Observability and Harish-Chandra Modules

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Abstract.

In an earlier note [10] we interpreted some questions of discrete observability of finite linear systems $dx/dt = Ax$ in terms of finite dimensional group representation theory. The main result said that a certain sort of observability can be cast into the language of group representation theory. Then, discrete observability comes down to whether the representation in question is cocyclic (dual to a cyclic representation) with the observation set up as a cocyclic vector (cyclic for the dual representation). Here we describe a setting in the representation theory of semisimple Lie groups where analogous results hold for infinite linear systems.

1. The Representation-Theoretic Interpretation of Observability.

In this section we recall the principal results of [10] connecting discrete observability and group representation theory.

1.1. Definition. Let π be a representation of a group G on a vector space V of dimension $n < \infty$. Fix a vector $x_0 \in V$, a (co)vector c' in the linear dual space V' of V, and a subset $S = \{g_1, ..., g_n\} \subset G$. The triple (π, c', S) is **discretely observable** if we can always solve for x_0 in the system of equations

$$c' \cdot \pi(g_i)x_0 = e_i, \qquad 1 \leqq i \leqq n \tag{1.2}$$

Discrete observability of (π, c', S) is equivalent to nonsingularity of the matrix

$$M = M(\pi, c', S) = \begin{pmatrix} c' \cdot \pi(g_1) \\ \vdots \\ c' \cdot \pi(g_n) \end{pmatrix} . \tag{1.3}$$

The notion of discrete observability for a linear system $dx/dt = Ax$ with constant coefficients, corresponds to the case of a 1-parameter linear group, where G is the additive group of real numbers, A is an $n \times n$ matrix,

$\pi(t) = exp(tA)$, and $g_i = t_i$ for some real numbers $t_1, \dots, t_n$, so that $\pi(g_i) = exp(t_i A)$. See [6].

This interpretation has a useful formulation [10]:

1.4. Theorem. Let π' denote the dual of π , representation of G on the linear dual V' of V. Let H denote the subgroup of G generated by S. If (π, c', S) is discretely observable then c' is a cyclic vector for $\pi'|_H$.

In particular, in Theorem 1.4, c' is a cyclic vector for π', so π' is a cyclic representation, i.e. π is a **cocyclic representation.**

1.5. Corollary. There exist $c' \in V'$ and $S \subset G$ such that (π, c', S) is discretely observable, if and only if the representation π is cocyclic.

In order to be able to use this result, we proved [10]

1.6. Theorem. Let π represent a group G on a finite dimensional vector space over a field $\mathbf{F}$. Then π is cocyclic if and only if every $\mathbf{F}$–irreducible summand of the maximal semisimple subrepresentation of π has multiplicity bounded by its $\mathbf{F}$–degree.

2. Harish-Chandra's K-Multiplicity Theorem.

In this section we describe certain results from the representation theory of semisimple[1] Lie groups. These results give a multiplicity bound much like that in Theorem 1.6.

Let G be a connected semisimple Lie group with finite center. Every compact subgroup of G is contained in a maximal compact subgroup, and any two maximal compact subgroups are conjugate. Now fix a maximal compact subgroup $K \subset G$; because of the conjugacy it doesn't matter which one we use.

2.1. Definitions. Let π be a representation of K on a complex vector space V. A vector $v \in V$ is called K-**finite** if $\pi(K) \cdot v$ is contained in a finite dimensional subspace of V. A subspace $U \subset V$ is called K-**isotypic** if it is $\pi(K)$-invariant, if the resulting action of K on U is a direct sum of copies of some irreducible representation of K, and if U is not properly contained in a larger subspace of V with those properties. If ψ is the irreducible representation of K in question, then U is called the ψ-**isotypic component** of V, and the representation of K on U is called the ψ-**isotypic component** of π.

Let $\mathfrak{g}_0$ denote the (real) Lie algebra of G and $\mathfrak{g}$ its complexification.

[1]The results of this section are true in somewhat greater generality than the setting described here. See the Appendix.

Similarly $\mathfrak{k}_0$ will be the (subalgebra of $\mathfrak{g}_0$ that is the) real Lie algebra of K and $\mathfrak{k}$ is the complexification of $\mathfrak{k}_0$.

2.2. Definition. A **$(\mathfrak{g}, K)$-module** is a complex vector space V that is simultaneously a $\mathfrak{g}$-module and a K-module, say through representations

$$\pi : \mathfrak{g} \longrightarrow End(V) \quad \text{and} \quad \pi : K \longrightarrow End(V)$$

in such a way that (i) every vector $v \in V$ is K-finite, (ii) the differential of π as a representation of K coincides with the $\mathfrak{k}$-restriction of π as a representation of $\mathfrak{g}$, and (iii) if $k \in K$ and $\xi \in \mathfrak{g}$ then $\pi[Ad(k)\xi] = \pi(k)\cdot\pi(\xi)\cdot\pi(k)^{-1}$.

2.3. Definitions. By **Harish-Chandra module** for $(\mathfrak{g}, K)$ we mean a $(\mathfrak{g}, K)$-module in which the K-isotypic subspaces are finite dimensional. A Harish-Chandra $(\mathfrak{g}, K)$-module V is **irreducible** if it is irreducible as a $\mathfrak{g}$-module, **indecomposable** if it is indecomposable as a $\mathfrak{g}$-module, **cyclic** if it is cyclic as a $\mathfrak{g}$-module, etc.

The point of these definitions is a celebrated series of foundational results of Harish-Chandra, a few of which can be summarized as follows.

2.4. Theorem. Let π be an irreducible unitary representation of G, say on the Hilbert space V_π, and let V be the space of all K-finite vectors in V_π. Then V dense in V_π and V is an irreducible Harish-Chandra module for $(\mathfrak{g}, K)$.

2.5. Theorem[2]**.** Let V be an irreducible Harish-Chandra module for $(\mathfrak{g}, K)$. Let π denote the representation of K on V. If ψ is any irreducible representation of K and if U is the ψ-isotypic component of V, then $\dim(U) \leqq \deg(\psi)^2$, that is, the multiplicity of ψ in π is bounded by the degree of ψ.

One needs somewhat more than plain topological irreducibility of a continuous representation π of G, say on a complete locally convex topological vector space (or even a Banach space) V_π, for the sort of result just described. The appropriate general notion is that of topologically completely irreducible (TCI) representation. One proves that π is TCI if and only if the space V of all K-finite vectors in V_π is an irreducible $(\mathfrak{g}, K)$ Harish-Chandra module and is dense in V_π. See [7] or [9]. In the context of semisimple groups it is usually more convenient to use the notion of admissible representation: π is **admissible** if V is dense in V_π and V is a $(\mathfrak{g}, K)$ Harish-Chandra module. One can prove that every $(\mathfrak{g}, K)$ Harish-Chandra module is the space of all K-finite vectors for an admissible representation of G.

[2]This is due to Harish-Chandra for linear groups as an easy consequence of his Subquotient Theorem [1]. For non-linear groups Harish-Chandra proved $\dim(U) \leqq c_\pi \cdot \deg(\psi)^2$, for some integer $c_\pi \geqq 1$. That is not quite good enough for our purposes. Later Lepowsky gave an algebraic argument [5] for Theorem 2.5, and more recently Casselman proved a Submodule Theorem [1] which strengthens the Subquotient Theorem so that Theorem 2.5 follows easily.

The connection between unitary representations, Harish-Chandra modules, and discrete observability, is given by comparing the multiplicity statements in Theorems 1.6 and 2.5. One concludes, for example,

2.6. Theorem. Let V be an irreducible Harish-Chandra module for $(\mathfrak{g}, K)$, let W be any finite dimensional K-invariant subspace, and let ϕ denote the representation of K on W. Then the representation ϕ is cocyclic. In other words, there exist $c' \in W'$ and $S \subset K$ such that $(\phi,\ c',\ S)$ is discretely observable.

3. Approximate Observability.

Let V be an irreducible $(\mathfrak{g}, K)$ Harish-Chandra module. Write $\widehat{K}$ for the unitary dual of K, i.e. the (set of equivalence classes of) irreducible unitary representations. Given a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ and a root ordering, $\psi \in \widehat{K}$ is specified by its highest weight $\nu \in \sqrt{-1}\mathfrak{t}_0^*$, which we abbreviate by $\psi = \psi_\nu$. Given $m \geqq 0$ we have the finite set

$$\widehat{K}_m = \{\psi_\nu \in \widehat{K} \quad | \quad ||\nu|| \leqq m\}$$

of representations of K. For each $\psi_\nu \in \widehat{K}$ let $V[\nu]$ denote the ψ_ν-isotypic subspace of V. Then $m \geqq 0$ specifies a finite dimensional K-invariant subspace

$$V_m = \sum\nolimits_{\psi_\nu \in \widehat{K}_m} V[\nu].$$

We are going to obtain a variation on Theorem 2.6 for V by applying that theorem to the V_m as $m \to \infty$.

We start by realizing V as the underlying Harish-Chandra module of a TCI Banach representation π of G on a Hilbert space V_π, in such a way that $\pi|_K$ is unitary. This is a standard procedure, using Casselman's Submodule Theorem [1] (which strengthens Harish-Chandra's Subquotient Theorem [2]) to locate V as a submodule of the Harish-Chandra module underlying a nonunitary principal series[3] representation of G. Let π' denote the dual representation. Its representation space is $V_{\pi'} = V_\pi'$, and the subspace V'

[3]The "principal series" or "unitary principal series" of G consists of the representations of the form $Ind_P^G(\mu \otimes \alpha)$ where $P = MAN$ is a minimal parabolic subgroup of G, where A is the vector group part of a maximally noncompact Cartan subgroup of G and α is a unitary character on A, where μ is an irreducible representation of the centralizer M of A in K, and where N is a certain nilpotent normal subgroup of P. Since M is compact, μ is finite dimensional and may be assumed to be unitary. Implicitly $\mu \otimes \alpha$ is extended from MA to $P = MAN$ by triviality on N. The "nonunitary principal series" is obtained by dropping the requirement that α be unitary, i.e. by taking α to be any 1-dimensional complex representation of A. In any case, $Ind\,_P^G(\mu \otimes \alpha)|_K = Ind_M^K(\mu)$ and thus is unitary.

of K-finite vectors is the Harish-Chandra module dual to V. The finite dimensional subspace $(V')_m$ is naturally identified with the dual $(V_m)'$ of V_m, so we simply denote it by V'_m.

The cardinality of $\widehat{K}_m$ is bounded by a polynomial $p(m)$ because highest weights ν are confined to a lattice in $\sqrt{-1}\mathfrak{t}_0^*$. So it is easy to see

3.1. Lemma. Choose cyclic vectors $c'_\nu \in V'[\nu]$, for every $\psi_\nu \in \widehat{K}$. Then the c'_ν can be rescaled so that $\sum c'_\nu$ converges absolutely in V'_π.

With this in mind, we define

3.2. Definition. Let π by a TCI Banach representation of G such that the space V of K-finite vectors in V_π is a $(\mathfrak{g}, K)$ Harish-Chandra module. A vector $c \in V_\pi$ is **approximately cyclic** for K if $c = \sum c_\nu$, absolutely convergent in V_π, where each c_ν is a K-cyclic vector in $V[\nu]$. A vector $c' \in V'_\pi$ is **approximately cocyclic** for K if $c' = \sum c'_\nu$, absolutely convergent in V'_π, where each c'_ν is a K-cyclic vector in $V'[\nu]$.

3.3. Definition. Let π be a TCI Banach representation of G. Fix $c' \in V'_\pi$. Then (π, c') is **approximately discretely observable** for K just when $c' = \lim c'_m$ absolutely convergent with $c'_m \in V'_m$, and we have an increasing sequence of subsets $S_m \subset K$ with cardinality $|S_m| = dim V'_m$, so that we can always solve the system of equations

$$c' \cdot \pi(g_i)x_0 = e_i, \qquad 1 \leqq i \leqq n$$

for $x_m \in V_m$.

The idea of Definition 3.3 is that, in a clearly measured way, one can come as close as desired to observability – at the price of sufficiently many observations. Now Theorem 2.6 and Lemma 3.1 combine to yield

3.4. Theorem. Let π be a TCI Banach representation of G. Then π' is approximately cocyclic. Let $c' \in V'_\pi$ be an approximately cocyclic vector. Then (π, c') is approximately discretely observable.

Appendix. K-Multiplicities for General Semisimple Groups.

In this Appendix we indicate how the results of §2 extend to a class of reductive Lie groups that contains all connected semisimple groups and all groups of Harish-Chandra class.

The **general semisimple groups** studied in [3], [4] and [8] are the reductive Lie groups G (i.e. $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ with $\mathfrak{s}$ semisimple and $\mathfrak{z}$ commutative) that satisfy the conditions

(A.1) G has a normal abelian subgroup Z which centralizes the identity component G^0 of G and such that $Z \cdot G^0$ has finite index in G, and

(A.2) if $x \in G$ then conjugation $Ad(x)$ is an **inner** automorphism on the complexified Lie algebra $\mathfrak{g}$.

This is a convenient class in which to do representation theory.

Fix a general semisimple group G. There is no loss of generality in expanding Z to $Z \cdot Z_{G^0}$ where Z_{G^0} is the center of G^0.

Let $Z_G(G^0)$ denote the centralizer of G^0 in G. Denote $G^\dagger = Z_G(G^0) \cdot G^0$. Many arguments for a general semisimple group G go from G^0 to $G^\dagger$ to G.

The analog of maximal compact subgroup for G^0 is just the full inverse image K^0 of a maximal compact subgroup in the connected linear semisimple Lie group G^0/Z_{G^0}. The analog of maximal compact subgroup for $G^\dagger$ is just $K^\dagger = Z_G(G^0) \cdot K^0$, which in fact is the full inverse image of a maximal compact subgroup in $G^0/Z_{G^0} = G^\dagger/Z_G(G^0)$. The analog of a maximal compact subgroup K for G can be equivalently defined as the G-normalizer of K^0, the G-normalizer of $K^\dagger$, or the full inverse image of a maximal compact subgroup in G/Z or in $G/Z_G(G^0)$. We refer to these groups K, $K^\dagger$ and K^0 respectively as **maximal compactly embedded subgroups** of G, $G^\dagger$ and G^0. If Z is compact, they are just the maximal compact subgroups.

By **Cartan involution** of G we mean an involutive automorphism whose fixed point set is a maximal compactly embedded subgroup. All the standard results hold: every maximal compactly embedded subgroup of G is the fixed point set of a unique Cartan involution, and every Cartan involution of $\mathfrak{g}_0$ extends uniquely to a Cartan involution of G. See [8].

A technique developed in [8] reduces the proofs of Theorems 2.4 and 2.5 for connected reductive Lie groups G^0 to the case where Z_{G^0} is compact, and there one can use Harish-Chandra's arguments without change.

Passage from G^0 to $G^\dagger$ is based on two straightforward facts.

(A.3) The irreducible representations of $G^\dagger$ are just the $\pi^\dagger = \xi \otimes \pi^0$ where ξ is an irreducible, necessarily finite dimensional, representation of $Z_G(G^0)$, where π^0 is an ireducible representation of G^0, and where ξ and π^0 agree on Z_{G^0}.

(A.4) The irreducible subrepresentations of $\pi^\dagger|_{K^\dagger}$ are just the $\psi^\dagger = \xi \otimes \psi^0$ where ξ is the irreducible finite dimensional representation of $Z_G(G^0)$ mentioned above, and where ψ^0 is an irreducible representation of $\pi^0|_{K^0}$.

In Theorem 2.4 now $V_{\pi^\dagger} = E_\xi \otimes V_{\pi^0}$. Since the representation space E_ξ of ξ is finite dimensional, the spaces of $K^\dagger$-finite and K^0-finite vectors are related by $V^\dagger = E_\xi \otimes V^0$. The validity of the assertion passes directly from G^0 to $G^\dagger$. In Theorem 2.5 the Harish-Chandra modules are related by $V^\dagger = E_\xi \otimes V^0$, so again the result for $(\mathfrak{g}, K^0)$ Harish-Chandra modules implies the result for $(\mathfrak{g}, K^\dagger)$ Harish-Chandra modules.

Passage from $G^\dagger$ to G uses a variation on the classical Schur's Lemma.

(A.5) If $\pi^\dagger$ is an irreducible unitary representation of $G^\dagger$ then the induced representation $Ind_{G^\dagger}^G(\pi^\dagger)$ is a finite sum of irreducible unitary representations of G. If π is an irreducible unitary representation of G then $\pi|_{G^\dagger}$ is a finite sum of irreducible unitary representations of $G^\dagger$. The multiplicity of π in $Ind_{G^\dagger}^G(\pi^\dagger)$ is equal to the multiplicity of $\pi^\dagger$ in $\pi|_{G^\dagger}$.

Let π be an irreducible unitary representation of G, say on a Hilbert space V_π, and let V be the space of K-finite vectors. Realize π as a subrepresentation of $Ind_{G^\dagger}^G(\pi^\dagger)$ for some irreducible unitary representation $\pi^\dagger$ of $G^\dagger$. The representation space of $Ind_{G^\dagger}^G(\pi^\dagger)$ is the space

$$Ind_{G^\dagger}^G(V_{\pi^\dagger}) = [L^2(G) \otimes V_{\pi^\dagger}]^{G^\dagger}$$

of $G^\dagger$-fixed vectors, where $G^\dagger$ acts on $L^2(G)$ by right translation and on $V_{\pi^\dagger}$ by $\pi^\dagger$. G acts on $Ind_{G^\dagger}^G(V_{\pi^\dagger})$ by left translation on the $L^2(G)$ factor. The subspace of K-finite vectors is

$$Ind_{G^\dagger}^G(V^\dagger) = [L^2(G)'' \otimes V^\dagger]^{G^\dagger}$$

where $L^2(G)''$ consists of the elements of $L^2(G)$ that are K-finite on the left and the right. If we assume Theorem 2.4 for the representation $\pi^\dagger$ then it follows that the space $Ind_{G^\dagger}^G(V^\dagger)$ of K-finite vectors for $Ind_{G^\dagger}^G(V_{\pi^\dagger})$ is dense and is a Harish-Chandra module, i.e. that Theorem 2.4 holds for π.

The restriction of ξ to Z_{G^0} is a multiple of a unitary character ζ. The left regular representations of the groups K^0, $K^\dagger$ and K relative to ζ are

$$\lambda^0 = Ind_{Z_{G^0}}^{K^0}(\zeta), \quad \lambda^\dagger = Ind_{Z_{G^0}}^{K^\dagger}(\zeta), \quad \lambda = Ind_{Z_{G^0}}^{K}(\zeta).$$

Induction by stages says that $\lambda = Ind_{K^\dagger}^K(\lambda^\dagger)$. Theorem 2.5 for the $(\mathfrak{g}, K^\dagger)$ Harish-Chandra module $V^\dagger$ just says that the representation $\pi^\dagger$ of $K^\dagger$ is equivalent to a subrepresentation of $\lambda^\dagger$. It follows that the induced representation of K is equivalent to a subrepresentation of λ. In other words, Theorem 2.5 follows for the $(\mathfrak{g}, K)$ Harish-Chandra module V.

Theorems 2.6 and 3.4 now hold for irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules and TCI Banach representations π of G, where G is a general semisimple group and K is a maximal compactly embedded subgroup.

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