Geometric Realizations of Discrete Series Representations in a Nonconvex Holomorphic Setting

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To Jacques Tits on his 60th birthday

Section 1. Introduction.

Let G be a semisimple Lie group that has discrete series representations. If W is a complex flag manifold of the complexification of G then G has only finitely many orbits on W, so it has open orbits. If Y is an open orbit, s is the complex dimension of its maximal compact subvarieties, and $\mathbf{V} \to Y$ is an appropriate homogeneous holomorphic Fréchet vector bundle, we see (Theorem 6.5 below) that the natural action of G on the Dolbeault cohomology $H^s(Y; \mathcal{O}(\mathbf{V}))$ is a discrete series representation specified explicitly by certain data defining $\mathbf{V} \to Y$, and we see that every discrete series representation of G arises this way. The result is particularly interesting (Theorem 5.5 below) when G acts transitively on a bounded symmetric domain D and W is the compact hermitian symmetric dual to D. There it exhibits every discrete series representation of G in a manner quite similar to the construction of the holomorphic discrete series.

We then discuss the question of using this geometric setting to construct the unitary structure of the discrete series representations in question.

Since the proofs are no more difficult than for Harish-Chandra class ([4], [5], [6]) I work with the class [16] of "general semisimple groups." These groups and their relative discrete series representations are described in §2. The connection with Dolbeault cohomology is recalled from [10] and [13] in §3. In §4 I review the orbit structure of hermitian symmetric spaces, using it in §5 for explicit realizations of relative discrete series representations in a setting analogous to that of the holomorphic discrete series. In §6 we see that this goes through essentially unchanged in the general setting of open orbits on complex flag manifolds. Then in §7 I try to indicate the connection with indefinite metric quantisation methods ([9], [18], [19]) for unitarizing representations of semisimple Lie groups.

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Section 2. Discrete Series for General Semisimple Groups.

In this Section we specify the class of Lie groups with which we work and recall the description of its (relative) discrete series representations. See [16] and [7] for details.

First recall the notion of relative discrete series. Let G be a unimodular locally compact group, Z a closed normal abelian subgroup. We write $\widehat{G}$ and $\widehat{Z}$ for their unitary duals. Z has left regular representation $\ell_Z = \int_{\widehat{Z}} \zeta \, d\zeta$, so the left regular representation of G is

$$\ell_G = \mathrm{Ind}_{\{1\}}^G(1) = \mathrm{Ind}_Z^G(\ell_Z) = \int_{\widehat{Z}} \mathrm{Ind}_Z^G(\zeta)\, d\zeta. \tag{2.1}$$

So $\ell_G = \int_{\widehat{Z}} \mathrm{Ind}_Z^G(\ell_{G,\zeta})\, d\zeta$ where $\ell_{G,\zeta} = \mathrm{Ind}_Z^G(\zeta)$. Here $\ell_{G,\zeta}$ is the left regular representation of G on the Hilbert space

$$L^2(G/Z,\zeta) = \{f : G \to \mathbf{C} \mid f(gz) = \zeta(z)^{-1} f(g) \text{ and } \int_{G/Z} |f(g)|^2\, d(gZ) < \infty\}. \tag{2.2}$$

Now $L^2(G) = \int_{\widehat{Z}} L^2(G/Z,\zeta)\, d\zeta$ and $\widehat{G}$ is the union of subsets

$$\widehat{G}_\zeta = \{[\pi] \in \widehat{G} \mid \zeta \text{ is a subrepresentation of } \pi|_Z\}. \tag{2.3}$$

We say that a class $[\pi] \in \widehat{G}$ is **ζ-discrete** if π is equivalent to a subrepresentation of $\ell_{G,\zeta}$. All such classes $[\pi]$ form the **ζ-discrete series** $\widehat{G}_{\zeta\text{-disc}}$ of G. The **relative** (to Z) **discrete series** of G is

$$\widehat{G}_{\mathrm{disc}} = \bigcup_{\zeta \in \widehat{Z}} \widehat{G}_{\zeta\text{-disc}}. \tag{2.4}$$

If Z is central in G then every class $[\pi] \in \widehat{G}$ specifies a character $\zeta \in \widehat{Z}$ by: $\pi|_Z$ is a multiple of ζ. In this case, $\widehat{G}$ is the disjoint union of the $\widehat{G}_\zeta$.

When Z is central in G and $\zeta \in \widehat{Z}$, one knows (see [16], §2) that the following are equivalent.

(2.5a) π is a ζ-discrete series representation of G,
(2.5b) every coefficient $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ belongs to $L^2(G/Z,\zeta)$, and
(2.5c) for some nonzero $u, v \in H_\pi$, the coefficient $f_{u,v} \in L^2(G/Z,\zeta)$.

Given (2.5) one has a number $deg(\pi) > 0$ such that the $L^2(G/Z,\zeta)$-inner product of coefficients of π is given by

$$\langle f_{u,v}, f_{s,t}\rangle = \frac{1}{deg(\pi)} \langle u,s\rangle \overline{\langle v,t\rangle} \text{ for } s,t,u,v \in H_\pi. \tag{2.6a}$$

Furthermore, if π' is another ζ-discrete series representation of G, and is not equivalent to π, then

$$\langle f_{u,v}, f_{u',v'}\rangle = 0 \text{ for } u,v \in H_\pi \text{ and } u',v' \in H_{\pi'}. \tag{2.6b}$$

These orthogonality relations come out of convolution formulae. With the usual

$$f * h(x) = [\ell_{G,\zeta}(f)h](x) = \int_{G/Z} f(y)h(y^{-1}x)\, d(yZ) \tag{2.7}$$

we have

$$f_{u,v} * f_{s,t} = \frac{1}{deg(\pi)} \langle u,t\rangle f_{s,v} \text{ for } s,t,u,v \in H_\pi \tag{2.8a}$$

for $[\pi] \in \widehat{G}_{\zeta\text{-disc}}$, and

$$f_{u,v} * f_{u',v'} = 0 \text{ for } u,v \in H_\pi \text{ and } u',v' \in H_{\pi'} \tag{2.8b}$$

whenever $[\pi] \neq [\pi']$ in $\widehat{G}_{\zeta\text{-disc}}$.

If G is a Lie group we write G^0 for its identity component, $\mathfrak{g}_0$ for its Lie algebra, and $\mathfrak{g}$ for the complexified Lie algebra $\mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$. Here we will work with the class of reductive Lie groups G such that

(2.9a) G has a normal abelian subgroup Z which centralizes the identity component G^0 of G and such that $Z \cdot G^0$ has finite index in G, and
(2.9b) if $x \in G$ then conjugation $Ad(x)$ is an inner automorphism on the complexified Lie algebra $\mathfrak{g}$.

This is a convenient class in which to do representation theory and harmonic analysis. It includes all connected semisimple Lie groups, and if a reductive Lie group G satisfies (2.9) so does the Levi component of every cuspidal parabolic subgroup. We will refer to reductive Lie groups G that satisfy (2.9) as **general semisimple groups.**

From now on, G is a general semisimple group. Without loss of generality we expand Z to $Z \cdot Z_{G^0}$ where Z_{G^0} is the center of G^0. In other words we assume that Z contains Z_{G^0}.

If $[\pi] \in \widehat{G}$ then Θ_π denotes its distribution character and χ_π is its infinitesimal character. The condition (2.9b) ensures that the latter exists.

Let $Z_G(G^0)$ denote the centralizer of G^0 in G. Denote $G^\dagger = Z_G(G^0) \cdot G^0$. Many constructions for a general semisimple group G go from G^0 to $G^\dagger$ to G.

The analog of maximal compact subgroup for G^0 is just the full inverse image K^0 of a maximal compact subgroup in the connected linear semisimple Lie group G^0/Z_{G^0}. The analog of maximal compact subgroup for $G^\dagger$ is just $K^\dagger = Z_G(G^0) \cdot K^0$, which in fact is the full inverse image of a maximal compact subgroup in $G^0/Z_{G^0} = G^\dagger/Z_G(G^0)$. The analog of a maximal compact subgroup K for G can be equivalently defined as the G-normalizer of K^0, the G-normalizer of $K^\dagger$, or the full inverse image of a maximal compact subgroup in G/Z or in $G/Z_G(G^0)$. We refer to these groups K, $K^\dagger$ and K^0 respectively as maximal **compactly embedded subgroups** of G, $G^\dagger$ and G^0. If Z is compact, they are just the maximal compact subgroups.

By **Cartan involution** of G we mean an involutive automorphism whose fixed point set $K = G^\theta$ is a maximal compactly embedded subgroup. All the standard results hold: every maximal compactly embedded subgroup of G is the fixed point set of a unique Cartan involution, and every Cartan involution of $\mathfrak{g}_0$ extends uniquely to a Cartan involution of G. See [16].

Fix a Cartan involution θ of G and let $K = G^\theta$. Every Cartan subgroup of G is $Ad(G^0)$-conjugate to a θ-stable Cartan subgroup. In particular, G has compactly embedded Cartan subgroups if and only if K contains a Cartan subgroup of G.

G has relative discrete series representations if and only if it has a compactly embedded Cartan subgroup. Suppose that G has such Cartan subgroups, and fix a Cartan subgroup $T \subset K$ of G. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system, $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ a choice of positive root system, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$, half the trace of $ad(\mathfrak{t})$ on $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

If π is a relative discrete series representation of G and Θ_π is its distribution character, then the equivalence class of π is determined by the restriction of Θ_π to $T \cap G'$ where G' is the regular set. So the relative discrete series of G is parameterized by parameterization of those restrictions. Here we follow [3], [16] and [4].

Let $G^\dagger = Z_G(G^0)G^0$ coincides with TG^0. The Weyl group $W^\dagger = W(G^\dagger, T)$ coincides with $W^0 = W(G^0, T^0)$ and is a normal subgroup of $W = W(G, T)$.

Let $\chi \in \widehat{T}$. Since T^0 is commutative, χ has differential $d\chi(\xi) = \lambda(\xi)I$ where $\lambda \in i\mathfrak{t}_0^*$ and where I is the identity on the representation space of χ. Suppose that $\lambda + \rho$ is regular, i.e., that $\langle\lambda + \rho, \alpha\rangle \neq 0$ for all $\alpha \in \Phi$. Then there are unique

(2.10a) $$[\pi_\chi^0] \in (\widehat{G^0})_{\zeta\text{-disc}} \text{ and } [\pi_\chi^\dagger] = [\chi|_{Z_G(G^0)} \otimes \pi_\chi^0] \in (\widehat{G^\dagger})_{\zeta\text{-disc}},\ \zeta|_{Z_{G^0}} = e^\lambda|_{Z_{G^0}}$$

whose distribution characters satisfy

(2.10b) $$\Theta_{\pi_\chi^0}(x) = \pm\frac{\sum_{w\in W^0} sign(w)e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \text{ and } \Theta_{\pi_\chi^\dagger}(zx) = \chi(z)\Theta_{\pi_\chi^0}(x)$$

for $z \in Z_G(G^0)$ and $x \in T^0 \cap G'$. These representations have infinitesimal character of Harish-Chandra parameter λ.

The same datum χ specifies a relative discrete series representation

(2.11a) $$\pi_\chi = \mathrm{Ind}_{G^\dagger}^G(\pi_\chi^\dagger) \in \widehat{G}_{\zeta\text{-disc}}$$

characterized by the fact that its distribution character

(2.11b) $$\Theta_{\pi_\chi} \text{ is supported in } G^\dagger \text{, and there } \Theta_{\pi_\chi} = \sum_{1\leq i\leq r} \Theta_{\pi_\chi^\dagger}\cdot\gamma_i^{-1}$$

where $\{\gamma_1, \ldots, \gamma_r\}$ is any system of coset representatives of W modulo $W^\dagger$.

Every relative discrete series representation of G is equivalent to a representation π_χ as just described, and $[\pi_\chi] = [\pi_{\chi'}]$ in $\widehat{G}$ if and only if $\chi' = \chi \cdot w^{-1}$ for some $w \in W$.

Section 3. Discrete Series and Dolbeault Cohomology.

We fix a Cartan subgroup $T \subset K$ of G. The root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ decomposes as the disjoint union of the **compact roots** $\Phi_K = \Phi(\mathfrak{k}, \mathfrak{t}) = \{\alpha \in \Phi : \mathfrak{g}_\alpha \subset \mathfrak{k}\}$ and the **noncompact roots** $\Phi_{G/K} = \Phi \setminus \Phi_K$. A choice $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of positive root system defines a G-invariant complex manifold structure on G/T such that $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ represents the holomorphic tangent space.

Fix a choice of Φ^+. Write Φ_K^+ for $\Phi^+ \cap \Phi_K$ and $\Phi_{G/K}^+$ for $\Phi^+ \cap \Phi_{G/K}$.

Let $\chi \in \widehat{T}$. Then $d\chi = \lambda I$ for some integral $\lambda \in i\mathfrak{t}_0^*$ where I is the identity transformation of the representation space E_χ. When λ is nonsingular, we have the relative discrete series representation π_χ.

The usual geometric realization of π_χ is on a space of square integrable harmonic forms. Let

(3.1) $\mathbb{E}_\chi \to G/T$: hermitian homogeneous holomorphic vector bundle associated to χ.

and

(3.2) $\square$: Kodaira-Hodge-Laplace operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $\mathbb{E}_\chi$.

Then we have spaces

(3.3) $\mathcal{H}^q(G/T; \mathbb{E}_\chi)$: harmonic L^2 $\mathbb{E}_\chi$-valued $(0, q)$-forms on G/T

on which G acts naturally and the natural action of G is a unitary representation. The basic fact here is the positive solution to the Kostant-Langlands Conjecture:

3.4. Theorem ([11], [12], [16]). *If $\lambda + \rho$ is singular then every $\mathcal{H}^q(G/T; \mathbb{E}_\chi) = 0$. If $\lambda + \rho$ is regular let*

$$q(\lambda + \rho) = |\{\alpha \in \Phi_K^+ : \langle\lambda + \rho, \alpha\rangle < 0\}| + |\{\beta \in \Phi_{G/K}^+ : \langle\lambda + \rho, \beta\rangle > 0\}|.$$

Then $\mathcal{H}^q(G/T; \mathbb{E}_\chi) = 0$ for $q \neq q(\lambda + \rho)$, and G acts irreducibly on $\mathcal{H}^{q(\lambda+\rho)}(G/T; \mathbb{E}_\chi)$ by the relative discrete series representation π_χ.

An important variation on the Kostant–Langlands Conjecture result — which in fact preceded its solution — is the Dolbeault cohomology realization [10]. Note that K/T is a maximal compact complex submanifold of G/T and denote $s = \dim_{\mathbb{C}} K/T$. Whenever

(3.5) $\lambda + \rho$ is antidominant: $\langle\lambda + \rho, \gamma\rangle < 0$ for all $\gamma \in \Phi^+$

we have $s = q(\lambda + \rho)$. This is the case where the bundle $\mathbb{E}_\chi \to G/T$ is negative.

If π is a relative discrete series representation of G, we can choose the positive root system Φ^+ so that $\pi = \pi_\chi$ where $\lambda = d\chi$ is such that $\lambda + \rho$ is antidominant. Thus there is no restriction on π_χ in

3.6. THEOREM ([10], [13]). *Suppose that $\lambda + \rho$ is antidominant, so $s = q(\lambda + \rho)$. Then $H^q(G/T; \mathcal{O}(\mathbb{E}_\chi)) = 0$ for $q \neq s$. $H^s(G/T; \mathcal{O}(\mathbb{E}_\chi))$ has a natural Fréchet space structure, G acts on $H^s(G/T; \mathcal{O}(\mathbb{E}_\chi))$ by a continuous representation, and this representation is infinitesimally equivalent to π_χ.*

This is the result that we will transport to the setting of open orbits and hermitian symmetric spaces.

SECTION 4. ORBIT STRUCTURE OF HERMITIAN SYMMETRIC SPACES.

Let G be a general semisimple Lie group, θ a Cartan involution, and $K = G^\theta$ a maximal compactly embedded subgroup. Suppose that the riemannian symmetric space $M^0 = G^0/K^0$ has a G^0-invariant complex structure, i.e., is an hermitian symmetric space. Then $\mathfrak{g}$ is a θ-stable direct sum of simple ideals $\mathfrak{g}_i$ and, using (2.9b), $M^0 \cong \prod M_i^0$ global product of irreducible hermitian symmetric spaces, where $M_i^0 = G_i^0/K_i^0$. The dual compact hermitian symmetric space also decomposes, $W \cong \prod W_i$ with W_i of the form $Int(\mathfrak{g}_i)/Q_i$ for an appropriate parabolic subgroup $Q_i \subset Int(\mathfrak{g}_i)$ and where M_i^0 is realized as a convex open G_i^0-orbit, the orbit of the identity coset in W_i. Thus the complex flag manifold W, the bounded symmetric domain M^0 sitting in W as a convex open G^0-orbit, more generally all G^0-orbits on W, and the representations we will construct for G, all break up as products.

We can view W as the flag variety of all parabolic subalgebras of $\mathfrak{g}$ that are $Int(\mathfrak{g})$-conjugate to $\mathfrak{q}$. In view of (2.9b), G acts on W by conjugation. Let $M = G \cdot \mathfrak{q}$, open G-orbit in W with $M^0 = G^0 \cdot \mathfrak{q}$ as one of its topological components. Also by (2.9b), no element of G can permute the W_i nontrivially.

Now we will suppose that the Lie group G is simple. Then hermitian symmetric space $M \subset W$ is irreducible. So the identity component $K^0 = K \cap G^0$ has 1-dimensional center Z_{K^0}. Let $Z_{K^0}^0$ denote the identity component of Z_{K^0}. Either

(4.1a) K is the centralizer $Z_G(Z_{K^0}^0)$ of $Z_{K^0}^0$ in G and M is connected

or

(4.1b) $Z_G(Z_{K^0}^0)$ has index 2 in K and M has two topological components.

See Lemma 4.8 below. Then we have

(4.2) $T \subset K$: Cartan subgroup of G and Φ^+ : positive root system such that $\mathfrak{q} = \mathfrak{k} + \mathfrak{p}_-$ where $\mathfrak{p}_- = \sum_{\beta \in \Phi^+_{G/K}} \mathfrak{g}_{-\beta}$

In other words ([1], [17], [18]) the simple root system has form $\Sigma = \{\sigma_0, \cdots, \sigma_\ell\}$ where σ_0 is the only noncompact simple root and every root has form $\alpha = \sum_{i=0}^{\ell} n_i\sigma_i$ with $n_0 = 0$ or ± 1. Then

$$\mathfrak{k} = \mathfrak{t} + \sum_{n_0=0} \mathfrak{g}_\alpha, \quad \mathfrak{p}_- = \sum_{n_0=-1} \mathfrak{g}_\alpha, \text{ and } \mathfrak{p}_+ = \overline{\mathfrak{p}_-} = \sum_{n_0=+1} \mathfrak{g}_\alpha \tag{4.3}$$

with $\mathfrak{p}_+$ representing the holomorphic tangent space of W.

Every $\beta \in \Phi_{G/K}$ determines a three dimensional simple (TDS) subalgebra $\mathfrak{g}[\beta] \subset \mathfrak{g}$, isomorphic to $\mathfrak{sl}(2; \mathbb{C})$ under

$$e_\beta \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_{-\beta} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h_\beta \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under θ, where $\mathfrak{p} = \mathfrak{p}_- + \mathfrak{p}_+$ and the corresponding $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Then $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0$ is the compact real form of $\mathfrak{g}$, and

$$\begin{aligned} &\mathfrak{p}_0 \text{ has } \mathbb{R}\text{-basis: the } x_{\beta,0} = e_\beta + e_{-\beta} \text{ and } y_{\beta,0} = \sqrt{-1}(e_\beta - e_{-\beta}),\; \beta \in \Phi^+_{G/K} \\ &\sqrt{-1}\mathfrak{p}_0 \text{ has } \mathbb{R}\text{-basis: the } x_\beta = \sqrt{-1}(e_\beta + e_{-\beta}) \text{ and } y_\beta = -(e_\beta - e_{-\beta}),\; \beta \in \Phi^+_{G/K} \end{aligned} \tag{4.4}$$

Recall that roots β' and β'' are **strongly orthogonal** if neither of $\beta' \pm \beta''$ is a root. Kostant's downward cascade construction for a maximal set $\Psi = \{\psi_1, \cdots, \psi_\ell\}$ of strongly orthogonal noncompact positive roots is: ψ_1 is the maximal root, and ψ_{i+1} is maximal among the noncompact positive roots orthogonal to $\{\psi_1, \cdots, \psi_i\}$. Here the $x_{\psi,0}$, $\psi \in \Psi$, span the $\mathfrak{a}_0$ of an Iwasawa decomposition $G = KAN$.

For every subset $\Gamma \subset \Psi$ we have the **partial Cayley transforms**

$$c_\Gamma = \prod_{\gamma \in \Gamma} c_\gamma \text{ where } c_\gamma = exp(\tfrac{\pi}{4} y_\gamma) \in Int(\mathfrak{g}). \tag{4.5a}$$

Let x_0 denote the identity coset $1 \cdot \mathfrak{q} \in W$ and define

$$x_{\Gamma,\Delta} = c_\Gamma c_\Delta^2(x_0) \text{ for } \Gamma, \Delta \subset \Psi. \tag{4.5b}$$

Then ([14], [15])

4.6. THEOREM. *The G^0-orbits on W are just the $G^0(x_{\Gamma,\Delta})$ where Γ and Δ are disjoint subsets of Ψ. An orbit $G^0(x_{\Gamma',\Delta'})$ is in the closure of $G^0(x_{\Gamma,\Delta})$ if and only if the cardinalities $|\Delta' \setminus \Gamma'| \leqq |\Delta \setminus \Gamma|$ and $|\Delta \cup \Gamma| \leqq |\Delta' \cup \Gamma'|$. In particular*

(i) *$G^0(x_{\Gamma,\Delta}) = G^0(x_{\Gamma',\Delta'})$ if and only if $|\Delta' \setminus \Gamma'| = |\Delta \setminus \Gamma|$ and $|\Delta \cup \Gamma| = |\Delta' \cup \Gamma'|$,*
(ii) *The number of G^0-orbits on W is $\frac{1}{2}(\ell + 1)(\ell + 2)$,*
(iii) *$G^0(x_{\Gamma,\Delta})$ is open in W if and only if Γ is empty, so there are $\ell + 1$ open orbits, and*
(iv) *the boundary of an open orbit $G^0(x_{\emptyset,\Delta})$ is the union of the orbits $G^0(x_{\Gamma',\Delta'})$ such that $\Delta' \setminus \Gamma' \subset \Delta \subset \Delta' \cup \Gamma'$.*

The Bergman-Shilov boundary of M^0 in W is $G^0(x_{\Psi,\emptyset})$; it is in the closure of every orbit and is the unique closed orbit.

Now let us look at the G-orbit structure, extending (4.1). Recall $G^\dagger = Z_G(G^0)\cdot G^0$ and $K^\dagger = Z_G(G^0)\cdot K^0 = K \cap G^\dagger$ from §2. Note that $Z_G(G^0)$ acts trivially on W because it acts trivially on $\mathfrak{g}$, so

(4.7) G^0 and $G^\dagger$ have the same orbits on W: $\quad G^0(x_{\Gamma,\Delta}) = G^\dagger(x_{\Gamma,\Delta})$ for all $\Gamma, \Delta \subset \Psi$.

On the other hand,

4.8. **Lemma.** *$K^\dagger = Z_G(Z^0_{K^0})$, the centralizer of $Z^0_{K^0}$ in G, and either $G^\dagger = G$ or $G^\dagger$ has index 2 in G.*

Proof. For the first assertion it suffices to prove $Z_G(Z^0_{K^0}) \subset G^\dagger$, because $Z_G(Z^0_{K^0})$ meets G^0 in K^0. Let $k \in Z_G(Z^0_{K^0})$. We have $k_1 \in K^0$ such that kk_1 normalizes T^0 and preserves Φ^+_K. Since kk_1 centralizes $Z^0_{K^0}$ it also preserves $\Phi^+_{G/K}$. Now kk_1 centralizes T^0, so $kk_1 \in T \subset T\cdot G^0 = G^\dagger$. As $k_1 \in K^0$ it follows that $k \in G^\dagger$. We have proved $K^\dagger = Z_G(Z^0_{K^0})$. It follows that $G^\dagger = Z_K(Z^0_{K^0})\cdot G^0$. But $Z^0_{K^0}$ is a connected 1-dimensional group normal in K, so either it is central in K or its K-centralizer has index 2 in K. So, finally, $G^\dagger$ either is all of G or is a subgroup of index 2 in G. This completes the proof. QED

4.9. **Theorem.** *If $G = G^\dagger$ then the G-orbits on W are just the G^0-orbits, $G(x_{\Gamma,\Delta}) = G^0(x_{\Gamma,\Delta})$, as given in Theorem 4.6. If $G \neq G^\dagger$ then $G^\dagger$ has index 2 in G and the G-orbits on W are the $G^0(x_{\Gamma,\Delta}) \cup G^0(x_{\Gamma,\Psi\setminus(\Gamma\cup\Delta)})$ where $\Gamma, \Delta \subset \Psi$ are disjoint.*

Proof. The case $G = G^\dagger$ is (4.7). Now assume $G \neq G^\dagger$. For the proof we may divide out G by the kernel $Z_G(G^0)$ of its action on W. In other words, we assume $G \subset Int(\mathfrak{g})$.

In view of Lemma 4.8, there is an element $k_0 \in K$ that normalizes $\mathfrak{t}$ and conjugates any element of the center of $\mathfrak{k}$ to its negative. As is already implicit in (4.1), $G(x_0) = G^0(x_0) \cup G^0(k_0x_0)$. The orbit $G^0(k_0x_0)$ must be of the form $G^0(x_{\Gamma,\Delta})$ for some $\Gamma, \Delta \subset \Psi$. The isotropy subgroup of G^0 at $x_{\Gamma,\Delta}$ is conjugate to the isotropy subgroup at k_0x_0, thus compact. It follows ([8], [20], [15]) that $G^0(x_{\Gamma,\Delta}) = G^0(x_{\emptyset,\Psi})$ and that the bounded symmetric domain M is of tube type.

Since M is of tube type, the square c^2_Ψ of the Cayley transform normalizes G^0 in $Int(\mathfrak{g})$ and $k_0^{-1}c^2_\Psi$ centralizes $Z^0_{K^0}$. The subgroup $G \subset Int(\mathfrak{g})$ has two components, and by the same reasoning the subgroup $(G \cup c^2_\Psi G) \subset Int(\mathfrak{g})$ has two components. Thus $G = G^0 \cup c^2_\Psi G^0 = G^0 \cup G^0 c^2_\Psi$.

In view of Theorem 4.6 the G-orbits on W are of the form $G^0(x_{\Gamma,\Delta}) \cup G^0(c^2_\Psi x_{\Gamma,\Delta})$. Suppose that Γ and Δ are disjoint. Then $c^2_\Psi x_{\Gamma,\Delta} = c^4_\Delta c^3_\Gamma c^2_{\Psi\setminus(\Gamma\cup\Delta)} x_0$. But [20] c^4_Δ, $c^4_\Gamma \in K^0$. Thus $G^0(c^2_\Psi x_{\Gamma,\Delta}) = G^0(x_{\Gamma,\Psi\setminus(\Gamma\cup\Delta)})$. This completes the proof of the Theorem. QED

4.10. **Corollary.** *If the bounded symmetric domain M^0 is not of tube type, then every G-orbit on W is connected.*

4.11. **Corollary.** *Suppose that the bounded symmetric domain M^0 is of tube type. If $G = G^\dagger$ then every G-orbit on W is connected. Now suppose $G \neq G^\dagger$. Then an orbit $G(x_{\Gamma,\Delta})$ is connected if and only if $\Gamma = \emptyset$ and $|\Delta| = \frac{1}{2}|\Psi|$.*

When G is not necessarily simple, we obtain the G-orbit structure more or less directly from Theorem 4.9 and its Corollaries, as follows. G acts on each of the flag manifolds W_i by conjugation, inducing a group $G_i \subset Int(\mathfrak{g}_i)$ of transformations of W_i. Theorem 4.9 and its Corollaries apply directly to give the G_i-orbit structure of W_i. Let $\overline{G}$ denote the subgroup of $Int(\mathfrak{g})$ induced by G, as in the proof of Theorem 4.9. Then $\overline{G}$ is a subgroup of $\prod G_i$ and its orbits sit accordingly in $W = \prod W_i$.

Section 5. Realization of the Discrete Series: Hermitian Case.

Retain the setup of §4, in particular the Cartan subgroup $T \subset K$ of G and the positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of (4.2). Since T is compactly embedded we have relative discrete series representations of G.

Let X denote the flag variety of all Borel subalgebras of $\mathfrak{g}$. The open G-orbits on X are the orbits $G\cdot\mathfrak{b}$ such that $\mathfrak{b} \cap \mathfrak{g}_0$ is the Cartan subalgebra of $\mathfrak{g}_0$ corresponding to a compactly embedded Cartan subgroup of G; see [14]. Any such Cartan subgroup of G is G^0-conjugate to T. Thus

5.1. **Lemma.** *The open G-orbits on X are the $G\cdot\mathfrak{b}$ where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ of the form*

$$\mathfrak{b}(\Phi^+_0) = \mathfrak{t} + \textstyle\sum_{\alpha\in\Phi^+_0} \mathfrak{g}_{-\alpha}$$

where Φ^+_0 is a positive root system for $(\mathfrak{g}, \mathfrak{t})$.

Let π be a relative discrete series representation of G. One has a positive root system $\Phi^+_\chi = \Phi^+_\chi(\mathfrak{g}, \mathfrak{t})$ such that $\pi = \pi_\chi$ where $\lambda = d\chi$ and $\lambda + \rho_\chi$ is antidominant. Here ρ_χ denotes half the sum of the elements of Φ^+_χ. Denote

(5.2) $\qquad \mathfrak{b}_\chi = \mathfrak{b}(\Phi^+_\chi)$: Borel subalgebra $\mathfrak{t} + \sum_{\alpha\in\Phi^+_\chi} \mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$

Then we have the open orbit $G\cdot\mathfrak{b}_\chi \cong G/T$ in X.

Consider the G-equivariant holomorphic fibration

(5.3) $\qquad \phi : X \to W$ defined by $g\cdot\mathfrak{b}(\Phi^+) \mapsto g\cdot\mathfrak{q}$ for all $g \in Int(\mathfrak{g})$.

As $\phi : X \to W$ is open and there are only finitely many G-orbits on X (see [14]), the open G-orbits in W are just the $\phi(G \cdot \mathfrak{b}(\Phi_0^+))$ where Φ_0^+ ranges over the set of all positive root systems for $(\mathfrak{g}, \mathfrak{t})$. Thus we have holomorphic fibrations

(5.4a) $\phi : G \cdot \mathfrak{b}_\chi \to G \cdot \mathfrak{q}_\chi = G(x_{\emptyset,\Delta})$ with $\mathfrak{q}_\chi$ of the form $c_\Delta^2 \cdot \mathfrak{q} = x_{\emptyset,\Delta}$ for some $\Delta \subset \Psi$

In view of the results of §4, the fibration can be expressed in terms of coset spaces of G:

(5.4b) $\phi : G/T \to G/K_\Delta$, fibre K_Δ/T, with $\mathfrak{k}_\Delta = c_\Delta^2 \cdot \mathfrak{k}$.

Here K_Δ^0 is determined by $\mathfrak{k}_\Delta$, which is specified by (5.4b) because (see [20]) $c_\Delta^2 \cdot \mathfrak{k}$ is the complexification of its intersection with $\mathfrak{g}_0$. The group $K_\Delta = Z_G(G^0) \cdot K_\Delta^0$ unless $G^\dagger \neq G$ and $|\Delta| = \frac{1}{2}|\Psi|$; in that case $Z_G(G^0) \cdot K_\Delta^0$ has index 2 in K_Δ. See Corollary 4.11 above.

Note that the total space and the typical fibre of (5.4) are open orbits in flag varieties of Borel subalgebras. That is an important ingredient in the proof of

5.5. **Theorem.** *Let $s_\Delta = dim_{\mathbb{C}}(K/(K \cap K_\Delta))$, dimension of the maximal compact subvariety of $G/K_\Delta = G(x_{\emptyset,\Delta})$; and let $t_\Delta = dim_{\mathbb{C}}(K \cap K_\Delta)/T$, dimension of the maximal compact subvariety of K_Δ/T. Let $\mathbb{V}_{\eta_\chi} \to G(x_{\emptyset,\Delta})$ denote the homogeneous holomorphic Fréchet bundle associated to the relative discrete series Fréchet representation η_χ of K_Δ on $V_{\eta_\chi} = H^t(K_\Delta/T; \mathcal{O}(\mathbb{E}_\chi|_{K_\Delta/T}))$. Then $H^q(G/K_\Delta; \mathcal{O}(\mathbb{V}_{\eta_\chi})) = 0$ for $q \neq s_\Delta$. $H^{s_\Delta}(G/K_\Delta; \mathcal{O}(\mathbb{V}_{\eta_\chi}))$ has a natural Fréchet structure, the natural action of G on $H^{s_\Delta}(G/K_\Delta; \mathcal{O}(\mathbb{V}_{\eta_\chi}))$ is a continuous representation, and this representation is infinitesimally equivalent to π_χ.*

Proof. The vanishing statement in Theorem 3.6 says that $H^v(K_\Delta/T; \mathcal{O}(\mathbb{E}_\chi|_{K_\Delta/T})) = 0$ for $v \neq t_\Delta$. So the Leray spectral sequence of $G/T \to G/K_\Delta$ collapses at E_2,

$$E_2^{u,v} = H^u(G/K_\Delta, \mathcal{O}(\mathbb{H}^v(K_\Delta/T; \mathcal{O}(\mathbb{E}_\chi|_{K_\Delta/T})))) \text{ and } d_2 : E_2^{u,v} \to E_2^{u+2,v-1},$$

and

$$H^w(G/T; \mathcal{O}(\mathbb{E}_\chi)) = \sum_{u+v=w} H^u(G/K_\Delta, \mathcal{O}(\mathbb{H}^v(K_\Delta/T; \mathcal{O}(\mathbb{E}_\chi|_{K_\Delta/T})))).$$

Again by the vanishing statement in Theorem 3.6, the left side vanishes for $w \neq s_\Delta + t_\Delta$ and the right side vanishes for $v \neq t_\Delta$. So the result of the spectral sequence is

$$H^{s_\Delta + t_\Delta}(G/T; \mathcal{O}(\mathbb{E}_\chi)) = H^{s_\Delta}(G/K_\Delta, \mathcal{O}(\mathbb{H}^{t_\Delta}(K_\Delta/T; \mathcal{O}(\mathbb{E}_\chi|_{K_\Delta/T})))).$$

Appealing to Theorem 3.6 once more, we see that this is the content of our assertion. QED

Theorem 5.5 exhibits any relative discrete series representation π_χ of G in way that is reminiscent of Harish-Chandra's construction [2] of the holomorphic discrete series. Interpret the latter as holomorphic sections of negative homogeneous vector bundles over $M = G/K_\emptyset$, where $s_\Delta = 0$, to see that it is the case $\Delta = \emptyset$ of Theorem 5.5.

Section 6. Realization of the Discrete Series: General Case.

G is a general semisimple Lie group, θ is a Cartan involution of G, and $K = G^\theta$ a maximal compactly embedded subgroup. We assume that

(6.1) G has a Cartan subgroup $T \subset K$, i.e. has relative discrete series representations.

Fix a positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ and denote

(6.2) $\mathfrak{b}$: Borel subalgebra $\mathfrak{t} + \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$.

So $G \cdot \mathfrak{b} \cong G/T$ is an open G-orbit in the flag variety X of Borel subalgebras of $\mathfrak{g}$.

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. Then [14] we may replace $\mathfrak{q}$ by an $Int(\mathfrak{g})$-conjugate and assume

(6.3a) $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{q}$ and $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{l}_0$ where $\mathfrak{l}$ is a Levi component of $\mathfrak{q}$.

In particular we have a G-equivariant $Int(\mathfrak{g})$-equivariant holomorphic fibration

(6.3b) $\phi : X \to W$ defined by $g \cdot \mathfrak{b} \mapsto g \cdot \mathfrak{q}$ for all $g \in Int(\mathfrak{g})$.

The double coset space $W(\mathfrak{k}, \mathfrak{t}) \backslash W(\mathfrak{g}, \mathfrak{t}) / W(\mathfrak{l}, \mathfrak{t})$ of Weyl groups parameterizes the open G-orbits on W. So the open orbits don't have as clean a parameterization in the general case as in the hermitian case. Still, $\phi : X \to W$ is open and the number of G-orbits is finite, so the open G-orbits in W are the ϕ-images of the open G-orbits in X. For example, the open orbit $G \cdot \mathfrak{q} \cong G/L$ is just $\phi(G \cdot \mathfrak{q}) \cong G/T$. Thus, we have holomorphic fibrations of open G-orbits in X as in (5.4),

(6.4a) $\phi : G \cdot \mathfrak{b}_\chi \to G \cdot \mathfrak{q}_\chi$ for all $\chi \in \widehat{T}$ with $\pi_\chi \in \widehat{G}_{disc}$.

Here we choose $w_\chi \in W(\mathfrak{g}, \mathfrak{t})$ such that $w_\chi \cdot \mathfrak{b}(\Phi^+) = \mathfrak{b}_\chi$ and define $\mathfrak{q}_\chi = w_\chi \cdot \mathfrak{q}$. Since $\mathfrak{t} \subset \mathfrak{q}_\chi$, the Levi component $w_\chi \cdot \mathfrak{l}$ of $\mathfrak{q}_\chi$ is the complexification if its intersection with $\mathfrak{g}_0$. It follows that (6.4a) has coset space expression

(6.4b) $\phi : G/T \to G/L_\chi$, fibre L_χ/T, where $\mathfrak{l}_\chi = w_\chi \cdot \mathfrak{l}$.

Here L_χ^0 is determined by $\mathfrak{l}_\chi$, which is specified by (6.4b) because (see [20]) $\mathfrak{l}_\chi$ is the complexification of its intersection with $\mathfrak{g}_0$. The group L_χ is the normalizer in G of the nilradical of $\mathfrak{q}$, so its components are just the cosets gL_χ^0 where $g \in G$ normalizes T and represents an element of the Weyl group that preserves the set of roots for the root spaces contained in the nilradical of $\mathfrak{q}$. See [14] for the techniques to make this explicit.

Now we have the general result corresponding to Theorem 5.5:

6.5. **Theorem.** *Let $s_\chi = \dim_{\mathbb{C}}(K/(K \cap L_\chi)$, dimension of the maximal compact subvarieties of G/L_χ; and let $t_\chi = \dim_{\mathbb{C}}(K \cap L_\chi)/T$, dimension of the maximal compact subvarieties of L_χ/T Let $\mathbb{V}_{\eta_\chi} \to G \cdot \mathfrak{q}_\chi$ denote the homogeneous holomorphic Fréchet vector bundle associated to the representation η_χ of L_χ on $V_{\eta_\chi} = H^{t_\chi}(L_\chi/T; \mathcal{O}(\mathbb{E}_\chi|_{L_\chi/T}))$. Then $H^q(G/L_\chi; \mathcal{O}(\mathbb{V}_{\eta_\chi}))$ vanishes for $q \neq s_\chi$. $H^{s_\chi}(G/L_\chi; \mathcal{O}(\mathbb{V}_{\eta_\chi}))$ has a natural Fréchet space structure, the natural action of G on $H^{s_\chi}(G/L_\chi; \mathcal{O}(\mathbb{V}_{\eta_\chi}))$ is a continuous representation, and this representation is infinitesimally equivalent to π_χ.*

The proof of Theorem 6.5 is the same as that of Theorem 5.5. This general result does not have the geometric interest of the hermitian case because the setting is not as natural. There is, however, a special case (described just below) that at least reflects the structure of the symmetric space G/K. And there is some reason to expect that Theorem 6.5 is of analytic interest as the general setting for constructions in indefinite metric geometric quantization [9].

Suppose that the Lie group G is simple but that the irreducible symmetric space M^0 is not necessarily hermitian. Then ([1], [17], [15]) there is a simple root system $\Sigma = \{\sigma_0, \cdots, \sigma_t\}$ where σ_0 is the only noncompact simple root. The maximal root is of the form $\mu = \sigma_0 + \sum_{1 \leq i \leq t} n_i \sigma_i$ if M^0 is hermitian, $\mu = 2\sigma_0 + \sum_{1 \leq i \leq t} n_i \sigma_i$ if M^0 is not hermitian. Write α for a root $\sum_{1 \leq i \leq t} \mathfrak{g}_\alpha$. We have decompositions

$$\text{(6.6)} \qquad \begin{aligned} &\mathfrak{l} = \mathfrak{t} + \sum\nolimits_{n_0 = 0} \mathfrak{g}_\alpha, \text{ and } \mathfrak{g} = \mathfrak{l} + \mathfrak{r} \text{ where } \mathfrak{r} = \sum\nolimits_{n_0 \neq 0} \mathfrak{g}_\alpha, \text{ and} \\ &\mathfrak{k} = \mathfrak{l} + \mathfrak{s} \text{ where } \mathfrak{s} = \sum\nolimits_{n_0 = \pm 2} \mathfrak{g}_\alpha, \text{ and } \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \text{ where } \mathfrak{p} = \sum\nolimits_{n_0 = \pm 1} \mathfrak{g}_\alpha. \end{aligned}$$

Of course $\mathfrak{k} = \mathfrak{l}$, $\mathfrak{p} = \mathfrak{r}$ and $\mathfrak{s} = 0$ in the hermitian case. In any case, these give us parabolic subalgebras

$$\text{(6.7)} \qquad \begin{aligned} &\mathfrak{q}_K = \mathfrak{l} + \mathfrak{s}_- \subset \mathfrak{k} \text{ where } \mathfrak{s}_\pm = \sum\nolimits_{n_0 = \pm 2} \mathfrak{g}_\alpha, \text{ and} \\ &\mathfrak{q} = \mathfrak{l} + \mathfrak{r}_- \subset \mathfrak{g} \text{ where } \mathfrak{r}_\pm = \mathfrak{s}_\pm + \sum\nolimits_{n_0 = \pm 1} \mathfrak{g}_\alpha. \end{aligned}$$

If we choose Φ^+ to be the positive root system corresponding to Σ, and use the parabolic $\mathfrak{q}$ of (6.7) to define the flag variety W, then the setting of Theorem 6.5 is related to the geometry of the riemannian symmetric space G/K.

Section 7. Indefinite Metric Quantization.

Theorems 5.5 and 6.5 provide an hermitian symmetric space version and a general flag manifold version of Theorem 3.6, the Dolbeault cohomology realization of the relative discrete series. Now we look at the possibility of proving an hermitian symmetric space version or a general flag manifold version of the L^2 cohomology realization, corresponding to Theorem 3.4.

Consider an open orbit $Y = G \cdot \mathfrak{q}_\chi \cong G/L$ as in Theorem 6.5. We want to study the square integrable $\mathbb{V}_\eta$-valued harmonic differential forms on G/L where η is a unitary representation of L, where V_χ is the representation space, and where $\mathbb{V}_\eta \to G/L$ is the associated holomorphic homogeneous hermitian vector bundle.

There are some serious problems here, solved only under restrictive additional conditions; see [9], [18] and [19]. The notion of square integrability is clear for sections of $\mathbb{V}_\eta$, but one must clarify it for $\mathbb{V}_\eta$-valued differential forms of higher degree. There are several obvious candidates for the notion of harmonic, and one must decide on the appropriate one. Then there are nondegeneracy and signature questions for the global inner product induced on spaces of square integrable harmonic $\mathbb{V}_\eta$-valued forms. Finally, the relation to Dolbeault cohomology is not obvious.

The problem of square integrability can be settled as in [9, §7]. The flag variety W in which Y sits as an open G-orbit has a positive definite hermitian metric that is invariant under the action of a compact real form of $Int(\mathfrak{g})$. This auxiliary positive definite hermitian metric is not G-invariant in general, but one can see that G does not distort it too much. Specifically, the argument of [9, Lemma 7.3] holds in our somewhat more general case and proves

7.1. **Lemma.** *If $g \in G$ then its tangent space maps on Y are uniformly bounded with respect to the auxiliary positive definite hermitian metric, with bound continuous in g.*

Let ω be a measurable $\mathbb{V}_\eta$-valued differential form on $Y \cong G/L$. Suppose that ω is square integrable in the sense that the pointwise norm, relative to the Hilbert space structure of V_η and the auxiliary positive definite hermitian metric on Y, is square integrable. Then its image by $g \in G$ has the same property, by the uniformity of the bound of Lemma 7.1. So Lemma 7.1 can be reformulated as

7.2. **Lemma.** *Let μ_Y be the measure on Y defined by the auxiliary positive definite hermitian metric. Given integers $p, q \geqq 0$, the natural action of G on bundle-valued (p,q)-forms induces a bounded representation of G on the Hilbert space $L_2^{(p,q)}(G/L; \mathbb{V}_\eta; \mu_Y)$ of $\mathbb{V}_\eta$-valued (p,q)-forms on Y that are square integrable with respect to μ_Y.*

The notion of "harmonic" is a little more subtle. This time we use the G-invariant indefinite-hermitian metric on Y and the corresponding Kodaira-Hodge-Laplace operator $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on $\mathbb{V}_\eta$-valued differential forms. Thus $\bar\partial^*$ is adjoint to $\bar\partial$ relative to the G-invariant metric, and $\square$ is G-invariant. In order of increasing severity, several candidates for harmonicity of a $\mathbb{V}_\eta$-valued form ω are

$$\text{(7.3)} \qquad \begin{aligned} &\square^n(\omega) = 0 \text{ for some integer } n > 0 && \text{(generalized weakly harmonic)} \\ &\square(\omega) = 0 && \text{(weakly harmonic)} \\ &\bar\partial(\omega) = 0 \text{ and } \bar\partial^*(\omega) = 0 && \text{(strongly harmonic)} \end{aligned}$$

Since our priority is construction of irreducible representations, we look for strongly harmonic $\mathbb{V}_\eta$-valued forms. So we look to

$$(7.4)\quad \begin{aligned}&\tilde{\mathcal{H}}^{(p,q)}(G/L;\mathbb{V}_\eta): \text{strongly harmonic } L^2(G/L,\mu_Y)\ \mathbb{V}_\eta\text{-valued } (p,q)\text{-forms on } G/L,\\ &\tilde{\pi}^{(p,q)}_\eta: \text{representation of } G \text{ on } \tilde{\mathcal{H}}^{(p,q)}(G/L;\mathbb{V}_\eta).\end{aligned}$$

Consider the map from harmonic forms to Dolbeault cohomology,

$$(7.5)\qquad \phi: \tilde{\mathcal{H}}^{(p,q)}(G/L;\mathbb{V}_\eta) \to H^{(p,q)}(G/L;\mathcal{O}(\mathbb{V}_\eta)) \text{ given by } \omega \mapsto [\omega].$$

In [9] we proved, in certain circumstances, that

(7.6a) ϕ maps the subspace $\tilde{\mathcal{H}}^s(G/L;\mathbb{V}_\eta)_{(K)}$ of K-finite L^2 harmonic $(0,s)$-forms onto the subspace $H^s(G/L;\mathcal{O}(\mathbb{V}_\eta))_{(K)}$ of K-finite Dolbeault cohomology classes,

(7.6b) the kernel of ϕ there is the kernel of the global inner product on $\tilde{\mathcal{H}}^s(G/L;\mathbb{V}_\eta)_{(K)}$,

(7.6c) the G-invariant global hermitian inner product induced on $H^s(G/L;\mathcal{O}(\mathbb{V}_\eta))_{(K)}$ is (positive or negative) definite.

When one has this, the geometric setting corresponding to Theorem 5.5 or Theorem 6.5 provides the unitary structure as well as the underlying Harish-Chandra module for the representation of G on Dolbeault cohomology.

7.7. PROPOSITION. *In the setting of Theorem 6.5, suppose that*

(i) *the G-invariant global hermitian inner product on $\tilde{\mathcal{H}}^{s_\chi}(G/L_\chi;\mathbb{V}_{\eta_\chi})_{(K)}$ is not identically zero and*

(ii) *$\psi \in L_2^{(0,s_\chi)}(G/L_\chi;\mathbb{V}_{\eta_\chi};\mu_Y), \bar\partial\psi = 0$ and $\Box\psi = 0$ imply $\bar\partial^*\psi = 0$ (i.e. weakly harmonic and $\bar\partial$-closed imply strongly harmonic in $L_2^{(0,s_\chi)}(G/L_\chi;\mathbb{V}_{\eta_\chi};\mu_Y)$).*

Then (7.6) holds, so the G-invariant global hermitian inner product on $\tilde{\mathcal{H}}^{s_\chi}(G/L_\chi;\mathbb{V}_{\eta_\chi})$ induces the pre Hilbert space structure on $H^{s_\chi}(G/L_\chi;\mathcal{O}(\mathbb{V}_{\eta_\chi}))_{(K)}$ with respect to which π_χ is unitary.

PROOF. Express Y as an increasing union of open submanifolds Y_n with compact closure. If $\omega \in \tilde{\mathcal{H}}^{s_\chi}(G/L_\chi;\mathbb{V}_\eta)$ and if $\psi \in \tilde{\mathcal{H}}^{s_\chi}(G/L_\chi;\mathbb{V}_\eta)$ represents the zero Dolbeault class then $\langle\psi,\omega\rangle = \lim_{n\to\infty}\langle\psi_n,\omega\rangle$ where $\psi = \bar\partial\beta$, β_n is the truncation of β on Y_n so $\beta = \lim_{n\to\infty}\beta_n$, and $\psi_n = \bar\partial\beta_n$. But $\langle\psi_n,\omega\rangle = \langle\bar\partial\beta_n,\omega\rangle = \langle\beta_n,\bar\partial^*\omega\rangle = 0$. Now $\langle\psi,\omega\rangle = 0$. We have shown that the kernel of ϕ is contained in the kernel of the G-invariant global hermitian inner product on $\tilde{\mathcal{H}}^{s_\chi}(G/L_\chi;\mathbb{V}_\eta)$. So that global inner product induces a well defined G-invariant inner product on the K-finite part $H^{s_\chi}(G/L_\chi;\mathcal{O}(\mathbb{V}_\eta))_{(K)}$ of the Dolbeault cohomology.

The irreducibility of Theorem 6.5 shows that the universal enveloping algebra $\mathfrak{G} = \mathcal{U}(\mathfrak{g})$ acts irreducibly on $H^{s_\chi}(G/L_\chi;\mathcal{O}(\mathbb{V}_\eta))_{(K)}$. So the induced global inner product is a real multiple of the positive definite inner product obtained by viewing $H^{s_\chi}(G/L_\chi;\mathcal{O}(\mathbb{V}_\eta))_{(K)}$ as the underlying Harish-Chandra module for the unitary representation π_χ. But (i) ensures that it is not identically zero. That simultaneously completes the proofs of (7.6) and of Proposition 7.7. QED

REFERENCES

1. A. Borel, J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221.
2. Harish–Chandra, *Representations of semisimple Lie groups*, VI, Amer. J. Math. **78** (1956), 564–628.
3. ————, *Discrete series for semisimple Lie groups*, II, Acta Math. **116** (1966), 1–111.
4. ————, *Harmonic analysis on real reductive groups*, I, J. Functional Analysis **19** (1975), 104–204.
5. ————, *Harmonic analysis on real reductive groups*, II, Inventiones Math. **36** (1976), 1–55.
6. ————, *Harmonic analysis on real reductive groups*, III, Annals of Math. **104** (1976), 117–201.
7. R. A. Herb and J. A. Wolf, *The Plancherel theorem for general semisimple Lie groups*, Compositio Math. **57** (1986), 271–355.
8. A. Korányi and J. A. Wolf, *Realization of hermitian symmetric spaces as generalized half-planes*, Annals of Math. **81** (1965), 265–288.
9. J. Rawnsley, W. Schmid and J. A. Wolf, *Singular unitary representations and indefinite harmonic theory*, J. Functional Analysis **51** (1983), 1–114.
10. W. Schmid, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Thesis, University of California at Berkeley, 1967.
11. ————, *On a conjecture of Langlands*, Annals of Math. **93** (1971), 1–42.
12. ————, *L^2-cohomology and the discrete series*, Annals of Math. **103** (1976), 375–394.
13. W. Schmid and J. A. Wolf, *Geometric quantization and derived functor modules for semisimple Lie groups*, J. Functional Analysis, to appear in 1990.
14. J. A. Wolf, *The action of a real semisimple Lie group on a complex manifold, I: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

15. ———, *Fine structure of hermitian symmetric spaces*, in "Symmetric Spaces: Short Courses Presented at Washington University," ed. Boothby & Weiss, 271-357, Marcel Dekker Inc., 1972.

16. ———, *The action of a real semisimple Lie group on a complex manifold, II: Unitary representations on partially holomorphic cohomology spaces*, Memoirs Amer. Math. Soc. **138** (1974).

17. ———, "Spaces of Constant Curvature (5th ed)," Publish or Perish, Wilmington, 1984.

18. ———, *Geometric quantization in the spirit of Gupta and Bleuler*, in "Differential Geometrical Methods in Mathematical Physics XI (Proceedings, Jerusalem, 1982)," 213–224, Reidel, Dordrecht, 1984.

19. ———, *Indefinite harmonic theory and unitary representations*, in "Applications of Group Theory in Physics and mathematical Physics (Proceedings, Chicago, 1982)," 289–294, Lectures in Applied Mathematics **21**, Amer. Math. Soc., 1985.

20. J. A. Wolf and A. Korányi, *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939.

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