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Geometric Quantization and Derived Functor Modules for Semisimple Lie Groups

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Let G be a semisimple Lie group, $\mathfrak{g}$ the complexified Lie algebra, and X the flag variety of $\mathfrak{g}$. The mechanism of geometric quantization suggests that the various G-orbits in X should give rise to representations of G. On the other hand, Zuckerman's derived functor construction attaches algebraic representations of $\mathfrak{g}$ to G-orbits. In this paper we show that geometric quantization leads to Fréchet representations of finite length, which are the maximal globalizations of the derived functor modules. We give two alternate realizations of the representations, as cohomology spaces of $\bar{\partial}_b$ complexes with hyperfunction coefficients, and as local cohomology groups along G-orbits in X. We use the latter realization to implement the duality between the derived functor modules and the Beilinson Bernstein modules, as cup product between local cohomology group followed by evaluation over the fundamental cycle.

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1. Introduction

In this paper we examine certain conjectures, first proposed in the sixties and seventies, on geometric realizations of representations of semisimple Lie groups.

To simplify the discussion in the Introduction, we suppose that G is a connected, linear, semisimple Lie group; these hypotheses will be relaxed in the main body of the paper. As a general rule, we denote the Lie algebra of a Lie group by the corresponding lower case German letter with subscript zero, e.g., $\mathfrak{g}_0$, and the complexified Lie algebra by the same German letter without subscript, e.g., $\mathfrak{g}$. The G-orbit through some λ in the dual space $\mathfrak{g}^*$ is said to be regular semisimple if it has a Cartan subgroup $H = G_\lambda$ as centralizer. The orbit can then be identified, as G-homogeneous real analytic manifold, with the quotient G/H. If the restriction of λ to $\mathfrak{h}$ lifts to a character $\chi: H \to \mathbb{C}^*$, one calls the orbit through λ integral. The datum of χ associates a G-equivariant line bundle $\mathbb{L}_\chi \to G/H$ to the principal bundle $H \to G \to G/H$. In the language of geometric quantization, the choice of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b}$ provides a G-invariant polarization for G/H, which turns out to carry a natural G-invariant symplectic structure. The line bundle and polarization determine a complex of differential forms on G/H, on which G acts by translation; its complex of global sections is isomorphic to

$$\{C^\infty(G) \otimes L_\chi \otimes \Lambda \mathfrak{n}^*\}^H, d_\mathfrak{n}, \tag{1.1}$$

via pullback from G/H to G. Here $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$; $C^\infty(G)$ is considered as H- and $\mathfrak{n}$-module by right translation; L_χ denotes the one-dimensional representation space of χ, endowed with the trivial action of $\mathfrak{n}$; the suprscript H refers to the space of H-invariants; and $d_\mathfrak{n}$ is the coboundary operator of the standard complex of Lie algebra cohomology with respect to $\mathfrak{n}$. A brief note of Kostant [14] suggests this construction as source of interesting representations of G.

If the Cartan subgroup H happens to be compact, the choice of polarization $\mathfrak{b}$ amounts to that of a G-invariant complex structure for G/H.

Moreover, the line bundle $\mathbb{L}_\chi \to G/H$ can be turned into a homogeneous holomorphic line bundle, i.e., a holomorphic line bundle to which the action of G lifts. In this situation, the complex (1.1) coincides with the Dolbeault complex of $\mathbb{L}_\chi$ and thus computes the sheaf cohomology groups $H^p(G/H, \mathcal{O}(\mathbb{L}_\chi))$. According to [18], under appropriate negativity conditions on the differential λ of χ, these groups vanish except in a single degree $p = s$, and the remaining group is a non-zero Fréchet G-module. As had been conjectured by Langlands [15], the L^2 cohomology of $\mathbb{L}_\chi$ is concentrated in the same degree s, and G acts on it according to a representation of the discrete series [21]. The natural map from L^2 to sheaf cohomology identifies the former with a dense subspace of the Fréchet G-module $H^s(G/H, \mathcal{O}(\mathbb{L}_\chi))$; this follows, for example, from the results of [19, 9]. In particular, $H^s(G/H, \mathcal{O}(\mathbb{L}_\chi))$ has the same Harish–Chandra module (in other words, the same underlying infinitesimal representation) as the L^2 cohomology of $\mathbb{L}_\chi$ in degree s.

Still in the case of a compact Cartan subgroup H, the cohomology of the complex (1.1) does not change if we replace $C^\infty(G)$ by the space of distributions $C^{-\infty}(G)$ or by the space of hyperfunctions $C^{-\omega}(G)$: the Dolbeault lemma remains valid in the context of distributions or hyperfunctions, so the Dolbeault complex with either distribution or hyperfunction coefficients provides a soft, respectively flabby, resolution of the sheaf $\mathcal{O}(\mathbb{L}_\chi)$.

At the opposite extreme, when H splits over $\mathbb{R}$, the polarization $\mathfrak{b}$ is the complexified Lie algebra of a Borel subgroup $B \subset G$. The inclusion $H \subset B$ induces a G-equivariant fibration of G/H over the compact homogeneous space G/B, with Euclidean fibres, and $\mathbb{L}_\chi$ drops to a homogeneous C^∞ line bundle $\mathbb{L}_\chi \to G/B$. In this setting, (1.1) can be interpreted as the complex of relative differential forms for the fibration, with values in $\mathbb{L}_\chi$. An application of the Rham's theorem along the fibres shows that (1.1) has no cohomology in positive degrees; in degree zero, the cohomology is precisely the space of C^∞ sections $C^\infty(G/B, \mathbb{L}_\chi)$. We can argue similarly if $C^\infty(G)$ in (1.1) is replaced by $C^{-\infty}(G)$ or $C^{-\omega}(G)$: the higher cohomology groups still vanish, but in degree zero one obtains the space of distribution sections $C^{-\infty}(G/B, \mathbb{L}_\chi)$, respectively the space of hyperfunction sections $C^{-\omega}(G/B, \mathbb{L}_\chi)$. Thus, unlike in the previous situation, the degree of regularity of the coefficients has an effect on the cohomology. However, in all three cases the underlying Harish–Chandra module is the same; it belongs to the principal series of representations.

If H is an arbitrary Cartan subgroup, we call a polarization $\mathfrak{b}$ for G/H "maximally real" if it maximizes the dimension of $\mathfrak{b} \cap \bar{\mathfrak{b}}$, subject to the condition $\mathfrak{h} \subset \mathfrak{b}$, of course. To identify the cohomology of the complex (1.1) in the case of a polarization of this type, one can combine the arguments from the previous two cases: under a suitable negativity condition, the cohomology vanishes in all but one degree and is a non-zero Fréchet G-module in that remaining degree; the resulting representations constitute the standard modules attached to the Cartan subgroup H [28].

Until now, the complex (1.1) has not been studied in the situation of an arbitrary polarization. It is not at all clear whether the coboundary operator has closed range. In order to circumvent problems of this sort, among other reasons, Zuckerman introduced his derived functor construction [26]. We shall show that certain complexes, closely related to the complex (1.1), produce Fréchet representations of G; the underlying Harish-Chandra modules are naturally and functorially isomorphic to those which Zuckerman's construction attaches to the same data. We also shall use these Fréchet generalizations of the Zuckerman modules to reinterpret the duality of Hecht *et al.* [7] as a cup product pairing between local cohomology groups.

In order to describe our results more precisely, we recall that there exist canonical ways of lifting Harish-Chandra modules to representations of G: the C^∞ and distribution globalizations constructed by Casselman and Wallach [27], as well as the minimal and maximal globalizations [22]. All four are exact functors from the category of Harish-Chandra modules to the category of representations of G on complete, locally convex, Hausdorff topological vector spaces. If the Cartan subgroup H is compact, the cohomology groups $H^p(G/H, \mathcal{O}(\mathbb{L}_\chi))$ of the complex (1.1) are known to coincide with the maximal globalizations of the underlying Harish-Chandra modules [22]. As was remarked earlier, the complex (1.1) with distribution or hyperfunction coefficients has the same cohomology as in the C^∞ case; in particular, the cohomology groups are maximal globalizations, regardless of whether we use C^∞, distribution, or hyperfunction coefficients. On the other hand, if H splits over $\mathbb{R}$, the cohomology of the complex (1.1), i.e., $C^\infty(G/B, \mathbb{L}_\chi)$, is the C^∞ globalization of the underlying Harish-Chandra module (essentially by definition of the C^∞ globalization). The cohomologies of the analogous complex with distribution or hyperfunction coefficients, i.e., $C^{-\infty}(G/B, \mathbb{L}_\chi)$ and $C^{-\omega}(G/B, \mathbb{L}_\chi)$, are, respectively, the distribution globalization and the maximal globalization of the same Harish-Chandra module; in other words, the three different choices of coefficients lead to three different globalizations.

The case of a maximally real polarization on G/H, for a general Cartan subgroup, lies somewhere between the extreme cases of a compact or a split Cartan subgroup. Typically, the cohomology of the complex (1.1) with C^∞, distribution, or hyperfunction coefficients does depend on the choice of coefficients, but C^∞ or distribution coefficients produce "mixed" topologies on the cohomology; only hyperfunction coefficients yield one of the four canonical globalizations—namely the maximal globalization. For polarizations which are not maximally real, the situation may be even more complicated: although examples are cumbersome to work out, there is evidence to suggest that coboundary operator of the complex (1.1)—with C^∞ coefficients—need not have closed range. In any event, one cannot expect uniform statements about the cohomology unless one works with hyperfunction coefficients. Thus, from now on, we replace the complex (1.1) by its hyperfunction analogue

$$\{C^{-\omega}(G)\otimes L_\chi\otimes\Lambda\mathfrak{n}^*\}^H, d_\mathfrak{n}. \tag{1.2}$$

The space of hyperfunctions on a non-compact manifold does not carry a natural Hausdorff topology. Nonetheless, the cohomology groups of the complex will turn out to be Fréchet spaces; the crux of the matter is a fibration of G/H over a compact homogeneous space, with the property that the restriction of $d_\mathfrak{n}$ to the fibres behaves partly like exterior differentiation, partly like the $\bar\partial$ operator.

The datum of the homogeneous space G/H along with a polarization $\mathfrak{b}$ is equivalent (up to an appropriate notion of conjugacy) to that of a G-orbit in X, the flag variety of the Lie algebra $\mathfrak{g}$: to $(G/H, \mathfrak{b})$, we associate the orbit $S = G\cdot\mathfrak{b}\subset X$. Since H normalizes $\mathfrak{b}$, there exists a natural G-invariant fibration $G/H\to S$. As homogeneous real analytic submanifold of the complex modifold X, S has the structure of CR manifold. The line bundle $\mathbb{L}_\chi\to G/H$ descends to a G-equivariant CR line bundle over S. Thus it makes sense to talk of the CR Dolbeault complex on S with hyperfunction coefficients and values in $\mathbb{L}_\chi$. Though $\mathbb{L}_\chi$ may not extend from S to all of X, it does extend to a $\mathfrak{g}$-equivariant holomorphic line bundle $\tilde{\mathbb{L}}_\chi$ over a germ $\tilde S$ of a neighborhood of S in X. Closely related to the CR Dolbeault complex on S is the Dolbeault complex on $\tilde S$, with values in $\tilde{\mathbb{L}}_\chi$ and coefficients which are hyperfunctions on $\tilde S$ supported on S; this latter complex computes the local cohomology groups $H^p_S(\tilde S, \mathcal{O}(\tilde{\mathbb{L}}_\chi))$.

Our first main result asserts that the following are isomorphic as G-modules: (i) the cohomology groups of the complex (1.2); (ii) those of the CR Dolbeault complex on S with hyperfunction coefficients and values in $\mathbb{L}_\chi$; (iii) the local cohomology groups $H^p_S(\tilde S, \mathcal{O}(\tilde{\mathbb{L}}_\chi))$, with a shift in degree by the real codimension of S in X; and (iv) the maximal globalizations of the Harish–Chandra modules which Zuckerman's derived functor construction assigns to the line bundle $\mathbb{L}_\chi\to G/H$ and the polarization $\mathfrak{b}$. This statement remains correct for any homogeneous CR vector bundle $\mathbb{E}\to S$ in place of $\mathbb{L}_\chi$. All the isomorphisms are functorial in $\mathbb{E}$ and have geometric descriptions. The main ingredients of our proof of these isomorphisms are the exactness of the maximal globalization [22] and certain intertwining operators between the cohomologies corresponding to neighboring G-orbits. The latter are the G-orbit analogues of the intertwining operators of Beilinson and Bernstein [3, 8]. Together, these allow us to reduce our statement, in several stages, to the geometric realization of the discrete series [18].

In the special case of a compact Cartan subgroup, the cohomology groups (i), (ii), (iii) coincide for trivial reasons, even on the level of complexes; the remaining isomorphism in this situation was proved by Aguilar Rodríguez [1]. Hecht and Taylor [10] have developed a notion of analytic localization, which leads to results similar—but not equivalent—to ours.

We already mentioned that Kostant [14] first called attention to the complex (1.1). Zuckerman introduced his derived functor construction as an algebraic analogue of geometric quantization; implicitly, at least, he conjectured a direct connection between his construction and the complex (1.1). Zuckerman also suggested a link between both of these and local cohomology along G-orbits in the flag variety X [30].

Let K be a maximal compact subgroup of G, and $K_{\mathbb{C}}$ the complexification of K. The Beilinson–Bernstein construction [2] attaches Harish-Chandra modules to $K_{\mathbb{C}}$-orbits in X. As was established in [7], these Beilinson–Bernstein modules are dual to the derived functor modules which correspond to G-orbits in X. The results of this paper lead to a more directly geometric description of the pairing constructed in [7]. Both types of modules can be described in terms of local cohomology—in one case along a G-orbit S, in the other case along a $K_{\mathbb{C}}$-orbit Q. If the two orbits are dual in the sense of Matsuki [17], the two cohomologies are paired by cup product into the local cohomology along $S \cap Q$, with values in the canonical sheaf Ω^n_X, $n = \dim_{\mathbb{C}} X$. This latter cohomology maps naturally to $H^n(X, \Omega^n_X) \cong \mathbb{C}$ (by evaluation over the cycle $[X]$). Our second main result asserts that the pairing between the two types of modules is given by cup product, followed by

$$H^n_{S \cap Q}(X, \Omega^n_X) \to H^n(X, \Omega^n_X) \cong \mathbb{C}.$$

In the case of the discrete series this is due to Žabčić [29].

Our first main result was announced in [24]. A more detailed discussion of the motivation for our project, along with a sketch of our arguments, can be found in [23].

We are indebted to Dragan Miličić and Kari Vilonen for helpful discussions.

2. Classical Constructions

In this paper G will be a real reductive Lie group such that

G has a closed normal abelian subgroup Z that centralizes the identity component G^0 and such that $|G/ZG^0| < \infty$, and if $x \in G$ then $\mathrm{Ad}(x)$ is an inner automorphism of the complexified Lie algebra $\mathfrak{g}$.

This is the class studied in [28]. Harish–Chandra class is specified by the additional conditions

$[G^0, G^0]$ has finite center and G/G^0 is finite.

For simplicity of exposition we will write as if G were of Harish–Chandra class, but everything will be valid for the larger class specified above, using the methods of [28, 7].

Recall the classical approach to geometric quantization of semisimple co-adjoint orbits. One can start with a *basic datum* $(H, \mathfrak{b}, \chi)$, where

$$\begin{aligned}&H \text{ is a Cartan subgroup of } G,\\ &\mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ with } \mathfrak{h} \subset \mathfrak{b}, \text{ and}\\ &\chi \text{ is a finite dimensional representation of } (\mathfrak{b}, H).\end{aligned} \tag{2.1}$$

In effect, $\mathfrak{b}$ is an invariant polarization on G/H and χ is a finite dimensional representation of H with a choice of $(\mathfrak{b}, H)$-module structure on the representation space. Denote the space by $E = E_\chi$. Since

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}, \text{ where } \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \text{ is the nilradical of } \mathfrak{b}, \tag{2.2}$$

necessarily $\mathfrak{n}$ acts trivially on E in the important case where H acts irreducibly on E.

We now have a homogeneous vector bundle

$$\mathbb{E} = \mathbb{E}_\chi \to G/H \text{ associated to } (H, \mathfrak{b}, \chi). \tag{2.3}$$

The right action of $\mathfrak{n}$ on $C^\infty(G)$ with the given action on E defines an action of $\mathfrak{n}$ on local C^∞ sections of $\mathbb{E}$. In the complex setting described below, one considers the G-modules that are the cohomologies of the sheaf

$$\begin{aligned}&\mathcal{C}_\mathfrak{n}(\mathbb{E})\text{: germs of } C^\infty \text{ sections } f \text{ of } \mathbb{E} \to G/H \text{ such that}\\ &f(x; \xi) + \chi(\xi) f(x) = 0 \text{ for all } x \in G \text{ and } \xi \in \mathfrak{b}.\end{aligned} \tag{2.4}$$

In order to accommodate "mixed polarizations" it is better to consider the cohomologies of the complex

$$C^\infty(G/H; \mathbb{E} \otimes \Lambda^\cdot \mathbb{N}^*), d_\mathfrak{n} \tag{2.5}$$

defined as follows. $\mathbb{N} \to G/H$ is the homogeneous vector bundle with fibre $\mathfrak{n}$ and $\mathbb{N}^*$ is its dual. Let $(\mathfrak{g}/\mathfrak{h})^* = \{\phi \in \mathfrak{g}^* : \phi(\mathfrak{h}) = 0\}$ represent the complexified cotangent space of G/H. Let q: $(\mathfrak{g}/\mathfrak{h})^* \to \mathfrak{n}^*$ denote the restriction dual to $\mathfrak{n} \cong \mathfrak{b}/\mathfrak{h} \subsetneq \mathfrak{g}/\mathfrak{h}$. Then the map

$$(\mathfrak{g}/\mathfrak{h})^* \otimes E \otimes \Lambda^p \mathfrak{n}^* \to E \otimes \Lambda^{p+1} \mathfrak{n}^* \qquad \text{by } (\phi, e, \omega) \mapsto e \otimes (q(\phi) \wedge \omega) \tag{2.6}$$

is the symbol of a unique first-order G-invariant operator

$$d_{\mathfrak{n}}: C^{\infty}(G/H; \mathbb{E} \otimes \Lambda^{p}\mathbb{N}^{*}) \rightarrow C^{\infty}(G/H; \mathbb{E} \otimes \Lambda^{p+1}\mathbb{N}^{*}). \tag{2.7}$$

The maps (2.7) specify the complex (2.5).

The bundles $\mathbb{E} \otimes \Lambda^{p}\mathbb{N}^{*} \rightarrow G/H$ pull back to trivial bundles on G. There, the complex (2.5) becomes

$$\{C^{\infty}(G) \otimes E \otimes \Lambda^{\cdot}\mathfrak{n}^{*}\}^{H}, d_{\mathfrak{n}}, \tag{2.8}$$

where $\{...\}^{H}$ denotes the space of H-invariants with H and $\mathfrak{n}$ acting on $C^{\infty}(G)$ from the right, and where $d_{\mathfrak{n}}$ is the coboundary olperator for Lie algebra cohomology of $\mathfrak{n}$.

Let $\mathfrak{n} \cap \bar{\mathfrak{n}} = 0$, i.e., suppose that the polarization $\mathfrak{b}$ is totally complex. This is always the case when H is compactly embedded in G, and in general it requires that H be as compact as possible in G. Then G has an invariant complex structure for which $\mathfrak{n}$ represents the antiholomorphic tangent space, $\mathbb{E} \rightarrow G/H$ has the structure of holomorphic vector bundle (see [25 or 19]), and $\mathscr{O}_{\mathfrak{n}}(\mathbb{E}) \rightarrow G/H$ is the sheaf of germs of holomorphic sections. Here $d_{\mathfrak{n}}$ is the Dolbeault operator $\bar{\partial}$ and the sheaf version of (2.5) is the Dolbeault resolution of $\mathscr{O}_{\mathfrak{n}}(\mathbb{E})$. Thus

$$H^{*}(C^{\infty}(G/H; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{N}^{*}), d_{\mathfrak{n}}) \cong H^{*}(G/H, \mathscr{O}_{\mathfrak{n}}(\mathbb{E})). \tag{2.9}$$

This leads, for example, to the C^{∞} fundamental series representations of G.

Let $\mathfrak{n} = \bar{\mathfrak{n}}$, i.e., suppose that the polarization $\mathfrak{b}$ is real. Thus H is as noncompact as possible in G, $\mathfrak{g}_0$ is quasi-split, and $\mathfrak{b}$ is the complexification of the Lie algebra $\mathfrak{b}_0$ of a minimal parabolic subgroup $B \subset G$. Suppose further that $\mathfrak{n}$ acts trivially on E (automatic if H is irreducible on E). Under the fibration $G/H \rightarrow G/B$, the bundle $\mathbb{E} \rightarrow G/H$ pushes down to a bundle $\mathbb{E} \rightarrow G/B$ and the sheaf $\mathscr{O}_{\mathfrak{n}}(\mathbb{E}) \rightarrow G/H$ pushes down to the sheaf $\mathscr{C}^{\infty}(\mathbb{E}) \rightarrow G/B$ of germs of C^{∞} sections over G/B. The Poincaré lemma for the euclidean fibres $\mathfrak{n} \cong H \backslash B$ of $G/H \rightarrow G/B$ implies

$$\begin{aligned} H^{p}(C^{\infty}(G/H; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{N}^{*}), d_{\mathfrak{n}}) &= H^{p}(G/H, \mathscr{O}_{\mathfrak{n}}(\mathbb{E})) \\ &= C^{\infty}(G/B; \mathbb{E}) \qquad \text{if} \quad p = 0, \\ &= 0 \qquad\qquad\qquad\quad \text{if} \quad p \neq 0. \end{aligned} \tag{2.10}$$

Now consider the general case of a maximally real polarization. Thus H is arbitrary but $\mathfrak{n} \cap \bar{\mathfrak{n}}$ is maximal for that choice of H. Choose a Cartan involution θ of G such that $\theta H = H$. It defines the maximal compactly embedded subgroup $K = \{x \in G : \theta x = x\}$ of G, we decompose $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ into ± 1 eigenspaces of θ, and we split $H = T \times A$, where $T = H \cap K$ and $A = \exp_G(\mathfrak{a}_0)$. Then $Z_G(\mathfrak{a}) = M \times A$ with $\theta M = M$. Let $P = MAN_H$ be an associated cuspidal parabolic subgroup of G. If $\eta = \psi \otimes \eta^0$ is a relative discrete series representation of $Z_M(M^0)M^0$, where η^0 has Harish–Chandra parameter $\nu + \rho_M \in i\mathfrak{t}_0^*$, and if $\sigma \in \mathfrak{a}_0^*$, then one has the H-series tempered representation

$$\pi_{\psi,\nu,\sigma} = \operatorname{Ind}_{Z_M(M^0)M^0AN_H}^{G}(\eta \otimes e^{i\sigma}).$$

See [28 or 11]. Suppose that the parameterization is such that ν is M-antidominant and let $s = \dim_{\mathbb{C}}(K \cap M)/T$. Then, combining [18, 19, 28],

$$H^p(C^\infty(G/H; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{N}^*), d_{\mathfrak{n}}) = 0 \qquad \text{for} \quad p \neq s,$$

and

$$G \text{ acts on } H^s(C^\infty(G/H; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{N}^*), d_{\mathfrak{n}}) \qquad \text{by } \pi_{\psi,\nu,\sigma}. \tag{2.11}$$

There are several serious problems with (2.5) for general $(H, \mathfrak{b})$. First, the complex (2.5) is not acyclic, in general, so it will compute the hypercohomology of a complex of sheaves rather than the cohomology of a single sheaf. Second, there is no reason, in general, to expect $d_{\mathfrak{n}}$ to have closed range, and, in fact, closed range is a delicate point in the cases described above. These problems are avoided in the algebraic version of (2.5), which (see Section 3 below) leads to the Zuckerman derived functor $(\mathfrak{g}, K)$-modules. In this paper we describe some geometric complexes, variations on (2.5), which effectively yield all standard representations of G, and we relate them to the Zuckerman modules.

3. Harish–Chandra Modules and Globalizations

By *representation* of G we will mean a continuous representation $(\pi, \tilde{V})$ of G, on a complete locally convex Hausdorff topological vector space, such that $(\pi, \tilde{V})$ has finite composition series. By *Harish–Chandra module* for G we mean a $\mathcal{U}(\mathfrak{g})$-finite K-semisimple $(\mathfrak{g}, K)$-module V in which every vector is K-finite and the K-multiplicities are finite.

If $(\pi, \tilde{V})$ is a representation of G, then $V = \{x \in \tilde{V} : x \text{ is } K\text{-finite}\}$ is dense in $\tilde{V}$, is a Harish–Chandra module for G, and consists of smooth vectors.

If V is a Harish–Chandra module for G, and if $(\pi, \tilde{V})$ is a representation of G such that V is $(\mathfrak{g}, K)$-isomorphic to the space of K-finite vectors in $\tilde{V}$, then we say that $(\pi, \tilde{V})$ is a *globalization* of V.

Zuckerman's derived functor is an algebraic version of $H^*(C^\infty(G/H; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{N}^*), d_{\mathfrak{n}})$ that results in Harish–Chandra modules for G. Let $\mathcal{M}(\mathfrak{g}, K)_{(K)}$ denote the category of K-finite $(\mathfrak{g}, K)$-modules, $\mathcal{M}(\mathfrak{g}, H \cap K)_{(H \cap K)}$ the same thing with K replaced by $H \cap K$, and

$$\Gamma : \mathcal{M}(\mathfrak{g}, H \cap K)_{(H \cap K)} \to \mathcal{M}(\mathfrak{g}, K)_{(K)}, \tag{3.1}$$

the functor that sends a module to its maximal $\mathfrak{k}$-finite $\mathfrak{k}$-semisimple submodule. Then [26], Γ is left exact and its right derived functors $R^p\Gamma$ are the Zuckerman functors. So the basic datum $(H, \mathfrak{b}, \chi)$ of (2.1) specifies $(\mathfrak{g}, K)$-modules

$$A^p(G, H, \mathfrak{b}, \chi) = (R^p\Gamma)\{\mathrm{Hom}_{\mathfrak{b}}(\mathscr{U}(\mathfrak{g}), E)_{(H \cap K)}\}. \tag{3.2}$$

They are Harish-Chandra modules for G.

To relate the Harish-Chandra modules (3.2) to the cohomologies of the complex (2.5), denote

$$C^{\mathrm{for}}: (H \cap K)\text{-finite formal power series sections at } 1 \cdot H \in G/H. \tag{3.3}$$

Evaluation at $1 \cdot H$ in the version (2.8) gives

$$C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*) \cong \mathrm{Hom}_{\mathfrak{b}}(\mathscr{U}(\mathfrak{g}), E \otimes \Lambda^p \mathfrak{n}^*)_{(H \cap K)}.$$

Here $d_{\mathfrak{n}}$ acts on the left-hand side giving a complex of $(\mathfrak{g}, H)$-modules. The kernel for $p = 0$ is isomorphic to $\mathrm{Hom}_{\mathfrak{b}}(\mathscr{U}(\mathfrak{g}), E)_{(H \cap K)}$. Resolve that by the complex $C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*), d_{\mathfrak{n}}$. Since this is an injective resolution, we conclude

$$A^p(G, H, \mathfrak{b}, \chi) \cong H^p(C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*)_{(K)}, d_{\mathfrak{n}}). \tag{3.4}$$

Note that (3.4) defines a map from the cohomology of the K-finite version of (2.5) to the derived functor module. That is the coefficient morphism,

$$H^p(C^{\infty}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*)_{(K)}, d_{\mathfrak{n}}) \to H^p(C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*)_{(K)}, d_{\mathfrak{n}}), \tag{3.5}$$

defined by the Taylor series expansion at $1 \cdot H$.

We will use (3.5) in showing that certain geometrically defined representations of G are a particular globalization of the Zuckerman modules (3.2), (3.4). There ar four functorial globalizations—the C^{∞} and distribution globalizations of Casselman and Wallach [27] and the minimal and maximal globalizations of [22]. Because of its topological properties, the maximal globalization is the one that is appropriate here.

Let V be a Harish-Chandra module for G and let $(\pi, \tilde{V})$ be any globalization. Every vector v' in the dual Harish-Chandra module

$$V': K\text{-finite vectors in the algebraic dual of } V$$

extends to a continuous linear functional on $\tilde{V}$. If $v \in V$ and $v' \in V'$, the coefficient,

$$f_{v,v'}: G \to \mathbb{C} \qquad \text{by} \quad f_{v,v'}(x) = (v', \pi(x)v),$$

is C^{ω} and its Taylor series at 1 depends only on the action of $\mathscr{U}(\mathfrak{g})$ on V. Any finite set $\{v_1', \ldots, v_n'\}$ of $\mathscr{U}(\mathfrak{g})$-generators of V' now defines an injection

$$V \hookrightarrow C^{\infty}(G)^n \qquad \text{by} \quad v \mapsto (f_{v,v_1'}, \ldots, f_{v,v_n'}). \tag{3.6}$$

The induced topology on V is independent of choice of $\{v_i'\}$ and

$$V_{\max}: \text{completion of } V \text{ in the induced topology} \tag{3.7}$$

is a globalization of V. We call it the *maximal globalization* because, if $\tilde{V}$ is any globalization, then the identity on V extends to a G-equivariant continuous injection $\tilde{V} \hookrightarrow V_{\max}$.

If $(\pi, \tilde{V})$ is a Banach globalization of V, then the subspace $\tilde{V}^{\omega}$ of analytic vectors has a natural complete locally convex topology and $\tilde{V}^{\omega}$ again is a globalization of V. Now let $\tilde{V}$ be a reflexive Banach space, $\tilde{V}'$ the dual Banach space, and π' the dual of π. Then

$$\tilde{V}^{-\omega}: \text{strong topological dual of } (\tilde{V}')^{\omega} \tag{3.8}$$

is the space of *hyperfunction vectors* of $\tilde{V}$. It is another globalization of V, and the point here is [22, p. 317] that

$$\text{the inclusion } \tilde{V}^{-\omega} \hookrightarrow V_{\max} \text{ is a topological isomorphism.} \tag{3.9}$$

Using this, one knows [22] that

$$V \to V_{\max} \text{ is an exact functor.} \tag{3.10}$$

Look back at the examples in Section 2. The Dolbeault lemma holds with any $(C^{\infty}, C^{-\infty}, C^{-\omega})$ coefficients, so (2.9) holds with C^{∞} replaced by $C^{-\omega}$, and thus each Dolbeault cohomology group (2.9) is the maximal globalization of its underlying Harish–Chandra module. In contrast, in the case (2.10) of real polarization, the maximal globalization of $C^{\infty}(G/B; \mathbb{E})_{(K)}$ is the space $C^{-\omega}(G/B; \mathbb{E})$ of hyperfunction sections.

4. Three Geometric Complexes

In view of (3.9) and the subsequent discussion, note that the differential $d_{\mathfrak{n}}$ of (2.5) extends naturally to hyperfunction sections, so we have a complex

$$C^{-\omega}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*), d_{\mathfrak{n}}. \tag{4.1}$$

The space of hyperfunctions on a noncompact manifold does not have a natural topology. We topologize the cohomology of the complex (4.1) by comparison of (4.7), (4.8) with the cohomology of the Cauchy–Riemann complex (4.5) described below. In order to identify the resulting G-modules we also need to study the local cohology groups (4.9). In fact, these three are related a *priori* because hyperfunctions can be defined as certain local cohomologies.

Let X denote the flag variety of Borel subalgebras of $\mathfrak{g}$ and consider the fibration

$$G/H \to S = G \cdot \mathfrak{b} \subset X. \tag{4.2}$$

X has a natural G-invariant complex structure and S is a homogeneous submanifold. Thus S has constant Cauchy–Riemann (CR) dimension. In particular,

$$\begin{aligned}\mathbb{N}_S\colon &\text{ intersection of the complexified tangent bundle of } S\\ &\text{ with the antiholomorphic tangent bundle of } X\end{aligned} \tag{4.3}$$

is a G-homogeneous vector bundle based on $\mathfrak{n}/(\mathfrak{n} \cap \bar{\mathfrak{n}})$. The part of the Dolbeault operator $\bar{\partial}_X$ contangent to S is the *Cauchy–Riemann operator*

$$\bar{\partial}_S\colon C^\infty(S; \Lambda^p \mathbb{N}_S^*) \to C^\infty(S; \Lambda^{p+1} \mathbb{N}_S^*). \tag{4.4}$$

See the Appendix for details. There $\mathbb{N}_S$ is denoted $\mathbb{T}^{0,1}(S)$. Extend χ to the G-stabilizer of $\mathfrak{b}$ by exponentiating the action of $\mathfrak{n} \cap \mathfrak{g}_0$ on the $(\mathfrak{b}, H)$-module E_χ. Then $\mathbb{E} \to G/H$ pushes down to a G-homogeneous bundle $\mathbb{E} \to S$ and we have the Cauchy–Riemann complex

$$C^{-\omega}(S; \mathbb{E} \otimes \Lambda^\cdot \mathbb{N}_S^*), \bar{\partial}_S \tag{4.5}$$

with hyperfunction coefficients. If we pull back to G as was done for (2.8), we see that it is isomorphic to the complex

$$\{C^{-\omega}(G) \otimes E \otimes \Lambda^\cdot (\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*\}^{\mathfrak{n} \cap \mathfrak{n}, H}, \delta_{\mathfrak{n}, \mathfrak{n} \cap \mathfrak{n}} \tag{4.6}$$

for relative Lie algebra cohomology of $(\mathfrak{n}, \mathfrak{n} \cap \bar{\mathfrak{n}})$ and hyperfunction coefficients.

$G/H \to S$ has euclidean space fibres. Apply the Poincaré lemma to those fibres to see that inclusion of the complex (4.6) in the complex (4.1) induces an isomorphism of cohomology. Thus

$$\begin{aligned} &H^*(C^{-\omega}(G/H; \mathbb{E} \otimes \Lambda^\cdot \mathbb{N}^*), d_{\mathfrak{n}})\\ &\quad \cong H^*(C^{-\omega}(S; \mathbb{E} \otimes \Lambda^\cdot \mathbb{N}_S^*), \bar{\partial}_S)\\ &\quad \cong H^*(\{C^{-\omega}(G) \otimes E \otimes \Lambda^\cdot (\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*\}^{\mathfrak{n} \cap \mathfrak{n}, H}, \delta_{\mathfrak{n}, \mathfrak{n} \cap \bar{\mathfrak{n}}}). \end{aligned} \tag{4.7}$$

Let $\tilde{S}$ denote the germ of an open neighborhood of S in X. Then $\mathbb{E} \to S$

has a unique holomorphic $\mathfrak{g}$-equivariant extension $\tilde{\mathbb{E}} \to \tilde{S}$. Consider the Dolbeault complex

$$C_S^{-\omega}(\tilde{S}; \tilde{\mathbb{E}} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}), \bar{\partial} \tag{4.8}$$

with coefficients that are hyperfunctions on $\tilde{S}$ with support in S. One knows [13] that

$$H^p(C_S^{-\omega}(\tilde{S}; \tilde{\mathbb{E}} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}), \bar{\partial}) \cong H_S^p(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}})), \tag{4.9}$$

where the right-hand side of (4.9) is local cohomology along S.

5. First Main Theorem

Fix a basic datum $(H, \mathfrak{b}, \chi)$ as in (2.1), let $S = G \cdot \mathfrak{b} \subset X$, and let $u = \operatorname{codim}_{\mathbb{R}}(S \subset X)$. The result is

5.1. THEOREM. *There are canonical isomorphisms*

$$H^p(C^{-\omega}(G/H; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}^*), d_{\mathfrak{n}}) \cong H^p(C^{-\omega}(S; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}_S^*), \bar{\partial}_S)$$
$$\cong H_S^{p+u}(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}})).$$

These cohomologies carry natural Fréchet topologies. The isomorphisms are topological and the action of G is continuous in these topologies. The resulting representations of G are canonically and topologically isomorphic to the action of G on the maximal globalization of $A^p(G, H, \mathfrak{b}, \chi)$.

5.2. *Remark.* The three complexes of the theorem do not have obvious reasonable topologies. The problem is that there is no reasonable topology for the space of hyperfunctions on a noncompact manifold. The topological part of the theorem must be understood in one of two equivalent ways. First, we will see that the cohomology of the Cauchy–Riemann complex can be calculated from a certain subcomplex that does have a good topology; that topology carries to the cohomology, carries over to the other two cohomologies by the isomorphisms, and makes Theorem 5.1 precise. See Section 7 below. Second, the topology is determined by the underlying Harish-Chandra module $A^p(G, H, \mathfrak{b}, \chi)$ because the topology of the maximal globalization can be defined purely in algebraic terms. See [22].

5.3. *Remark.* The identification of the cohomologies will be seen, in the proof, to be explicit and geometric, in fact induced by the natural map (3.5).

The proof of Theorem 5.1 is distributed over Sections 6 through 13 of this paper.

6. Isomorphisms of the Cohomologies

In this section we start the proof of Theorem 5.1 by showing

6.1. Proposition. *There are canonical isomorphisms*

$$H^p(C^{\infty}(G/H; \mathbb{E}\otimes \Lambda^{\cdot}\mathbb{N}^*), d_n) \cong H^p(C^{\infty}(S; \mathbb{E}\otimes \Lambda^{\cdot}\mathbb{N}_S^*), \bar{\partial}_S)$$
$$\cong H_S^{p+u}(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}}))$$

as G-modules without topology, where $u = \operatorname{codim}_{\mathbb{R}}(S \subset X)$.

Proof. The first isomorphism is (4.7), so we only need to show that the complexes (4.1) and (4.8) have naturally isomorphic cohomologies with a shift of degree by u.

As before, we fix a base point $\mathfrak{b}$ in X, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, with $\mathfrak{h}$ stable under both θ and complex conjugation. Let Y denote the variety of ordered Cartan subalgebras. As homogeneous space,

$$Y \simeq G_{\mathbb{C}}/H_{\mathbb{C}}, \tag{6.2}$$

where $G_{\mathbb{C}}$ is the adjoint group of $\mathfrak{g}$, and $H_{\mathbb{C}}$ is the connected subgroup with Lie algebra $\mathfrak{h}$. Since $H_{\mathbb{C}}$ normalizes $\mathfrak{b}$, there is a natural projection

$$p: Y \to X, \tag{6.3}$$

with fibre

$$p^{-1}(\mathfrak{b}) \simeq \exp \mathfrak{n} \tag{6.4}$$

over the base point $\mathfrak{b}$. We let $\mathbb{T}_{Y|X}$ denote the complexified relative tangent bundle of the fibration p, and $\mathbb{T}^{1,0}_{Y|X}$, $\mathbb{T}^{0,1}_{Y|X}$ the subbundle of holomorphic, respectively antiholomorphic, relative tangent vectors. Because of (6.4),

$$\mathbb{T}^{1,0}_{Y|X} \text{ is modeled on } \mathfrak{n}, \tag{6.5}$$

as homogeneous vector bundle over $Y \simeq G_{\mathbb{C}}/H_{\mathbb{C}}$.

The G-orbit S_Y through the base point in Y,

$$S_Y = G \cdot \mathfrak{h} \subset Y, \tag{6.6}$$

lies over $S = G \cdot \mathfrak{b}$, and

$$p: S_Y \to S \text{ has fibre } \exp(\mathfrak{n} \cap \mathfrak{g}_0) = \exp(\mathfrak{n} \cap \bar{\mathfrak{n}} \cap \mathfrak{g}_0) \tag{6.7}$$

at the base point. We note that $S_Y \simeq G/H$, so

$$S_Y \text{ is a real form of the complex manifold } Y. \tag{6.8}$$

In view of (6.4), $\dim_{\mathbb{C}} Y = 2\dim_{\mathbb{C}} X = 2n$, hence $\dim_{\mathbb{R}} S_Y = \dim_{\mathbb{C}} Y = \dim_{\mathbb{R}} X = \dim_{\mathbb{R}} S + \operatorname{codim}_{\mathbb{R}}(S \subset X)$. This shows:

$$u = \dim_{\mathbb{C}}(\mathfrak{n} \cap \bar{\mathfrak{n}}) \text{ is the fibre dimension of } p\colon S_Y \to S. \tag{6.9}$$

Let $\mathscr{C}_{S_Y}{}^{\omega}(Y; ...)$ denote the sheaf of hyperfunction sections of ..., with support in S_Y; analogously $\mathscr{C}_S{}^{\omega}(X)$ is the sheaf of hyperfunctions on X with support in S. Exterior differentiation along the fibres turns $\mathscr{C}_{S_Y}{}^{\omega}(Y; \Lambda^{\cdot}\mathbb{T}^*_{Y|X})$ into a complex. We claim

$$\mathscr{H}^k(p_*\mathscr{C}_{S_Y}{}^{\omega}(Y; \Lambda^{\cdot}\mathbb{T}^*_{Y|X})) = \begin{cases} 0, & k \neq 2n-u \\ \mathscr{C}_S{}^{\omega}(X), & k = 2n-u. \end{cases} \tag{6.10}$$

This is really a local statement. We may view the variables on the base as parameters, which reduces the statement to a problem along the fibres. Over a point of S, $p\colon Y \to X$ has fibre $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, with the fibre of $p\colon S_Y \to S$ corresponding to the subspace $\mathbb{R}^u \subset \mathbb{R}^{2n}$. Thus (6.10) comes down to the following statement: *the de Rham complex on $\mathbb{R}^{2n}$, with hyperfunction coefficients supported along $\mathbb{R}^u$, has cohomology only in degree $2n-u$; in that degree, the cohomology has dimension* 1. This is standard in the two extreme cases $u = 2n$ and $u = 0$ [13]. The general case follows from an argument which combines the two extreme cases.

Interpret $\mathscr{C}_S{}^{\omega}(X)$ as a complex of sheaves concentrated in degree zero. Then (6.10) becomes a quasi-isomorphism between complexes of sheaves,

$$\mathscr{C}_S{}^{\omega}(X)[u-2n] \simeq p_*\mathscr{C}_{S_Y}{}^{\omega}(Y; \Lambda^{\cdot}\mathbb{T}^*_{Y|X}); \tag{6.11}$$

here $[u-2n]$ denotes a shift of indexing by $2n-u$. Twisting the quasi-isomorphism with the Dolbeault complex of the vector bundle $\tilde{\mathbb{E}}$ over the germ $\tilde{S}$ of a neighborhood of S gives

$$\begin{aligned} &\mathscr{C}_S{}^{\omega}(\tilde{S}; \tilde{\mathbb{E}} \otimes \Lambda^{\cdot}\mathbb{T}^{0,1*}_X)[u-2n] \\ &\quad \simeq p_*\mathscr{C}_{S_Y}{}^{\omega}(p^{-1}(\tilde{S}); p^*\tilde{\mathbb{E}} \otimes \Lambda^{\cdot}\mathbb{T}^*_{Y|X} \otimes \Lambda^{\cdot}\mathbb{T}^*_{Y|X} \otimes \Lambda^{\cdot}p^*\mathbb{T}^{0,1*}_X). \end{aligned} \tag{6.12}$$

The $H_{\mathbb{C}}$-invariant splitting $\mathfrak{g} = \mathfrak{b} + \mathfrak{n}_+$, with $\mathfrak{n}_+$ equal to the sum of the positive root spaces, puts a $G_{\mathbb{C}}$-invariant holomorphic connection on the fibre bundle

$$\exp \mathfrak{n} \to G_{\mathbb{C}}/H_{\mathbb{C}} \simeq Y \to X;$$

this connection is flat since $[\mathfrak{n}_+, \mathfrak{n}_+] \subset \mathfrak{n}_+$. In other words, there exists a $G_{\mathbb{C}}$-invariant isomorphism

$$p^*\mathbb{T}_X \oplus \mathbb{T}_{Y|X} \simeq \mathbb{T}_Y, \tag{6.13}$$

which is compatible with the complex structure and Lie bracket. Henceforth we shall write $\mathbb{T}_X$ instead of $p^*\mathbb{T}_X$, to simplify the notation. Then

$$\begin{aligned}\Lambda^{\cdot}\mathbb{T}^*_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_X &\simeq \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_X\\ &\simeq \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_Y\end{aligned}$$

so the right-hand side of (6.12) can be identified with

$$p_*\mathscr{C}_{S_Y}{}^{\omega}(p^{-1}(\tilde{S}); p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_Y). \tag{6.14}$$

The local cohomology sheaves of $\mathcal{O}_Y$ along the real form S_Y vanish except in degree $\dim_{\mathbb{R}} S_Y = 2n$, and coincide with $\mathscr{C}^{-\omega}(S_Y)$, the sheaf of the hyperfunctions on S_Y, in degree $2n$ [13]. Equivalently,

$$\mathscr{C}_{S_Y}{}^{\omega}(p^{-1}(\tilde{S}); \Lambda^{\cdot}\mathbb{T}^{0,1*}_Y)[2n]\simeq\mathscr{C}^{-\omega}(S_Y) \tag{6.15}$$

(here we identify $\mathscr{C}^{-\omega}(S_Y)$ with a sheaf on Y, or on $p^{-1}(\tilde{S})$, having trivial stalks at points outside S_Y). We take cohomology with respect to the second index in (6.14), and use (6.15), to find

$$\begin{aligned}&\mathscr{C}_{S_Y}{}^{\omega}(p^{-1}(\tilde{S}); p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_Y)[2n]\\ &\quad\simeq\mathscr{C}^{-\omega}(S_Y; p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X});\end{aligned}$$

hence (6.12) becomes

$$\mathscr{C}_S{}^{\omega}(\tilde{S}; \tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_X)\simeq p_*\mathscr{C}^{-\omega}(S_Y; p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X})[-u]. \tag{6.16}$$

The higher direct images of the flabby sheaves $\mathscr{C}^{-\omega}(S_Y; ...)$ vanish. Thus, by Leray and (6.16),

$$\begin{aligned}H^*_S(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}})) &= H^*(C_S{}^{\omega}(\tilde{S}; \tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_X))\\ &= \mathbb{H}^*(\tilde{S}, \mathscr{C}_S{}^{\omega}(\tilde{S}; \tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{0,1*}_X))\\ &= \mathbb{H}^*(\tilde{S}, p_*\mathscr{C}^{-\omega}(S_Y; p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0}_{Y|X})[-u])\\ &= \mathbb{H}^*(p^{-1}(\tilde{S}), \mathscr{C}^{-\omega}(S_Y; p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X})[-u])\\ &= H^{*-u}(C^{-\omega}(S_Y; p^*\tilde{\mathbb{E}}\otimes\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X})).\end{aligned} \tag{6.17}$$

At the second step and at the last step, we used th fact that the hypercohomology of a complex of flabby sheaves is computed by its complex of global sections. The homogeneous vector bundle $p^*\mathbb{E}$ over $S_Y\simeq G/H$ is modeled on the H-module E. Recall (6.5) and observe that the isomorphism (6.15) relates differentiation by a real analytic vector field ξ

on S_Y to Lie derivative along the holomorphic extension of ξ on Y. Thus $C^{-\omega}(S_Y; p^*\tilde{\mathbb{E}} \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X})$ coincides with the complex (4.1) (Here recall (6.5).) Now (6.17) implies Proposition (6.1).

7. A Subcomplex of the CR Complex

In this section we describe a subcomplex of the Cauchy–Riemann complex (4.5) that has several key properties. First, it has the same cohomology as (4.5). Second, it has a natural Fréchet topology. Third (we will see this in later sections), its differentials have closed range, so the cohomology inherits a Fréchet structure.

Fix a Cartan involution θ of G with $\theta H = H$. Thus $H = T \times A$ with $T = H \cap K$ and $A = \exp(\mathfrak{a}_0)$, where

$$K = G^{\theta}\text{: maximal compactly embedded subgroup of } G$$

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a}\text{: } \pm 1 \text{ eigenspaces of } \theta|_{\mathfrak{h}}.$$

We have our orbit $S = G \cdot \mathfrak{b}$, $\mathfrak{h} \subset \mathfrak{b}$. We are going to define a related orbit $S_{\max} = G \cdot \mathfrak{b}_{\max}$, where $\mathfrak{h} \subset \mathfrak{b}_{\max}$ and $\mathfrak{b}_{\max}$ is maximally real for that condition. Then, as described before (2.11), G has a cuspidal parabolic subgroup $P = MAN_H$, $Z_G(\mathfrak{a}_0) = M \times A$, $\theta M = M$, and $\mathfrak{b}_{\max} \subset \mathfrak{p}$. We will do this in such a way that

7.1. Proposition. *$G \cap B \subset G \cap B_{\max}$, so $g\mathfrak{b} \mapsto g\mathfrak{b}_{\max}$ defines a fibration $S \to S_{\max}$. $G \cap B_{\max} \subset P$, so $g \cdot \mathfrak{b}_{\max} \mapsto gP$ defines a fibration $S_{\max} \to G/P$. The fibres of each of the three fibrations are complex submanifolds of X:*

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
S_{\max} \arrow[rr] & & G/P
\end{tikzcd}$$

Our subcomplex of ((4.5) depends on $S \to G/P$, and we will need $S \to S_{\max}$ in Section 10.

7.2. Lemma. *Let Φ^+ be a positive root system for $(\mathfrak{g}, \mathfrak{h})$. If there is no complex*[1] *simple root α such that $\bar{\alpha} \in -\Phi^+$, then Φ^+ is maximally real.*

Proof. Let $\xi \in \mathfrak{a}_0$ and $\eta \in i\mathfrak{t}_0$ such that $\Phi^+ = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h}) : \alpha(\xi + \eta) > 0\}$. If $\alpha \in \Phi^+$ is simple, then either $\alpha|_{\mathfrak{a}} = 0$ and $\alpha(\eta) > 0$, or $\alpha(\xi) > |\alpha(\eta)|$. So, if $\alpha \in \Phi^+$, either $\alpha|_{\mathfrak{a}} = 0$ and $\alpha(\eta) > 0$, or $\alpha(\xi) > 0$. In other words Φ^+ is the

[1] A root α is called *complex* if $\bar{\alpha} \neq \pm\alpha$.

maximally real system defined by $\Phi^+(\mathfrak{m}, \mathfrak{t}) = \{\alpha \in \Phi : \alpha|_{\mathfrak{a}} = 0 \text{ and } \alpha(\eta) > 0\}$ and $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Phi : \alpha(\xi) > 0\}$. Q.E.D.

7.3. LEMMA. *Let α be a complex simple root for Φ^+ such that $\bar{\alpha} \in -\Phi^+$, and let s_α be the Weyl reflection. Then*

$$(s_\alpha \Phi^+) \cap \overline{(s_\alpha \Phi^+)} = (\Phi^+ \cap \overline{\Phi^+}) \cup \{\alpha, \bar{\alpha}\}.$$

Proof. Note $s_\alpha \Phi^+ = (\Phi^+ \setminus \{\alpha\}) \cup \{-\alpha\}$. Q.E.D.

Combine Lemmas 7.2 and 7.3:

7.4. LEMMA. *Let Φ^+ be a positive root system for $(\mathfrak{g}, \mathfrak{h})$. Then there are sequences $\{\alpha_0, ..., \alpha_q\}$ of roots and $\{\Phi^+ = \Phi_0^+, ..., \Phi_{q+1}^+ = \Phi_{\max}^+\}$ of positive root systems, such that*

(i) *α_i is complex simple for Φ_i^+ with $\bar{\alpha}_i \in -\Phi_i^+$,*

(ii) *$s_{\alpha_i} \Phi_i^+ = \Phi_{i+1}^+$,*

(iii) *$\Phi_{q+1}^+ = \Phi_{\max}^+$ is maximally real, and*

(iv) *$\{\alpha \in \Phi_i^+ : \alpha|_{\mathfrak{a}} = 0\}$ is independent of i.*

Now the Borel $\mathfrak{b}_{\max} = \mathfrak{h} + \sum_{\alpha \in \Phi_{\max}^+} \mathfrak{g}_{-\alpha}$ and the orbit $S_{\max} = G \cdot \mathfrak{b}_{\max} \subset X$ are defined. Note from Lemmas 7.3 and 7.4 that $\Phi^+ \cap \overline{\Phi^+} \subset \Phi_{\max}^+ \cap \overline{\Phi_{\max}^+}$. Thus

$$\mathfrak{b} \cap \bar{\mathfrak{b}} \subset \mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}}, \qquad \text{i.e., } \mathfrak{g}_0 \cap \mathfrak{b} \subset \mathfrak{g}_0 \cap \mathfrak{b}_{\max}. \tag{7.5}$$

Each topological component of $G \cap B$, or of $G \cap B_{\max}$, contains every coset of T modulo T^0. Thus

$$G \cap B \subset G \cap B_{\max}, \qquad \text{so } S \text{ fibres over } S_{\max}. \tag{7.6}$$

The fibre of $S \to S_{\max}$ is a complex submanifold of X, as a consequence of

7.7. LEMMA. *Define $\mathfrak{q} = \mathfrak{b} \cap \mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}}$. Then $\mathfrak{q}$ is a complex subalgebra of $\mathfrak{g}$, normalized by $G \cap B$ and contained in $\mathfrak{b}$, such that*

$$\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}} \qquad \textit{and} \qquad \mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{b} \cap \bar{\mathfrak{b}}.$$

In other words [5] the fibre of $S \to S_{\max}$ is a complex submanifold of X and $\mathfrak{q}/(\mathfrak{b} \cap \bar{\mathfrak{b}})$ represents its antiholomorphic tangent space.

Proof. We defined $\mathfrak{q}$ as an intersection of complex subalgebras of $\mathfrak{g}$ that contain $\mathfrak{h}$ and, by (7.6), are normalized by $G \cap B$. So $\mathfrak{q}$ is a complex subalgebra, $G \cap B$ normalizes $\mathfrak{q}$, and $\mathfrak{h} \subset \mathfrak{q}$.

Write $\Psi = \Phi^+ \cap \Phi^+_{\max} \cap \overline{\Phi^+_{\max}}$, so that $\mathfrak{q} = \mathfrak{h} + \sum_\Psi \mathfrak{g}_{-\alpha}$. Now $\Phi^+ \cap \overline{\Phi^+} \subset \Phi^+_{\max} \cap \overline{\Phi^+_{\max}}$ gives $\Psi \cap \bar{\Psi} = \Phi^+ \cap \overline{\Phi^+}$, so $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{b} \cap \bar{\mathfrak{b}}$.

Look at the negative system. That gives $(-\Phi^+) \cap (-\overline{\Phi^+}) \subset (-\Phi^+_{\max}) \cap (-\overline{\Phi^+_{\max}})$. Take complements: $\Phi^+ \cup \overline{\Phi^+} \supset \Phi^+_{\max} \cup \overline{\Phi^+_{\max}}$. Thus $\Psi \cup \bar{\Psi} = (\Phi^+ \cup \overline{\Phi^+}) \cap (\Phi^+_{\max} \cap \overline{\Phi^+_{\max}}) = \Phi^+_{\max} \cap \overline{\Phi^+_{\max}}$, so $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}}$. Q.E.D.

The assertions of Proposition 7.1 regarding $S \to S_{\max}$ now are proved. As for $S_{\max} \to G/P$, note that $\mathfrak{b}_{\max} + \overline{\mathfrak{b}_{\max}} = \mathfrak{p}$, so $\mathfrak{b}_{\max}$ is a totally complex polarization for the fibre ($\cong M/T$) of $S_{\max} \to G/P$. Now we combine these.

7.8. LEMMA. *Define $\mathfrak{r} = \mathfrak{q} + (\mathfrak{b} \cap \mathfrak{m})$. Then $\mathfrak{r}$ is a complex subalgebra of $\mathfrak{g}$ such that $\mathfrak{r} + \bar{\mathfrak{r}} = \mathfrak{p}$ and $\mathfrak{r} \cap \bar{\mathfrak{r}} = \mathfrak{b} \cap \bar{\mathfrak{b}}$. In other words, $\mathfrak{r}$ is a totally complex polarization for the fibre of the composition $S \to S_{\max} \to G/P$, which thus is a complex submanifold of X, and the fibre of $S \to S_{\max}$ is a complex submanifold of the fibre of $S \to G/P$.*

Proof. Note from Lemma 7.4(iv) that $\mathfrak{b} \cap \mathfrak{m} = \mathfrak{b}_{\max} \cap \mathfrak{m}$. Since $\mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}}$ is the sume of $\mathfrak{h}$ with the $\mathfrak{g}_{-\alpha}$ such that $\alpha(\xi) > 0$ for a certain $\xi \in \mathfrak{a}_0$, now $[\mathfrak{b} \cap \mathfrak{m}, \mathfrak{q}] \subset \mathfrak{q}$, and it follows that $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}$. Now compute

$$\begin{aligned}
\mathfrak{r} + \bar{\mathfrak{r}} &= \mathfrak{q} + \bar{\mathfrak{q}} + (\mathfrak{b} \cap \mathfrak{m}) + \overline{(\mathfrak{b} \cap \mathfrak{m})} \\
&= \mathfrak{q} + \bar{\mathfrak{q}} + \mathfrak{m} \\
&= \mathfrak{m} + (\mathfrak{b}_{\max} \cap \overline{\mathfrak{b}_{\max}}) \\
&= \mathfrak{m} + \mathfrak{h} + \mathfrak{n}_{//} = \mathfrak{p}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{r} \cap \bar{\mathfrak{r}} &= (\mathfrak{q} \cap \bar{\mathfrak{q}}) + (\mathfrak{q} \cap \overline{\mathfrak{b} \cap \mathfrak{m}}) + (\bar{\mathfrak{q}} \cap \mathfrak{b} \cap \mathfrak{m}) + \mathfrak{t} \\
&= (\mathfrak{b} \cap \bar{\mathfrak{b}}) + (\text{subspaces of } \mathfrak{b} \cap \bar{\mathfrak{b}}) \\
&= \mathfrak{b} \cap \bar{\mathfrak{b}}.
\end{aligned}$$

That completes the argument. Q.E.D.

The proof of Proposition 7.1 is complete. Now we proceed to describe a certain subcomplex of the Cauchy–Riemann complex (4.5). For that, we want to talk about hyperfunctions that are smooth along the fibre of $S \to G/P$.

7.9. LEMMA. *Let $W \to U$ be a C^ω fibration. Then the sheaf of germs of hyperfunctions on W, which are C^∞ along the fibres, is well defined.*

Proof. The basic ingredients of the theory of hyperfunctions make good sense with C^∞ dependence on parameters. See Komatsu [13]. In particular, hyperfunctions on sets $U_1 \times U_2$, U_i open in $\mathbb{R}^{n_i}$, with C^∞ dependence on the $\mathbb{R}^{n_2}$ variable, can be patched together. In other words, such hyperfunctions constitute a sheaf.

Now we sheck that change of local trivialization preserves the sort of hyperfunction just described. So, let $w(u_1, u_2)$ be a hyperfunction on $U_1 \times U_2$ that is C^∞ in u_2. Let $F: U_1 \times U_2 \to U_1 \times U_2$ be a C^ω diffeomorphism of the form $F(u_1, u_2) = (u_1, f(u_1, u_2))$. We want to see that $w(u_1, f(u_1, u_2))$ is $C^{-\omega}$ in (u_1, u_2) and C^∞ in u_2.

Let $\tilde{U}_1 \subset \mathbb{C}^{n_1}$ be a Stein neighborhood of U_1. Let $\tilde{w}(\tilde{u}_1, u_2)$ be a cocycle on $\tilde{U}_1$, representing the hyperfunction $w(u_1, u_2)$, C^∞ parameter u_2. Shrinking U_1, $\tilde{U}_1$, and U_2, we can suppose that f extends to $\tilde{f}: \tilde{U}_1 \times U_2 \to U_2$ holomorphic in $\tilde{U}_1$. Then $\tilde{w}(\tilde{u}_1, \tilde{f}(\tilde{u}_1, u_2))$ represents a hyperfunction with C^∞ dependence on u_2. Q.E.D.

Combining (7.1) and (7.9) we have a well-defined sheaf,

$$\mathscr{C}^{-\omega}_{G/P}(S)\text{: germs of hyperfunctions on } S \text{ that are } C^\infty \text{ along the fibres of } S \to G/P. \tag{7.10}$$

Caution: this notation should not be confused with the notation of Section 6, where the subscript refers to support. (7.10) defines (using $\bar{\partial}_S$) a complex of sheaves

$$\mathscr{C}^{-\omega}_{G/P}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_S)\text{: germs of sections of } \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_S \to S, \text{ coefficients in } C^{-\omega}_{G/P} S. \tag{7.11}$$

Taking global sections we arrive at a subcomplex

$$C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \Lambda^\cdot \mathbb{N}^*_S), \bar{\partial}_S \tag{7.12}$$

of the Cauchy–Riemann complex (4.5). The basic facts about the complex (7.12) are

7.13. PROPOSITION. *The inclusion of* (7.12) *in the Cauchy–Riemann complex* (4.5) *induces isomorphisms of cohomology.*

7.14. PROPOSITION. *The* $C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_S)$ *have natural Fréchet topologies. In those topologies,* $\bar{\partial}_S$ *is continuous and the actions of* G *are Fréchet representations.*

We first prove Proposition 7.13.

7.15. LEMMA. *Let $\mathbb{V} \to Z$ be a holomorphic vector bundle. Then the sheaf $\mathcal{O}(\mathbb{V}) \to Z$ of germs of holomorphic sections has resolution by the complex $\{\mathscr{C}^{-\omega}(Z; \mathbb{V} \otimes \Lambda^{\cdot} \mathbb{T}^{0,1}(Z)^*), \bar{\partial}\}$ of germs of $\mathbb{V}$-valued $(0, \cdot)$-forms with hyperfunction coefficients.*

(This is proved in Komatsu [13].)

7.16. LEMMA. *The inclusions $\mathscr{C}_{G/P}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*) \to \mathscr{C}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*)$ of the sheaves* (7.11) *into the sheaves corresponding to the Cauchy–Riemann complex, induce isomorphisms of cohomology sheaves.*

Proof. The bundle is irrelevant; we need only prove this for scalar forms on an open set $U \times V \subset \mathbb{R}^a \times \mathbb{R}^b$ that are $C^{-\omega}$ on U and C^{∞} on V. Komatsu's argument [13] for the hyperfunction Dolbeault lemma is valid with C^{∞} parameters. This reduces the proof to the usual Dolbeault lemma. Q.E.D.

7.17. LEMMA. *The sheaves $\mathscr{C}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*)$ and $\mathscr{C}_{G/P}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*)$ are soft.*

Proof. Softness is local. We must show that every $x \in S$ has a neighborhood U such that any section over a closed subset of U extends to a global section. For this we may assume that U has compact closure, that $\mathbb{E} \to S$ and $\mathbb{N}_S \to S$ are trivial over U, and that U is the inverse image of a locally trivializing open set for $S \to G/P$. With respect to the latter we write $U = U_1 \times U_2$.

Let $F \subset U$ be a closed set. Since F is compact, any section of $\mathscr{C}^{-\omega}$ over F extends to a neighborhood W of F, $\mathrm{cl}(W) \subset U$. It is standard that a hyperfunction on W extends (not uniquely) to a hyperfunction on U with support in $\mathrm{cl}(W)$, then of course by zero to all of S. That is the (standard) argument that $\mathscr{C}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*)$ is soft. We modify it for $\mathscr{C}_{G/P}^{-\omega}(S; \mathbb{E} \otimes \Lambda^p \mathbb{N}_S^*)$.

First suppose $F = F_1 \times F_2$, $F_i \subset U_i$. Apply the above argument to $F_1 \subset U_1$ with C^{∞} parameters in F_2. Thus any section of $\mathscr{C}_{G/P}^{-\omega}$ over F extends to a global section. For the general case, let σ be a section of $\mathscr{C}_{G/P}^{-\omega}$ over F, and $\tilde{\sigma}$ an extension of σ to an open neighborhood W of F with $\mathrm{cl}(W) \subset U$. Cover F by a finite number of sets $F_\alpha \times V_\alpha$, F_α compact, V_α open, $F_\alpha \times V_\alpha \subset W$. Let $\tilde{\sigma}_\alpha$ be a section over $U_1 \times V_\alpha$ that agrees with $\tilde{\sigma}$ on $F_\alpha \times V_\alpha$ and has support inside $\mathrm{cl}(W)$. Let $\{\phi_\alpha\}$ be a C^{∞} partition of 1 for the covering $\{V_\alpha\}$ of $\bigcup V_\alpha$. Let $\tilde{\phi}_\alpha(u_1, u_2)$ denote $\phi_\alpha(u_1)$. Then $\sum \tilde{\phi}_\alpha \tilde{\sigma}_\alpha$ extends σ to U with support in $\mathrm{cl}(W)$, so it extends further by 0 to a global section of $\mathscr{C}_{G/P}^{-\omega}$. Q.E.D.

A general result in sheaf cohomology [6] says that, from Lemma 7.16, the inclusion of sheaves induces an isomorphism of hypercohomology.

Since both complexes consist of soft sheaves by Lemma 7.17, the hypercohomology is just the cohomology of the associated complex of global sections. That completes the proof of Proposition 7.13.

Now we deal with Proposition 7.14. First notice that the theory of hyperfunctions with values in a reflexive Banach space is developed in exactly the same way as for complex valued hyperfunctions. Thus,

7.18. LEMMA. *Let $\mathbb{V} \to M$ be a C^{ω} vector bundle over a compact C^{ω} manifold M, whose typical fibre is a reflexive Banach space V, and let $\mathbb{V}^* \to M$ be the dual bundle, modeled on V^*. Then the space $C^{-\omega}(M; \mathbb{V})$ of hyperfunction sections has a natural Fréchet topology, which is the strong dual to the natural topology on $C^{\omega}(M; \mathbb{V}^*)$.*

Now let F denote the typical fibre of $S \to G/P$. So $C^{\infty}(F)$ is a Fréchet space , limit of Sobolev spaces V_n of functions on F. That expresses $C^{\infty}(F)$ as a topological limit of Hilbert spaces V_n. The corresponding bundle $\mathbb{C}^{\infty}(F)$ thus is a limit of Hilbert bundles $\mathbb{V}_n$. Lemma 7.18 applies to the $\mathbb{V}_n \to G/P$. Now

$$C_{G/P}^{\omega}(S) = C^{-\omega}(G/P; \mathbb{C}^{\infty}(F)) = \lim C^{-\omega}(G/P; \mathbb{V}_n) \qquad (7.19)$$

has a natural G-invariant topology in which $\bar{\partial}_S$ is continuous. The limit topology here is given by the union of an increasing family of seminorms, so it is Fréchet. Proposition 7.14 follows.

8. THE TENSORING ARGUMENT

In this section we use tensoring arguments to reduce Theorem 5.1 to a special case.

8.1. DEFINITION. An admissible Fréchet G-module *has property* (MG) if it is the maximal globalization of its underlying Harish–Chandra module.

8.2. DEFINITION. A complex $(C^{\cdot}, d)$ of Fréchet G-modules *has property* (MG) if (i) d has closed range, (ii) the cohomolgies $H^p(C^{\cdot}, d)$ are admissible and of finite length, and (iii) each $H^p(C^{\cdot}, d)$ has property (MG).

8.3. DEFINITION. Given a basic datum $(H, \mathfrak{b}, \chi)$, the corresponding homogeneous vector bundle $\mathbb{E} \to S$ *has property* (MG) if the partially smooth Cauchy–Riemann complex (7.12) has property (MG).

In view of Section 6, we want to prove that, for every $(H, \mathfrak{b}, \chi)$, the homogeneous bundle $\mathbb{E} \to S$ has property (MG). But Theorem 5.1 requires more. Denote

$$H^p(S; \mathbb{E}) = H^p(C^{\omega}(S; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}_S^*), \bar{\partial}_S). \tag{8.4}$$

Proposition 7.13 shows that $H^p(S; \mathbb{E})$ is calculated by a Fréchet complex. As $\bar{\partial}_S$ and projection to K-isotypic subspaces commute, $H^p(S; \mathbb{E})_{(K)}$ is calculated by the subcomplex of K-finite forms in that Fréchet complex. These forms are smooth because they are smooth in certain directions by definition and because K is transitive on the transverse directions. In particular, K-finite forms in that Fréchet complex have formal power series at $1 \cdot H$, and so we can use (3.5) to define morphisms

$$H^p(S; \mathbb{E})_{(K)} \to A^p(G, H, \mathfrak{b}, \chi). \tag{8.5}$$

The last aissertion of Theorem 5.1 is that these are isomorphisms.

8.6. Definition. The bundle $\mathbb{E} \to S$ *has property* (Z) if the maps (8.5) are isomorphisms.

Tensoring has to start somewhere. So we consider the following condition on a pair $(H, \mathfrak{b})$.

There exist a positive root system Φ^+ and a number $C > 0$ such that: if $\mathbb{E} \to S$ is irreducible, $\lambda = d\chi \in \mathfrak{h}^*$, $\lambda_{\mathbb{R}}$ is the restriction of λ to the real form $\mathfrak{h}_{\mathbb{R}}$ on which roots take real values, and $\langle \lambda_{\mathbb{R}}, \alpha \rangle > C$ for all $\alpha \in \Phi^+$, then $\mathbb{E} \to S$ has both properties (Z) and (MG). (8.7)

In this section we prove

8.8. Proposition. *Fix* $(H, \mathfrak{b})$. *If* (8.7) *is true then, for arbitrary basic data of the form* $(H, \mathfrak{b}, \chi)$, $\mathbb{E} \to S$ *has both properties* (Z) *and* (MG).

For any $(H, \mathfrak{b})$, Proposition 8.8 reduces the proof of Theorem 5.1 to the proof of (8.7). That will be done in Sections 9 through 13.

8.9. Lemma. *Let* $T: U \to V$ *be a continuous* G*-equivariant map of admissible Fréchet* G*-modules. If* U *and* V *have property* (MG), *then* T *has closed range and both* Ker T *and* Coker T *have property* (MG).

Proof. If Ker $T = 0$, i.e., T is an injection, this is the fact [22] that "maximal globalization" is an *exact* functor on the category of Harish Chandra modules for G.

The case $\operatorname{Ker} T \subsetneq U$ now shows that $U/\operatorname{Ker} T$ has (MG), and then the case $U/\operatorname{Ker} T \subsetneq V$ shows that T has closed range and $V/\operatorname{Im} T$ has (MG).

Q.E.D.

Now consider the category of complexes of Fréchet G-modules with G-equivariant differentials. Morphisms of such complexes are, by definition, continuous and G-equivariant. The content of Proposition 7.14 is that the partially smooth subcomplex (7.12) of the Cauchy–Riemann complex (4.5) belongs to this category.

8.10. LEMMA. *Given a short exact sequence of complexes of Fréchet G-modules, the maps in the corresponding long exact cohomology sequence are continuous.*

Proof. Let $0 \to C' \to^{\alpha} C \to^{\beta} C'' \to 0$ be the short exact sequence in question. Since α and β consist of continuous maps, the induced maps

$$H^p(C') \xrightarrow{\alpha} H^p(C) \xrightarrow{\beta} H^p(C'')$$

are continuous. The connecting maps $\delta\colon H^p(C'') \to H^{p+1}(C')$ are induced by $\alpha^{-1} \circ d \circ \beta^{-1}$, where d is the differential of C. So it suffices to show that if $S \subset (C')^{p+1}$ is a closed subset and is a union of cosets of the kernel of the differential d' of C', then $\beta d^{-1}\alpha(S)$ is closed in $(C'')^p$. $\operatorname{Im} \alpha = \operatorname{Ker} \beta$ is Fréchet so the open mapping theorem says that the $\alpha\colon (C')^q \to \alpha(C')^q$ are homeomorphisms. Now $\alpha(S)$ is closed in C^{p+1}. As d is continuous, $d^{-1}\alpha(S)$ is closed in C^p. Since $d^{-1}\alpha(S)$ is a union of cosets of $\alpha(C')^p$, β maps the complement of $d^{-1}\alpha(S)$ onto the complement of $\beta d^{-1}\alpha(S)$. The open mapping theorem for $\beta\colon C/\operatorname{Ker} \beta \to C''$ says that β is open, so now $\beta d^{-1}\alpha(S)$ is closed. Q.E.D.

8.11. LEMMA. *Let $0 \to C' \to^{\alpha} C \to^{\beta} C'' \to 0$ be a short exact sequence of complexes of Fréchet G-modules. If two of them have* (MG) *then the third has* (MG).

Proof. It suffices to show that the differential of the third complex has closed range. Once that is done, the cohomologies are admissible and have finite length because they fit into an exact sequence where the surrounding terms have these two properties. Also, the long exact cohomology sequence maps naturally into the corresponding sequence of maximal globalizations of underlying Harish–Chandra modules, so exactness of the maximal globalization functor says that the cohomologies of the third complex have (MG).

Write d, d', d'' for the differentials; B, B', B'' for the coboundaries; Z, Z', Z'' for the cocycles.

Let C' and C'' have (MG). The $H^p(C'')$ are Hausdorff so $\mathrm{Ker}\{Z^p \to H^p(C'')\} = \beta^{-1}(B'')^p \cap Z^p = \alpha(Z')^p + dC^{p-1}$ is closed and

$$\alpha \oplus d : (Z')^p \oplus C^{p-1} \to \beta^{-1}(B'')^p \cap Z^p$$

is an open map. If $\omega \in Z^p$ is in the closure of B^p, it is in $\beta^{-1}(B'')^p$ now, so there are nets $\{\phi_n\} \subset C^{p-1}$ and $\{\psi_n\} \subset (Z')^p$ with $\{\phi_n\} \to \phi$, $\{\psi_n\} \to \psi$, where $\omega = \alpha\psi + d\phi$ and $\alpha\psi_n + d\phi_n \in B^p$. Now $\alpha\psi_n \in B^p$ so $[\psi_n] \in H^p(C')$ is annihilated by $\alpha_* : H^p(C') \to H^p(C)$, hence in the image of $H^{p-1}(C'') \to H^p(C')$. That image is closed by Lemma 8.9, so it contains $[\psi]$. Now $\alpha_*[\psi] = 0$, i.e., $[\omega] = 0$, i.e. $\omega \in B^p$.

Let C and C' have (MG). Then an analogous argument shows that the $(B'')^p$ are closed.

Let C and C'' have (MG). Then the image of $H^{p-1}(C) \to H^{p-1}(C'')$ is closed by Lemma 8.1, its pre-image $\beta Z^{p-1} + (B'')^{p-1}$ in $(Z'')^{p-1}$ is closed there, and thus $Z^{p-1} + \alpha(C'')^{p-1} = \beta^{-1}\{\beta Z^{p-1} + (B'')^{p-1}\}$ is closed in C^{p-1}. Now let ω be in the closure of $(B')^p$. Then $\omega \in (Z')^p$ and there is a net $\{\phi_n\} \subset (C')^{p-1}$ with $\{d\phi_n\} \to \omega$. Since $H^p(C)$ is Hausdorff and $H^p(C')^{p-1}$ with $\{d\phi_n\} \to \omega$. Since $H^p(C)$ is Hausdorff and $H^p(C') \to H^p(C)$ is continuous, $\alpha\omega$ is of the form $d\psi$, $\psi \in C^{p-1}$. Now $\{d\alpha\phi_n\} \to d\psi$. B^p is closed, so d: $C^{p-1} \to B^p$ is an open map, and thus one has $\{\psi_n\} \subset C^{p-1}$ with $\{\psi_n\} \to \psi$ and $d\psi_n = d\alpha\phi_n$. Now the ψ_n, hence ψ, are in the closed subspace $Z^{p-1} + \alpha(C')^{p-1}$. So $\alpha\omega = d\psi \in d\alpha(C')^{p-1} = \alpha d'(C')^{p-1} = \alpha(B')^p$. Since α is injective, now $\omega \in (B')^p$. Q.E.D.

Let $0 \subset E_1 \subset \cdots \subset E_k = E$ be a composition series for the $(H, \mathfrak{b})$-module E. In other words, the corresponding homogeneous bundles $0 \subset \mathbb{E}_1 \subset \cdots \subset \mathbb{E}_k = \mathbb{E}$ over S satisfy: the quotient bundles $\mathbb{L}_j = \mathbb{E}_j/\mathbb{E}_{j-1} \to S$ are irreducible. By induction on k, Lemma 8.11 gives us

8.12. Corollary. *If all the* $\mathbb{L}_j \to S$ *have* (MG) *then* $\mathbb{E} \to S$ *has* (MG). *If* $\mathbb{E} \to S$ *and all but one of the* $\mathbb{L}_j \to S$ *have* (MG), *then the remaining* $\mathbb{L}_i \to S$ *has* (MG).

The corresponding result holds for property (Z):

8.13. Lemma. *If all the* $\mathbb{L}_j \to S$ *have* (Z) *then* $\mathbb{E} \to S$ *has* (Z). *If* $\mathbb{E} \to S$ *and all but one of the* $\mathbb{L}_j \to S$ *have* (Z), *then the remaining* $\mathbb{L}_i \to S$ *has* (Z).

Proof. The morphisms based on (3.5) induce morphisms (8.5) of long exact cohomology sequences corresponding to a short exact sequence of bundles. If two of these bundles have (Z) then the five-lemma ensures that the third has (Z). That proves the version of Lemma 8.11, where (Z) replaces (MG), and now the assertion follows as in (8.12). Q.E.D.

Let F be a finite dimensional G-module, $\mathbb{F} \to S$ the associated homogeneous vector bundle. The $\mathfrak{g}$-equivariant extension $\tilde{\mathbb{F}} \to \tilde{S}$ to a germ of a neighborhood is holomorphically trivial, so

$$\text{the complex } \{C^{\infty}(S; \mathbb{E} \otimes \mathbb{F} \otimes \Lambda^{\cdot} \mathbb{N}_S^*), \bar{\partial}_S\} \text{ is the direct sum of } \dim F \text{ copies of } \{C^{\infty}(S; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}_S^*), \bar{\partial}_S\}. \tag{8.14}$$

Now, if $\bar{\partial}_S$ has closed range for $\mathbb{E}$ then it has closed range for $\mathbb{E} \otimes \mathbb{F}$. It follows that

8.15. LEMMA. *If* $\mathbb{E} \to S$ *has* (MG) *then* $\mathbb{E} \otimes \mathbb{F} \to S$ *has* (MG).

Similarly, from a glance at the origin (3.5) of (8.5),

8.16. LEMMA. *If* $\mathbb{E} \to S$ *has* (Z) *then* $\mathbb{E} \otimes \mathbb{F} \to S$ *has* (Z).

Now we have the analytic tools for the proof of Proposition 8.8. We still need a geometric tool.

8.17. DEFINITION. Let $F_1, ..., F_l$ be the irreducible finite dimensional $\mathfrak{g}$-modules corresponding to the fundamental highest weights. Let $B \subset \mathfrak{h}_{\mathbb{R}}^*$ be an open ball of some radius with center at 0. Then $\mu \in \mathfrak{h}_{\mathbb{R}}^*$ is *accessible from* B if there exist $\mu_0 \in B$, $j \in \{1, ..., l\}$ and an extremal weight ν of F_j such that $\mu = \mu_0 + \nu$, and $\mu_0 + \nu' \in B$ for any weight $\nu' \neq \nu$ of F_j.

The idea is that the weight system of F_j is in the convex hull of the extremal weights, and appropriate translations $\nu' \mapsto \mu_0 + \nu'$ will keep all the weights inside a large radius ball B except for one extremal weight. In fact,

8.18. LEMMA. *There exists* $r_0 > 0$ *with the following property: If* $B \subset \mathfrak{h}_{\mathbb{R}}^*$ *is an open ball centered at* 0, *with radius* $r \geq r_0$, *then*

$$B \cup \{\mu \in \mathfrak{h}_{\mathbb{R}}^* : \mu \text{ is accessible from } B\}$$

contains an open ball centered at 0 *with radius* $> r$.

Proof. We first check that it suffices to prove

if $\mathfrak{s}$ is a hyperplane in $\mathfrak{h}_{\mathbb{R}}^*$ then there exist a parallel translate $\mathfrak{s}'$ of $\mathfrak{s}$, an integer $j \in \{1, ..., l\}$ and an extremal weight ν of F_j such that ν is on one side of $\mathfrak{s}'$ and the other weights of F_j are on the other side. (8.19)

In effect, a unit vector η in $\mathfrak{h}_{\mathbb{R}}^*$ determines a parallelism class of hyperplanes, and if $\mathfrak{s}$ is in this class then replace η by $-\eta$ if necessary so that $\mathfrak{s}' = \{\xi \in \mathfrak{h}_{\mathbb{R}}^* : \xi \cdot \eta = r\}$ for some $r > 0$. We can deform η slightly, using the same j and ν in (8.19), and the $\mathfrak{s}'$ defined by $\xi \cdot \eta = r$ will still satisfy (8.19). Now we have (j, ν, r) associated to a neighborhood of η in the unit sphere. A finite number of these neighborhoods cover. Let r_0 be the maximum of their numbers $r > 0$. Now (8.19) implies the lemma.

We prove (8.19). Choose $\eta \in \mathfrak{h}^*_{\mathbb{R}}$, unit vector orthogonal to $\mathfrak{s}$ with $\mathfrak{s}$ given by $\xi \cdot \eta = r$ for some $r > 0$. Since (8.19) is invariant under the Weyl group we may assume η in the positive Weyl chamber. So $\eta \cdot \alpha \geqslant 0$ for all simple roots α, $\eta \cdot \alpha > 0$ for some simple root α. Choose $j \in \{1, ..., l\}$ so the the corresponding fundamental highest weight, say ξ_j, has $\xi_j \cdot \alpha > 0$ for a simple root α with $\eta \cdot \alpha > 0$. If ν' is another weitht of F_j then $\eta \cdot \nu' < \eta \cdot \xi_j$. Now (8.19) follows with $\nu = \xi_j$. Q.E.D.

8.20. *Proof of Proposition* 8.8. Assume (8.7). If Proposition 8.8 fails, then in view of Corollary 8.12 and Lemma 8.13, Proposition 8.8 must fail for a basic datum $(H, \mathfrak{b}, \chi)$ with $\mathbb{E} \to S$ irreducible.

Let $\mathbb{E} \to S$ be irreducible, so $d\chi = \lambda \in \mathfrak{h}^*$. Let C be as in (8.7), r_0 as in Lemma 8.18, and $\lambda_0 \in \mathfrak{h}^*_{\mathbb{R}}$. If $\langle \lambda_0, \alpha \rangle$ is suffciently large for all $\alpha \in \Phi^+$, then $\|\lambda_{\mathbb{R}} - \lambda_0\| < r_0$ forces all $\langle \lambda_{\mathbb{R}}, \alpha \rangle > C$ for $\alpha \in \Phi^+$, so by (8.7) $\mathbb{E} \to S$ has both (Z) and (MG). Fix one such λ_0 and let

$$r_1 = \sup\{r > 0 \colon \|\lambda_{\mathbb{R}} - \lambda_0\| < r \text{ implies (Z) and (MG) for } \mathbb{E}\}.$$

Then $r_1 \geqslant r_0$. If $r_1 < \infty$ then Lemma 8.18 provides $r_2 > r_1$ such that the open ball $B(r_2)$, radius r_2 and center 0, is contained in $B(r_1) \cup \{\text{accessible from } B(r_1)\}$. So (Z) and (MG) carry over from $B(r_1)$ to $B(r_2)$ by Lemmas 8.15 and 8.16. That contradicts the choice of r_1. Thus r_1 is infinite. That proves Proposition 8.8. Q.E.D.

9. Maximally Real Polarizations

In this section we show that (8.7) holds for maximally real polarizations. So the results of Sections 7, 8, and 9 combine to give

9.1. Proposition. *Theorem* 5.1 *holds when the basic datum* $(H, \mathfrak{b}, \chi)$ *is such that* $\mathfrak{b}$ *is a maximally real polarization.*

We break the argument into small steps.

9.2. Lemma (Aguilar–Rrodríguez [1]). *If G is connected and H is compactly embedded in G, then the* (Z) *part of* (8.7) *is true.*

Proof. As H is compactly embedded, $\mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{h}$, $S = G/H$ is open in X, and the complexes (4.1), (4.5), (4.8), and (7.12) all coincide, yielding Dolbeault cohomology $H^{\cdot}(S; \mathbb{E})$.

Let $\mathbb{E} \to S$ be irreducible, $\lambda = d\chi \in \mathfrak{h}^*$.

Suppose first that G is relatively compact, i.e., $G = K$. If $\mathbb{E} \to S$ is positive then the Borel–Weil theorem and its Zuckerman module analog say that both sides of (8.5) are zero for $p \neq 0$, irreducible and isomorphic for $p = 0$. So it suffices to show the morphism of (8.5) not identically zero for $p = 0$. That is clear: the Taylor series of a nonzero holomorphic section is nonzero.

We have just proved (Z) for irreducible positive bundles when $G = K$. It follows for irreducible negative bundles when $G = K$, either by Lemma 8.16 or by passage to dual bundles.

Now we are in the general case of Lemma 9.2. If the irreducible bundle $\mathbb{E} \to S$ is sufficiently negative, i.e., if $\langle \lambda, \alpha \rangle \ll 0$ whenever $\alpha > 0$, then [18] $H^p(S; \mathbb{E}) = 0$ for $p \neq \dim_{\mathbb{C}} K/H$, and for $p = \dim_{\mathbb{C}} K/H$ is the underlying Harish-Chandra module for the relative discrete series representation $\pi_{\lambda + \rho}$ of Harish-Chandra parameter $\lambda + \rho$. It is also known [26] that $A^p(G, H, \mathfrak{h}, \chi)$ is 0 for $p \neq \dim_{\mathbb{C}} K/H$, and for $p = \dim_{\mathbb{C}} K/H$ is the Harish-Chandra module for $\pi_{\lambda+\rho}$. Thus again we need only check that the morphism of (8.5) is not identically zero for $p = \dim_{\mathbb{C}} K/H$. So, consider

$$\begin{tikzcd} H^p(G/H; \mathbb{E})_{(K)} \arrow[r, "r_1"] \arrow[d, "m_G"] & H^p(K/H; \mathbb{E}|_{K/H}) \arrow[d, "m_K"] \\ A^p(G, H, \mathfrak{h}, \chi) \arrow[r, "r_2"] & A^p(K, H, \mathfrak{h} \cap \mathfrak{k}, \chi), \end{tikzcd}$$

where m_G, m_K are the morphisms (8.5) for G and K, and r_1, r_2 are restrictions to the maximal compact subvariety. The diagram commutes, the r_i are surjective, and we just saw that m_K is nonzero. So m_G is nonzero. Q.E.D.

9.3. LEMMA. *If H is compactly embedded in G then the* (Z) *part of* (8.7) *is true.*

Proof. Let $Z = Z_G(G^0)$, centralizer of the identity component G^0 in the full group G. Then $ZG^0 = HG^0$ and $H = ZH^0$. Note that $\chi = \zeta \otimes (\chi|_{H^0})$ where $\zeta = \chi|_Z$. Let $\mathbb{E} \to S$ be irreducible and $\lambda = d\chi \in \mathfrak{h}^*$ as before.

Lemma 9.2 says that there is a positive root system Φ^+ such that, for λ deep in the corresponding chamber, the

$$H^p(G^0/H^0; \mathbb{E}^0)_{(K^0)} \to A^p(G^0, H^0, \mathfrak{h}, \chi|_{H^0})$$

are $(\mathfrak{g}, K^0)$-isomorphisms. Here $\mathbb{E}^0 = \mathbb{E}|_{S^0}$ for some component $S^0 \cong G^0/H^0$ of S. Now tensor with ζ to see that the $H^p(ZG^0/H; \mathbb{E}^0)_{(ZK^0)} \to A^p(ZG^0, H, \mathfrak{h}, \chi)$ are $(\mathfrak{g}, ZK^0)$-isomorphisms. Induction from $(\mathfrak{g}, ZK^0)$-modules to $(\mathfrak{g}, K)$-modules carries these to isomorphisms $H^p(G/H; \mathbb{E})_{(K)} \to A^p(G, H, \mathfrak{h}, \chi)$. Q.E.D.

9.4. LEMMA. *If* $\mathfrak{b}$ *is maximally real, then the* (Z) *part of* (8.7) *is true.*

Proof. As described before, G has a cuspidal parabolic subgroup $P=MAN_H$ associated to H, such that $\mathfrak{b}\subset\mathfrak{p}$. Here $H=T\times A$ with $T=H\cap K$ and $A=\exp(\mathfrak{a}_0)$, $S\cong G/H\cdot N_H$, and S fibres over G/P with holomorphic fibres M/T.

Let $\mathbb{E}\to S$ be irreducible. Note that $\chi=\chi_T\otimes\chi_A$, where

$$d\chi=\lambda\in\mathfrak{h}^*,\qquad \chi_T=\chi|_T,\qquad d\chi_T=\lambda|_{\mathfrak{t}},\qquad \chi_A=\exp(\chi|_{\mathfrak{a}}).$$

If $\mathbb{E}|_{M/T}$ is sufficiently negative then, from Lemma 9.3, the

$$H^p(M/T;\mathbb{E}|_{M/T})_{(K\cap M)}\to A^p(M,T,\mathfrak{b}\cap\mathfrak{m},\chi_T) \tag{9.5a}$$

are isomorphisms of $(\mathfrak{m},K\cap M)$-modules; those modules are nonzero just for $p=\dim_{\mathbb{C}}(K\cap M)/T$. So any $(K\cap M)$-finite $\mathbb{E}$-valued closed form on M/T with formal power series coefficients at $1\cdot T$ is cohomologous to a smooth one, and a smooth closed $(K\cap M)$-finite $\mathbb{E}$-valued form on M/T is exact just when its Taylor series at $1\cdot T$ is cohomologous to zero.

Let Z^p and B^p denote the respective spaces of closed and exact smooth $(K\cap M)$-finite $\mathbb{E}$-valued $(0,p)$-forms on M/T, and let ${}^{\circ}Z^p$ and ${}^{\circ}B^p$ denote the corresponding spaces with "smooth" replaced by "formal power series" for the coefficients. So (9.5a) says

$$Z^p/B^p\cong{}^{\circ}Z^p/{}^{\circ}B^p \text{ as } (\mathfrak{m},K\cap M)\text{-modules}, \tag{9.5b}$$

where the isomorphism is induced by $Z^p\to{}^{\circ}Z^p$, Taylor series at $1\cdot T$.

We compute $A^p(G,H,\mathfrak{b},\chi)$ from the complex of K-finite $\mathbb{E}$-valued forms on G/H, formal power series coefficients at $1\cdot H$. Apply the Poincaré lemma (valid in the context of formal power series) to the fibres $\cong N_H$ of $G/H\to G/H\cdot N_H\cong S$. So we can compute $A^p(G,H,\mathfrak{b},\chi)$ from the subcomplex of forms constant along right N_H cosets that annihilate N_H directions, i.e., forms that push down to S. Since these forms are K-finite, thus well-defined functions on the K-orbit of the base point, we can compute $A^p(G,H,\mathfrak{b},\chi)$ from the complex of left K-finite, right $K\cap M$ invariant, functions from K to the Zuckerman complex for M/T. In other words,

$$A^p(G,H,\mathfrak{b},\chi)\cong C^{\infty}(K;{}^{\circ}Z^p)^{K\cap M}_{(K)}/C^{\infty}(K;{}^{\circ}B^p)^{K\cap M}_{(K)}. \tag{9.6a}$$

From (9.5b) we may take the values to be in the spaces of smooth forms,

$$A^p(G,H,\mathfrak{b},\chi)\cong C^{\infty}(K;Z^p)^{K\cap M}_{(K)}/C^{\infty}(K;B^p)^{K\cap M}_{(K)}. \tag{9.6b}$$

Any such function $K\to Z^p$ coms from a closed global K-finite form on S, because M/T is fibre and K is transitive on the base of $S\to G/P\cong K/K\cap M$.

Conclusion: $H^p(S; \mathbb{E})_{(K)} \to A^p(G, H, \mathfrak{b}, \chi)$ is surjective. But here each side is induced from the corresponding side of (9.5a), so the surjection is an isomorphism. Q.E.D.

9.7. LEMMA. *If H is compactly embedded in G, then the* (MG) *part of* (8.7) *is true.*

Proof. As before we may assume that $\mathbb{E}$ is irreducible. Let $\lambda = d\chi$ be deep in the negative Weyl chamber. Then the assertion is essentially contained in [18]. There, it is shown that $H^p(S; \mathbb{E}) = 0$ for $p \neq p_0 = \dim_{\mathbb{C}} K/H$, that $\bar\partial$ has closed range, and that $H^{p_0}(S; \mathbb{E})$ is a certain representation described above in Lemmas 9.2 and 9.3. This last is done by exhibiting a topological isomorphism with the space of sections, of a certain vector bundle over G/K, annihilated by a certain first-order operator $\mathscr{D}$. The maximal globalization maps to this space of sections by $v \mapsto F_v$, where $F_v(x) = p(\pi(x)^{-1}v)$, p projection to the minimal K-type of π. By maximality, this map is an isomorphism. Now use the tensoring argument. Q.E.D.

9.8. LEMMA. *If* $\mathfrak{b}$ *is maximally real then the* (MG) *part of* (8.7) *is true.*

Proof. Again, we may assume that $\mathbb{E}$ is irreducible. Let $\lambda = d\chi = \nu + i\sigma$, $\nu \in i\mathfrak{t}_0^*$ deep in the negative Weyl chamber of $\Phi^+(\mathfrak{m}, \mathfrak{t})$. Then

$$H^p(S; \mathbb{E}) = H^p(C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{N}_S^*), \bar\partial_S)$$

vanishes except in degree $p_0 = \dim_{\mathbb{C}} K_M/T$, $K_M = K \cap M$, and $W = H^{p_0}(S; \mathbb{E})$ is the $C^{-\omega}$ induced representation $\mathrm{Ind}^G_{MAN_H}(\eta \otimes e^{i\sigma})$. This is as in (2.11), except that in the induction step one considers hyperfunctions $G \to V_\eta$ rather than measurable functions that are L^2 on G/P. The induced module W has finite length because η is irreducible. It also has a good Fréchet topology, which one sees as follows.

As $\mathfrak{b}$ is maximally real, $G \cap B = H \cdot N_H$ and $S \to G/P$ has fibre $P/H \cdot N_H = M/T$. So the complex $C^{-\omega}_{G/P}$ consists of forms which, as "functions" on G, are smooth in the MAN_H directions and are hyperfunctions in the transverse directions. The Leray spectral sequence argument relating Dolbeault cohomology of $\mathbb{E}|_{M/T}$ to sections of a vector bundle over M/K_M, which is done explicitly in [18], is constructive and works with hyperfunction parameters. Thus the closed range argument of [18] proves closed range here. In particular, W inherits a Fréchet topology from $C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \Lambda^{p_0} \mathbb{N}_S^*)$.

It remains only to show that W is (MG). The Casselman submodule theorem [4] realizes the underlying Harish-Chandra module of V_η as a subpresentation of a (non-unitary) principal series $(\mathfrak{m}, K_M)$-module. The latter has maximal globalization that is just the appropriate induced representation of M in the context of hyperfunctions [22]. Since V_η is (MG) by Lemma 9.7, now it is a closed subspace of a $C^{-\omega}$ principal series representation space of M. Thus W is (MG). Q.E.D.

Lemmas 9.4 and 9.5 combine to prove (8.7) for maximally real $\mathfrak{b}$. Proposition 8.8 thus says that $\mathbb{E} \to S$ satisfies both (Z) and (MG) when $\mathfrak{b}$ is maximally real. Combine this with Propositions 6.1, 7.6, and 7.14 to obtain Proposition 9.1.

10. Change of Polarization; Statement

In this section we formulate a statement on change of polarization, (10.9), and show how it completes the proof of Theorem 5.1. The statement itself will be proved in Sections 11, 12, and 13.

The Cartan subgroup $H \subset G$ is fixed. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra such that

$$\mathfrak{h} \subset \mathfrak{b} \qquad \text{and} \qquad \mathfrak{b} \text{ is } \textit{not} \text{ maximally real.} \tag{10.1}$$

Lemma 7.2 gives us a complex simple root α such that $\bar{\alpha} \notin \Phi^+$. Denote

$$\Phi_0^+ = s_\alpha \Phi^+, \qquad \mathfrak{b}_0 = s_\alpha \mathfrak{b}, \qquad S_0 = G \cdot \mathfrak{b}_0. \tag{10.2}$$

As in the proof of Proposition 7.1, $G \cap B \subset G \cap B_0$, so S fibres over S_0. More precisely, $\mathfrak{q}_\alpha = \mathfrak{b} + \mathfrak{b}_0 + \bar{\mathfrak{b}}_0$ is normalized by $G \cap B$ and satisfies

$$\mathfrak{q}_\alpha + \bar{\mathfrak{q}}_\alpha = \mathfrak{b}_0 \cap \bar{\mathfrak{b}}_0 \qquad \text{and} \qquad \mathfrak{q}_\alpha \cap \bar{\mathfrak{q}}_\alpha = \mathfrak{b} \cap \bar{\mathfrak{b}}$$

so the fibres of p_0: $S \to S_0$ are complex submanifolds of X. In fact, Lemma 7.3 shows that the fibre

$$p_0^{-1}(1 \cdot B_0) = \exp(\mathfrak{g}_\alpha) \cdot B \cong \mathbb{C} = P^1(\mathbb{C}) \backslash \{0\}. \tag{10.3}$$

Let $\mathfrak{b}_\alpha = \mathfrak{b} + \mathfrak{g}_\alpha = \mathfrak{b}_0 + \mathfrak{g}_{-\alpha}$, let X_α denote the flag manifold of parabolic subalgebras of $\mathfrak{g}$ that are $\mathrm{Int}(\mathfrak{g})$-conjugate to $\mathfrak{b}_\alpha$, and consider the orbit $S_\alpha = G \cdot \mathfrak{b}_\alpha \subset X_\alpha$. The natural projection $p_\alpha : X \to X_\alpha$ is holomorphic, so its restrictions

$$p_\alpha : S \to S_\alpha \text{ (fibre } \mathbb{C}) \qquad \text{and} \qquad p_\alpha : S_0 \to S_\alpha \text{ (bijective)}$$

are CR maps. Note that

$$p_\alpha^{-1} S_\alpha = S \cup S_0 \qquad \text{and} \qquad p_0 = (p_\alpha|_{S_0})^{-1} \circ (p_\alpha|_S).$$

Despite that, p_0 is not CR. Now observe

10.4. LEMMA. *The respective nilradicals* $\mathfrak{n}$, $\mathfrak{n}_0$, *and* $\mathfrak{n}_\alpha$ *of* $\mathfrak{b}$, $\mathfrak{b}_0$, *and* $\mathfrak{b}_\alpha$ *satisfy*

$$\mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{g}_{-\alpha} \qquad \textit{and} \qquad \mathfrak{n}_0 = \mathfrak{n}_\alpha + \mathfrak{g}_\alpha$$

and

$$\mathfrak{n} \cap \bar{\mathfrak{n}} = \mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha \qquad \textit{and} \qquad \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0 = (\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \mathfrak{g}_\alpha + \bar{\mathfrak{g}}_\alpha.$$

Choose a generator $\omega^\alpha \in (\mathfrak{g}_{-\alpha})^*$. We can view ω^α as an element of $\mathfrak{g}^*$ by $\omega^\alpha(\mathfrak{h}) = 0$ and $\omega^\alpha(\mathfrak{g}_\beta) = 0$ for $\beta \neq -\alpha$. In either case ω^α has $\mathfrak{h}$-weight α. Since $\Lambda(\mathfrak{n}_0 + \mathfrak{g}_{-\alpha})^* \simeq \Lambda(\mathfrak{n}_0)^* \otimes \Lambda(\mathfrak{g}_{-\alpha})^*$, exterior product with ω^α and restriction from $\mathfrak{n}_0 + \mathfrak{g}_{-\alpha}$ to $\mathfrak{n}$ define a linear map

$$e(\omega^\alpha) \colon \Lambda\mathfrak{n}_0^* \to \Lambda\mathfrak{n}^*. \tag{10.5}$$

10.6. LEMMA. *Let V be a $\mathfrak{g}$-module. Then the*

$$(-1)^p\, e(\omega^\alpha) \colon V \otimes \Lambda^p \mathfrak{n}_0^* \to V \otimes \Lambda^{p+1}\mathfrak{n}^* \tag{10.7a}$$

define a morphism of complexes (for the Lie algebra cohomologies of $\mathfrak{n}_0$ *and* $\mathfrak{n}$*). This morphism restricts to an injective map of the subcomplexes for relative Lie algebra cohomology of* $(\mathfrak{n}_0, \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0)$ *and* $(\mathfrak{n}, \mathfrak{n} \cap \bar{\mathfrak{n}})$,

$$(V \otimes \Lambda^{\cdot}(\mathfrak{n}_0/\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0)^*)^{\mathfrak{n}_0 \cap \mathfrak{n}_0} \to (V \otimes \Lambda^{\cdot}(\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*)^{\mathfrak{n} \cap \mathfrak{n}}, \tag{10.7b}$$

for which the image

$$e(\omega^\alpha) \cdot (V \otimes \Lambda^p(\mathfrak{n}_0/\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0)^*)^{\mathfrak{n}_0 \cap \mathfrak{n}_0} \subset (V \otimes \Lambda^{p+1}(\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}} + \bar{\mathfrak{g}}_\alpha)^*)^{\mathfrak{n} \cap \mathfrak{n} + \mathfrak{g}_\alpha}. \tag{10.7c}$$

Proof. Let $\phi \in V \otimes \Lambda^p \mathfrak{n}_0^*$. Let $\xi_0, ..., \xi_{p+1} \in \mathfrak{n}$ such that each ξ_i belongs to a root space. As α is simple we cannot have $0 \neq [\xi_i, \xi_j] \in \mathfrak{g}_{-\alpha}$. So $(e(\omega^\alpha) \cdot d\phi)(\xi_0, ..., \xi_{p+1}) = 0 = d(e(\omega^\alpha)\phi)(\xi_0, ..., \xi_{p+1})$ unless some $\xi_i \in \mathfrak{g}_{-\alpha}$. And if $\xi_0 \in \mathfrak{g}_{-\alpha}$ and the other $\xi_i \in \mathfrak{n}_\alpha$ then

$$\begin{aligned}(e(\omega^\alpha) \cdot d\phi)(\xi_0, ..., \xi_{p+1}) &= \omega^\alpha(\xi_0)\, d\phi(\xi_1, ..., \xi_{p+1}) \\ &= -d(e(\omega^\alpha)\phi)(\xi_0, ..., \xi_{p+1}).\end{aligned}$$

Thus (10.7a) is a morphism of complexes.

Lemma 10.4 says $\mathfrak{g}_{-\alpha} \subset \mathfrak{n} \cap \bar{\mathfrak{n}} \subset \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$, so $e(\omega^\alpha)$ maps the annihilator of $\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$ in $\Lambda\mathfrak{n}_0^*$ into the annihilator of $\mathfrak{n} \cap \bar{\mathfrak{n}}$ in $\Lambda\mathfrak{n}^*$. Note that ω^α is $\mathfrak{n}$-invariant, for $\omega^\alpha[\mathfrak{n}, \mathfrak{n}] = 0$ because α is simple. But $e(\omega^\alpha)$ maps $\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$ invariants to $\mathfrak{n} \cap \bar{\mathfrak{n}}$ invariants because $\mathfrak{n} \cap \bar{\mathfrak{n}} \subset \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$. Now the $(-1)^p\, e(\omega^\alpha)$ restrict as asserted to (8.7b). And (10.7b) is injective because $\mathfrak{g}_{-\alpha} \subset \mathfrak{n}_0$, $\mathfrak{g}_{-\alpha} \subset \mathfrak{n}$, and $\mathfrak{g}_\alpha \subset \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$. Thus also $\mathfrak{g}_\alpha \subset \mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0$, and (10.7c) follows. Q.E.D.

Given $\beta \in \Phi(\mathfrak{g}, \mathfrak{h})$ we associate a representation $e^{\beta}: H \to \mathbb{C}^*$ as follows: Recall $\mathrm{Ad}(G) \subset \mathrm{Int}(\mathfrak{g})$ and let J_H be the Cartan subgroup of $\mathrm{Int}(\mathfrak{g})$ that contains $\mathrm{Ad}_G(H)$. Then β exponentiates to a quasi-character on the complex torus J_H, and e^{β} is the lift of its restriction to $\mathrm{Ad}_G(H)$.

The bundle $\mathbb{E}_\chi \to G/H$ pushes down separately to bundles $\mathbb{E}_\chi \to S$ and $\mathbb{E}_\chi \to S_0$. Similarly the bundle $\mathbb{L}_\beta \to G/H$ associated to e^{β} pushes down separately to line bundles $\mathbb{L}_\beta \to S$ and $\mathbb{L}_\beta \to S_0$.

We now apply Lemma 10.6 with $V = C^{-\omega}(G)$, using the identification between the complexes (4.5) and (4.6). Thus the $(-1)^p e(\omega^\alpha)$ induce G-equivariant morphisms of complexes

$$C^{-\omega}(G/H; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \to C^{-\omega}(G/H; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S) \qquad (10.8a)$$

which induces morphisms of subcomplexes

$$C^{-\omega}(S_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \to C^{-\omega}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S). \qquad (10.8b)$$

The change from $\mathbb{E}$ to $\mathbb{E} \otimes \mathbb{L}_{-\alpha}$ in (10.8) is due to the fact that ω^α has weight α. The content of (10.7c) is that the image of (10.8b) consists of forms that are holomorphic along the fibres of $S \to S_0$.

We can now formulate the statement mentioned at the beginning of this section.

> Suppose that χ is irreducible, so $d\chi = \lambda \in \mathfrak{h}^*$, and suppose further that $2\langle \lambda + \rho - \alpha, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not a positive integer. Then (10.8b) induces an isomorphism of cohomology groups. In particular, $H^0(C^{-\omega}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha}), \bar{\partial}_S) = 0$. (10.9)

The main point of this section is

10.10. PROPOSITION. *Fix H, and suppose that* (10.9) *is true whenever χ is irreducible and* $\mathfrak{b}$ *is not maximally real. Then, for arbitrary basic data of the form* $(H, \mathfrak{b}, \chi)$, $\mathbb{E} \to S$ *has both properties* (Z) *and* (MG). *In other words, Theorem* 5.1 *holds for arbitrary basic data of the form* $(H, \mathfrak{b}, \chi)$.

Since the image of (10.8b) consists of forms holomorphic on the fibres of $S \to S_0$, and since $S \to G/P$ factors through $S_0 \to G/P$ by Proposition 7.1, now (10.8b) restricts to a morphism of subcomplexes

$$(-1)^p e(\omega^\alpha): C^{-\omega}_{G/P}(S_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \to C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S). \qquad (10.11)$$

Similarly we can apply Lemma 10.6 with $V = C^{\mathrm{for}}(G)$ as in (3.3), to obtain

$$(-1)^p\, e(\omega^\alpha)\colon C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \to C^{\mathrm{for}}(G/H: \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S). \tag{10.12}$$

Since $e(\omega^\alpha)$ commutes with the Taylor series map, we have a commutative diagram

$$\begin{array}{ccc} C^{-\omega}_{G/P}(S_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0})_{(K)} & \longrightarrow & C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S)_{(K)} \quad (10.13\mathrm{a}) \\ \big\downarrow & & \big\downarrow \\ C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0})_{(K)} & \longrightarrow & C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S)_{(K)} \quad (10.13\mathrm{b}) \end{array}$$

of morphisms of K-finite subcomplexes.

The same orbit $S_{\max}$ serves for both S and S_0 in Proposition 7.1, and by Proposition 9.1 every $\mathbb{E} \to S_{\max}$ has both (Z) and (MG). Thus we may assume by induction on $\dim S - \dim S_{\max}$ that every $\mathbb{E} \to S_0$ has both (Z) and (MG). According to Corollary 8.12 and Lemma 8.13, we need only prove (Z) and (MG) for irreducible $\mathbb{E} \to S$. Finally, the cohomologies and maps that occur in Theorem 5.1 all are compatible with coherent continuation, so we may assume that $2\langle \lambda + \rho - \alpha, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not a positive integer, where $d\chi = \lambda \in \mathfrak{h}^*$. Now, by (10.9), $e(\omega^\alpha)$ induces $H^p(S_0; \mathbb{E}) \cong H^{p+1}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha})$.

The diagram

$$\begin{array}{ccc} C^{-\omega}_{G/P}(S_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) & \longrightarrow & C^{-\omega}_{G/P}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S) \quad (10.14\mathrm{a}) \\ \big\downarrow & & \big\downarrow \\ C^{-\omega}(S_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) & \longrightarrow & C^{-\omega}(S; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S) \quad (10.14\mathrm{b}) \end{array}$$

commutes, and Proposition 7.13 says that the vertical inclusions induce isomorphisms in cohomology. We just saw that (10.14b) induces an isomorphism in cohomology. Now (10.14a) also induces an isomorphism in cohomology.

10.15. Lemma. *The map* (10.14a) *is continuous, injective, and has closed range.*

Proof of Lemma 10.15. A glance at the definition (Lemma 7.20) of the topologies shows that (10.14a) is continuous. It is injective by Lemma 10.6.

Use the identification of complexes (4.5) and (4.6) to write (10.14b) as $\{C^{-\omega}(G) \otimes E \otimes \Lambda^p(\mathfrak{n}_0/\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0)^*\}^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0, H} \to \{C^{-\omega}(G) \otimes E \otimes \mathbb{C}_{-\alpha} \otimes \Lambda^{p+1}(\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*\}^{\mathfrak{n} \cap \bar{\mathfrak{n}}, H}$. Using Lemma 10.4, this exhibits the image of

(10.14a) as all forms in $C_{G/P}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{p+1}\mathbb{N}_S^*)$ which, as sums of monomials in the $\omega^{\gamma}\in(\mathfrak{g}_{-\gamma})^*$, (i) involve ω^{α}, (ii) do not involve $\omega^{-\alpha}$, and (iii) are annihilated by the right action of $\mathfrak{g}_\alpha$ and $\bar{\mathfrak{g}}_\alpha$. So (10.14a) has closed image. Q.E.D.

In view of Lemma 10.15 we have an exact sequence of Fréchet complexes, from sequences

$$0\to C_{G/P}^{-\omega}(S_0; \mathbb{E}\otimes\Lambda^p\mathbb{N}_{S_0}^*)\to C_{G/P}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{p+1}\mathbb{N}_S^*)\to Q^p\to 0,$$

where the Q^p are the quotient Fréchet spaces. Since (10.14a) is an isomorphism in cohomology, all $H^p(Q^{\cdot})=0$, and $Q^{\cdot}$ trivially has property (MG). By induction on $\dim S-\dim S_{\max}$, the complex $C_{G/P}^{-\omega}(S_0; \mathbb{E}\otimes\Lambda^{\cdot}\mathbb{N}_{S_0}^*)$ has (MG). Now Lemma 8.11 tells us that $C_{G/P}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)$ has property (MG).

The morphism (10.13b) induces maps $A^p(G, H, \mathfrak{b}_0, \chi)\to A^{p+1}(G, H, \mathfrak{b}, \chi\otimes e^{-\alpha})$. Consider the dual map of the dual Beillinson–Bernstein modules [7]. In [8] it is proved to be an isomorphism. Thus (10.13b) induces an isomorphism of cohomology. So does (10.13a), by our assumption of (10.9) and passage to the K-finite subcomplex. Also, by induction on $\dim S-\dim S_{\max}$, the first vertical arrow in (10.13) is an isomorphism on cohomology. Now the second vertical arrow in (10.13) is a cohomology isomorphsim. Thus $C_{G/P}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)$ has property (Z).

This completes the proof of Proposition 10.10.

We end this section with a variation on (10.9). Let $C_{S_0}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)$ denote the subcomplex of $C^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)$ consisting of forms ϕ such that ϕ and $\bar{\partial}_S\phi$ vanish on (0, 1) vectors tangent to the fibres of $S\to S_0$. The condition on $\bar{\partial}_S\phi$ says that the coefficients of ϕ are holomorphic along these fibres. Apply the Dolbeault lemma fibre by fibre to see that

$$C_{S_0}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)\subsetneq C^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*) \qquad (10.16)$$

induces isomorphisms on cohomology. But (10.7c) says that $e(\omega^\alpha)\cdot C^{-\omega}(S_0; \mathbb{E}\otimes\Lambda^{\cdot}\mathbb{N}_{S_0}^*)$ is contained in $C_{S_0}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{\cdot}\mathbb{N}_S^*)$. Thus (10.9) is equivalent to

> Suppose that χ is irreducible, so $d\chi=\lambda\in\mathfrak{h}^*$, and suppose further that $2\langle\lambda+\rho-\alpha, \alpha\rangle/\langle\alpha, \alpha\rangle$ is not a positive integer. Then the morphism of complexes defined by the
>
> $$(-1)^p e(\omega^\alpha)\colon C^{-\omega}(S_0; \mathbb{E}\otimes\Lambda^p\mathbb{N}_{S_0}^*)\to C_{S_0}^{-\omega}(S; \mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{p+1}\mathbb{N}_S^*)$$
>
> induces isomorphisms of cohomology. (10.17)

The point is that we can localize (10.17) with respect to S_α. Let $U_\alpha \subset S_\alpha$ be an S_α-open subset whose S_α-closure $\bar{U}_\alpha$ is compact and has an X_α-open neighborhood over which $p_\alpha : X \to X_\alpha$ is holomorphically trivial. Let $U_0 = S_0 \cap p_\alpha^{-1} U_\alpha$ and $U = S \cap p_\alpha^{-1} U_\alpha$. Then $(-1)^p e(\omega^\alpha)$ localizes to maps

$$C^{\omega}(U_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \to C_{S_0}{}^{\omega}(U; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S). \qquad (10.18)$$

Some generalities on sheaf theory—see the paragraph after the proof of Lemma 7.17—say that if the morphisms (10.18) induce isomorphisms in cohomology as U_α ranges over a basis for the topology of S_α, then the map in (10.17) induces isomorphism in cohomology. Thus (10.9) and (10.17) are equivalent to

> Suppose that χ is irreducible, so $d\chi = \lambda \in \mathfrak{h}^*$, and suppose further that $2\langle \lambda + \rho - \alpha, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not a positive integer. Then every $x_\alpha \in S_\alpha$ has an open neighborhood U_α as just before (10.18), such that (10.18) induces isomorphisms of cohomology. (10.19)

11. Change of Polarization: Duality on the Fibre

We are going to prove (10.19), and thus complete the proof of Theorem 5.1, by reducing (10.19) to a certain dual statement in this section and the next, and proving the dual statement in Section 13.

We identify $(P^1(\mathbb{C}), \{0\})$ with the fibre of the pair of projections

$$S \cup S_0 = p_\alpha^{-1} S_\alpha \to S_\alpha \qquad \text{and} \qquad S_0 \to S_\alpha \qquad (11.1)$$

considered in Section 10. The isotropy subgroup $G_{\mathfrak{b}_\alpha}$ of G at $\mathfrak{b}_\alpha \in X_\alpha$ acts transitively on $P^1(\mathbb{C}) \setminus \{0\}$, the fibre of $S \to S_\alpha$, with isotropy subgroup $G_{\mathfrak{b}}$ at $\mathfrak{b}$. Then $\mathfrak{g}_\alpha$ represents the holomorphic tangent space of the fibre, and the polarization is given by $\mathfrak{q} = \mathfrak{h} + (\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \mathfrak{g}_\alpha$. Here $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{b}_\alpha \cap \bar{\mathfrak{b}}_\alpha$ and $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{b} \cap \bar{\mathfrak{b}}$ as in the proof of Lemma 7.7.

In this section it will be convenient to distinguish between the homogeneous vector bundles over S_0, S, and S_α associated by χ, in a somewhat more explicit way. We will write $\mathbb{E}_0 \to S_0$, $\mathbb{E} \to S$, and $\mathbb{E}_\alpha \to S_\alpha$ for the bundles that come from the respective basic data $(H, \mathfrak{b}_0, \chi)$, $(H, \mathfrak{b}, \chi)$, and $(H, \mathfrak{b}_\alpha, \chi)$. Also, if χ is integral, i.e., $\mathbb{E}_0$ extends to $\mathbb{E}_X \to X$, then we note that $\mathbb{E}_{X|S}$ comes from the basic datum $(H, \mathfrak{b}, \chi \circ s_\alpha)$.

Now define the $(\mathfrak{b}_\alpha, G_{\mathfrak{b}_\alpha})$-module,

$$V = H^0(P^1(\mathbb{C}) \setminus \{0\}; \mathcal{O}(\mathbb{E}^*)) = \{C^\infty(G_{\mathfrak{b}_\alpha}) \otimes E^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \mathfrak{g}_\alpha : H}. \qquad (11.2)$$

For the moment let us assume

11.3. LEMMA. *The strong topological dual V' is naturally identified with the $(\mathfrak{b}_\alpha, G_{\mathfrak{b}_\alpha})$-module $\mathcal{O}_{\{0\}}(\tilde{\mathbb{E}}_0|_{\text{germ of nbhd of } 0 \text{ in } P^1(\mathbb{C})})$, where $\tilde{\mathbb{E}}_0$ is a $\mathfrak{g}$-equivariant extension of $\mathbb{E}_0$ to a germ of a neighborhood of S_0 in $S \cup S_0$.*

Since V' is a $(\mathfrak{b}_\alpha, G_{\mathfrak{b}_\alpha})$-module it defines an (infinite dimensional) homogeneous CR bundle $\mathbb{V}'_\alpha \to S_\alpha$. That pulls back to a homogeneous CR-bundle $\mathbb{V}'_0 = p^*_\alpha \mathbb{V}'_\alpha \to S_0$ such that the identity on V' induces a CR bundle map from $\mathbb{V}'_\alpha$ to $\mathbb{V}'_0$. Evaluation at 0 defines a $(\mathfrak{b}_0, G_{\mathfrak{b}_0})$ equivariant map $V' \to E$, and thus a CR bundle map $\mathbb{V}'_0 \to \mathbb{E}_0$. Combining these two steps we have

11.4. LEMMA. *Pull back from S_α to S_0, and evaluation $V' \to E$ at 0, combine to give a CR bundle map $\mathbb{V}'_\alpha \to S_\alpha$ to $\mathbb{E}_0 \to S_0$.*

The map $E^* \to V$ dual to $V' \to E$, is $(\mathfrak{b}_0, G_{\mathfrak{b}_0})$-invariant but not $(\mathfrak{b}_\alpha, G_{\mathfrak{b}_\alpha})$-invariant; in fact E^* is not a $\mathfrak{b}_\alpha$-module. So the induced bundle map from $\mathbb{E}^*_0 \to S$ to $\mathbb{V} \to S_\alpha$ is not CR. Nevertheless we need to be more precise about this map, and we will show that $E^* \to V$ is given by

$$
\begin{aligned}
E^* &\cong \{\mathfrak{g}_\alpha\text{-invariants in } (C^\infty(G_{\mathfrak{b}_\alpha}) \otimes E)^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \bar{\mathfrak{g}}_\alpha; H}\} \\
&\hookrightarrow \{C^\infty(G_{\mathfrak{b}_\alpha}) \otimes E\}^{(\mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha) + \mathfrak{g}_\alpha; H} \cong V.
\end{aligned}
\tag{11.5}
$$

In order to prove Lemma 11.3 and (11.5), we look at a single fibre for (11.1) in the abstract.

Let $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, standard generators for $\mathfrak{sl}(2; \mathbb{C})$. They act on the inhomogeneous coordinate z of $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by $hz = -2z$, $ez = -1$, $fz = z^2$. Consider the $SL(2; \mathbb{C})$-homogeneous holomorphic line bundle of degree n, $n \in \mathbb{Z}$,

$$
\mathbb{L}_n \to P^1(\mathbb{C}), \text{ fibre }
\begin{cases}
\text{at 0 associated to } \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mapsto a^n \\
\text{at } \infty \text{ associated to } \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto a^{-n}.
\end{cases}
\tag{11.6}
$$

It has meromorphic sections σ_n and τ_n, related by $\tau_n = z^n \sigma_n$, specified (to constant multiple) by their divisors

$$
(\sigma_n) = n \cdot \infty \qquad \text{and} \qquad (\tau_n) = n \cdot 0. \tag{11.7}
$$

Now $h \cdot \sigma_n = n\sigma_n$ and $e \cdot \sigma_n = 0$ from the definitions of $\mathbb{L}_n$ and σ_n. The commutation relations in $\mathfrak{sl}(2; \mathbb{C})$ and the action of $\mathfrak{sl}(2; \mathbb{C})$ on z now give

$$
\begin{gathered}
h(z^k \sigma_n) = (n - 2k)\, z^k \sigma_n, \qquad e(z^k \sigma_n) = -kz^{k-1} \sigma_n, \\
f(z^k \sigma_n) = (k - n)\, z^{k+1} \sigma_n
\end{gathered}
\tag{11.8a}
$$

and, similarly,

$$h(z^k\tau_n) = (-2k-n)\, z^k\tau_n, \qquad e(z^k\tau_n) = (-k-n)\, z^{k-1}\tau_n,$$
$$f(z^k\tau_n) = kz^{k+1}\tau_n \tag{11.8b}$$

for $k \in \mathbb{Z}$. In fact, for $n = \lambda \in \mathbb{C}$ these relations define $\mathfrak{sl}(2;\mathbb{C})$-equivariant holomorphic line bundles

$$\mathbb{L}_\lambda \to P^1(\mathbb{C})\setminus\{\infty\} = \mathbb{C}, \qquad \text{which has spanning section } \sigma_\lambda, \tag{11.9a}$$

$$\mathbb{L}_\lambda \to P^1(\mathbb{C})\setminus\{0\}, \qquad \text{which has spanning section } \tau_\lambda. \tag{11.9b}$$

As in (11.6), the fibre at 0 of (11.9a) is an $h\mathbb{C} + f\mathbb{C}$ module, while the fibre at ∞ of (11.9b) is the opposite $h\mathbb{C} + e\mathbb{C}$ module. The respective spaces of h-finite holomorphic sections are irreducible $\mathfrak{sl}(2;\mathbb{C})$-modules, $\mathfrak{sl}(2;\mathbb{C})$-invariantly paired with dual bases $\{z^{-k}\tau_\lambda\}$ and $\{(-1)^k\binom{\lambda}{k} z^k\sigma_\lambda\}$. Now if $\lambda \notin \{0, 1, 2, \dots\}$ then

$$H^0(P^1(\mathbb{C})\setminus\{0\}; \mathcal{O}(\mathbb{L}_\lambda)) = \left\{ \sum_{k=0}^{\infty} a_k z^{-k}\tau_\lambda : \limsup |a_k|^{1/k} = 0 \right\} \tag{11.10}$$

has strong topological dual

$$\left\{ \sum_{k=0}^{\infty} b_k z^k\sigma_\lambda : \limsup |b_k|^{1/k} < \infty \right\} = \mathcal{O}_{\{0\}}(\mathbb{L}_\lambda). \tag{11.11}$$

The condition $\lambda \notin \{0, 1, 2, \dots\}$ ensures that the modules (11.10) and (11.11) are irreducible; when they are reducible the duality fails. In the irreducible case, relations (10.8) show that (11.10) and (11.11) are dual on the h-finite level, and then the duality follows by looking at convergence.

Evaluation of germs at 0 now gives an h-invariant map

$$\mathcal{O}_{\{0\}}(\mathbb{L}_\lambda) \to \mathbb{L}_\lambda|_0 = L_\lambda \tag{11.12}$$

which is dual to the inclusion

$$L_{-\lambda} = \operatorname{Ker}\{f \text{ on } H^0(P^1(\mathbb{C})\setminus\{0\}; \mathcal{O}(\mathbb{L}_\lambda))\} \hookrightarrow H^0(P^1(\mathbb{C})\setminus\{0\}; \mathcal{O}(\mathbb{L}_\lambda)). \tag{11.13}$$

In the duality between (11.12) and (11.13), ∞ does not really play a special role. For $\mathbb{L}_\lambda$ is a well-defined $\mathfrak{sl}(2;\mathbb{C})$-homogeneous line bundle over $P^1(\mathbb{C})\setminus K$ for any compact subset $K \subset P^1(\mathbb{C})\setminus\{0\}$, and $\mathcal{O}_{\{0\}}(\mathbb{L}_\lambda) = \varinjlim\{\mathcal{O}_{P^1(\mathbb{C})\setminus K}(\mathbb{L}_\lambda) : K \subset P^1(\mathbb{C})\setminus\{0\} \text{ compact}\}$.

We now identify $(P^1(\mathbb{C}), \{0\})$ with the fibre of $S \cup S_0 \to S_\alpha$ and $S_0 \to S_\alpha$ as in (11.1), with $\{\infty\}$ corresponding to the base point of S. Then Lemma 11.3 follows from the duality between (11.10) and (11.11), and (11.5) follows from the duality between (11.12) and (11.13).

Some comments are in order. The root space $\mathfrak{g}_\alpha$ stabilizes the fibre $P^1(\mathbb{C})$ of $X \to X_\alpha$ over $\mathfrak{b}_\alpha$, operates nontrivially, and fixes $\mathfrak{b}_0 = \mathfrak{h} + \mathfrak{g}_\alpha + \mathfrak{n}_\alpha$, which corresponds to $0 \in P^1(\mathbb{C})$. Similarly, f in the $\mathfrak{sl}(2; \mathbb{C})$ considerations preserves 0. So $\mathfrak{g}_\alpha$ corresponds to $\mathbb{C}f$. Second, α is simple so $\mathfrak{h} + (\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \bar{\mathfrak{g}}_\alpha$ is an ideal in $\mathfrak{b}_\alpha \cap \bar{\mathfrak{b}}_\alpha = \mathfrak{h} + (\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \bar{\mathfrak{g}}_\alpha + \mathfrak{g}_\alpha$ so it does not matter whether $\mathfrak{g}_\alpha$ acts from the left or the right in (11.5). Third, χ^* lifts from H to $G_{\mathfrak{b}_\alpha} = H \cdot \exp_G(\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0)$ and induces

$$E^* \cong \{C^{\chi}(G_{\mathfrak{b}_\alpha}) \otimes E^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha) + \bar{\mathfrak{g}}_\alpha + \mathfrak{g}_\alpha; H}, \tag{11.14}$$

which is implicit in (11.5).

12. Change of Polarization: Reduction to Dual Statement

The CR canonical bundles $\mathbb{K}_S \to S$, $\mathbb{K}_{S_0} \to S_0$, and $\mathbb{K}_{S_\alpha} \to S_\alpha$ are discussed in the Appendix. Following (6.4) and (6.6), their typical fibres at the base points $\mathfrak{b}$, $\mathfrak{b}_0$ and $\mathfrak{b}_\alpha$, are the respective highest exterior powers of $(\mathfrak{g}/\mathfrak{b})^*$, $(\mathfrak{g}/\mathfrak{b}_0)^*$, and $(\mathfrak{g}/\mathfrak{b}_\alpha)^*$. In view of Lemma 10.4 those CR canonical bundles are

$$\mathbb{L}_{-2\rho} \to S, \qquad \mathbb{L}_{-2\rho+2\alpha} \to S_0, \qquad \mathbb{L}_{-2\rho+\alpha} \to S_\alpha. \tag{12.1}$$

In particular, the analog of (A.9) for S_0 is

$$C^{-\omega}(U_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \cong \frac{C^\omega(\mathrm{cl}(U_0); \mathbb{F} \otimes \Lambda^{c-p} \mathbb{N}^*_{S_0})'}{C^\omega(\mathrm{bd}(U_0); \mathbb{F} \otimes \Lambda^{c-p} \mathbb{N}^*_{S_0})'} \tag{12.2a}$$

where $c = \dim_{CR} S_0$ and

$\mathbb{F} = \mathbb{E}^* \otimes \mathbb{L}_{-2\rho+2\alpha}$, bundle over S_0 associated to the basic datum $(H, \mathfrak{b}_0, \chi^* \otimes e^{-2\rho+2\alpha})$. (12.2b)

Forms in $C_{S_0}^{-\omega}(U; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S)$ can be viewed as forms in $C^{-\omega}(U_\alpha; \mathbb{H} \otimes \Lambda^{p+1} \mathbb{N}^*_{S_\alpha})$, where $\mathbb{H} \to S_\alpha$ has fibre over $x \in S_\alpha$ the space of homomorphic sections of $(\mathbb{E} \otimes \mathbb{L}_{-\alpha})|_{S \cap \pi^{-1}(x)}$. The analog of (A.9) for $C^{-\omega}(U_\alpha; \mathbb{H} \otimes \Lambda^{p+1} \mathbb{N}^*_{S_\alpha})$ and the fibre duality (11.5) combine as follows. Let $\mathrm{cl}(U_0)^\sim$ and $\mathrm{bd}(U_0)^\sim$ denote germs of neighborhoods of $\mathrm{cl}(U_0)$ and $\mathrm{bd}(U_0)$ in $S \cup S_0$. Let $C^\omega_{S_0}(\mathrm{cl}(U_0)^\sim; ...)$ denote the space of C^ω forms on $\mathrm{cl}(U_0)^\sim$ that vanish on sets of vectors, one of which is tangent to the fibre and which are holomorphic along the fibre. Use the same convention for $\mathrm{bd}(U_0)^\sim$. Let $\tilde{\mathbb{F}}$ denote $\mathfrak{g}$-equivariant extension of $\mathbb{F}$. Then

$$C_{S_0}^{-\omega}(U; \mathbb{E} \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_S) \cong \frac{C^\omega_{S_0}(\mathrm{cl}(U_0)^\sim; \tilde{\mathbb{F}} \otimes \Lambda^{c-p} \mathbb{N}^*_S)'}{C^\omega_{S_0}(\mathrm{bd}(U_0)^\sim; \tilde{\mathbb{F}} \otimes \Lambda^{c-p} \mathbb{N}^*_S)'} \tag{12.3}$$

with c and $\mathbb{F}$ as in (12.2). Here we went from $p+1$ to $c-p=(c+1)-(p+1)$ because $c+1=\dim_{\mathbf{CR}} S_\alpha$, and from $\mathbb{E}\otimes\mathbb{L}_{-\alpha}$ to $\mathbb{F}=(\mathbb{E}\otimes\mathbb{L}_{-\alpha})^*\otimes\mathbb{L}_{-2\rho+\alpha}$ because of the S_α part of (12.1).

Now consider three statements, as follows:

> Restriction from $\mathrm{cl}(U_0)^\sim$ to $\mathrm{cl}(U_0)$,
>
> $$C^\omega_{S_0}(\mathrm{cl}(U_0)^\sim;\tilde{\mathbb{F}}\otimes\Lambda^\cdot\mathbb{N}^*_S)\to C^\omega(\mathrm{cl}(U_0);\mathbb{F}\otimes\Lambda^\cdot\mathbb{N}^*_{S_0})$$
>
> and restriction from $\mathrm{bd}(U_0)^\sim$ to $\mathrm{bd}(U_0)$,
>
> $$C^\omega_{S_0}(\mathrm{bd}(U_0)^\sim;\tilde{\mathbb{F}}\otimes\Lambda^\cdot\mathbb{N}^*_S)\to C^\omega(\mathrm{bd}(U_0);\mathbb{F}\otimes\Lambda^\cdot\mathbb{N}^*_{S_0})$$
>
> are continuous and surjective. (12.4)

> The maps $(-1)^p\, e(\omega^\alpha)$ of (10.18),
>
> $$C^{-\omega}(U_0;\mathbb{E}\otimes\Lambda^p\mathbb{N}^*_{S_0})\to C_{S_0}{}^{-\omega}(U;\mathbb{E}\otimes\mathbb{L}_{-\alpha}\otimes\Lambda^{p+1}\mathbb{N}^*_S)$$
>
> are dual via (12.2) and (12.3) to the restriction maps (12.4). (12.5)

The restriction maps (12.4) induce isomorphism in cohomology. (12.6)

The result of this section is

12.7. PROPOSITION. *Let χ be irreducible, so $d\chi=\lambda\in\mathfrak{h}^*$, and suppose that $2\langle\lambda+\rho-\alpha,\alpha\rangle/\langle\alpha,\alpha\rangle\notin\{1,2,3,\ldots\}$. Then statements* (12.4) *and* (12.5) *hold, and if* (12.6) *holds then Theorem* 5.1 *holds for* $(H,\mathfrak{b},\chi)$.

At first we assume (12.4), (12.5), and (12.6), and we will deduce Theorem 5.1 for $(H,\mathfrak{b},\chi)$. According to Proposition 10.10 it suffices to prove (10.9), and for that we need only prove (10.19).

By (12.4) and functoriality of restriction, we have a commutative diagram of complexes,

$$\begin{array}{ccccccccc}
0\to & J^\cdot(\mathrm{cl}(U_0)^\sim) & \to & C^\omega_{S_0}(\mathrm{cl}(U_0)^\sim;\tilde{\mathbb{F}}\otimes\Lambda^\cdot\mathbb{N}^*_S) & \to & C^\omega(\mathrm{cl}(U_0);\mathbb{F}\otimes\Lambda^\cdot\mathbb{N}^*_{S_0}) & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & J^\cdot(\mathrm{bd}(U_0)^\sim) & \to & C^\omega_{S_0}(\mathrm{bd}(U_0)^\sim;\tilde{\mathbb{F}}\otimes\Lambda^\cdot\mathbb{N}^*_S) & \to & C^\omega(\mathrm{bd}(U_0);\mathbb{F}\otimes\Lambda^\cdot\mathbb{N}^*_{S_0}) & \to 0,
\end{array}$$

where the horizontal sequences are topologically exact. These spaces are complete locally convex and Hausdorff, so the horizontal sequences in the dual diagram

$$\begin{array}{ccccccccc} 0 \to & C^{\omega}(\mathrm{cl}(U_0); \mathbb{F} \otimes \Lambda^{\cdot} \mathbb{N}^*_{S_0})' & \to & C^{\omega}_{S_0}(\mathrm{cl}(U_0)^{\sim}; \tilde{\mathbb{F}} \otimes \Lambda^{\cdot} \mathbb{N}^*_{S})' & \to & J^{\cdot}(\mathrm{cl}(U_0)^{\sim})' & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & C^{\omega}(\mathrm{bd}(U_0); \mathbb{F} \otimes \Lambda^{\cdot} \mathbb{N}^*_{S_0})' & \to & C^{\omega}_{S_0}(\mathrm{bd}(U_0)^{\sim}; \tilde{\mathbb{F}} \otimes \Lambda^{\cdot} \mathbb{N}^*_{S})' & \to & J^{\cdot}(\mathrm{bd}(U_0)^{\sim})' & \to 0 \end{array}$$

also are exact. By (12.6), the complexes $J^{\cdot}(\mathrm{cl}(U_0)^{\sim})$ and $J^{\cdot}(\mathrm{bd}(U_0)^{\sim})$ have all cohomologies zero, so their differentials have closed range. Now $H^*(J^{\cdot}(\mathrm{cl}(U_0)^{\sim})') = H^*(J^{\cdot}(\mathrm{cl}(U_0)^{\sim}))' = 0$ and, similarly, $H^*(J^{\cdot}(\mathrm{bd}(U_0)^{\sim})' = 0$. Now the horizontal arrows $C^{\omega}(...)' \to C^{\omega}_{S_0}(...)'$ in the dual diagram induce isomorphisms of cohomology. By (12.5) those horizontal arrows given by $(-1)^p e(\omega^{\alpha})$. Now (12.2), (12.3), and the five-lemma say that (10.18) induces an isomorphism in cohomology. That is the statement of (10.19).

For small U_0, (12.4) is just the fact that restriction of real analytic functions from an open set $W \subset \mathbb{R}^n$ to a set of the form $W \cap \mathbb{R}^k$ is surjective.

Now Proposition 12.7 is reduced to the proof of (12.5).

Let V be as in (11.2) and $\mathbb{V}'_{\alpha} \to S_{\alpha}$ the CR bundle associated to V'. Just as noted after (12.2), we can identify $C^{\omega}_{S_0}(U; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_S)$ with $C^{\omega}(U_{\alpha}; \mathbb{V}'_{\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_{S_{\alpha}})$. That gives us $C^{\omega}_{S_0}(\tilde{U}_0; \tilde{\mathbb{E}} \otimes \Lambda^p \mathbb{N}^*_S) \cong C^{\omega}(U_{\alpha}; \mathbb{V}'_{\alpha} \otimes \Lambda^{p+1} \mathbb{N}^*_{S_{\alpha}})$. With this identification the restriction maps

$$C^{\omega}_{S_0}(\tilde{U}_0; \tilde{\mathbb{E}} \otimes \Lambda^p \mathbb{N}^*_S) \to C^{\omega}(U_0; \mathbb{E} \otimes \Lambda^p \mathbb{N}^*_{S_0}) \tag{12.8}$$

are given as follows. Let $\psi_1 \in C^{\omega}(U_{\alpha}; \mathbb{V}'_{\alpha})$ and $\psi_2 \in C^{\omega}(U_{\alpha}; \Lambda^{p+1} \mathbb{N}^*_{S_{\alpha}})$. Then ψ_2 pulls back to $p^*_{\alpha}\psi_2 \in C^{\omega}(U_0; \Lambda^{p+1} \mathbb{N}^*_{S_0})$ and ψ_1 evaluates at 0, as in Lemma 11.4, to $\psi_1(0) \in C^{\omega}(U_0; \mathbb{E}_0)$. Now $\psi_1 \otimes \psi_2$ maps to $\psi_1(0) \otimes p^*_{\alpha}\psi_2 \in C^{\omega}(U_0; \mathbb{E}_0 \otimes \Lambda^{p+1} \mathbb{N}^*_{S_0})$.

The dual of $\psi_1 \mapsto \psi_1(0)$, i.e., of evaluation $\mathbb{V}'_{\alpha} \to \mathbb{E}_0$ at 0, is the inclusion $\mathbb{E}^*_0 \subsetneq \mathbb{V}_{\alpha}$ given by (11.5). Thus, to dualize (12.8), we only need to dualize pullback $p^*_{\alpha}: C^{\omega}(U_{\alpha}; \Lambda^{\cdot} \mathbb{N}^*_{S_{\alpha}}) \to C^{\omega}(U_0: \Lambda^{\cdot} \mathbb{N}^*_{S_0})$ on scalar forms.

Since C^{∞} forms are dense in $C^{-\omega}$ forms, we may at this point simply argue that the $(-1)^p e(\omega^{\alpha})$ on C^{∞} forms are dual to the restrictions as in (12.4) of C^{∞} forms. This is in sharp contrast to the situation of (12.6), which is specific to the C^{ω} setting and will be proved in Section 13.

Formal self duality of the Cauchy–Riemann complex, (A.8) for S_{α}, says that $C^{\infty}(U_{\alpha}; \Lambda^{\cdot} \mathbb{N}^*_{S_{\alpha}})$ is formally dual in complementary degrees to

$$\begin{aligned} &C^{\infty}(U_{\alpha}; \mathbb{L}_{-2\rho+\alpha} \otimes \Lambda^{\cdot} \mathbb{N}^*_{S_{\alpha}}) \\ &\quad \cong \{C^{\infty}(\pi^{-1}_{\alpha} U_{\alpha}) \otimes L_{-2\rho+\alpha} \otimes \Lambda^{\cdot}(\mathfrak{b}_{\alpha}/(\mathfrak{b}_{\alpha} \cap \bar{\mathfrak{b}}_{\alpha}))^*\}^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0; H} \end{aligned} \tag{12.9a}$$

and $C^{\infty}(U_0; \Lambda^{\cdot}\mathbb{N}^*_{S_0})$, similarly, is formally dual in complementary degrees to

$$\begin{aligned} &C^{\infty}(U_0; \mathbb{L}_{-2\rho+2\alpha} \otimes \Lambda^{\cdot}\mathbb{N}^*_{S_0}) \\ &\quad \cong \{C^{\infty}(\pi_0^{-1}U_0) \otimes L_{-2\rho+2\alpha} \otimes \Lambda^{\cdot}(\mathfrak{n}_0/(\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0))^*\}^{\mathfrak{n} \cap \bar{\mathfrak{n}}_0; H}. \qquad (12.9\mathrm{b}) \end{aligned}$$

Here $\pi: G \to S = G/G_{\mathfrak{b}}$, $\pi_0: G \to S_0 = G/G_{\mathfrak{b}_0}$, and $\pi_\alpha: G \to S_\alpha = G/G_{\mathfrak{b}_\alpha}$ are the natural projections. Since $\pi^{-1}(U) = \pi_\alpha^{-1}(U_\alpha) = \pi_0^{-1}(U_0)$, and since

$$(\mathfrak{b}_\alpha/(\mathfrak{b}_\alpha \cap \bar{\mathfrak{b}}_\alpha))^* \cong [(\mathfrak{n}/(\mathfrak{n} \cap \bar{\mathfrak{n}})) + \bar{\mathfrak{g}}_\alpha]^* \hookrightarrow (\mathfrak{n}/(\mathfrak{n} \cap \bar{\mathfrak{n}}))^*,$$

Lemma 10.6 says that $e(\omega^\alpha)$ maps the right-hand side of (12.9b) to the right-hand side of (12.9a),

$$\begin{aligned} &(-1)^p e(\omega^\alpha): \{C^{\infty}(\pi^{-1}U) \otimes L_{-2\rho+2\alpha} \otimes \Lambda^p(\mathfrak{n}_0/(\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0))^*\}^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0; H} \\ &\quad \to \{C^{\infty}(\pi^{-1}U) \otimes L_{-2\rho+\alpha} \otimes \Lambda^{p+1}(\mathfrak{b}_\alpha/(\mathfrak{b}_\alpha \cap \bar{\mathfrak{b}}_\alpha))^*\}^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0; H}. \qquad (12.10) \end{aligned}$$

We assert that

$$\text{The maps (12.10) are formally dual to the} \quad p_\alpha^*: C^{\infty}(U_\alpha; \Lambda^{\cdot}\mathbb{N}^*_{S_\alpha}) \to C^{\infty}(U_0; \Lambda^{\cdot}\mathbb{N}^*_{S_0}). \qquad (12.11)$$

Before proving (12.11), let us show how it implies (12.5) and thus completes the proof of Proposition 12.7.

Asuming (12.11), the formal dual of the restriction map (12.8) is given by $j \otimes (-1)^p e(\omega^\alpha)$, where j is the map $E^* \hookrightarrow V$ of (11.5) on values of forms, dual to evaluation at 0. Thus the formal dual of (12.8) is given by

$$\begin{aligned} &C^{\infty}(U_0; \mathbb{E}^* \otimes \mathbb{L}_{-2\rho+2\alpha} \otimes \Lambda^p\mathbb{N}^*_{S_0}) \\ &\quad \cong [C^{\infty}(\pi^{-1}U) \otimes \{C^{\infty}(G_{\mathfrak{b}_\alpha}) \otimes E^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha)+\mathfrak{g}_\alpha+\bar{\mathfrak{g}}_\alpha; H} \\ &\qquad\qquad \otimes L_{-2\rho+2\alpha} \otimes \Lambda^p(\mathfrak{n}_0/(\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0))^*]^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0} \\ &\quad \hookrightarrow [C^{\infty}(\pi^{-1}U) \otimes \{C^{\infty}(G_{\mathfrak{b}_\alpha}) \otimes E^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha)+\mathfrak{g}_\alpha; H} \\ &\qquad\qquad \otimes L_{-2\rho+2\alpha} \otimes \Lambda^p(n_0/(\mathfrak{n}_0/(\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0))^*]^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0; H} \\ &\quad \xrightarrow{(-1)^p e(\omega^\alpha)} [C^{\infty}(\pi^{-1}U) \otimes \{C^{\infty}(G_{\mathfrak{b}_\alpha}) \otimes E^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha)+\mathfrak{g}_\alpha; H} \\ &\qquad\qquad \otimes L_{-2\rho+\alpha} \otimes \Lambda^{p+1}(\mathfrak{b}_\alpha/(\mathfrak{b}_\alpha \cap \bar{\mathfrak{b}}_\alpha))^*]^{\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0; H} \\ &\quad \cong \{C^{\infty}(\pi^{-1}U) \otimes E^* \otimes L_{-2\rho+\alpha} \otimes \Lambda^{p+1}((\mathfrak{n}/(\mathfrak{n} \cap \bar{\mathfrak{n}})) + \bar{\mathfrak{g}}_\alpha)^*\}^{(\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha)+\mathfrak{g}_\alpha; H} \\ &\quad \cong C^{\infty}_{S_0}(U; \mathbb{E}^* \otimes \mathbb{L}_{-2\rho+\alpha} \otimes \Lambda^{p+1}\mathbb{N}^*_S). \end{aligned}$$

This proves (12.5) for C^{∞} forms, and by continuity it follows as stated for $C^{-\omega}$ forms.

Finally we prove (12.11). We may assume U_α small so that $p_\alpha^{-1}U_\alpha = U_\alpha \times P^1(\mathbb{C})$ holomorphically with $U_0 = \{(w, t) \in U_\alpha \times P^1(\mathbb{C}) : t = \phi(w)\}$. Here $\phi \in C^\omega(U_\alpha)$ and a glance at CR dimensions shows that $\bar{\partial}_\alpha \phi \neq 0$ everywhere, $\bar{\partial}_\alpha = \bar{\partial}_{S_\alpha}$. Now every CR form on U_α has unique expression $\omega + (\bar{\partial}_\alpha \phi) \wedge \psi$, where ω, ψ are CR forms on U_0 transported to U_α. As $\phi(w) = t$ on U_0, $\bar{\partial}\phi = \bar{\partial}t = 0$ on U_0, so $p_\alpha : U_0 \to U_\alpha$ pulls back $p_\alpha^*(\omega + (\bar{\partial}_\alpha \phi) \wedge \psi)|_{S_0} = \omega$.

Let Ω_α be a generating section of the CR canonical bundle $\mathbb{L}_{2\rho+\alpha} \to S_\alpha$. Then $\Omega_0 = (dt \wedge p_\alpha^* \Omega_\alpha)|_{S_0}$ is a generating section of the CR canonical bundle $\mathbb{L}_{2\rho+2\alpha} \to S_0$. Let $\omega_1 \wedge \Omega_0$ denote an $\mathbb{L}_{2\rho+2\alpha}$-valued CR forms on U_0, either ω_1 or both ω and ψ compactly supported. Then we pair

$$\begin{aligned}
&\langle p_\alpha^*(\omega + \bar{\partial}_\alpha \phi \wedge \psi), \omega_1 \wedge \Omega_0 \rangle_{S_0} \\
&= \langle \omega, \omega_1 \wedge \Omega_0 \rangle_{S_0} = \int_{S_0} \omega \wedge \omega_1 \wedge \Omega \\
&= \int_{S_0} \omega \wedge \omega_1 \wedge dt \wedge \Omega_\alpha = \int_{S_\alpha} \omega \wedge \omega_1 \wedge d\phi \wedge \Omega_\alpha \\
&= \int_{S_\alpha} \omega \wedge \omega_1 \wedge \bar{\partial}_\alpha \phi \wedge \Omega_\alpha \\
&= (-1)^{\deg \omega_1} \int_{S_\alpha} (\omega + \bar{\partial}_\alpha \phi \wedge \psi) \wedge (\bar{\partial}_\alpha \phi \wedge \omega_1 \wedge \Omega_\alpha) \\
&= (-1)^{\deg \omega_1} \langle \omega + \bar{\partial}_\alpha \phi \wedge \psi, \bar{\partial}_\alpha \phi \wedge \omega_1 \wedge \Omega_\alpha \rangle_{S_\alpha}.
\end{aligned}$$

So $p_\alpha^* : C^\infty(U_\alpha; \Lambda^{\cdot} \mathbb{N}_{S_\alpha}^*) \to C^\infty(U_0; \Lambda^{\cdot} \mathbb{N}_{S_0}^*)$ is formally dual to $\pm e(\bar{\partial}_\alpha \phi \otimes \partial/\partial t)$: $C^\infty(U_0; \mathbb{L}_{2\rho+2\alpha} \otimes \Lambda^{\cdot} \mathbb{N}_{S_0}^*) \to C^\infty(U_\alpha; \mathbb{L}_{2\rho+\alpha} \otimes \Lambda^{\cdot} \mathbb{N}_{S_0}^*)$. But $\bar{\partial}_\alpha \phi \otimes \partial/\partial t$ and $\omega^\alpha \in C^\infty(S_\alpha; \mathbb{L}_\alpha \otimes \mathbb{N}_{S_\alpha}^*)$ are of the same type and ω^α is G-invariant. Thus, to complete the proof of (12.11) we need only show that

$$\bar{\partial}_\alpha \phi \otimes \frac{\partial}{\partial t} \text{ extends } G\text{-invariantly to all of } S_\alpha, \tag{12.12a}$$

$$\bar{\partial}_\alpha \phi \otimes \frac{\partial}{\partial t} \text{ and } \omega^\alpha \text{ agree up to a constant multiple at the base point } \mathfrak{b}_\alpha \in S_\alpha. \tag{12.12b}$$

Notice that $\bar{\partial}_\alpha \phi$ vanishes on $T^{0,1}(S_0)^*$ which at each point has codimension 1 in $T^{0,1}(S_\alpha)^*$. A change of fibre coordinate in $X \to X_\alpha$ must be of the form $s = F(w, t)$, U_0 given by $s = \psi(w)$, $\psi(w) = F(w, \phi(w))$. As F is holomorphic,

$$\frac{\partial \psi}{\partial \bar{w}^j} = \frac{\partial F}{\partial t} \frac{\partial \phi}{\partial \bar{w}^j}.$$

Also

$$\frac{\partial}{\partial t}=\frac{\partial s}{\partial t}\frac{\partial}{\partial s}=\frac{\partial F}{\partial t}\frac{\partial}{\partial s}.$$

So $\bar\partial_\alpha\psi\otimes\partial/\partial s=\bar\partial_\alpha\phi\otimes\partial/\partial t$. Thus $\bar\partial_\alpha\phi\otimes\partial/\partial t$ has invariant meaning. That proves (12.12a).

$\bar\partial_\alpha\phi$ annihilates $T^{0,1}(S_0)^*$. In view of (12.12a) and invariance of ω^α, now (12.12b) is just a matter of whether ω^α annihilates $T^{0,1}(S_0)^*$. But ω^α is dual to $\mathfrak{g}_{-\alpha}$, and $\mathfrak{n}_0$ is a sum of root spaces $\mathfrak{g}_\beta$ with $\beta\neq-\alpha$. That proves (12.12b).

With (12.12) we have completed the proof of Proposition 12.7.

13. Change of Polarization: Proof of Dual Statement

In this section we prove (12.6), completing the proof of Theorem 5.1.

Retain the notation used in the proof of (12.11). Consider a CR form on U_α and express it as $\omega+\bar\partial_\alpha\phi\wedge\psi$, where ω and ψ are CR forms on U_0 viewed as forms on U_α.

13.1. Lemma. *$\bar\partial_\alpha(\omega+\bar\partial_\alpha\phi\wedge\psi)=\bar\partial_0\omega+\bar\partial_\alpha\phi\wedge(d_\eta\omega-\bar\partial_0\psi)$, where η is the vector field on U_α such that* (i) *$\eta(\phi)=1$ and* (ii) *$[\eta,\xi]=0$ whenever ξ is the p_α-image of a* (0, 1) *vector field on S_0, and where d_η is Lie derivative along η.*

Proof. This is just $\bar\partial_0\gamma=\bar\partial_\alpha\gamma+d\bar t\wedge d_\eta(\gamma)$. In effect, $\bar\partial_\alpha(\omega+\bar\partial_\alpha\phi\wedge\psi)=\bar\partial_\alpha\omega-\bar\partial_\alpha\phi\wedge\bar\partial_\alpha\psi=\bar\partial_0\omega+\bar\partial_\alpha\phi\wedge d_\eta\omega-\bar\partial_\alpha\phi\wedge\bar\partial_0\psi$. Q.E.D.

Let W_0 be either of $\mathrm{cl}(U_0)$ or $\mathrm{bd}(U_0)$. We must understand the restriction maps $C^\omega_{S_0}(\tilde W_0;\tilde{\mathbb{F}}\otimes\Lambda^\cdot\mathbb{N}^*_S)\to C^\omega(W_0;\mathbb{F}\otimes\Lambda^\cdot\mathbb{N}^*_{S_0})$ of (12.4). Since we may always shrink W_0, we assume all bundles trivial and only consider the case of scalar valued forms, $C^\omega_{S_0}(\tilde W_0;\Lambda^\cdot\mathbb{N}^*_S)\to C^\omega(W_0;\Lambda^\cdot\mathbb{N}^*_{S_0})$. Now let $\Omega\in C^\omega_{S_0}(\tilde W_0;\Lambda^\cdot\mathbb{N}^*_S)$. In other words Ω is a CR form on a germ of an S-neighborhood $\{(w,t)\in W_\alpha\times P^1(\mathbb{C})\colon |t-\phi(w)|<\varepsilon\}$ of W_0 which does not involve $d\bar t$ and whose coefficients are holomorphic in t. So there is a power series expansion $\Omega=\sum_{n\geq0}(t-\phi)^n\gamma_n$ with $\gamma_n\in C^\omega(W_\alpha;\Lambda^\cdot\mathbb{N}_{S_\alpha})$. Expand $\gamma_n=\omega_n+\bar\partial_\alpha\phi\wedge\psi_n$ where ω_n, ψ_n are CR on W_0:

$$\Omega=\sum_{n\geq0}(t-\phi)^n(\omega_n+\bar\partial_\alpha\phi\wedge\psi_n). \tag{13.2}$$

Since $t-\phi(w)=0$ on S_0, (13.2) gives us

$$\Omega|_{W_0}=\omega_0. \tag{13.3}$$

In order to use (13.3) to prove (12.6), we apply Lemma 13.1 to (13.2) and obtain

$$\bar{\partial}_S \Omega = \sum_{n \geq 0} (t-\phi)^n \{\bar{\partial}_0 \omega_n + \bar{\partial}_\alpha \phi \wedge [d_\eta \omega_n - (n+1)\,\omega_{n+1} - \bar{\partial}_0 \psi_n]\}. \quad (13.4)$$

Now, if $\omega_0 \in C^\omega(W_0; \Lambda^{\cdot} \mathbb{N}^*_{S_0})$ is $\bar{\partial}_0$-closed, we set $\omega_{n+1} = (1/(n+1))\, d_\eta \omega_n$ to obtain a convergent $\bar{\partial}_S$-closed $\sum (t-\phi)^n \omega_n \in C^\omega_{S_0}(\tilde{W}_0; \Lambda^{\cdot} \mathbb{N}^*_S)$. In other words, using the fact that $\bar{\partial}_0$ and d_η commute,

13.5. Lemma. *If $\omega_0 \in C^\omega(W_0; \Lambda^{\cdot} \mathbb{N}^*_{S_0})$ is $\bar{\partial}_0$-closed, then*

$$\Omega = \exp((t-\phi)\,d_\eta) \cdot \omega_0 \in C^\omega_{S_0}(\tilde{W}_0; \Lambda^{\cdot} \mathbb{N}^*_S)$$

is defined, $\bar{\partial}_S$-closed, and satisfies $\Omega|_{W_0} = \omega_0$.

If $\Omega \in C^\omega_{S_0}(\tilde{W}_0; \Lambda^{\cdot} \mathbb{N}^*_S)$ has $\bar{\partial}_S \Omega = 0$ and $\Omega|_{S_0} = \omega_0$ with $\omega_0 = \bar{\partial}_0 \beta_0$, we define $\beta_{n+1} = (1/(n+1))(d_\eta \beta_n - \psi_n)$ and $\Phi = \sum_{n \geq 0} (t-\phi)^n \beta_n$. Then $\omega_{n+1} = \bar{\partial}_0 \beta_{n+1} = \bar{\partial}_0 \beta_{n+1}$ by induction: $\bar{\partial}_0 \beta_{n+1} = (1/(n+1))(d_\eta \omega_n - \psi_n)$, which is ω_{n+1} by (13.4) since $\bar{\partial}_S \Omega = 0$. And $d_\eta \beta_n - (n+1)\beta_{n+1} = \psi_n$ by construction, so $\Omega = \bar{\partial}_S \Phi$:

13.6. Lemma. *If $\Omega \in C^\omega_{S_0}(\tilde{W}_0; \Lambda^{\cdot} \mathbb{N}^*_S)$ is $\bar{\partial}_S$-closed and $\Omega|_{W_0}$ is $\bar{\partial}_0$-exact, then Ω is $\bar{\partial}_S$-exact.*

The restriction maps of (12.4) are surjective on cohomology by Lemma 13.5 and injective on cohomology by Lemma 13.6. That proves (12.6). In view of Proposition 12.7, the proof of Theorem 5.1 is complete.

14. Second Main Theorem

Theorem 5.1 can be used to reformulate the duality theorem of [7] in more directly geometric terms. To simplify the discussion, we treat the case of a group G in Harish–Chandra's class. As in the rest of the paper, all arguments go through for the larger class of groups described in Section 2; see [28, 7] for the necessary techniques.

We fix a maximal compact subgroup $K \subset G$, and let θ denote the corresponding Cartan involution. The complexification $K_{\mathbb{C}}$ of K acts on the flag variety X. A G-orbit $S \subset X$ and a $K_{\mathbb{C}}$-orbit $Q \subset X$ are dual in the sense of Matsuki [17] if $S \cap Q$ contains a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, with nilpotent radical $\mathfrak{n}$ and Levi component $\mathfrak{h}$, such that (i) $\mathfrak{h}$ is the complexified Lie algebra of a Cartan subgroup $H \subset G$, and (ii) $\theta\mathfrak{h} = \mathfrak{h}$. In this situation, K acts transitively on $S \cap Q$.

Let $\chi: H \to G(E)$ be an irreducible, finite dimensional representation. We extend $d\chi$ from $\mathfrak{h}$ to $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, by letting $\mathfrak{n}$ act trivially. Since χ can be continued from $K \cap H$ to an algebraic representation of the complexification $(K \cap H)_{\mathbb{C}}$, and since $(K \cap H)_{\mathbb{C}}$ is a Levi component of the isotropy subgroup of $K_{\mathbb{C}}$ at $\mathfrak{b}$, E is the fibre at $\mathfrak{b}$ of a homogeneous, algebraic vector bundle $\mathbb{E}$ over the algebraic variety $Q = K_{\mathbb{C}} \cdot \mathfrak{b}$. The fact that $d\chi$ is defined not only on $\mathfrak{k} \cap \mathfrak{b}$, but on all of $\mathfrak{b}$, implies that $\mathbb{E}$ extends to a $\mathfrak{g}$-equivariant holomorphic vector bundle over the germ $\tilde{Q}$ of a Hausdorff neighborhood of Q in X. Relative to appropriate local trivializations, the transition functions have logarithmic derivatives which are algebraic. Consequently $\mathbb{E} \to Q$ extends also as a $\mathfrak{g}$-equivariant algebraic vector bundle over a formal neighborhood $\tilde{\tilde{Q}}$ of Q in X. For simplicity, we will denote both the holomorphic extension to $\tilde{Q}$ and the algebraic extension to $\tilde{\tilde{Q}}$ by the original symbol $\mathbb{E}$.

The $(H, \mathfrak{b})$-module E also determines a G-invariant CR vector bundle over the G-orbit $S = G \cdot \mathfrak{b}$. As was discussed in Section 4, this latter bundle extends to a $\mathfrak{g}$-equivariant holomorphic vector bundle $\tilde{\mathbb{E}}$ over the germ of a Hausdorff neighborhood $\tilde{S}$ of S. We write $\tilde{\mathbb{E}}^*$ for the dual vector bundle.

To define the local cohomology group along Q of an algebraic vector bundle, it suffices to know this bundle over the formal neighborhood $\tilde{\tilde{Q}}$. In particular, it makes sense to talk of the local cohomology sheaves, in the algebraic context, of $\mathbb{E} \to \tilde{\tilde{Q}}$. We denote these by $\mathscr{H}^*_Q(\mathscr{O}(\mathbb{E}))_{\mathrm{alg}}$. Similarly, the holomorphic bundle $\mathbb{E} \to \tilde{Q}$ gives rise to local cohomology sheaves in the analytic context, which we denote by $\mathscr{H}^*_Q(\mathscr{O}(\mathbb{E}))$. Both $\mathscr{H}^*_Q(\mathscr{O}(\mathbb{E}))$ and $\mathscr{H}^*_Q(\mathscr{O}(\mathbb{E}))_{\mathrm{alg}}$ are concentrated in degree $d = \operatorname{codim}_{\mathbb{C}}(Q \subset X)$.

In a natural manner, $\mathscr{H}^d_Q(\mathscr{O}(\mathbb{E}))_{\mathrm{alg}}$ is a $K_{\mathbb{C}}$-equivariant sheaf of $\mathscr{D}_{\lambda+\rho}$-modules; here we are using the noltation of [7], except that we let $\mathfrak{n}$ correspond to the set of negative roots, rather than to the set of positive roots as in [7]. Up to a shift by the top exterior power of the canonical bundle, $\mathscr{H}^d_Q(\mathscr{O}(\mathbb{E}))_{\mathrm{alg}}$ is the $\mathscr{D}$-module direct image of $\mathscr{O}_Q(\mathbb{E}|_Q)$ via the inclusion $j: Q \hookrightarrow X$. Thus

$$H^p(X, \mathscr{H}^d_Q(\mathscr{O}(\mathbb{E}))_{\mathrm{alg}})_{\mathrm{alg}} \cong H^{p+d}_Q(\tilde{\tilde{Q}}, \mathscr{O}(\mathbb{E}))_{\mathrm{alg}} \tag{14.1}$$

are the Beilinson–Bernstein modules attached to (Q, χ).

According to the duality theorem of [7], the Harish–Chandra modules (14.1) are naturally dual to the derived functor modules $A^{s-p}(G, H, \mathfrak{b}, \chi^* \otimes e^{-2\rho})$, where χ^* is the dual representation of χ, and

$$s = \dim_{\mathbb{R}}(Q \cap S) - \dim_{\mathbb{C}} Q. \tag{14.2}$$

In other words, there exists a natural pairing

$$H^{p+d}_Q(\tilde{\tilde{Q}}, \mathscr{O}(\mathbb{E}))_{\mathrm{alg}} \times A^{s-p}(G, H, \mathfrak{b}, \chi^* \otimes e^{-2\rho}) \to \mathbb{C}, \tag{14.3}$$

which exhibits each of the factors as the dual (in the category of Harish–Chandra modules) of the other. By Theorem 5.1, the right factor is the K-finite part of the local cohomology group $H_S^{s-p+u}(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n)$, where Ω_X^n is the canonical sheaf of X. Thus (14.3) can be re-interpreted as a pairing

$$H_Q^{p+d}(\tilde{\tilde{Q}}, \mathcal{O}(\mathbb{E}))_{\mathrm{alg}} \times H_S^{s-p+u}(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n)_{(K)} \to \mathbb{C}.$$

We shall see, as a consequence of Lemma 15.11 below, that $s+d+u=n=\dim_{\mathbb{C}} X$. Thus, re-labelling the indices, (14.3) becomes a pairing,

$$H_Q^p(\tilde{\tilde{Q}}, \mathcal{O}(\mathbb{E}))_{\mathrm{alg}} \times H_S^{n-p}(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n)_{(K)} \to \mathbb{C}. \tag{14.4}$$

Cup product maps the local cohomology groups along Q and S, both taken in the analytic category, into local cohomology along $Q \cap S$:

$$H_Q^p(\tilde{Q}, \mathcal{O}(\mathbb{E})) \times H_S^{n-p}(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n) \to H_{Q \cap S}^n(X, \Omega_X^n). \tag{14.5}$$

Local cohomology maps naturally to cohomology on all of X,

$$H_{Q \cap S}^*(X, \Omega_X^n) \to H^*(X, \Omega_X^n). \tag{14.6}$$

Also, by Borel–Weil–Bott,

$$H^n(X, \Omega_X^n) \cong \mathbb{C}. \tag{14.7}$$

Combining (14.5)–(14.7), we obtain a pairing

$$H_Q^p(\tilde{Q}, \mathcal{O}(\mathbb{E})) \times H_S^{n-p}(\tilde{S}, \mathcal{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n) \to \mathbb{C}. \tag{14.8}$$

It is $(\mathfrak{g}, K)$-invariant because of the canonical nature of the definition.

14.9. THEOREM. *Via the natural map*

$$H_Q^p(\tilde{\tilde{Q}}, \mathcal{O}(\mathbb{E}))_{\mathrm{alg}} \to H_Q^p(\tilde{Q}, \mathcal{O}(E)),$$

the cup product pairing (14.8) *induces the pairing* (14.4). *In particular, the cup product pairing realises the duality between the Beilinson–Bernstein modules and the derived functor modules.*

In the special case of the discrete series, this is due to Žabčić. A sketch of the proof of the theorem will occupy the next section.

15. PROOF OF THEOREM 14.9

We begin with a general discussion of local cohomology and cup products. For the moment, let X be an arbitrary topological space, $S \subset X$

a closed subspace, $\mathcal{O}$ a sheaf of commutative $\mathbb{C}$-algebras, and $\mathscr{E}$ a sheaf of $\mathcal{O}$-modules. To these data, one can associate a "local cohomology sheaf" $\mathscr{H}_S(\mathscr{E})$ in the derived category of sheaves of $\mathcal{O}$-modules, as follows. Let $\mathscr{E}^{\cdot}$ be a flabby resolution of $\mathscr{E}$, and $\mathscr{E}^{\cdot}_S$ the subcomplex of sections which vanish outside S. Then $\mathscr{E}^{\cdot}_S$ is determined up to quasi-isomorphism, and thus represents a well-defined object $\mathscr{H}_S(\mathscr{E})$ in the derived category. The cohomology groups of $\mathscr{H}_S(\mathscr{E})$—equivalently, the hypercohomology groups of any complex representing $\mathscr{H}_S(\mathscr{E})$ are precisely the local cohomology groups of $\mathscr{E}$ along S, $H_S^*(X, \mathscr{E})$ [13].

If $T \subset X$ is a second closed subspace and $\mathscr{F}$ a second sheaf of $\mathcal{O}$-modules on X, the local version of cup product defines a morphism

$$\mathscr{H}_S(\mathscr{E}) \otimes^L \mathscr{H}_T(\mathscr{F}) \to \mathscr{H}_{S \cap T}(\mathscr{E} \otimes^L \mathscr{F}); \tag{15.1}$$

here $\otimes^L$ denotes the tensor product, over $\mathcal{O}$, in the derived category. This local product pairing can be described most easily in terms of the canonical flabby resolutions of $\mathscr{E}$ and $\mathscr{F}$ [6]. Via the identifications $R\Gamma\mathscr{H}_S(\ldots) = H_S^*(X, \ldots)$, the induced map

$$(R\Gamma\mathscr{H}_S(\mathscr{E})) \otimes_{\mathbb{C}} (R\Gamma\mathscr{H}_T(\mathscr{F})) \to R\Gamma\mathscr{H}_{S \cap T}(\mathscr{E} \otimes^L \mathscr{F})$$

realizes the global cup product pairing

$$H_S^*(X, \mathscr{E}) \otimes H_T^*(X, \mathscr{F}) \to H_{S \cap T}^*(X, \mathscr{E} \otimes^L \mathscr{F}). \tag{15.2}$$

Here, as in the following, we suppress the subscript $\mathbb{C}$ to the tensor product when this causes no ambiguity.

We now suppose that X is a complex manifold, $\mathcal{O} = \mathcal{O}_X$ the sheaf of holomorphic functions, S and T are C^∞ submanifolds of X, $\mathscr{E} = \mathcal{O}(\mathbb{E})$ and $\mathscr{F} = \mathcal{O}(\mathbb{F})$ the sheaves of sections of holomorphic vector bundles $\mathbb{E}$, $\mathbb{F}$. Since these sheaves are locally free over $\mathcal{O}$,

$$\mathcal{O}(\mathbb{E}) \otimes^L \mathcal{O}(\mathbb{F}) = \mathcal{O}(\mathbb{E} \otimes \mathbb{F}). \tag{15.3}$$

In particular, the cup product pairing (15.2) takes values in the local cohomology of $\mathcal{O}(\mathbb{F} \otimes \mathbb{F})$:

$$H_S^*(X, \mathcal{O}(\mathbb{E})) \otimes H_T^*(X, \mathcal{O}(\mathbb{F})) \to H_{S \cap T}^*(X, \mathcal{O}(\mathbb{F} \otimes \mathbb{F})). \tag{15.4}$$

The Dolbeault resolution with hyperfunction coefficients is flabby, so

$$\mathscr{C}_S{}^{\omega}(X; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{T}_X^{0,1*}) = \mathscr{H}_S(\mathcal{O}(\mathbb{E})) \tag{15.5}$$

in the derived category, where $\mathscr{C}_S{}^{\omega}(X, \ldots)$ is the sheaf of hyperfunction sections of..., with support in S. In particular, the complex of global sections computes the local cohomology along S,

$$H_S^*(X, \mathcal{O}(\mathbb{E})) = H^*(C_S{}^{\omega}(X; \mathbb{E} \otimes \Lambda^{\cdot}\mathbb{T}_X^{0,1*})). \tag{15.6}$$

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To calculate $\mathscr{H}_S(\mathcal{O}(\mathbb{E})) \otimes^L \mathscr{H}_T(\mathcal{O}(\mathbb{F}))$, one must pass to a resolution of $\mathscr{C}_S{}^{\omega}(X;...)$ which is flat over $\mathcal{O}$. No geometric $\mathcal{O}$-flat resolution of the sheaf of hyperfunctions exists. This explains why the cup product (15.4) cannot be expressed directly in terms of the description (15.6) of the local cohomology groups.

For lack of better notation, we let $\mathscr{C}_S^{\infty}(X;...)$ denote the space of distribution sections of..., with support in S, and smooth along S: locally, these are derivatives of various orders in directions normal to S, applied to smooth measures on S. The inclusion

$$\mathscr{C}_S^{\infty}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \subsetneq \mathscr{C}_S{}^{\omega}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \tag{15.7}$$

determines a morphism in the derived category

$$\mathscr{C}_S^{\infty}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \to \mathscr{H}_S(\mathcal{O}(\mathbb{E})). \tag{15.8}$$

We shall say that a local or global section ϕ of $\mathscr{C}_S{}^{\omega}(X, \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*})$ is smooth along S if it lies in the image of the inclusion (15.7). In that case, the singular support of ϕ is contained in the conormal bundle of S. Analogously, the singular support of a section ψ of $\mathscr{C}_T{}^{\omega}(X; \mathbb{F} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*})$ lies in the conormal bundle of T if ψ is smooth along T. Any two hyperfunctions whose singular supports are linearly disjoint can be multiplied [12]. Consequently, if

$$S \text{ and } T \text{ have linearly disjoint conormal bundles,} \tag{15.9}$$

it makes sense to take the wedge product $\phi \wedge \psi$ of any two forms ϕ, ψ as above, over their common domain of definition. The description of products of hyperfunctions in terms of boundary values of holomorphic functions [12] can be translated back into cocycles with respect to a relative covering [13], which shows that the wedge product $\phi \wedge \psi$ represents the local cup product of the classes of ϕ and ψ. In other words, under the hypothesis (15.9), the morphism (15.8), and the corresponding morphisms for T and $S \cap T$ relate the local cup product

$$\mathscr{H}_S(\mathcal{O}(\mathbb{E})) \otimes \mathscr{H}_T(\mathcal{O}(\mathbb{F})) \to \mathscr{H}_{S \cap T}(\mathcal{O}(\mathbb{E} \otimes \mathbb{F})) \tag{15.10}$$

to the wedge product of forms which are smooth along S and T, respectively. This observation will be crucial to our proof of Theorem 14.9.

We specialize the preceding discussion to the situation of Theorem 14.9. Thus S is a G-orbit in the flag variety X. The role of T will be played by Q, the $K_{\mathbb{C}}$-orbit dual to S. Then S and Q satisfy the hypothesis (15.9):

15.11. LEMMA. *The conormal bundles of S and Q in X are linearly disjoint at every point of $Q \cap S$.*

Proof. Since K acts transitively on $Q \cap S$, it suffices to check the statement at the base point $\mathfrak{b}$. We identify the complexified tangent space of X at $\mathfrak{b}$ with $\mathfrak{g}/\mathfrak{b} \oplus \mathfrak{g}/\bar{\mathfrak{b}}$. Via this identification, the complexified tangent spaces of Q and S correspond to the images of, respectively, $\mathfrak{k} \oplus \mathfrak{k}$ and the diagonal $\Delta\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$. The Killing form identifies $\mathfrak{n}$ with the dual of $\mathfrak{g}/\mathfrak{b}$, and the (-1)-eigenspace $\mathfrak{p}$ of θ with the annihilator of $\mathfrak{k}$. Thus we can identify the complexified cotangent space of X with $\mathfrak{n} \oplus \bar{\mathfrak{n}}$, the complexified conormal space of Q with $(\mathfrak{n} \cap \mathfrak{p}) \oplus (\bar{\mathfrak{n}} \cap \mathfrak{p})$, and the complexified conormal space of S with $\{(\xi, -\xi) \mid \xi \in \mathfrak{n} \cap \bar{\mathfrak{n}}\}$. Thus $\{(\xi, -\xi) \mid \xi \in \mathfrak{n} \cap \bar{\mathfrak{n}} \cap \mathfrak{p}\}$ corresponds to the intersection of the conormal spaces. For any root α of $(\mathfrak{g}, \mathfrak{h})$, $\theta\alpha = -\bar{\alpha}$. Hence $\mathfrak{n} \cap \bar{\mathfrak{n}} \cap \mathfrak{p} = 0$, which proves the lemma.

As one consequence of the lemm, the real codimensions of Q and S add up to the real codimension of $Q \cap S$,

$$2n - \dim_{\mathbb{R}}(Q \cap S) = 4n - \dim_{\mathbb{R}} S - 2 \dim_{\mathbb{C}} Q.$$

Hence, by (14.2),

$$s + d + u = n, \tag{15.12}$$

as had been asserted in Section 14.

The local cohomology sheaves of $\mathcal{O}(\mathbb{E})$ along Q are concentrated in degree $d = \operatorname{codim}_{\mathbb{C}}(Q \subset X)$—this is true both for the analytic and the algebraic local cohomology. Thus we may think of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))$ and $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_{\mathrm{alg}}$ as single sheaves, in degree d. We let $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_f$ denote the subsheaf of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))$ generated by $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_{\mathrm{alg}}$ over the sheaf of holomorphic functions. Near any point of Q, we can introduce algebraic coordinate functions $\{z_1, ..., z_{n-d}; w_1, ..., w_d\}$ such that Q is the variety defined by $w_1 = \cdots = w_d = 0$. Let $\{s_1, ..., s_n\}$ be a local holomorphic frame for $\mathbb{E} \to \tilde{Q}$, such that the s_j restrict to algebraic sections of $\mathbb{E} \to \tilde{\tilde{Q}}$. Local sections of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_{\mathrm{alg}}$ can be represented uniquely as principal parts of algebraic sections of $\mathbb{E}$ which are regular outside the union of d divisors $\{w_j = 0\}_{1 \leq j \leq d}$—equivalently, as finite sums

$$\sum_{j,I} f_{j,I}(z)\, s_j w^{-I}; \tag{15.13}$$

here $I = (i_1, i_2, ..., i_d)$ ranges over d-tuples of strictly positive integers, w^{-I} denotes the monomial $w_1^{-i_1} w_2^{-i_2} \cdots w_d^{-i_d}$, and the $f_{j,I}$ are regular algebraic functions in the variables z_i. Local sections of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_f$ can be represented in the same way, except that the $f_{j,I}$ need only be holomorphic, not algebraic. Sections of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))$, finally, may involve infinite but convergent series.

The inclusion $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_f \subsetneq \mathscr{H}_Q(\mathcal{O}(\mathbb{E}))$ factors through

$$\mathscr{C}_Q^{\infty}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \to \mathscr{C}_Q^{\,\omega}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \simeq \mathscr{H}_Q(\mathcal{O}(\mathbb{E})) \qquad (15.14)$$

(cf. (15.5), (15.7)), as follows. Let ϕ be a sum of the type (15.13), with holomorphic coefficient functions $f_{j,I}$, and ψ an $\mathbb{E}^*$-valued $(n, n-d)$-form with smooth coefficients and compact support inside the (z, w)-coordinate neighborhood. For $r > 0$,

$$C_r = \{(z, w) \mid |w_j| = r,\ 1 \leqslant j \leqslant d\} \qquad (15.15)$$

is a $(2n-d)$-cycle, and it makes sense to integrate the contracted product $\phi\psi$ over C_r. A small calculation in $\mathbb{C}^n$ shows that the pairing

$$\langle \phi, \psi \rangle = \lim_{r \to 0} \int_{C_r} \phi\psi \qquad (15.16)$$

is continuous in the variable ψ and vanishes on exact forms. Thus we may view ϕ as $\bar{\partial}$-closed, $\mathbb{E}$-valued, distribution coefficient $(0, d)$-form, with support along Q. If ϕ only involves a complete intersection of first-order poles—i.e., if only the single multi-index $I = (1, ..., 1)$ occurs in the local representation (15.13)—one checks readily that this form is independent of the choice of local coordinate system, that it is smooth along S, and that it represents the local cohomology class ϕ via the morphism (15.8). Both $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_f$ and $\mathscr{C}_Q^{\infty}(X; \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*})$ are modules over the sheaf of holomorphic differential operators acting on $\mathcal{O}(\mathbb{E})$, the former is generated over this sheaf of differential operators by sections having only first-order poles, and the pairing (15.16) respects the $\mathscr{D}$-module structures. This shows that the differential form which we have associated to ϕ has all the required properties.

To avoid complicated notation, we tacitly identify any local section ϕ of $\mathscr{H}_Q(\mathcal{O}(\mathbb{E}))_f$ with the section of $\mathscr{C}_Q^{\infty}(X; \mathbb{E} \otimes \Lambda^d \mathbb{T}_X^{0,1*})$ which it defines. Now let ψ be a local section of $\mathscr{C}_S^{\infty}(\tilde{S}; \tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{0,1*} \otimes \Lambda^p \mathbb{T}_X^{0,1*})$. Since ϕ and ψ are smooth along Q and S, respectively, the contracted wedge product $\phi \wedge \psi$ makes sense as $(n, d+p)$ form with distribution coefficients supported on $Q \cap S$, and smooth along $Q \cap S$. According to (15.10),

$$\phi \otimes \psi \mapsto \phi \wedge \psi \qquad \text{(contracted wedge product)} \qquad (15.17)$$

represents the local cup product pairing followed by contraction $\mathbb{E} \otimes \tilde{\mathbb{E}}^* \to \mathbb{C}$,

$$\mathscr{H}_Q(\mathcal{O}(\mathbb{E})) \otimes^L \mathscr{H}_S(\Omega^n(\tilde{\mathbb{E}}^*)) \to \mathscr{H}_{S \cap Q}(\Omega^n(\mathbb{E} \otimes \tilde{\mathbb{E}}^*)) \to \mathscr{H}_{S \cap Q}(\Omega^n), \qquad (15.18)$$

via the morphism (15.8) and the analogous morphisms for Q and $S \cap Q$.

The arguments which establish the quasi-isomorphism (6.16) apply also in the context of forms with distribution coefficients which are smooth along S. Thus, replacing $\tilde{\mathbb{E}}$ by $\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{0,1*}$, we obtain a quasi-isomorphism

$$\begin{aligned} &\mathscr{C}_S^{\infty}(\tilde{S}; \tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}^{0,1*}) \\ &\quad \simeq p_* \mathscr{C}^{\infty}(S_Y; p^* \tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*})[-u]. \qquad (15.19) \end{aligned}$$

Let $Q_Y \subset Y$ denote the $K_{\mathbb{C}}$-orbit through the base point in $Y \simeq G_{\mathbb{C}}/H_{\mathbb{C}}$; in [8], Q_Y is denoted by $\tilde{Q}$. Recall (6.7), and note that $p: Q_Y \to Q$ has typical fibre $\exp(\mathfrak{k} \cap \mathfrak{n})$. The intersection $\mathfrak{n} \cap \bar{\mathfrak{n}} \cap \mathfrak{k}$ reduces to zero since $\theta\alpha = -\bar{\alpha}$ for any root α, and K acts transitively on $Q \cap S$, hence

$$p: Q_Y \cap S_Y \xrightarrow{\sim} Q \cap S. \qquad (15.20)$$

We now let $Q \cap S$ and $Q_Y \cap S_Y$ play the roles of S and S_Y in the proof of (6.16) and replace $\tilde{\mathbb{E}}$ by the canonical bundle of X. Since the real form S_Y of Y contains $Q_Y \cap S_Y$,

$$\mathscr{C}^{\omega}_{S_Y \cap Q_Y}(Y; \Lambda^{\cdot} \mathbb{T}_Y^{0,1*})[2n] \simeq \mathscr{C}^{-\omega}_{S_Y \cap Q_Y}(S_Y)$$

is the appropriate analogue of (6.15). Also, the shift by $u - 2n$ in (6.12), i.e., the shift by minus the real codimension of the fibre of $p: S_Y \to S$ in the fibre of $p: Y \to X$, becomes a shift by $-2n$ because of (15.20). The two shifts by $\pm 2n$ cancel, hence

$$\mathscr{C}^{-\omega}_{Q \cap S}(X; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \simeq p_* \mathscr{C}^{-\omega}_{Q_Y \cap S_Y}(S_Y; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}) \qquad (15.21)$$

and, similarly,

$$\mathscr{C}^{-\infty}_{Q \cap S}(X; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}) \simeq p_* \mathscr{C}^{-\infty}_{Q_Y \cap S_Y}(S_Y; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}). \qquad (15.22)$$

The proof of the duality theorem [8] hinges on the quasi-isomorphism

$$\mathscr{H}_Q(\mathscr{O}(\mathbb{E}))_{\text{alg}} \simeq p_* \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^* \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}))_{\text{alg}}[2n - 2s], \qquad (15.23)$$

which can be deduced from the description (15.13) of the local cohomology sheaves. Here $\mathscr{H}_Q(\mathscr{O}(\mathbb{E}))_{\text{alg}}$ is viewed as a complex of sheaves concentrated in degree $d = \operatorname{codim}_{\mathbb{C}}(Q \subset X)$, whereas the other side is a double complex concentrated in degree $n + d - s = \operatorname{codim}_{\mathbb{C}}(Q_Y \subset Y)$ with respect to one of the two gradings. Analogously, for the same reasons, we have

$$\mathscr{H}_Q(\mathscr{O}(\mathbb{E}))_f \simeq p_* \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^* \mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}))_f [2n - 2s]. \qquad (15.24)$$

Via the quasi-isomorphism (15.19), (15.22), (15.24), the contracted wedge product (15.17) corresponds to a morphism

$$\begin{aligned} p_* \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}))_f[n+d-s] \\ \otimes p_* \mathscr{C}^{-\infty}(S_Y; p^*\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}) \\ \to p_* \mathscr{C}^{-\infty}_{Q_Y \cap S_Y}(S_Y; \Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}) \end{aligned} \tag{15.25}$$

(note the shifts in (15.19), (15.24), and observe that $2n-2s+u=n+d-s$ by (15.12)). We shall now describe an intrinsically defined pairing

$$\begin{aligned} \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}))_f[n+d-s] \\ \otimes \mathscr{C}^{-\infty}(S_Y; p^*\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}) \\ \to \mathscr{C}^{-\infty}_{Q_Y \cap S_Y}(S_Y; \Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}); \end{aligned} \tag{15.26}$$

Proposition 15.28 below will assert that this intrinsic pairing induces the pairing (15.25).

Since $\mathbb{E}$ and $\tilde{\mathbb{E}}^*$ pair into the trivial bundle, we may as well suppose $\mathbb{E} = \tilde{\mathbb{E}}^* =$ trivial line bundle. Also, $\Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^n \mathbb{T}^{1,0*}_{Y|X} \simeq \Lambda^{2n}\mathbb{T}^{1,0*}_Y$ restricts to the bundle of forms of top degree on the real form S_Y of Y. This sets up a duality between distribution sections of $\Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^p \mathbb{T}^{1,0*}_{Y|X}$ on S_Y and compactly supported C^∞ sections of $\Lambda^{n-p}\mathbb{T}^{1,0*}_{Y|X}$. Thus (15.26) amounts to a pairing into $\mathbb{C}$, between sections of $\mathscr{H}_{Q_Y}(\mathscr{O}_Y(\Lambda^{\cdot}\mathbb{T}^{1,0*}_{Y|X}))_f[n+d-s]$ and compactly supported C^∞ sections of $\Lambda^n \mathbb{T}^{1,0*}_X \otimes \Lambda^{n-\cdot}\mathbb{T}^{1,0*}_{Y|X}$ on S_Y. Because of the algebra structure of $\Lambda \mathbb{T}^{1,0*}_{Y|X}$, the definition of (15.26) now comes down to a map

$$\mathscr{H}_{Q_Y}(\Omega^{2n}_Y)_f[n+d-s] \to \mathscr{C}^{-\infty}_{Q_Y \cap S_Y}(S_Y; \Lambda^{2n}\mathbb{T}^{1,0*}_Y). \tag{15.27}$$

Recall that $n+d-s$ is the codimension of Q_Y in Y, so $\mathscr{H}_{Q_Y}(\Omega^{2n}_Y)_f[n+d-s]$ is concentrated in degree zero. This sheaf can be identified with the $\mathscr{D}$-module direct image, in the holomorphic category, of the canonical sheaf of Q_Y. Analogously, the right-hand side of (15.27) is the $\mathscr{D}$-module direct image, in the C^∞ category, of the sheaf of C^∞ forms of top degree on $Q_Y \cap S_Y$. By restriction from Q_Y to the real form $Q_Y \cap S_Y$ ($Q_Y \subset Y$ is defined over $\mathbb{R}$), the canonical sheaf of Q_Y maps to the sheaf of C^∞ forms of top degree on $Q_Y \cap S_Y$. On the level of the $\mathscr{D}$-module direct images, this induces the map (15.27), by restriction from Y to the real form S_Y. As was explained before, (15.27) induces the pairing (15.26).

15.28. Proposition. *The canonically defined pairing* (15.26) *induces the morphism* (15.25), *which corresponds to the local cup product via the quasi-isomorphisms* (15.19), (15.22), (15.24).

Before turning to the proof of the proposition, we shall explain how it implies Theorem 14.9).

Recall (6.17). By exactly the same reasoning, using (15.19) instead of (6.16), we find

$$\begin{aligned}
&H^*(C_S^\infty(\tilde{S}, \tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}))\\
&\quad = \mathbb{H}^*(\tilde{S}, \mathscr{C}_S^\infty(\tilde{S}; \tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_X^{0,1*}))\\
&\quad = \mathbb{H}^*(\tilde{S}, p_* \mathscr{C}^\infty(S_Y; p^*\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*})[-u])\\
&\quad = H^{*-u}(C^\infty(S_Y; p^*\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*})).
\end{aligned} \tag{15.29}$$

Analogously, because of (15.24),

$$\begin{aligned}
\mathbb{H}^*(X, \mathscr{H}_Q(\mathscr{O}(\mathbb{E}))_f) &= \mathbb{H}^*(X, p_* \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}))_f[2n-s])\\
&= \mathbb{H}^{*+u}(Y; \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}))_f[n+d-s])\\
&= \mathbb{H}^{*+u}(\Gamma \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{\cdot} \mathbb{T}_{Y|X}^{1,0*}))_f[n+d-s]).
\end{aligned} \tag{15.30}$$

Here we have used the equality (15.12), the fact that p is an affine morphism—which makes the higher direct images vanish—and the fact that Y is affine, hence cohomologically trivial; cf. [8]. Now let

$$\begin{aligned}
&[\phi] \in \operatorname{Im}\{H_Q^q(\tilde{\tilde{Q}}, \mathscr{O}(\mathbb{E}))_{\mathrm{alg}} \to H_Q^q(\tilde{Q}, \mathscr{O}(\mathbb{E})\},\\
&[\psi] \in H_S^{n-q}(\tilde{S}, \mathscr{O}(\tilde{\mathbb{E}}^*) \otimes \Omega_X^n)_{(K)}
\end{aligned} \tag{15.31}$$

be given. The contracted cup product $[\phi] \wedge [\psi]$ takes values in $H_{Q \cap S}^n(X, \Omega_X^n)$. Via (15.29), (15.30), $[\phi]$ can be represented by a cycle

$$\phi \in \Gamma \mathscr{H}_{Q_Y}(\mathscr{O}_Y(p^*\mathbb{E} \otimes \Lambda^{q+u} \mathbb{T}_{Y|X}^{1,0*}))_f[n+d-s], \tag{15.32}$$

and $[\psi]$ by a K-finite cycle

$$\psi \in C^\infty(S_Y; p^*\tilde{\mathbb{E}}^* \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{n-u-q} \mathbb{T}_{Y|X}^{1,0*}). \tag{15.33}$$

Let $\{\phi, \psi\} \in C_{Q_Y \cap S_Y}^\infty(S_Y; \Lambda^{2n} \mathbb{T}_Y^{1,0*})$ denote the image of $\phi \otimes \psi$ under the pairing (15.26). We now use Proposition 15.28: the identifications (15.29) and (15.30) are canonical, hence compatible with cup products, so

$$\{\phi, \psi\} \text{ represents the contracted cup product } [\phi] \wedge [\psi]. \tag{15.34}$$

Recall (14.6)–(14.7). Evaluating $[\phi] \wedge [\psi]$ over the fundamental cycle $[X]$ amounts to pairing[2] any representative form in $C_{Q \cap S}^\omega(X; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^n \mathbb{T}_X^{0,1*})$

[2] Forms of top degree with hyperfunction coefficients on the compact manifold X are naturally dual to real analytic functions.

against the constant function 1. Since $\{\phi, \psi\}$ represents $[\phi] \wedge [\psi]$ via the quasi-isomorphism (15.22), we can perform this operation already on S_Y. Specifically, we can pair the compactly supported form $\{\phi, \psi\}$—of top degree, with distribution coefficients—against the function 1 on S_Y, to obtain the value of $[\phi] \wedge [\psi]$ on $[X]$. Symbolically, "pairing against 1" is integration. We have shown:

$$([\phi] \wedge [\psi])[X] = \int_X \{\phi, \psi\}. \tag{15.35}$$

From the definition of the pairing (15.26) it follows that $\{\phi, \psi\}$ depends only on the $(k-1)$st jet of ψ along $Q_Y \cap S_Y$, where k is the largest order of poles appearing in any local expression (15.13) for the coefficients of ϕ. In particular, via the natural map

$$C^\infty(S_Y; p^*\tilde{\mathbb{E}} \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^\cdot \mathbb{T}_{Y|X}^{1,0*}) \to C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^\cdot \mathbb{N}^*)_{(K)} \tag{15.36}$$

(recall (3.3)–(3.5) and (6.5)), the pairing $\phi \otimes \psi \mapsto \{\phi, \psi\}$ extends to $C^{\mathrm{for}}(G/H; \mathbb{E} \otimes \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^\cdot \mathbb{N}^*)_{(K)}$. This extended pairing, followed by "evaluation against 1," is precisely the pairing which effects the duality theorem of [8]. Thus (15.35) embodies the assertion of Theorem 14.9.

It remains to prove Proposition 15.28. The assertion is local with respect to the fibrations $Y \to X$, $S_Y \to S$, so we may as well assume that $\mathbb{E}$ is the trivial line bundle,

$$\mathbb{E} = \mathbb{E}^* = \mathbb{C}. \tag{15.37}$$

As in the Appendix, we let $\mathbb{T}^{1,0}(S)$, $\mathbb{T}^{0,1}(S)$ denote the "holomorphic" and "antiholomorphic" parts of the tangent bundle of the CR manifold S. Since S has constant CR dimension,

$$(\mathbb{T}^{1,0}(S) + \mathbb{T}^{0,1}(S))^\perp, \qquad \text{the annihilator of } (\mathbb{T}^{1,0}(S) + \mathbb{T}^{0,1}(S)), \tag{15.38}$$

is a C^∞ subbundle of the cotangent bundle $\mathbb{T}_S^*$.

15.39. Lemma. *The complex structure operator $J = J_X$, followed by the natural surjection $\mathbb{T}_X^*|_S \to \mathbb{T}_S^*$, defines an isomorphism of the conormal bundle $\mathbb{T}_S^\perp$ onto the bundle $(\mathbb{T}^{1,0}(S) + \mathbb{T}^{0,1}(S))^\perp$.*

Proof. The image of the map in question is the annihilator of the largest J-invariant subspace of $\mathbb{T}_S$, i.e., the annihilator of $\mathbb{T}^{1,0}(S) + \mathbb{T}^{0,1}(S)$. This annihilator has rank $\dim_{\mathbb{R}} S - 2c = 2n - u - 2c$, with $u = \operatorname{codim}_{\mathbb{R}}(S \subset X)$, $c = \dim_{\mathrm{CR}}(S)$, as before. The conormal bundle, on the other hand, has rank u. Hence the lemma comes down to the equality

$n=c+u$. But $\mathbb{T}^{0,1}(S)$ and $\mathbb{T}_S$ are modeled, respectively, on $\mathfrak{b}/\mathfrak{b}\cap\bar{\mathfrak{b}}$ and $\mathfrak{g}/\mathfrak{b}\cap\bar{\mathfrak{b}}$. Thus,

$$\begin{aligned} c+u &= \dim_{\mathbb{C}}(\mathfrak{b}/\mathfrak{b}\cap\bar{\mathfrak{b}})+2n-\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{b}\cap\bar{\mathfrak{b}}) \\ &= 2n-\dim_{\mathbb{C}}\mathfrak{g}+\dim_{\mathbb{C}}\mathfrak{b}=n, \end{aligned}$$

as had to be shown.

Because of the lemma, the image of the conormal bundle $\mathbb{T}_S^{\perp}$ under the bundle map $v\mapsto v+iJv$ constitutes a C^{∞} subbundle $\mathbb{M}\subset\mathbb{T}_X^{0,1*}|_S$, of rank u, such that

$$(\mathbb{T}_X^{0,1*}|_S)/\mathbb{M}\simeq\mathbb{T}^{0,1}(S)^*. \tag{15.40}$$

We now define a subsheaf

$$\mathscr{C}_S^{\infty}(\tilde{S};\Lambda^n\mathbb{T}_X^{1,0*}\otimes\Lambda^{\cdot}\mathbb{T}_X^{0,1*})^0\subset\mathscr{C}_S^{\infty}(\tilde{S};\Lambda^n\mathbb{T}_X^{1,0*}\otimes\Lambda^{\cdot}\mathbb{T}_X^{0,1*}) \tag{15.41}$$

as follows. The sheaf $\mathscr{C}_S^{\infty}(\tilde{S};...)$ has a natural increasing filtration, according to the number derivatives; it is dual to the filtration of $\mathscr{C}^{\infty}(\tilde{S};...)$ according to the order of vanishing along S. The subsheaf (15.41) consists of all forms which (i) lie in the lowest level of the filtration and (ii) are divisible by $\Lambda^u\mathbb{M}$. In the local formula for $\bar{\partial}$, non-tangential derivatives occur paired with sections of $\mathbb{M}$, so (15.41) is actually a subcomplex of sheaves.

15.42. Lemma. *The inclusion* (15.41) *induces a quasi-isomorphism.*

Proof. Let ϕ be a $\bar{\partial}$-closed local section of $\mathscr{C}_S^{\infty}(\tilde{S};...)$. Shrinking the domain, if necessary, we may assume that the coefficients involve only a finite number, say l, normal derivatives. The polynomial Poincaré lemma, applied fibre-by-fibre in the normal directions, implies that we can reduce l by one by adding a boundary to ϕ—unless $l=0$, of course. If $l=0$, the condition $\bar{\partial}\phi=0$ forces ϕ to be divisible by $\Lambda^u\mathbb{M}$. More generally, $\bar{\partial}$ increases the order of normal derivatives of any form which is not divisible by $\Lambda^u\mathbb{M}$. It follows that (15.41) induces a bijection on the level of cohomology.

The lemma allows us to replace $\mathscr{C}_S^{\infty}(\tilde{S};...)$ by its subcomplex $\mathscr{C}_S^{\infty}(\tilde{S};...)^0$ in the course of proving Proposition 15.28.

Let ϕ be a local section of $\mathscr{H}_Q(\mathcal{O}_X)_f[d]$ and ψ a local section of $\mathscr{C}_S^{\infty}(\tilde{S};\Lambda^n\mathbb{T}_X^{1,0*}\otimes\Lambda^q\mathbb{T}_X^{0,1*})^0$, both defined on some open set $U\subset X$. As explained earlier, $\phi\wedge\psi$ is a scalar form of type $(n, q+d)$ with distribution coefficients, supported on $S\cap Q$ and smooth along $S\cap Q$. To know this form is to know its effect on any compactly supported C^{∞} $(0, n-q-d)$-form η on U, in other words, the expression $\int_U\phi\wedge\psi\wedge\eta$, for any such test form η. On the level of the subcomplex (15.41), the quasi-isomorphism (15.19) is induced by a map

$$\psi \mapsto \tilde{\psi} \in \Gamma(p^{-1}U; \mathscr{C}^{-\infty}(S_Y; \Lambda^n \mathbb{T}_X^{1,0*} \otimes \Lambda^{q-u} \mathbb{T}_{Y|X}^{1,0*})), \tag{15.43}$$

which we shall describe below. Similarly, we shall describe an explicit map

$$\phi \mapsto \tilde{\phi} \in \Gamma(p^{-1}U; \mathscr{H}_{Q_Y}(\mathcal{O}_Y(\Lambda^{n-s} \mathbb{T}_{Y|X}^{1,0*}))_f[n+d-s], \tag{15.44}$$

which induces the quasi-isomorphism (15.24). Let

$$\{\tilde{\phi}, \tilde{\psi}\} \in \Gamma(p^{-1}U; \mathscr{C}^{-\infty}_{Q_Y \cap S_Y}(S_Y; \Lambda^{n+d+q} \mathbb{T}_Y^{1,0*})) \tag{15.45}$$

denote the image of $\tilde{\phi} \otimes \tilde{\psi}$ under the pairing (15.26). Here recall (6.13) and (15.12). The restriction of $\mathbb{T}_Y^{1,0*}$ to the real form $S_Y \subset Y$ is naturally isomorphic to the cotangent bundle of S_Y, so we may view $\{\tilde{\phi}, \tilde{\psi}\}$ as scalar-valued $(n+d+q)$-form on $p^{-1}U \cap S_Y$, with distribution coefficients, supported and smooth along $Q_Y \cap S_Y$. In particular, $\{\tilde{\phi}, \tilde{\psi}\}$ can be integrated against the smooth, compactly supported $(n-q-d)$-form $p^*\eta|_{S_Y}$. Unraveling the statement of Proposition 15.28, we find the assertion comes down to the equality

$$\int_U \phi \wedge \psi \wedge \eta = \int_{p^{-1}U \cap S_Y} \{\tilde{\phi}, \tilde{\psi}\} \wedge (p^*\eta|_{S_Y}), \tag{15.46}$$

for every ϕ, ψ, η as above.

By assumption, the coefficients of ψ are distributions on U, supported on S, smooth along S, and not involving any normal derivatives. Formally, any distribution with these properties can be written as $h\sigma^{-1}$, where h is a smooth function on $U \cap S$ and σ is a generating section of $\Lambda^u \mathbb{T}_S^{\perp}$, the top exterior power of the conomral bundle of S in X. Now let τ be a generating section of the top exterior power of the conormal bundles of the fibres of $S_Y \to S$ in the fibres of $Y \to X$; these top exterior powers constitute a rank one subbundle in $\Lambda^{2n-u} \mathbb{T}^*_{Y|X}|_{S_Y}$. If h and σ are as before, $(p^*h)\sigma^{-1}\tau^{-1}$ defines a distribution on Y, supported and smooth along S_Y. Thus we may divide the distribution coefficients of $p^*\psi$ by τ, to obtain a form on Y with distribution coefficients, supported and smooth along S_Y; we may then take the wedge product[3] of this form with τ. The map

$$\psi \mapsto \frac{p^*\psi}{\tau} \wedge \tau$$

[3] Strictly speaking, we should take the wedge product with some extension of τ to a section of $\Lambda^{2n-u} \mathbb{T}^*_{Y|X}$ over a full neighborhood of S_Y in Y. The particular choice of the extension does not matter since the coefficients of ψ, and hence those of $\tau^{-1}p^*\psi$, involve no normal derivatives.

represents the quasi-isomorphism (6.12) on the level of the subcomplex (15.41). By definition of the subcomplex, the values of ψ are divisible by $\Lambda^u\mathbb{M}$. But $\mathbb{M}$ is congruent to $\mathbb{T}_S^\perp$ modulo $\mathbb{T}_X^{1,0*}$, so the values of ψ are divisible by σ. Dividing the values of ψ by σ while multiplying the distribution coefficients by σ turns ψ into a smooth, scalar-valued $(n+q-u)$-form on $S_Y \cap U$, which we denote by $r_S\psi$. Note that $\sigma \wedge \tau$ is a generating section of the top exterior power of the conormal bundle of $S_Y \subset Y$ via (6.13); moreover,

$$\frac{p^*\psi}{\tau} \wedge \tau = \frac{p^* r_S \psi}{\sigma \wedge \tau} \wedge \sigma \wedge \tau.$$

A statement similar to Lemma 15.42 is the reason for the quasi-isomorphism (6.15) and the analogous quasi-isomorphism in the context of distribution forms. We conclude: the assignment $\psi \mapsto \tilde{\psi}$, with

$$\tilde{\psi} = p^* r_S \psi, \tag{15.47}$$

represents the quasi-isomorphism (15.19) when we make the natural identifications (6.13) and

$$\mathbb{T}_Y^{1,0}|_{S_Y} \simeq \mathbb{T}_{S_Y}. \tag{15.48}$$

For future reference, we observe:

> the values of $r_S\psi$ along $Q \cap S$ are divisible by $d\omega|_S$, for any local section ω of the ideal sheaf $\mathscr{I}_S \subset \mathcal{O}_X$. (15.49)

This follows from Lemmas 15.11 and 15.39 and from the fact that ψ has bidegree $(n, \cdot)$.

Shrinking U, if necessary, we can introduce holomorphic coordinates

$$(t, v, w, z) = (t_1, ..., t_s, v_1, ..., v_{n-s}, w_1, ..., w_d, z_1, ..., z_{n-d}) \tag{15.50}$$

on $p^{-1}U$, such that (w, z) are coordinates on U, (t, v) are coordinates along the fibres of $p^{-1}U \to U$, and

$$\begin{aligned} U \cap Q &= \{(z, w) \in U \mid w = 0\}, \\ p^{-1}U \cap Q_Y &= \{(t, v, w, z) \in p^{-1}U \mid v = w = 0\}. \end{aligned} \tag{15.51}$$

Symbolically, we write dw for $dw_1 \wedge \cdots \wedge dw_d$, w^{-1} for $(w_1 \cdots w_d)^{-1}$, etc. Note that $v = 0$ defines the fibre of $Q_Y \to Q$ as a subvariety of any given fibre F of $p^{-1}U \to U$. Thus,

$$\frac{dv}{v} \in \Gamma(F; \mathscr{H}_{F \cap Q_Y}(\Omega_F^{n-s})_f[n-s]) \tag{15.52}$$

represents what might be called the "fundamental cycle of local cohomology" of the pair $(F, F \cap Q_Y)$. In particular, this section is canonically attached to the pair $(F, F \cap Q_Y)$. We now use the flat connection (6.13) to turn these sections, corresponding to the various fibres of $p^{-1}U \to U$, into a section

$$\nu \in \Gamma(p^{-1}U; \mathscr{H}_{Q_Y}(\Omega^n_{Y|X})_r[n+d-s]). \tag{15.53}$$

The very definition of ν shows that $\phi \mapsto \tilde{\phi}$, with

$$\tilde{\phi} = \phi\nu, \tag{15.54}$$

induces the quasi-isomorphism (15.24). Since the connection was used to describe ν, we cannot conclude that $\nu = dv/v$ as meromorphic differential forms on $p^{-1}U$; however,

$$\nu \equiv dv/v \text{ modulo the linear span of the } dw_j, \tag{15.55}$$

since the v_i and w_j generate the ideal sheaf of Q_Y.

We now have all the ingredients of the identity (15.46) which is yet to be verified. Before doing so, we make one minor and one major reduction of the problem. From the explicit description of the morphism (15.43), we conclude

$$(\psi \wedge \eta)^{\sim} = \tilde{\psi} \wedge (p^*\eta|_{S_Y}),$$

and, from the description of the pairing (15.26),

$$\{\tilde{\phi}, \tilde{\psi} \wedge (p^*\eta|_{S_Y})\} = \{\tilde{\phi}, \tilde{\psi}\} \wedge (p^*\eta|_{S_Y}).$$

Thus we may replace ψ by $\psi \wedge \eta$ and η by the constant function 1. Concretely, we suppose

$$q = n - d, \qquad \text{and } \phi \text{ has compact support in } U. \tag{15.56}$$

In this situation, (15.46) becomes

$$\int_U \phi \wedge \psi \wedge 1 = \int_{p^{-1}U \cap S_Y} \{\tilde{\phi}, \tilde{\psi}\} \wedge 1. \tag{15.57}$$

By infinitesimal left translation, $\mathfrak{g}$ acts on X and Y as a Lie algebra of homomorphic vector fields. The realm form $\mathfrak{g}_0 \subset \mathfrak{g}$ also acts a a Lie algebra of real, C^∞ vector fields. Complexifying this latter action, we obtain a second action of $\mathfrak{g}$, by complex C^∞ vector fields. These two actions agree on holomorphic or meromorphic objects, by the Cauchy–Riemann equations. Since S and S_Y are G-orbits, the action of $\mathfrak{g}_0$ and its complexification induce actions on any G-invariant complex of sheaves, in particular on the subcomplex (15.41). Though not globally invariant, the local cohomology sheaves along $Q \subset X$ and $Q_Y \subset Y$ are infinitesimally invariant. The various pairings and morphisms which enter the formula (15.57) are canonically defined, hence preserved by the infinitesimal action. This allows us to shift differentiation along any $\xi \in \mathfrak{g}$ from ϕ (where we may use either action) to ψ (where we must use the second action). As $\mathscr{U}(\mathfrak{g})$-module (via the holomorphic action), $\mathscr{H}_Q(\mathscr{O}_X)_f[d]$ is generated by sections which involve only complete intersections of first-order poles, since $\mathfrak{g}$ maps onto the fibres of the holomorphic conormal bundle of $Q \subset X$. Thus it suffices to verify (15.57) when

$$\phi \text{ is a complete intersection of first-order poles,} \tag{15.58}$$

as we shall assume from now on.

In terms of the local coordinate system (15.50), we have $\phi = f(z)\, w^{-1}$, for some holomorphic function $f(z)$ on $U \cap Q$. Since $dw \wedge d\bar{w}$ generates the top exterior power of the conormal bundle of $Q \subset X$, we may view $(dw \wedge d\bar{w})^{-1} f(z)$ as a distribution on U, supported and smooth along Q, not involving any normal derivatives. The discussion around (15.16) shows that ϕ corresponds to the distribution-coefficient $(0, d)$-form $(dw \wedge d\bar{w})^{-1} f(z)\, d\bar{w}$. Recall the relationship between the differential forms ψ and $r_S\psi$: $\psi = (\sigma^{-1} r_S \psi) \wedge \sigma$. Here $r_S\psi \wedge \sigma$ is a smooth section of $\Lambda^{2n-d}\mathbb{T}^*_X|_S$, and division by σ turns the coefficients, which are smooth functions on $U \cap S$, into distributions on U, supported and smooth along S, not involving any normal derivatives. Thus $f(z)\, d\bar{w} \wedge r_S\psi \wedge \sigma$ becomes a smooth section of $\Lambda^{2n}\mathbb{T}^*_X|_{Q \cap S}$. When we divide the coefficient functions by the generating section $dw \wedge d\bar{w} \wedge \sigma$ of the top exterior power of the conormal bundle of $(Q \cap S) \subset X$, they become distributions on U, supported and smooth along $Q \cap S$, not involving any normal derivatives. In that sense,

$$\phi \wedge \psi = (dw \wedge d\bar{w} \wedge \sigma)^{-1} f(z)\, d\bar{w} \wedge r_S\psi \wedge \sigma. \tag{15.59}$$

Since this form has top degree, we can divide its values by $dw \wedge d\bar{w} \wedge \sigma$, to obtain a smooth, compactly supported form ξ on $U \cap Q \cap S$, of top degree. The left-hand side of the relation (15.57) is precisely the integral of the form ξ over $Q \cap S$.

Because of (15.47), (15.49), (15.54), (15.55), we have the formal equality

$$\tilde{\phi} \wedge \tilde{\psi} = f(z)\, w^{-1} v^{-1}\, dv \wedge p^* r_S \psi. \tag{15.60}$$

Moreover, the values of this form are divisible by dw. This expression only involves first-order poles. The definition of the pairing (15.26) therefore amounts to interpreting $w^{-1}v^{-1}dw \wedge dv$ as evaluation along $w = v = 0$, i.e., along Q_Y. The result is a smooth, compactly supported $(n-2d-u)$-form on $Q_Y \cap S_Y$, which coincides with the form ξ of the previous paragraph via the diffeomorphism $p: Q_Y \cap S_Y \xrightarrow{\sim} Q \cap S$. Like any smooth form of top degree, $p^*\xi$ maps forward to a form of top degree on S_Y, with distribution coefficients supported along $Q_Y \cap S_Y$. That distribution coefficient form on S_Y is the image $\{\tilde{\phi}, \tilde{\psi}\}$ of $\tilde{\phi} \otimes \tilde{\psi}$ under the pairing (15.26). Hence

$$\int_{p^{-1}U \cap S_Y} \{\tilde{\phi}, \tilde{\psi}\} \wedge 1 = \int_{p^{-1}U \cap Q_Y \cap S_Y} p^*\xi = \int_{U \cap Q \cap S} \xi = \int_U \phi \wedge \psi \wedge 1,$$

as was to shown.

APPENDIX: CAUCHY–RIEMANN COMPLEX

We collect some material on induced $\bar{\partial}$ complexes for which there appears to be no convenient reference.

Let X be a complex manifold, $n = \dim_{\mathbb{C}} X$, and let $S \subset X$ be a C^∞ submanifold, $m = \dim_{\mathbb{R}} S$. Write $\mathbb{T}(\ldots)$ for complexified tangent bundle, $\mathbb{T}_x(\ldots)$ for its fibre at x. If $x \in S$ then $\dim_{\mathbb{C}}\{\mathbb{T}_x(S) \cap \mathbb{T}^{1,0}(X)\}$ is the *holomorphic* or *Cauchy–Riemann* dimension of S at x. From now on we assume that S has constant Cauchy–Riemann dimension, say c. In other words,

$$\mathbb{T}^{1,0}(S) = \mathbb{T}(S) \cap \mathbb{T}^{1,0}(X) \qquad \text{and} \qquad \mathbb{T}^{0,1}(S) = \mathbb{T}(S) \cap \mathbb{T}^{0,1}(X)$$

are smooth sub-bundles in $\mathbb{T}(S)$, of fibre dimension c.

Note that $\mathbb{T}_x^{0,1}(S)$ consists of all $\xi \in \mathbb{T}_x(S)$ that annihilate germs of functions on X holomorphic at x. So

$$\mathscr{T}^{0,1}(S)\text{: sheaf of germs of } C^\infty \text{ sections of } \mathbb{T}^{0,1}(S)$$

is a sheaf of Lie algebras under Poisson bracket. Dually,

$$\mathscr{A}^{\cdot}(S)\text{: sheaf of germs of } C^\infty \text{ sections of } \Lambda^{\cdot}(\mathbb{T}^{0,1}(S))^* \tag{A.1}$$

is a differential graded sheaf. The differential $\bar{\partial}_S: \mathscr{A}^p(S) \to \mathscr{A}^{p+1}(S)$ is given as follows. The germs $\phi \in \mathscr{A}_x^p(S)$ are just the germs of restrictions $\omega|_{T^{0,1}(S)}$, where ω is a C^∞ $(0, p)$-form on X in a neighborhood of x. Then

$$\bar{\partial}_S(\phi) = (\bar{\partial}\omega)|_{T^{0,1}(S)}, \qquad \text{where} \quad \phi = \omega|_{T^{0,1}(S)}. \tag{A.2}$$

We say that a vector bundle $\mathbb{E} \to S$ is *Cauchy–Riemann* (CR) if its transition functions are annihilated by sections of $\mathbb{T}^{0,1}(S)$. Then the transition functions are annihilated by $\bar{\partial}_S$. As in the Dolbeault case, now $\bar{\partial}_S$ acts on the sheaf

$$\mathscr{A}^{\cdot}(S;\mathbb{E}): C^{\infty} \text{ section germs of } \mathbb{E}\otimes \Lambda^{\cdot}(\mathbb{T}^{0,1}(S))^*, \tag{A.3}$$

which is the twist of $\mathscr{A}^{\cdot}(S)$ by $\mathbb{E}$. With the $\bar{\partial}_S: \mathscr{A}^p(S;\mathbb{E}) \to \mathscr{A}^{p+1}(S;\mathbb{E})$, (A.3) is a differential graded sheaf.

The CR analog of the canonical bundle is $\mathbb{K}_S = \Lambda^{m-c}(\mathbb{T}(S)/\mathbb{T}^{0,1}(S))^*$. Note that $m-c$ is the fibre dimension of $\mathbb{T}(S)/\mathbb{T}^{0,1}(S)$.

We are going to prove that

$$\mathbb{K}_S \to S \text{ is a CR line bundle,} \tag{A.4}$$

in order to discuss a pairing

$$\mathscr{A}^p(S;\mathbb{E}) \otimes \mathscr{A}^q(S;\mathbb{E}^*\otimes\mathbb{K}_S) \to \mathscr{C}^{p+q+m-c}(S), \tag{A.5}$$

where the latter is the sheaf of germs of C^{∞} differential forms of degree $p+q+m-c$. When $p+q=c$, (A.5) plus integration over S gives a duality which is the starting point for certain isomorphisms of cohomology.

A.6. LEMMA. *Let $x\in S$. Then there exist functions $w^1, ..., w^{m-c}$ holomorphic on an X-neighborhood of x such that the $dw^i|_S$ span the fibres of $\{\mathbb{T}(S)/\mathbb{T}^{0,1}(S)\}^*$ in an S-neighborhood of x.*

Proof. Let z be a complex local coordinate on X near x such that the $(\partial/\partial z^i)|_x$, $1\leqslant i\leqslant c$, span $\mathbb{T}^{1,0}_x(S)$. Then the $dz^i|_S$, $d\bar{z}^j|_S$, $1\leqslant i, j\leqslant c$, have common annihilator $\mathbb{N}\to S$ (near x) which is a smooth sub-bundle of $\mathbb{T}(S)$ such that $\mathbb{T}(S) = \mathbb{T}^{1,0}(S)\oplus\mathbb{T}^{0,1}(S)\oplus\mathbb{N}$. The almost complex structure operator J of X preserves $\mathbb{T}^{1,0}(S)\oplus\mathbb{T}^{0,1}(S)$, and $\mathbb{T}(S)\cap J\mathbb{N} = 0$. Now $\{\xi - iJ\xi : \xi\in\mathbb{N}\}$ is a rank $m-2c$ sub-bundle of $\mathbb{T}^{1,0}(X)|_S$ disjoint from $\mathbb{T}(S)$. So we can modify $\{z^{c+1}, ..., z^n\}$ and assume that the $\mathrm{Re}(\partial/\partial z^i)|_x$, $c<i\leqslant m-c$, span $\mathbb{N}_x$. Now the $dz^i|_S$, $1\leqslant i\leqslant m-c$, are linearly independent near x with common annihilator $\mathbb{T}^{0,1}(S)$, so the lemma follows with $w^i = z^i$. Q.E.D.

Now we can prove (A.4). Given $\{w^i\}$ as in Lemma A.6, $(dw^1\wedge\cdots\wedge dw^{m-c})|_S$ spans the fibre of $\mathbb{K}_S$ near x. Given also $\{u^j\}$ as in Lemma A.6, now $(du^1\wedge\cdots\wedge du^{m-c})|_S$ is a multiple of $(dw^1\wedge\cdots\wedge dw^{m-c})|_S$, so $du^j|_S = \sum a^j_i\, dw^i|_S$ for some C^{∞} functions a^j_i, and

$$(du^1\wedge\cdots\wedge du^{m-c})|_S = \det(a^j_i)\cdot(dw^1\wedge\cdots\wedge dw^{m-c})|_S. \tag{A.7}$$

Evaluate $0 = ddu^j|_S = \sum da^j_i\wedge dw^i|_S$ on (ξ, η_l) with $\xi\in\mathbb{T}^{0,1}(S)$ and $\eta_l\in\mathbb{T}(S)$ such that $dw^i(\eta_l) = \delta^i_l$. Note $dw^i(\xi) = 0$ because w^i is holomorphic.

holomorphic. So $da_I^j(\xi)=0$. Thus the determinant in (A.7) is annihilated by every $\xi \in \mathbb{T}^{0,1}(S)$. That proves (A.4).

Since $m-c=\dim_{\mathbb{C}} \mathbb{T}_x(S)/\mathbb{T}_x^{0,1}(S)$ and $\mathbb{K}_S = \Lambda^{m-c}((\mathbb{T}(S)/\mathbb{T}^{0,1}(S))^*)$, we have a well-defined map

$$\Lambda^{p+q}\mathbb{T}_x^{0,1}(S)^* \otimes (\mathbb{K}_S)_x \to \Lambda^{p+q+m-c}\mathbb{T}_x^*(S)$$

specified by exterior product. Composing it with

$$\{\mathbb{E}_x \otimes \Lambda^p \mathbb{T}_x^{0,1}(S)^*\} \otimes \{\mathbb{E}_x^* \otimes \Lambda^q \mathbb{T}_x^{0,1}(S)^*\} \to \Lambda^{p+q}\mathbb{T}_x^{0,1}(S)^*,$$

we have (A.5). For $p+q=c$, we follow it by integration over S. That gives an identification

$$C^{-\infty}(S; \mathbb{E} \otimes \Lambda^p \mathbb{T}^{0,1}(S)^*) \cong C_c^{\infty}(S; \mathbb{E}^* \otimes \mathbb{K}_S \otimes \Lambda^{c-p}\mathbb{T}^{0,1}(S)^*)' \quad (A.8)$$

of the space of distribution sections of $\mathbb{E} \otimes \Lambda^p \mathbb{T}^{0,1}(S)^*$ with the strong topological dual of the space of compactly supported C^{∞} sections of $\mathbb{E} \otimes \mathbb{K}_S \otimes \Lambda^{c-p}\mathbb{T}^{0,1}(S)^*$.

Similarly, if S is a C^{ω} submanifold of X and the CR bundle $\mathbb{E} \to S$ is C^{ω}, then (A.5) induces a formal duality between real analytic and hyperfunction sections. Let U be open in S with closure $\mathrm{cl}(U)$ compact. Then

$$C^{-\omega}(U; \mathbb{E} \otimes \Lambda^p \mathbb{T}^{0,1}(S)^*) \cong \frac{C^{\omega}(\mathrm{cl}(U); \mathbb{E}^* \otimes \mathbb{K}_S \otimes \Lambda^{c-p}\mathbb{T}^{0,1}(S)^*)'}{C^{\omega}(\mathrm{bd}(U); \mathbb{E}^* \otimes \mathbb{K}_S \otimes \Lambda^{c-p}\mathbb{T}^{0,1}(S)^*)'}. \quad (A.9)$$

The dualities (A.5), (A.8), (A.9) are compatible with $\bar{\partial}_S$:

A.10. LEMMA. *If ω_1, ω_2 are sections of $\mathbb{E} \otimes \Lambda^p \mathbb{T}^{0,1}(S)^*$, $\mathbb{E}^* \otimes \mathbb{K}_S \otimes \Lambda^q \mathbb{T}^{0,1}(S)^*$ over an open $U \subset S$, then $d(\omega_1 \wedge \omega_2) = (\bar{\partial}_S \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (\bar{\partial}_S \omega_2)$.*

Proof. As in the argument of (A.6), we have a complex local coordinate z on X near $x \in S$ such that the $dz^i|_S$, $1 \leqslant i \leqslant c$, span the fibres of $\mathbb{T}^{1,0}(S)^*$ and the $dz^i|_S$, $1 \leqslant i \leqslant s-c$, span the fibres of $\{\mathbb{T}(S)/\mathbb{T}^{0,1}(S)\}^*$. Locally $\omega_1 = \sum_{|I|=p} f_I \sigma_I \, d\bar{z}^I|_S$ and $\omega_2 = \sum_{|J|=q} h_J \tau_J \, dz^1 \wedge \cdots \wedge dz^{m-c} \wedge d\bar{z}^J|_S$, where f_I and h_J are scalar valued, σ_I and h_J are local CR sections of $\mathbb{E}$ and $\mathbb{E}^*$, and $d\bar{z}^I$ and $d\bar{z}^J$ are monomials in the $d\bar{z}^k$. Now, on S,

$$\bar{\partial}_S \omega_1 = \sum (\bar{\partial}_S f_I)\, \sigma_I \, d\bar{z}^I|_S$$

and

$$\bar{\partial}_S \omega_2 = \sum (\bar{\partial}_S h_J)\, \tau_J \, dz^1 \wedge \cdots \wedge dz^{m-c} \wedge d\bar{z}^J|_S$$

because σ_j, τ_j are CR. Thus,

$$(\bar{\partial}_S \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (\bar{\partial}_S \omega_2)$$
$$= (d\omega_1 \wedge \omega_2) + (-1)^p \omega_1 \wedge d\omega_2 = d(\omega_1 \wedge \omega_2)$$

as asserted. Q.E.D.

Note that Lemma A.10 carries the dualities (A.8) and (A.9) to the level of cohomology.

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