a sequence of points $\{x_1, x_2, \cdots\}\subset S^n$ and we are allowed to sample (observe) the $f(x_i:t)$ at some time $t>0$. We would like to see whether it is possible to deduce $b(x)=f(x:0)$ for all $x\in S^n$ from this data. In particular we would like to impose conditions on the x_i that make this possible. These conditions should be appropriate. For example, it is not reasonable to choose a countable dense subset of S^n for the sequence $\{x_1, x_2, \cdots\}$. Our conditions will be such that, given $x_0\in S^n$ and a neighborhood of x_0 in S^n, the sequence $\{x_i\}$ will be contained in the neighborhood.

Background material for this problem can be found in Sakawa [17], in Gilliam, Li and Martin [9], in Gilliam and Martin [10], in Martin and Wallace [15], and in Wolf ([21], [22]).

Separating variables in the usual way we first consider functions $u(x:t)=k(x)h(t)$. Since $\Delta_x u=(\Delta k)h$ and $\frac{\partial}{\partial t}u=kh'$, we look for eigenfunctions $\phi: S^n\to \mathbf{C}$, say $\Delta\phi=\lambda\phi$. Any such eigenfunction ϕ defines a solution $u(x:t)=e^{-t\lambda}\phi(x)$ to (1.1).

On general grounds, each eigenvalue $\lambda\geqq 0$, each eigenspace $A(\lambda)\subset L^2(S^n)$ is finite dimensional, and there is a polynomial function $p(r)$ that bounds the number (with multiplicity) of eigenvalues $< r^2$. See [11] for example. We use that to formalize our notion of observability, as follows.

Definition 1.1 *The heat equation* (1.1) *is* **discretely observable at** $x_0\in S^n$ *if, for* $t\geqq 0$ *and every neighborhood* V *of* x_0 *in* S^n*, there is a countable subset* $\{x_1, x_2, \cdots\}\subset V$ *with the following property. Let* $E_r(S^n)=\sum_{\lambda=0}^{r}A(\lambda)$ *and let* $n_r=dimE_r(S^n)$*. If* f *is the solution to the heat equation* (1.1) *for initial data* $b\in L^2(S^n)$*, then the* $f(x_i:t)$, $1\leqq i\leqq n_r$*, determine the orthogonal projection of* $b(x)$ *to* $E_r(S^n)$*.*

In our specific situation, with a certain normalization of the riemannian metric on S^n (from the negative of the Killing form of $SO(n+1)$), the

[1] We use the sign of the Laplace-Beltrami operator corresponding to the Laplacian $\Delta=-\sum\frac{\partial^2}{\partial x_i{}^2}$ on euclidean space, because this both usual and natural in differential geometry and in group representation theory.

eigenvalues and multiplicities are given as follows [1].

n	eigenvalues λ_m	multiplicities $d_m = dimA(\lambda_m)$	for
$2\ell - 1$	$\frac{m^2+2m(\ell-1)}{4(\ell-1)}$	$\frac{m+\ell-1}{\ell-1}\prod_{k=1}^{2\ell-3}\frac{m+k}{k}$	$\ell \geqq 2, m \geqq 0$
2ℓ	$\frac{m^2+m(2\ell-1)}{4\ell-2}$	$\frac{2m+2\ell-1}{2\ell-1}\prod_{k=1}^{2\ell-2}\frac{m+k}{k}$	$\ell \geqq 1, m \geqq 0$

(1.2)

Now $L^2(S^n) = \sum_m A(\lambda_m)$, Hilbert space direct sum of the eigenspaces of Δ as indicated in table (1.2). For each $m \geqq 0$ choose

$$\{\phi_{m,1}, \cdots, \phi_{m,d_m}\} : \text{ orthonormal basis of } A(\lambda_m). \tag{1.3}$$

Then the general solution to (1.1) is of the form

$$f(x:t) = \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} c_{m,j}e^{-t\lambda_m}\phi_{m,j}(x), \qquad \sum_{m,j}|c_{m,j}|^2 < \infty \tag{1.4}$$

for $x \in S^n$ and $t \geqq 0$. This corresponds to the choice of initial temperature function $b(x) = f(x:0) = \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} c_{m,j}\phi_{m,j}(x)$.

The question of discrete observability comes down to that of recovering the coefficients $c_{m,j}$ in (1.4) from data $f(x_k : t)$. Fix $t \geqq 0$ and $x_0 \in S^n$. Our sample points will be of the form $x_k = s_k^{-1}(x_0)$ for an appropriate subset $\{s_k\} \subset U$ where U is an arbitrary neighborhood of the identity in $SO(n+1)$. So our observations or measurements will be of the form

$$\begin{aligned} f(x_k:t) &= \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} c_{m,j}e^{-t\lambda_m}\phi_{m,j}(x_k) \\ &= \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} c_{m,j}e^{-t\lambda_m}[\pi(s_k)\phi_{m,j}](x_0) \end{aligned} \tag{1.5}$$

where π is the left regular representation, $[\pi(g)\phi](x) = \phi(g^{-1}x)$ for $g \in SO(n+1)$ and $x \in S^n$, of $SO(n+1)$ on $S^n = SO(n+1)/SO(n)$. Specifically, for each number $r > 0$ we consider the question of recovering the finite set of coefficients $c_{m,j}$, $1 \leqq j \leqq d_m$, $m \leqq r$, from the observations (1.5).

The left regular representation π of $SO(n+1)$ on $L^2(S^n)$ breaks up as

$$\pi = \sum_{m=0}^{\infty}\pi_m \ , \quad \text{direct sum of the distinct irreducibles }, \tag{1.6}$$

where π_m is the action of $SO(n+1)$ on $A(\lambda_m)$. We now write block form matrices using the bases (1.3) of the $A(\lambda_m)$. The observation (1.5) is given by

$$f(x_k:t) = (C_0, C_1, \cdots)\cdot\pi(s_k)\cdot\begin{pmatrix} e^{-t\lambda_0}\Phi_0(x_0) \\ e^{-t\lambda_1}\Phi_1(x_0) \\ \vdots \end{pmatrix} \tag{1.7}$$

where

$$C_m = (c_{m,1}\ ,\cdots,\ c_{m,d_m}) \text{ and } \Phi_m = \begin{pmatrix} \phi_{m,1} \\ \vdots \\ \phi_{m,d_m} \end{pmatrix}. \tag{1.8}$$

In other words, the summand of (1.5) up to a given value of m is

$$\begin{aligned} f_r(x_k : t) &= \sum_{m=0}^{r} \sum_{j=1}^{d_m} c_{m,j} e^{-t\lambda_m} \phi_{m,j}(x_k) \\ &= (C_0\ ,\cdots, C_r) \cdot \begin{pmatrix} \pi_0(s_k) & & \\ & \ddots & \\ & & \pi_r(s_k) \end{pmatrix} \times \\ & \quad \begin{pmatrix} e^{-t\lambda_0}\Phi_0(x_0) \\ \vdots \\ e^{-t\lambda_r}\Phi_r(x_0) \end{pmatrix} \\ &= \sum_{m=0}^{r} C_m \cdot \pi_m(s_k) \cdot e^{-t\lambda_m} \Phi_m(x_0). \end{aligned} \tag{1.9}$$

Note that f_r corresponds to orthogonal projection of f from $L^2(S^n)$ to $E_r(S^n)$, for every fixed value of t. In summary, we have verified

Lemma 1.1 *Fix $x_0 \in S^n$ and a time $t \geqq 0$. Then heat equation (1.1) is discretely observable at x_0 if, and only if, the following condition holds. For every neighborhood U of the identity in $SO(n+1)$ and every $m \geqq 0$, there exist $\{s_{m,1}\ ,\cdots,\ s_{m,d_m}\} \subset U$ such that, if f is the solution to (1.1) for $L^2(S^n)$ initial data $b = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} c_{m,j}\phi_{m,j}(x)$, then we can always solve for $(C_1\ ,\cdots,\ C_r)$ in the system of equations*

$$\begin{aligned} & f(s_{m,j}^{-1} x_0 : t) = \\ & (C_0\ ,\cdots,\ C_r) \cdot \begin{pmatrix} \pi_0(s_{m,j}) & & \\ & \ddots & \\ & & \pi_r(s_{m,j}) \end{pmatrix} \times \\ & \begin{pmatrix} e^{-t\lambda_0}\Phi_0(x_0) \\ \vdots \\ e^{-t\lambda_r}\Phi_r(x_0) \end{pmatrix}, \end{aligned} \tag{1.10}$$

for $0 \leqq m \leqq r$ and $1 \leqq j \leqq d_m$.

Now we check the condition in Lemma 1.1.

The column vector $e^{-t\lambda_m}\Phi_m(x_0) \in \mathbb{C}^{d_m}$ cannot be zero. For otherwise, each $\phi_{m,j}$ vanishes at x_0. Then every $\phi \in A(\lambda_m)$ vanishes at x_0. Since

$[\pi_m(g)\phi](x) = \phi(g^{-1}x)$ and $SO(n+1)$ is transitive on S^n, now every $\phi \in A(\lambda_m)$ vanishes at every $x \in S^n$. That says $A(\lambda_m) = 0$, which is not the case.

Let W be a neighborhood of x_0 in S^n. Express $W = U(x_0) = \{u(x_0) \mid u \in U\}$ where U is a neighborhood of the identity in $SO(n+1)$. Any neighborhood of the identity generates the group $SO(n+1)$. Using the bases (1.3) to identify $A(\lambda_m)$ with $\mathbf{C}^{d_m}$, and using irreducibility of π_m, we see that $\Phi_m(x_0)$ is a cyclic vector for π_m in $\mathbf{C}^{d_m}$. In other words, $\pi_m(SO(n+1)) \cdot \Phi_m(x_0)$ spans $\mathbf{C}^{d_m}$. So if U' is a neighborhood of the identity in $SO(n+1)$ then $\pi_m(U')$ cannot stabilize any proper subspace of $\mathbf{C}^{d_m}$ that contains $\Phi_m(x_0)$. Now choose U' small so that $U' = (U')^{-1}$ and $(U')^k \subset U$ for $1 \leqq k \leqq d_m + 1$, and let F_k denote the subspaces of $\mathbf{C}^{d_m}$ defined by

$$F_1 \text{ is the span of } \pi_m(U') \cdot \Phi_m(x_0) \text{ and } F_{k+1} \text{ is the span of } \pi_m(U') \cdot F_k \ .$$

Since $dim F_{d_m+1} \leqq d_m$ we must have some $F_{k+1} = F_k$. There, necessarily $F_k = \mathbf{C}^{d_m}$. We have verified that $\pi_m(U) \cdot \Phi_m(x_0)$ spans $\mathbf{C}^{d_m}$.

Now there is a set $\{s_{m,1}, \cdots, s_{m,d_m}\} \subset U$ such that the column vectors $\pi_m(s_{m,j}) \cdot \Phi_m(x_0)$ span $\mathbf{C}^{d_m}$. Of course we can multiply by a nonzero scalar. So $\mathbf{C}^{d_m}$ has a basis of the form $\{\pi_m(s_{m,j}) \cdot e^{-t\lambda_m}\Phi_m(x_0)$ where $\{s_{m,1}, \cdots, s_{m,d_m}\} \subset U$.

Now the column vectors

$$\begin{pmatrix} \pi_0(s_{m,j}) & & \\ & \ddots & \\ & & \pi_r(s_{m,j}) \end{pmatrix} \cdot \begin{pmatrix} e^{-t\lambda_0}\Phi_0(x_0) \\ \vdots \\ e^{-t\lambda_r}\Phi_r(x_0) \end{pmatrix}, \qquad (1.11)$$

with

$$0 \leqq m \leqq r, \ 1 \leqq j \leqq d_m$$

form a basis of the space $\mathbf{C}^{n_r}$, $n_r = \sum_{m=0}^{r} d_m$. So any $1 \times n_r$ row vector $(C_0, \cdots, C_r)$ is determined by its inner product with the column vectors (1.11). That is the condition of Lemma 1.1. So we have just proved

Theorem 1.1 *The heat equation $\Delta_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0$, $x \in S^n$ and $t \geqq 0$, is discretely observable at every point $x_0 \in S^n$.*

2 Invariant Evolution Equations: General Theory

Let X be a homogeneous space G/K where G is a compact Lie group, and let

$$D : L^2(X) \to L^2(X) \qquad (2.1)$$

be a closed[2] densely defined operator that commutes with the action

$$[L(g)f](x) = f(g^{-1}x) \tag{2.2}$$

which is the left regular representation G on $L^2(X)$. The corresponding evolution equation, analog of the heat (diffusion) equation, is

$$D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0 \tag{2.3}$$

on $X \times \mathbf{R}$. The usual heat equation is the case where D is the Laplace-Beltrami operator for a G-invariant riemannian metric on X. See Friedman [7, Part 2] for the basic facts on evolution equations, and any standard reference such as Riesz and Sz.-Nagy [16] for the few basic facts we use concerning normal operators.

The following result is well known in the case of the heat equation; Gilkey [8, Ch. 1, §6] and Yosida [23, Ch. XIV, §2] are convenient references.

Proposition 2.1 *If D is a normal operator[3] on $L^2(X)$, then there is a complete orthonormal set $\{\phi_j\}$ in $L^2(X)$ of eigenfunctions of D. If $\{\phi_j\}$ is any such orthonormal set, $D\phi_j = \lambda_j\phi_j$, then the $L^2(X)$ solutions to (2.3) are just the functions of the form*

$$f(x:t) = \sum\nolimits_j c_j e^{-t\lambda_j}\phi_j(x), \quad c_j \in \mathbf{C} \tag{2.4}$$

for $x \in X$ and for $t \in \mathbf{R}$ in the range [4] such that $\sum_j |c_j e^{-t\lambda_j}|^2 < \infty$.

Proof: The point is to use G-invariance and the compactness implicit in the Peter-Weyl Theorem to replace compactness in the argument which shows that the Laplacian Δ has discrete spectrum.

The Peter-Weyl Theorem gives a decomposition

$$L^2(G) = \sum\nolimits_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^* \text{ and } L^2(X) = \sum\nolimits_{\pi\in\widehat{G}} V_\pi \otimes (V_\pi^*)^K . \tag{2.5}$$

Here $\widehat{G}$ is the set of (equivalence classes of) irreducible unitary representations of G and V_π is the (finite dimensional) vector space on which G is

[2] Recall the definition: if $\{f_n\}$ is a sequence in the domain of D, if $\{f_n\}$ converges in $L^2(X)$ to some element f, and if $\{D(f_n)\}$ converges in $L^2(X)$ to some element h, then f is in the domain of D and $D(f) = h$. In other words, the graph of D is closed in $L^2(X) \oplus L^2(X)$.

[3] As usual, we say that D is normal if $DD^* = D^*D$. Since D is closed and densely defined, its adjoint D^* also is closed and densely defined. Here $DD^* = D^*D$ means that the domains of the compositions DD^* and D^*D are the same and are dense in $L^2(X)$, and that $D(D^*(f)) = D(d^*(f))$ for every function f in that common dense domain.

[4] If all but finitely many of the c_j vanish for $\mathrm{Re}\lambda_j < 0$, then this range includes $t \geqq 0$. Similarly, if all but finitely many of the c_j vanish for $\mathrm{Re}\lambda_j > 0$, this range includes $t \leqq 0$. In particular, if only finitely many c_j are nonzero, this range is all of $\mathbf{R}$.

represented by π. We identify $V_\pi \otimes V_\pi^*$ with the complex span of the matrix coefficient functions for π,

$$v \otimes w^* \text{ corresponds to the function } x \mapsto \langle v\ ,\ \pi(x)w\rangle$$

where $w^* \in V^*$ corresponds to inner product with $w \in V$. Thus $V_\pi \otimes (V_\pi^*)^K$ is the subspace consisting of functions $f \in V_\pi \otimes V_\pi^*$ such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. Those are the functions in $V_\pi \otimes V_\pi^*$ that can be (and will be) viewed as functions on G/K.

In the left regular representation (2.2) of G, the G-module structure implicit in (2.5) is

$$L^2(G) = deg(\pi)\sum\nolimits_{\pi\in\widehat{G}} V_\pi \text{ and } L^2(X) = mult(1_K, \pi|_K)\sum\nolimits_{\pi\in\widehat{G}} V_\pi \quad (2.6)$$

where $deg(\pi)$ is the degree of the representation π and $mult(1_K, \pi|_K)$ is the multiplicity of the trivial representation 1_K in the restriction $\pi|_K$.

The linear operator D preserves each of the finite dimensional summands

$$A(\pi) = V_\pi \otimes (V_\pi^*)^K \subset L^2(X). \quad (2.7)$$

Its adjoint D^* is also G-invariant and thus preserves each $A(\pi)$. As D is normal, $D|_{A(\pi)}$ commutes with $D^*|_{A(\pi)}$ and thus is diagonal relative to some orthonormal basis of $A(\pi)$. Diagonalizing each $D|_{A(\pi)}$ we have a complete orthonormal set $\{\phi_j\}$ in $L^2(X)$ consisting of eigenfunctions of D.

The remaining assertion follows as usual by Fourier expansion. □

Express $\widehat{G}$ as an increasing union of finite subsets $\widehat{G}_r$:

$$\widehat{G}_r = \{\pi_\nu \in \widehat{G} \mid ||\nu|| < r\} \quad (2.8)$$

where $\pi_\nu \in \widehat{G}$ has highest weight[5] ν. Any irreducible representation maps the (second order) Casimir operator ω from G to a scalar multiple of the identity. Identifying that scalar multiple with the scalar one has $\pi_\nu(\omega) = ||\nu + \rho||^2 - ||\rho||^2$ where ρ is half the sum of the positive roots. So the filtration of $\widehat{G}$ by the sets $\widehat{G}_r$ of (2.8) is equivalent to the filtration of $\widehat{G}$ by the sets

$$\widehat{G}_{(r)} = \{\pi_\nu \in \widehat{G} \mid \pi_\nu(\omega) < r^2\}.$$

Suppose that the riemannian metric on X comes from an invariant positive definite inner product on the Lie algebra $\mathfrak{g}$ that extends the negative of the Killing form. Then $L(\omega)$ gives the action of the Laplacian Δ of X

[5] Irreducible unitary representations of compact connected Lie groups, more generally irreducible finite dimensional complex representations of reductive Lie algebras, are parameterized by their "highest weights." This is É. Cartan's highest weight theory ([2], [3]). As we will see in a moment in the formula for $\pi_\nu(\omega)$, and as will be seen in §§3 and 4, the highest weight theory is extremely useful. There are a number of good expositions, for example those of Dynkin [6, Appendix], Humphreys [12] and Varadarajan [18].

on $L^2(X)$. (Here we use the sign such that Δ is a positive semidefinite operator.) So the filtration of $L^2(X)$ given by (2.8), as an increasing union of finite dimensional subspaces, is

$$L^2(X) = \bigcup_{r>0} E_r(X) \tag{2.9}$$

where

$$E_r(X) = \sum_{\|\nu\|<r} V_{\pi_\nu} \otimes (V^*_{\pi_\nu})^K = \sum_{\|\nu\|<r} mult(1_K, \pi|_K) \sum_{\pi\in\widehat{G}} V_\pi. \tag{2.10}$$

It is equivalent to the filtration of $L^2(X)$ by eigenvalues of Δ.

Note that the complete orthonormal set $\{\phi_j\}$ in $L^2(X)$ of eigenfunctions of D, which we constructed in the proof of Proposition 2.1, is an increasing union of orthonormal bases of the finite dimensional subspaces $E_r(X) \subset L^2(X)$ of (2.9) and (2.10). So we can fix one such complete orthonormal set $\Phi = \bigcup_{r>0} \Phi_r$ where

$$\Phi_r = \{\phi_1, \cdots, \phi_{n_r}\} \text{ is an orthonormal basis of } E_r(X) \text{ with } D\phi_j = \lambda_j \phi_j. \tag{2.11}$$

The finite dimensional spaces $E_r(X)$ are discretely observable in the sense of [21] because the action of G on $E_r(X)$ is a subrepresentation of the left regular representation of G on $L^2(G)$. More precisely,

Proposition 2.2 *Fix $x_0 \in X$. Then G has a countable subset $S = \bigcup_{r>0} S_r$ with $S_r = \{s_1, \cdots, s_{n_r}\}$ such that the function evaluations $\psi_j : f \mapsto f(s_j^{-1}x_0)$, $1 \leqq j \leqq n_r$, form a basis of the linear dual space of $E_r(X)$. If U is an open subset of G that meets every connected component, then we can find $S \subset U$.*

Proof: Any point evaluation $\psi_{(x)}(f) = f(x)$ can be viewed as a linear functional on $E_r(X)$. Let L^* denote the dual of the action L of G on $E_r(X)$. Then $L^*(g)(\psi_{(x)}) : f \mapsto f(g^{-1}x)$.

Let U be an open subset of G that meets every topological component. Define $F_{f,x} : G \to \mathbf{C}$ by $F_{f,x}(g) = f(g^{-1}x)$. Since the summand $V_\pi \otimes V_\pi^*$ of $L^2(G)$ consists of real analytic functions for each $\pi \in \widehat{G}$, and f is constrained to $E_r(X)$, now $F_{f,x}$ is real analytic on G and thus determined by its restriction to U. So $\{\psi_{(u^{-1}x_0)}|_{E_r(X)} \mid u \in U\}$ spans the dual space of $E_r(X)$, and we have $S_r = \{s_1, \cdots, s_{n_r}\} \subset U$ such that the $\psi_{(s_j^{-1}x_0)}|_{E_r(X)}$, $1 \leqq j \leqq n_r$, form a basis of $E_r(X)$.

If $r < v$ then we can choose $S'_{r,v} = \{s_{n_r+1}, \cdots, s_{n_v}\} \subset U$ so that

$$\psi_{(s_j^{-1}x_0)}|_{E_r(X)^\perp \cap E_v(X)},\ n_r < j \leqq n_v, \text{ is a basis of } E_r(X)^\perp \cap E_v(X).$$

Then we may take $S_v = S_r \cup S'_{r,v}$. Since $\{v \in \mathbf{R} \mid dim\ E_r(X) < dim\ E_v(X)$ for all $r < v\}$ is a discrete set $r_0 < r_1 < \cdots$ of non-negative real numbers that goes to infinity, this process of enlarging S_r defines the countable set $S = \bigcup_{r>0} S_r$ required by the proposition. □

Propositions 2.1 and 2.2 combine with (2.9) and (2.10) and (2.11) to give a particular type of observability for the evolution equation (2.3).

Theorem 2.1 *Let X be a homogeneous space G/K where G is a compact Lie group. Let D be a closed densely defined G-invariant normal operator on $L^2(X)$. Choose a complete orthonormal set $\Phi = \bigcup_{r>0} \Phi_r$ in $L^2(X)$ as in (2.11). Then the $L^2(X)$ solutions to the evolution equation (2.3) are just the functions*

$$f(x:t) = lim_{r\to\infty} \sum\nolimits_{1\leqq j\leqq n_r} c_j e^{-t\lambda_j}\phi_j(x) \tag{2.12}$$

for $x \in X$ and for $t \in \mathbf{R}$ in the range such that $\sum_j |c_j e^{-t\lambda_j}|^2 < \infty$. The solution $f(x:t)$ of the evolution equation is observable at any time $t_0 \geqq 0$ on any open subset of X that meets every topological component. In other words, let U be an open subset of G that meets every topological component. Then there is an increasing union $S = \bigcup_{r>0} S_r \subset G$ with each $S = \{s_1, \cdots, s_{n_r}\} \subset U$ such that each partial sum

$$f_r(x:t) = \sum\nolimits_{1\leqq j\leqq n_r} c_j e^{-t\lambda_j}\phi_j(x) \tag{2.13}$$

is determined by the "observations" $f_r(s_j^{-1}x_0 : t_0)$, $1 \leqq j \leqq n_r$.

3 The Invariant Heat Equation

We specialize the results of Section 2 to the case where D is the Laplace-Beltrami operator Δ with respect to a G-invariant riemannian metric on $X = G/K$, in a situation in which the spectrum of Δ has an explicit description.

Let κ denote the Cartan–Killing form on the Lie algebra $\mathfrak{g}$ of the compact Lie group G. Recall $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}'$ is the derived algebra. Then $\kappa(\mathfrak{z}, \mathfrak{g}) = 0$ and κ is negative definite on $\mathfrak{g}'$. Suppose that the G-invariant riemannian metric on X is induced by a bi-invariant positive definite bilinear form β on $\mathfrak{g}$ such that

$$\beta(\mathfrak{z}, \mathfrak{g}') = 0 \text{ and } \beta = -\kappa \text{ on } \mathfrak{g}'. \tag{3.1}$$

In other words, $X = G/K$ is a normal homogeneous space with a particular normalization of riemannian metric; see [5, §1] and compare [13] and [14].

Let $\mathfrak{G} = \mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra associated to G. The Casimir element $\omega \in \mathfrak{G}$ corresponding to the extension -β of the Cartan–Killing form, is given by

$$\omega = -\sum \xi_j \xi_j^* \text{ where } \xi_j \text{ is any basis of } \mathfrak{g}_{\mathbf{C}} \text{ and } \xi_j^* \text{ is the } \beta\text{-dual basis.} \tag{3.2}$$

The point of the setup (3.1) and (3.2) is that

$$\Delta \text{ is just the action of } \omega \text{ on } L^2(X). \tag{3.3}$$

In effect, one just has to check this at the base point $x_0 = 1 \cdot K$ of G/K. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the β-orthogonal decomposition. Choose a β-orthonormal basis ξ_i of $\mathfrak{p}$ and a β-orthonormal basis η_j of $\mathfrak{k}$. If f is C^2 at x_0 then $\omega(f)(x_0) = -\sum \xi_i^2(f)(x_0) = \Delta(f)(x_0)$, the latter equality because the ξ_i generate geodesics from x_0.

Now we can describe the spectrum of Δ by means of highest weights of representations of G, as in [1], at least when G is connected. But this becomes slightly clumsy when G is not connected. Let π_ν^0 denote the unique irreducible representation of the identity component G^0 that has highest weight ν. One has a finite set of highest weights ν_i, the images of ν under graph automorphisms of the Schläfli-Dynkin diagram of the Lie algebra $\mathfrak{g}$, and the highest weights of the irreducible summands of the induced representation $\mathrm{Ind}_{G^0}^G(\pi_\nu^0)$ are just the orbit $G\cdot\nu$ of ν in ν_i. Every irreducible representation of G with highest weight ν is a summand of the induced representation $\mathrm{Ind}_{G^0}^G(\pi_\nu^0)$. To avoid this sort of nuisance, we now assume that G is connected.

Fix a maximal torus $T \subset G$. The corresponding subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ denote the root system. Fix a positive root system Φ^+. Let Ψ denote the corresponding system of simple roots. Let $\Lambda \subset \sqrt{-1}\mathfrak{t}$ denote the lattice of integral weights, i.e. of those linear functionals on $\mathfrak{t}$ of the form $d\chi$, $\chi \in \widehat{T}$. Now

$$\Lambda^+ = \left\{ \nu \in \Lambda \mid \frac{2\langle \nu, \psi\rangle}{\langle \psi, \psi\rangle} \geqq 0 \text{ for all } \psi \in \Psi \right\} \tag{3.4}$$

parameterizes $\widehat{G}$ by highest weight. For every $\nu \in \Lambda^+$ we have the degree

$$d(\nu) = deg(\pi_\nu) = \prod\nolimits_{\alpha\in\Phi^+} \frac{\langle \nu + \rho, \alpha\rangle}{\langle \rho, \alpha\rangle}, \tag{3.5}$$

the multiplicity

$$m(\nu) = mult(1_K, \pi_\nu|_K) = \text{ multiplicity of } 1_K \text{ as a summand of } \pi_\nu|_K\ , \tag{3.6}$$

and the eigenvalue

$$\Delta\phi = (||\nu + \rho||^2 - ||\rho||^2)\phi \text{ for all } \phi \in A(\pi_\nu) \tag{3.7}$$

where, as before, ρ is half the sum of the positive roots.

We modify (2.11) to fit our situation. Let $\{v_{1,\nu}\ , \cdots, \ v_{d(\nu),\nu}\}$ be an orthonormal basis of V_{π_ν} such that the subset $\{v_{1,\nu}^*\ , \cdots, \ v_{m(\nu),\nu}^*\}$ of the dual basis of $V^*_{\pi_\nu}$ spans $(V^*_{\pi_\nu})^K$. The corresponding orthonormal basis of $A(\pi_\nu) = V_{\pi_\nu} \otimes (V^*_{\pi_\nu})^K$ is

$$\Phi_\nu = \{\phi_{i,j,\nu} \mid 1 \leqq i \leqq d(\nu),\ 1 \leqq j \leqq m(\nu)\} \tag{3.8}$$

where

$$\phi_{i,j,\nu}(x) = d(\nu)\langle v_{i,\nu}\ ,\ \pi_\nu(x)v_{j,\nu}\rangle.$$

Decompose the set Λ^+ of (3.4) according to (2.8) and (2.9) and (2.10):

$$\Lambda^+ = \bigcup\nolimits_{r>0} \Lambda^+_r \tag{3.9}$$

where

$$\Lambda^+_r = \left\{ \nu \in \Lambda \mid ||\nu|| < r \text{ and } \frac{2\langle \nu, \psi\rangle}{\langle \psi, \psi\rangle} \geqq 0 \text{ for all } \psi \in \Psi \right\}.$$

Thus (2.12) and (2.13) become

$$f(x:t) = lim_{r\to\infty} f_r(x:t) \tag{3.10}$$

with $f_r(x:t)$ given by

$$\sum\nolimits_{\nu\in\Lambda^+_r} d(\nu) \sum\nolimits_{1\leqq i\leqq d(\nu),\ 1\leqq j\leqq m(\nu)} c_{i,j,\nu} e^{-t(||\nu+\rho||^2 - ||\rho||^2)} \langle v_{i,\nu}\ ,\ \pi_\nu(x)v_{j,\nu}\rangle. \tag{3.11}$$

Now we can deal explicitly with the ordinary heat equation

$$\Delta_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0 \tag{3.12}$$

on $X \times \mathbf{R}$. Fix a complete orthonormal set $\Phi = \bigcup_{\nu\in\Lambda^+} \Phi_\nu$ in $L^2(X)$ as in (3.8). Then Φ is the increasing union of finite subsets

$$\Phi_r = \bigcup\nolimits_{\nu\in\Lambda^+_r} \Phi_\nu = \{\phi_1, \cdots, \phi_{n_r}\},\ n_r = \sum\nolimits_{\nu\in\Lambda^+_r} m(\nu)d(\nu) \tag{3.13}$$

as in (2.11).

Theorem 3.1 *Let $X = G/K$ be a normal homogeneous riemannian manifold with metric normalized as in* (3.1). *Fix $b \in L^2(X)$, say*

$$b(x) = \sum\nolimits_{\nu\in\Lambda^+} d(\nu) \sum\nolimits_{1\leqq i\leqq d(\nu),\ 1\leqq j\leqq m(\nu)} c_{i,j,\nu} \langle v_{i,\nu}\ , \pi_\nu(x)v_{j,\nu}\rangle \tag{3.14}$$

with $\sum_{\nu\in\Lambda^+} |c_{i,j,\nu}|^2 < \infty$. Then the heat equation (3.12) *has a unique solution with initial data $b(x)$; it is given by* (3.10) *and* (3.11). *In fact* (3.10) *converges to a real analytic function $f(x:t)$ on $X \times \{t \in \mathbf{R} \mid t > 0\}$. This solution $f(x:t)$ to the heat equation is observable at any time $t_0 \geqq 0$ on any neighborhood of an arbitrary point $x_0 \in X$: If U is any open subset of G there is an increasing set $S = \bigcup_{r>0} S_r \subset G$ with each $S = \{s_1, \cdots, s_{n_r}\} \subset U$ such that each partial sum $f_r(x:t)$, as in* (3.11), *is determined by the observations $f_r(s_j^{-1}x_0 : t_0)$, $1 \leqq j \leqq n_r$.*

Proof: For convergence of (3.10) to a real analytic function $f(x:t)$ on $X \times \{t \in \mathbf{R} \mid t > 0\}$ see any discussion of the heat equation on a compact manifold, for example [23, Ch. XIV, §2] or [8, Ch. 1, §6]. Now our assertions are a special case of Theorem 2.1. □

4 Symmetric Space Theory

We now specialize the results of Section 2 to the case where $X = G/K$ is a riemannian symmetric space. For the same reasons as in §3, we assume that the riemannian metric is normalized as in (3.1).

We recall some standard facts about symmetric spaces; see [11]. The algebra $\mathcal{D}(G/K)$ of G-invariant differential operators on X is commutative. If $\pi \in \widehat{G}$ recall the π-isotypic subspace $A(\pi) \subset L^2(G)$ of (2.7). There are two possibilities: either the multiplicity $m(\pi)$ of 1_K in $\pi|_K$ is zero and $A(\pi) = 0$, or $m(\pi) = 1$ and $A(\pi) \cong V_\pi$ as G-module. In the latter case π is a **class 1** representation of G, the algebra $\mathcal{D}(G/K)$ acts on $A(\pi)$ by scalars, and the corresponding associative algebra homomorphism $\chi_\pi : \mathcal{D}(G/K) \to \mathbf{C}$ specifies $A(\pi)$.

Lemma 4.1 *If $D \in \mathcal{D}(G/K)$, with domain $C^\infty(G/K)$, then D has unique closure $\widetilde{D}$ as a densely defined linear operator on $L^2(G/K)$, and $\widetilde{D}$ is a normal operator on $L^2(X)$.*

Proof: Observe that $A(\pi^*) = \{\, \overline{f} \mid f \in A(\pi)\}$. Compute

$$\begin{aligned} \int_G f(g)\overline{D^*h(g)}d(gK) &= \int_G Df(g)\overline{h(g)}d(gK) \\ &= \int_G \chi_\pi(D)f(g)\overline{h(g)}d(gK) \\ &= \int_G f(g)\overline{\overline{\chi_\pi(D)}h(g)}d(gK) \end{aligned}$$

for $D \in \mathcal{D}(G/K)$ and $f, h \in A(\pi)$. This shows $\overline{\chi_\pi(D)} = \chi_{\pi^*}(D) = \chi_\pi(D^*)$. It follows that D^* is a well defined element of $\mathcal{D}(G/K)$. Note D and D^* commute on each $A(\pi)$. So their closures, from the dense domain of finite linear combinations of functions in the $A(\pi)$, must be given by $\widetilde{D} = D^{**}$ and $(\widetilde{D})^* = \widetilde{D^*} = D^*$. These are normal operators on $L^2(X)$. □

From now on, we will identify any $D \in \mathcal{D}(G/K)$ with its $L^2(G/K)$-closure $\widetilde{D}$. Then Theorem 2.1 applies directly to every $D \in \mathcal{D}(G/K)$. A little bit of symmetric space theory makes this rather explicit.

The symmetry of $X = G/K$ at the base point $1 \cdot K = x_0$ defines an involutive automorphism θ of the Lie algebra $\mathfrak{g}$. Decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, sum of the ± 1-eigenspaces of θ. Here $\mathfrak{k}$ is the Lie algebra of K. Choose

$$\mathfrak{a} : \text{ maximal abelian subspace of } \mathfrak{p}. \qquad (4.1)$$

It is unique up to conjugation by an element of K, and $\mathfrak{a}$ extends uniquely to

$$\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} + \mathfrak{a} : \text{ Cartan subalgebra of } \mathfrak{g} \tag{4.2}$$

where $\mathfrak{t}_{\mathfrak{k}} = \mathfrak{t} \cap \mathfrak{k}$ is a Cartan subalgebra of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

The root system $\Phi = \Phi(\mathfrak{t}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ defines the **restricted root system**

$$\Phi_{\mathfrak{a}} = \Phi_{\mathfrak{a}}(\mathfrak{a}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Phi \text{ and } \alpha|_{\mathfrak{a}} \neq 0\}. \tag{4.3}$$

Every choice of **positive restricted root system** $\Phi_{\mathfrak{a}}^{+}$ is of the form

$$\Phi_{\mathfrak{a}}^{+} = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Phi^{+} \text{ and } \alpha|_{\mathfrak{a}} \neq 0\} \tag{4.4}$$

for an appropriate choice of positive root system $\Phi^{+} = \Phi^{+}(\mathfrak{t}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$. We fix one such choice of positive restricted root system (4.5).

Consider the lattice

$$\Lambda_{\mathfrak{a}} = \left\{ \nu \in \sqrt{-1}\mathfrak{a}^{*} \mid \frac{\langle \nu, \psi \rangle}{\langle \psi, \psi \rangle} \in \mathbf{Z} \text{ for all } \psi \in \gneqq \right\} \tag{4.5}$$

and the subset of dominant linear functionals

$$\Lambda_{\mathfrak{a}}^{+} = \left\{ \nu \in \Lambda_{\mathfrak{a}} \mid \frac{\langle \nu, \psi \rangle}{\langle \psi, \psi \rangle} \geqq 0 \text{ for all } \psi \in \Psi \right\} \tag{4.6}$$

A famous theorem of Cartan [4], made precise by Helgason [11], says that $\Lambda_{\mathfrak{a}}^{+}$ parameterizes the class 1 representations of G in case G is simply connected and K is connected. See [20, Ch. III] for a concise proof. We formulate the result to take account of the possibility that G may be non-simply connected:

Theorem 4.1 (Cartan-Helgason Theorem) *Suppose that G and K are connected. Then the irreducible representation π_ν of G, with highest weight ν relative to Φ^{+}, is of class 1 if and only if* (i) $\nu|_{\mathfrak{t}_{\mathfrak{k}}} = 0$ *and* (ii) $\nu_{\mathfrak{a}} \in \Lambda_{\mathfrak{a}}^{+}$.

Now we proceed more or less as in §3. Denote

$$\Lambda_{\mathfrak{a},r}^{+} = \{\nu \in \Lambda_{\mathfrak{a}}^{+} \mid ||\nu|| < r\}. \tag{4.7}$$

Let $\chi(\nu)$ denote the associative algebra homomorphism $\chi_{\pi_\nu} : \mathcal{D}(G/K) \to \mathbf{C}$ which gives the joint eigenvalue of $\mathcal{D}(G/K)$ on $A(\pi)$. The degree of π_ν is given by the polynomial

$$d(\nu) = deg(\pi_\nu) = \prod_{\alpha \in \Phi^{+}} \frac{\langle \nu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}. \tag{4.8}$$

Denote

$$v_\nu : \text{ a choice of } \pi_\nu(K)\text{–fixed unit vector in } V_{\pi_\nu}. \tag{4.9}$$

It is unique up to multiplication by a complex number of absolute value 1. Choose

$$\{v_{1,\nu}\ ,\cdots,\ v_{d(\nu),\nu}\}:\ \text{any orthonormal basis of } V_{\pi_\nu}. \tag{4.10}$$

Then the corresponding space $A(\pi_\nu)$ of functions on G/K has orthonormal basis

$$\Phi_\nu = \{\phi_{i,\nu} \mid 1 \leqq i \leqq d(\nu)\} \text{ where } \phi_{i,\nu}(x) = d(\nu)\langle v_{i,\nu}\ ,\ \pi_\nu(x)v_\nu\rangle. \tag{4.11}$$

Thus (3.10) and (3.11) become

$$f(x:t) = lim_{r\to\infty} f_r(x:t) \tag{4.12}$$

with $f_r(x:t)$ given by

$$\sum\nolimits_{\nu\in\Lambda^+_{s,r}} d(\nu)\sum\nolimits_{1\leqq i\leqq d(\nu)} c_{i,\nu}e^{-t\chi(\nu)}\langle v_{i,\nu}\ ,\pi(x)v_{j,\nu}\rangle. \tag{4.13}$$

Now we deal explicitly with the evolution equation

$$D_x f(x:t) + \frac{\partial}{\partial t} f(x:t) = 0\ ,\ d\in \mathcal{D}(G/K) \tag{4.14}$$

on $G/K\times \mathbf{R}$. We have the complete orthonormal set $\Phi = \bigcup_{\nu\in\Lambda^+}\Phi_\nu$ in $L^2(G/K)$. It is the increasing union of finite subsets

$$\Phi_r = \bigcup\nolimits_{\nu\in\Lambda^+_{s,r}} \Phi_\nu = \{\phi_1,\cdots,\phi_{n_r}\},\ n_r = \sum\nolimits_{\nu\in\Lambda^+_{s,t}} d(\nu) \tag{4.15}$$

as in (3.13). Now the analog of Theorem 3.1 is

Theorem 4.2 *Let $X = G/K$ be a compact riemannian symmetric space with metric normalized as in* (3.1). *Fix $b\in L^2(X)$, say*

$$b(x) = \sum\nolimits_{\nu\in\Lambda^+_s} d(\nu)\sum\nolimits_{1\leqq i\leqq d(\nu)} c_{i,\nu}\langle v_{i,\nu}\ ,\pi_\nu(x)v_\nu\rangle \tag{4.16}$$

with $\sum_{\nu\in\Lambda^+_s}|c_{i,\nu}|^2<\infty$. Then the evolution equation (4.14) *has a unique solution with initial data $b(x)$; it is given by* (4.12) *and* (4.13). *In fact* (4.12)*converges to a function $f(\cdot:t)\in L^2(X)$ for every $t\in\mathbf{R}$ in the range $J_{b,D}\subset\mathbf{R}$ given by*

$$\sum\nolimits_{\nu\in\Lambda^+_s}\sum\nolimits_{1\leqq i\leqq d(\nu)} |c_{i,\nu}e^{-t\chi(\nu)}|^2 < \infty. \tag{4.17}$$

This solution $f(x:t)$ to (4.14) *is observable at any time $t\in J_{b,D}$ on any neighborhood of an arbitrary point $x_0\in X$: If U is any open subset of G there is an increasing set $S=\bigcup_{r>0}S_r\subset G$ with each $S=\{s_1,\cdots,s_{n_r}\}\subset U$ such that each partial sum $f_r(x:t)$, as in* (4.13), *is determined by the observations $f_r(s_j^{-1}x_0:t_0),\ 1\leqq j\leqq n_r$.*

This is a strong form of discrete observability for invariant evolution equations on compact riemannian symmetric spaces.

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