The Schwartz Space of a General Semisimple Lie Group. I. Wave Packets of Eisenstein Integrals

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1. INTRODUCTION

Suppose $G$ is a semisimple Lie group with infinite center. As in the finite center case, the tempered spectrum of $G$ consists of families of representations induced from cuspidal parabolic subgroups $P = MAN$. However, in the infinite center case, the representations of $M$ to be induced are not discrete series, but are relative discrete series which occur in continuous families. In two previous papers [8, 9] we studied matrix coefficients of relative discrete series representations for a connected simple group with

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infinite center and showed how to combine them into wave packets along the continuous parameter to construct Schwartz functions on the group.

In this paper we construct Schwartz class wave packets of matrix coefficients corresponding to the various induced series of tempered representations for any connected reductive Lie group $G$. In the special case that $G = P = M$, these are exactly the wave packets of relative discrete series matrix coefficients constructed in [8, 9]. To study induced representations on $G$ we must be able to define continuous families of relative discrete series representations on the Levi components $M$ of arbitrary cuspidal parabolic subgroups. These groups $M$ need not be connected, but lie in the class of reductive groups studied in [11]. (See (10.1) for the precise definition.) Since we do not prove our wave packets are Schwartz by induction on $G$, but rather by induction on parabolic subgroups of $G$, we do not need to assume $G$ is a general group in the class (10.1). This simplifies the constructions of the machinery needed to define wave packets. In Section 10, we indicate how the results for connected groups can be extended to arbitrary groups in the class (10.1).

To construct wave packets we proceed as follows. We start with continuous families of relative discrete series matrix coefficients on $M$ of the type constructed in [9]. We use these to form Eisenstein integrals similar to those defined by Harish-Chandra in [3]. Thus our Eisenstein integrals have two types of continuous parameters, those corresponding to unitary characters of $A$ which occur in Harish-Chandra's Eisenstein integrals, plus additional continuous parameters coming from families of relative discrete series representations on $M$. Wave packets must be taken along both types of continuous parameters to obtain Schwartz class functions on the group $G$. Roughly speaking, the wave packets considered will be integrals of the form

$$F_x(x) = \int \int E(P: h: v: x) \alpha(h: v) m(h: v) \, dh \, dv, \quad x \in G,$$

where $E(P: h: v: x)$ is an Eisenstein integral corresponding to $v \in \hat{A}$ and to a continuous family of matrix coefficients for relative discrete series representations $\pi_h$ of $M$: $m(h: v) \, dh \, dv$ is the Plancherel measure corresponding to the associated family of induced representations $\pi_{h, v} = \text{Ind}_P^G(\pi_h \otimes v \otimes 1)$ of $G$; and $\alpha(h: v)$ is a suitable Schwartz function in the parameter variables.

The space $\hat{A}$ is a Euclidean space, and Harish-Chandra proved in the finite center case that a necessary and sufficient condition for the wave packet $F_x$ to be a Schwartz function on $G$ is that $\alpha$ be an ordinary Schwartz function on $\hat{A}$ [5]. The infinite center situation is more complicated. First, $\alpha$ must be a jointly smooth function of $h$ and $v$ which decays rapidly at
infinity as for an ordinary Schwartz function. However, as a function of $h$, $\alpha$ must also decay rapidly, in the sense of having a zero of infinite order, as $h$ approaches values on walls where $\pi_h$ is a limit of discrete series representation. Finally, there are conditions on $\alpha$ at points $(h_0, v_0)$ for which the induced representation $\pi_{h_0, v_0}$ is reducible. One way to phrase this condition is to require that the product $\alpha(h; v) m(h; v)$ be jointly smooth in $h$ and $v$. This is a restriction only at points $(h_0, v_0)$ as above where the Plancherel function $m(h; v)$, which is separately smooth in each variable, fails to be jointly continuous.

In this paper we make a slightly stronger assumption on $\alpha$, namely, that $\alpha(h; v) m_R(h; v)$ is jointly smooth in $h$ and $v$, where $m_R(h; v)$ is part of the Plancherel function. (See (9.12) for the definition.) Points $(h_0, v_0)$ at which $m_R(h; v)$ is not smooth, but $m(h; v)$ is, correspond to induced representations $\pi_{h_0, v_0}$ which only fail to be reducible because certain limits of discrete series are zero. For such $\alpha$, we are able to prove that $F_\alpha$ is a Schwartz function using Harish-Chandra's theory of the $c$-function, in particular the result which says that for a fixed discrete series representation of $M$, the Plancherel measure cancels the poles of the $c$-function considered as a meromorphic function of $v$. In order to remove this extra assumption on $\alpha$, we will need to know more about the $c$-function as a meromorphic function of $h$ and $v$ jointly. These results will also be needed to study the "mixed wave packets" described in the next paragraph, and so are deferred to the paper in which we will study the mixed wave packets.

In the finite center case, Harish-Chandra proved that every $K$-finite Schwartz function on $G$ is a finite sum of Schwartz wave packets of Eisenstein integrals coming from the various series of tempered representations [5]. (Of course, discrete series representations have no continuous parameters so the degenerate wave packets in this case are just single matrix coefficients.) In the infinite center case, the analogue of $K$, the maximal compact subgroup, is non-compact, and there are no $K$-finite functions in the Schwartz space of $G$. However, there is a dense subspace of the Schwartz space consisting of "$K$-compact" functions, that is, ones for which the $K$-types are restricted to lie in a compact subset of $\hat{K}$. (See [9].) However, it is not true in the infinite center case that a $K$-compact Schwartz function will be a finite sum of Schwartz wave packets of the type described above. The problem comes from the fact that the different series of tempered representations interfere where a reducible principal series representation breaks up as a sum of limits of relative discrete series representations. As a result of this interference between series, not all Schwartz functions on the group decompose as sums of Schwartz wave packets. The typical $K$-compact Schwartz function on $G$ breaks up as a sum of pieces from different series of representations which individually are not Schwartz functions, but which "patch together" at reducible principle
series and limits of discrete series to form a Schwartz function. The wave packets studied in this paper are the ones which patch together with the zero wave packet from all other series. In another paper we will study the mixed wave packets which have non-trivial patches.

The organization of this paper is as follows. In Section 2 we develop some structural information about $G$ and reductive components of its cuspidal parabolic subgroups. We also recall the basic definitions of the Schwartz space.

In Section 3 we discuss the parameterization of relative discrete series representations on $M$ and the corresponding continuous families of induced representations on $G$.

In Section 4 we extend results on holomorphic families of relative discrete series matrix coefficients which were proved in [9] for the case that $M$ is a simple, simply connected group of hermitian type, to all Levi components $M$ of cuspidal parabolic subgroups.

In Section 5 we reformulate the results of Section 4 as results on holomorphic families of spherical functions, and extend the growth estimates proved in [8, 9] to our general class of groups $M$.

In Section 6 we define holomorphic families of Eisenstein integrals and check that they are eigenfunctions of the center of the enveloping algebra.

In Sections 7 and 8 we extend the machinery developed by Harish-Chandra to study growth properties of abstract families of functions generalizing Eisenstein integrals to include dependence on the continuous parameters coming from the relative discrete series. Specifically, in Section 7 we use differential equations to sharpen a priori estimates, and in Section 8 we use these estimates to show that wave packets formed from a certain class of functions are Schwartz.

In Section 9 we show that the Eisenstein integrals defined in Section 6 are members of the abstract family studied in Section 7 and use the results of Section 8 to show that wave packets of Eisenstein integrals of the type described in this section are Schwartz functions. Further, we show that when these Schwartz functions are written in terms of tempered characters using the Plancherel formula, only the series of representations used to form the Eisenstein integrals occurs in the expansion.

In Section 10 we show how to extend the results of Section 9 from the case of connected groups to arbitrary groups in the class (10.1).

2. GROUP STRUCTURE

Throughout the first nine sections of this paper, $G$ is a connected reductive Lie group. Fix a Cartan involution $\theta$ of $G$ as in [11] and let $K$ denote the fixed point set of $\theta$. It is the full inverse image of a maximal compact
subgroup of the linear group $G/Z_G$, but is compact only when the center $Z_G$ of $G$ is compact.

**Proposition 2.1.** $K$ has a unique maximal compact subgroup $K_1$ and has a closed normal vector subgroup $V$ such that

(a) $K = K_1 \times V$.

(b) $Z = Z_G \cap V$ is co-compact in both $V$ and $Z_G$.

**Proof.** Let $p: G \to G$ be the universal covering and $K = p^{-1}(K)$. Then $K$ is direct product of the compact semisimple group $[K, K]$ and a vector group $\mathcal{W}$. Let $Z = Z_G \cap \mathcal{W}$; it has finite index in $Z_G$ and is co-compact in $\mathcal{W}$. Let $\mathcal{U}$ be the subspace of $\mathcal{W}$ spanned by $\text{Ker}(p) \cap \mathcal{W}$ and define $K_1 = p([K, K] \times \mathcal{U})$. Then $K_1$ is the unique maximal compact subgroup of $K$.

Decompose $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ such that $Z_G \cap \mathcal{V}$ is co-compact in $\mathcal{V}$. Then $p|_{\mathcal{V}}$ is one to one, so $V = p(\mathcal{V})$ is a closed vector subgroup of $K$, and $K = K_1 \times V$.

For (b), $(\text{Ker}(p) \cap \mathcal{W}) \times (Z_G \cap \mathcal{V})$ is co-compact in $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$, so $p(Z_G \cap \mathcal{V}) = Z_G \cap V$ is co-compact in $V$. Also, $Z_G \cap \mathcal{W}$ has finite index in $Z_G$, and $(\text{Ker}(p) \cap \mathcal{W}) \times (Z_G \cap \mathcal{V})$ has finite index in $Z_G \cap \mathcal{W}$, so $Z_G \cap V = p(Z_G \cap \mathcal{V})$ has finite index in $p(Z_G) = Z_G$. Q.E.D.

Let $P$ be a cuspidal parabolic subgroup of $G$. In other words, up to $G$-conjugacy, $P$ is given as follows. Start with a $\theta$-stable Cartan subgroup $H \subset G$. Then $H = T \times A$, where $T = H \cap K$ and $A = \exp a$, $a = h \cap p$, where $g = \mathfrak{g} + p$ under $\theta$. Then $Z_G(A) = M \times A$, where $\theta M = M$, we can choose a positive system $\Phi^+ = \Phi^+(\mathfrak{g}_a, A)$ of restricted roots, and $N = \exp(n)$, where $n = \sum_{\varphi \in \Phi^+} \mathfrak{g}_\varphi$. Now $P = MAN$. Note that $Z \subset M$ and that although $M$ need not be connected, it is in the class of reductive groups studied in [11, 6, 7]. (See (10.1) for the precise definition.)

2.2 **Remark.** Write $K_0^0 = M^0 \cap K$. Our structural results (2.1) for $K$ can be applied to $K_0^0$ to write $K_0^0 = K_{0,M}^0 \times V_M$, where $K_{0,M}^0 = K_0 \cap M^0$ is maximal compact in $K_0^0$. Note that $V_M \not\subset V$ in general. Write $M^* = Z_M(M^0)^0 \cap K^0_0$ and $K^0_M = K \cap M^*.$

In order to define spherical functions we will need to consider the larger group $K_{M}^1 = K_{M}^1 \times V$. Note this is a group since $V$ centralizes $K$. We can write $K_{M}^1 = K_{M,1}^1 \times V$, where $K_{M,1}^1 = K_M^1 \cap K_1$ is the unique maximal compact subgroup of $K_M^1$.

Let $g = f + p$, $\pm 1$ eigenspace decomposition under $\theta$. Choose a maximal abelian subspace $a_0 \subset p$ and a positive restricted root system $\Phi^+ = \Phi^+(\mathfrak{g}, a_0)$. As usual, $\rho = \frac{1}{2} \sum_{\varphi \in \Phi^+} m(\varphi) \varphi$, where $m(\varphi) = \dim \mathfrak{g}_\varphi$. The Iwasawa decomposition

$$G = N_0 A_0 K, \quad x = n(x) \cdot \exp H(x) \cdot k(x)$$

(2.3)
specifies the zonal spherical function on \( G \) for \( 0 \in a_0^* \),

\[
\Xi(x) = \int_{KZ} e^{-iH(kx)} \, d(kZ).
\]  

(2.4)

It is the lift of the corresponding function on the linear semisimple group \( G/Z_G \), and thus if \( \sigma \) is defined as in (2.7) there are constants \( C, q \geq 0 \) so that

\[
e^{-q(a)} \Xi(a) \leq C(1 + \sigma(a))^q e^{-q(a)}
\]  

(2.5)

for all \( a \in A_0^+ = \{ a \in A_0 : x(\log a) > 0 \text{ for all } x \in \Phi^+ \} \).

Growth in \( G \) is determined by a function \( \tilde{\sigma} : G \to \mathbb{R}^+ \) which is defined as follows. Choose an \( \text{Ad}_G(K) \)-invariant positive definite inner product on \( V \). If \( x \in G \) we decompose

\[
x = v(x) \cdot k(x) \cdot \exp \xi(x) \in VK_1 \cdot \exp(p)
\]  

(2.6)

and then we set

\[
\sigma_1(x) = \|v(x)\|, \quad \sigma(x) = \|\xi(x)\|, \quad \text{and} \quad \tilde{\sigma}(x) = \sigma_1(x) + \sigma(x).
\]  

(2.7)

The main properties of \( \sigma \) are

\[
\sigma(k_1 x k_2) = \sigma(x) \quad \text{for all } x \in G, \quad k_1, k_2 \in K;
\]  

(2.8a)

and

\[
\sigma(xy) \leq \sigma(x) + \sigma(y) \quad \text{for all } x, y \in G.
\]  

(2.8b)

The corresponding properties of \( \tilde{\sigma} \) are

\[
\tilde{\sigma}(k x k^{-1}) = \tilde{\sigma}(x) \quad \text{for all } x \in G, \quad k \in K;
\]  

(2.9a)

\[
\tilde{\sigma}(k_1 x k_2) = \tilde{\sigma}(x) \quad \text{for all } x \in G, \quad k_1, k_2 \in K_1;
\]  

(2.9b)

\[
\tilde{\sigma}(xy) \leq 3(\tilde{\sigma}(x) + \tilde{\sigma}(y)) \quad \text{for all } x, y \in G.
\]  

(2.9c)

Let \( W \) be a Banach space and \( f \in C^\infty(G; W) \). If \( D_1, D_2 \in \mathcal{U}(g) \) and \( r \in \mathbb{R} \), we define

\[
p_1 \|f\|_{r, D_1} = \sup_{x \in G} (1 + \sigma(x))^r \Xi(x)^{-1} \|f(D_1 : x; D_2)\|_W.
\]  

(2.10a)

The Schwartz space is

\[
\mathcal{S}(G; W) = \{ f \in C^\infty(G; W) : p_1 \|f\|_{r, D_1} < \infty \text{ for all } D_1, D_2 \in \mathcal{U}(g) \text{ and all } r \in \mathbb{R} \}.
\]  

(2.10b)
It is a complete locally convex topological vector space with the topology defined by the seminorms (2.10a). And of course the most important case is

\[ C(G) = C(G; \mathbb{C}). \]  

**Proposition 2.11.** Define \( \kappa_v = G \to V \) by \( x \in \kappa_v(x) K_1 A_0 N_0 \), where \( G = KA_0 N_0 = VK_1 A_0 N_0 \) is the Iwasawa decomposition. Then \( \kappa_v(\exp p) \) is bounded.

It suffices to prove Proposition 2.11 in the case where \( G \) is simply connected, non-compact, and of hermitian type.

**Lemma 2.12.** There exists \( C > 0 \) such that \( \sigma_v(xy) \leq C + \sigma_v(x) + \sigma_v(y) \) for all \( x, y \in G \).

**Lemma 2.13.** There exists \( C' > 0 \) such that \( 0 \leq \sigma_v(n) \leq C' \) for all \( n \in N_0 \).

**Proof of Proposition from Lemmas.** Let \( \psi \in \mathfrak{p} \), \( \exp \psi = \kappa(\exp \xi) \cdot a \cdot n \) with \( \kappa(\exp \xi) \in K, a \in A_0 \), and \( n \in N_0 \). Using Lemma 2.12 \( \| \kappa_v(\exp \xi) \| = \sigma_v(\kappa_v(\exp \xi)) = \sigma_v(\kappa(\exp \xi)) = \sigma_v((\exp \xi)n^{-1}a^{-1}) \leq C + \sigma_v(\exp \xi) + \sigma_v(n^{-1}a^{-1}) = C + \sigma_v(n^{-1}) + \sigma_v(a^{-1}) = 2C + \sigma_v(n^{-1}). \) Now \( \| \kappa_v(\exp \xi) \| \leq 2C + C' \) by Lemma 2.13. Q.E.D.

**2.14 Proof of (2.12).** Write \( x = \lambda h \exp \psi \) and \( y = \lambda' k' \exp \psi' \), where \( \lambda, \lambda' \in V, k, k' \in K_1 \), and \( \psi, \psi' \in \mathfrak{p} \). Then

\[ xy = (v'v')(kk')(\exp \psi')(\exp \psi') \text{ where } \psi'' = \text{Ad}(v'k')^{-1}(\xi). \]

Write \( (\exp \psi')(\exp \psi') = w \exp \eta \), where \( w \in V, h \in K_1 \), and \( \eta \in \mathfrak{p} \). Then

\[ xy = (v'v')(kk'h)(\exp \eta), \text{ so } \]

\[ \sigma_v(xy) = \| vv' \| \leq \| v \| + \| v' \| + \| w \| = \| w \| + \sigma_v(x) + \sigma_v(y). \]

Use \( G \subset P_+ K_{\xi} P \) from [8, Theorem 2.17], \( b = p \cdot (h) \cdot (k) \cdot p \cdot (b) \), and let \( \tilde{k}_v(h) \) denote the \( V_{\xi} \) projection of \( K_{\xi} = V_{\xi} \times (K_1)_{\xi} \). Then

\[ \| w \| = \sigma_v((\exp \psi''(\exp \psi')) = \| \text{Im} \tilde{k}_v((\exp \psi''(\exp \psi')) \| \]

by [9, Lemma 10.6], and that is bounded by some constant \( C \) according to [9, Lemma 10.7]. Q.E.D.

**2.15 Proof of (2.13).** Let \( \lambda \in \Phi_{\alpha_0}, \xi \in g_{\lambda}, \) and \( \eta \in \theta \xi \in g_{\lambda} \). Assume \( \xi \neq 0 \) so that \( \xi \), \( \eta \), and \( h_{\lambda} = [\xi, \eta] \in a_0 \) span a three-dimensional simple algebra \( g[\lambda] \). Then \( \xi - \xi + \eta \) spans \( f[\lambda] = f \cap g[\lambda], h_{\lambda} \) spans \( a[\lambda] = a_0 \cap g[\lambda], \xi \) spans \( n[\lambda] = n_0 \cap g[\lambda], \) and \( \{ h_{\lambda}, \xi - \eta \} \) spans \( p[\lambda] = p \cap g[\lambda]. \) The analytic subgroup \( G[\lambda] \) for \( g[\lambda] \) has form \( G[\lambda] = \)

K[\lambda]\cdot \exp p[\lambda], \text{ so } \exp \zeta = \exp(\zeta) \exp(\zeta), \text{ where } \zeta \in \mathfrak{t}[\lambda] \text{ and } \zeta \in \mathfrak{p}[\lambda]. \text{ If } \zeta, \zeta' \in \mathfrak{t}, \mathfrak{t}' \text{ then } \zeta \in \mathfrak{t}_1 \text{ and } \sigma_{\lambda}(\exp \zeta, \zeta') = 0. \text{ Now write } \zeta = \zeta_0 + \zeta_1, \zeta_0 \in \mathfrak{u}, \text{ and } \zeta_1 \in \mathfrak{t}_1, \text{ so } \sigma_{\lambda}(\exp \zeta) = \|\zeta_0\| \leq \|\zeta\|. \text{ An } \widetilde{SL}(2; \mathbb{R}) \text{ calculation shows that } \cos(r\|\zeta\|) \neq 0, \text{ where } r \text{ is a constant that depends only on } g \text{ and } g[\lambda], \text{ so } \|\zeta\| < \pi/2r.

Summary: given \lambda \in \Phi^+_\alpha, \text{ we have } C > 0 \text{ such that } \sigma_\lambda(\exp \zeta) \leq C \text{ for all } \zeta \in \mathfrak{g}_\alpha.

Let \eta \in N_\alpha. \text{ Then we can write } \eta = \prod_{\zeta \in \Phi^+_{\alpha_0}} \exp(\zeta), \text{ where } \zeta \in \mathfrak{g}_\alpha. \text{ Using Lemma 2.12, } \sigma_\lambda(\eta) \leq |\Phi^+_{\alpha_0}| C + \sum_{\zeta \in \Phi^+_{\alpha_0}} \sigma_{\lambda}(\exp \zeta) \leq |\Phi^+_{\alpha_0}| C + \sum C_\alpha.

Q.E.D.

3. Discrete Series Parameters

Fix a \theta-stable Cartan subgroup \(H \subset G\) and an associated cuspidal parabolic subgroup \(P = MAN\). Here \(H = T \times A\), where \(T = H \cap K\) and \(A = \exp(a), a = b \cap p, \text{ and } Z_G(A) = M \times A\) with \(\theta M = M\). In this section we discuss holomorphic families of coefficients of relative discrete series representations of \(M\).

Fix a positive root system \(\Phi^+_M = \Phi^+(m, t)\) such that

if \(m_1\) is a non-compact simple ideal of hermitian type in \(m\)
then \(\Phi^+(m_1, t \cap m_1)\) contains a unique non-compact simple root. \(3.1\)

We denote \(\Phi^+_{M, K} = \Phi^+(m, t)\) and write \(\rho_M\) and \(\rho_{M, K}\) for the half sums over \(\Phi^+_M\) and \(\Phi^+_{M, K}\). Define

\(A_M: \text{ all } \lambda \in i\mathfrak{t}^* \text{ such that}\)

\(\lambda - \rho_M \text{ is integral, i.e., } e^{\lambda} \rho_M \text{ is defined on } T^0\)

\(\lambda \text{ is } \Phi^+_M \text{-non-singular, i.e., } \langle \lambda, \alpha \rangle \neq 0 \text{ for } \alpha \in \Phi_M\)

\(\lambda \text{ is } \Phi^+_{M, K} \text{-dominant, i.e., } \langle \lambda, \alpha \rangle \geq 0 \text{ for } \alpha \in \Phi^+_{M, K}\).

Write \(t = t_1 \oplus \mathfrak{u}_M\) where \(t_1 = t \cap \mathfrak{t}_1\), and set \(i\mathfrak{u}^*_M = \{ \lambda \in i\mathfrak{t}^* : \lambda(t_1) = 0 \}\). Then \(A_M\) is the disjoint union of subsets

\(\{ \hat{\lambda}_h = \hat{\lambda}_0 + h_M : h_M \in i\mathfrak{u}^*_M \text{ and } \beta(\hat{\lambda}_h) \neq 0 \text{ for } \beta \in \Phi^+_M \setminus \Phi^+_{M, K} \}\). \(3.2b\)

where \(\hat{\lambda}_0\) belongs to the discrete set

\(A_{M, 0}: \text{ all } \hat{\lambda}_0 \in i\mathfrak{t}^* \text{ such that } \hat{\lambda}_0 - \rho_{M, K} \text{ is integral} \)

\(\hat{\lambda}_0 \text{ is } \Phi^+_{M, K} \text{-dominant, and } \Phi^+_{M, K} \text{ non-singular.}\)

and \(\hat{\lambda}_0(\mathfrak{u}_M) = 0\). \(3.3\)
If \( \lambda \in \Lambda_M \) then \( \pi_\lambda \) denotes the corresponding relative discrete series representation of \( M^0 \). Thus the relative discrete series of \( M^0 \) is the disjoint union of \( \text{continuous families} \)

\[
\{ \lambda_0 + h_M : h_M \in \mathcal{D}_M = \mathcal{D}_M(\lambda_0) \},
\]

(3.4a)

where \( \lambda_0 \in \Lambda_{M,0} \) and \( \mathcal{D}_M \) is a topological component of

\[
\{ h_M \in \mathfrak{i} \mathfrak{v}^*_M : \beta(h_M) \neq -\beta(\lambda_0) \text{ for } \beta \in \Phi_{M,0}^+ \setminus \Phi_{M,K}^+ \}.
\]

(3.4b)

To define continuous families of relative discrete series representations on \( M^1 \) and \( M \) we proceed as follows. Recall the decomposition \( K = K_1 \times V \) of (2.1). Since \( V \) is central in \( K \), each \( h \in \mathfrak{i} \mathfrak{v}^* \) gives a unitary character \( e^h \) of \( K \) which is trivial on \( K_1 \). Consider \( i \mathfrak{v}^* \subset \mathfrak{i} \mathfrak{t}^* \) by \( i \mathfrak{v}^* = \{ h \in i \mathfrak{t}^* : h(\mathfrak{t}_1) = 0 \} \). Then for each \( h \in i \mathfrak{v}^* \), we can define \( h_M(h) = h_{|1} \). Since \( \mathfrak{t}_1 \subset \mathfrak{t} \), \( h_M(h) \in i \mathfrak{v}^*_M \). Further, any \( h_M \in i \mathfrak{v}^*_M \) can be extended to \( \mathfrak{v}_M \oplus \mathfrak{t}_1 \) by making it zero on \( \mathfrak{t}_1 \), and then extended arbitrarily from \( \mathfrak{v}_M \oplus \mathfrak{t}_1 \) to \( \mathfrak{t} \) to give a linear functional \( h \in i \mathfrak{v}^* \) with \( h_M(h) = h_{|M} \). Thus

\[
i \mathfrak{v}^*_M = \{ h_M(h) = h_{|1} : h \in i \mathfrak{v}^* \}. \]

(3.5)

We can now reparametrize our continuous families on \( M^0 \) as follows. Fix \( \lambda_0 \in \Lambda_{M,0} \) and let \( \mathcal{D} = \{ h \in i \mathfrak{v}^* : h_M(h) \in \mathcal{D}_M \} \). Then \( \{ \lambda_0 + h_M : h_M \in \mathcal{D}_M \} = \{ \lambda_0 + h_M(h) : h \in \mathcal{D}_M \} \). Further, the representation \( \pi_{\lambda_0 + h_M(h)} \) has \( Z_{M^0} \)-character \( e^{\lambda_0 - \rho_M + h_M(h)} \) of \( Z_{M^0} = e^{\lambda_0 - \rho_M} \otimes e^h |_{Z_{M^0}} \). Let \( \chi(0) \) be any element of \( Z_M(M^0)^\vee \) with \( Z_{M^0} \)-character \( e^{\lambda_0 - \rho_M} \). Since \( Z_M(M^0) \subset K \) and \( e^h \) is a character of all of \( K \), \( e^h |_{Z_M(M^0)} \) is a character, which we will also denote by \( e^h \), of \( Z_M(M^0) \). Set

\[
\chi(h) = \chi(0) \otimes e^h, \quad h \in i \mathfrak{v}^*.
\]

(3.6)

Then \( \chi = \{ \chi(h) : h \in i \mathfrak{v}^* \} \) is a continuous family of irreducible unitary representations of \( Z_M(M^0) \), and \( \chi(h) \) has the same \( Z_{M^0} \)-character \( e^{\lambda_0 - \rho_M + h_M(h)} \) as \( \pi_{\lambda_0 + h_M(h)}^0 \). Thus we can define \( \text{continuous families of relative discrete series representations of } M^1 = Z_M(M^0)M^0 \) by

\[
\pi = \{ \pi(h) = \chi(h) \otimes \pi_{\lambda_0 + h_M(h)}^0 : h \in \mathcal{D} \}
\]

(3.7a)

and \( \text{continuous families of relative discrete representations of } M \) by

\[
\pi^M = \{ \pi^M(h) = \text{Ind}_{M^1}^M \pi(h) : h \in \mathcal{D} \}. \]

(3.7b)

The parametrization (3.7) of the relative discrete series of \( M \) extends in the obvious way to a parametrization of the tempered series of representations of \( G \) associated to \( H \):

\[
\pi_{h,v} = \text{Ind}_{MAN}^G (\pi^M(h) \otimes e^{iv} \otimes 1_N).
\]

(3.8a)
Here we use normalized induction, so $\pi_{h,v}$ is unitary just when $v \in a^*$. The corresponding continuous families of $H$-series representations of $G$ are the sets

$$\pi_G = \{ \pi_{h,v} : h \in \mathcal{H}, v \in a^* \}. \tag{3.8b}$$

Since $\pi^M(h)$ is obtained via induction from $M^1$ to $M$, we can obtain $\pi_{h,v}$ directly as

$$\pi_{h,v} = \text{Ind}_{M^1}^{M} (\pi(h) \otimes e^{iv} \otimes 1_N). \tag{3.9}$$

We will use this fact to avoid extending results from $M^1$ to $M$.

We will also use $i\nu^*$ to parametrize families of $K^0_M$-types of $\pi_{\mathfrak{a}^1 + h \mathfrak{u}(h)}$ and $K^+_M$-types of $\pi(h)$. Let

$$\sigma_{h,0}^0 : \text{irreducible unitary representation of } K^0_M \tag{3.10a}$$

with $Z_{\mathfrak{a}^0}$-character $\mu^{\mathfrak{a}^0}$:

$$\sigma_{h,0}^0 = \sigma_0^0 \otimes e^{h \mathfrak{u}} : \text{representation of } K^0_M \tag{3.10b}$$

with $Z_{\mathfrak{a}^0}$-character $e^{\langle \cdot \rangle \mu^{\mathfrak{a}^0}}$:

$$\sigma_h = \sigma_{h,0}^0 \otimes \chi(h) : \text{representation of } K^+_M. \tag{3.10c}$$

Note that

$$\sigma_h = \sigma_{h,0}^0 \otimes e^h \quad \text{for all } h. \tag{3.10d}$$

The representations $\sigma_h$ are well defined for $h \in \mathfrak{u}^*_\mathbb{C}$. We will denote

$$\sigma = \{ \sigma_h : h \in \mathfrak{u}^*_\mathbb{C} \} \tag{3.11}$$

and will refer to $\sigma$ as a holomorphic family of irreducible representations of $K^+_M$.

Let $\tau_0$ be an irreducible unitary representation of $K$. Set

$$\tau_h = \tau_0 \otimes e^h \tag{3.12a}$$

$$\tau = \{ \tau_h : h \in \mathfrak{u}^*_\mathbb{C} \} \tag{3.12b}$$

Then $\tau$ is called a holomorphic family of irreducible representations of $K$.

### 4. Discrete Series Coefficients

Because our continuous families of $H$-series representations can be induced directly from $M^1 \mathcal{A}N$, we will be able to define Eisenstein integrals of matrix coefficients for the family $\pi_\nu$ of induced representations directly
from continuous families of matrix coefficients for the family $\pi$ of relative discrete series representations on $M^\dagger$.

Fix a continuous family $\pi$ of relative discrete series representations of $M^\dagger$ as in (3.7), and two holomorphic families $\sigma_1$ and $\sigma_2$ of irreducible representations of $K^\dagger_M$ as in (3.10). If $h \in \mathrm{bd} \mathcal{D}$ we understand $\pi(h)$ to be the limit, by coherent continuation, of representations $\pi(h')$, $h' \in \mathcal{D}$. Denote

$$\mathcal{H}(\pi; \sigma_1; h) = \mathcal{H}(\pi(h); \sigma_{1,h}):$$ the $\sigma_{1,h}$-isotypic subspace of the representation space $\mathcal{H}(\pi; h) = \mathcal{H}(\pi(h))$. (4.1)

$$\mathcal{T}(\pi; \sigma_1; \sigma_2; h):$$ The linear span in $C\text{'}(M^\dagger)$ of the $x \rightarrow \langle \pi(h)(x) w_2, w_1 \rangle$, $w_i \in \mathcal{H}(\pi; \sigma_i; h)$. (4.2)

It is an easy consequence of [9, Theorem 4.1] that $\dim \mathcal{H}(\pi; \sigma_j; h)$ is constant for $h \in \mathrm{cl}(\mathcal{D})$; see Lemma 4.9 below.

First we will construct a family $\mathcal{F}(\pi; \sigma_1; \sigma_2)$ of functions $f \in C\text{'}(v^*_C \times M^\dagger)$ such that

$$h \rightarrow f(h; x)$$ is holomorphic on $v^*_C$ for all $x \in M^\dagger$ (4.3a)

and

$$f(h) \in \mathcal{T}(\pi; \sigma_1; \sigma_2; h)$$ when $h \in \mathrm{cl} \mathcal{D}$. (4.3b)

Functions in the family $\mathcal{F}(\pi; \sigma_1; \sigma_2)$ will be called holomorphic families of discrete series coefficients. In the course of the construction we will prove

**Theorem 4.4.** Fix $h' \in \mathrm{cl} \mathcal{D}$ and $w_i \in \mathcal{H}(\pi; \sigma_j; h)$. Then there exists $f \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ such that $f(h'; x) = \langle \pi(h')(x) w_2, w_1 \rangle$ for all $x \in M^\dagger$.

**Theorem 4.5.** Let $f \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ and $D_1, D_2 \in \mathcal{H}(m)$. Then $f(h; D_2; x; D_1) = \sum_{i=1}^n p_i(h) f_i(h; x)$, where the the $p_i$ are polynomials and $f_i \in \mathcal{F}(\pi; \sigma_{i_1}; \sigma_{i_2})$ for appropriate holomorphic families $\sigma_{i}$ of irreducible representations of $K^\dagger_M$.

Second, we will work out a number of consequences of the construction and of Theorems 4.4 and 4.5.

We construct $\mathcal{F}(\pi; \sigma_1; \sigma_2)$ by reducing to the case [9, Sects. 5 and 6] of a non-compact simply connected simple group of hermitian type.

**4.6. Case.** $M^\dagger_N$ is connected and simply connected. Then $M^\dagger_N = M_0 \times M_1 \times \cdots \times M_j$, where $M_0$ is a vector group and the other $M_j$ are simple, connected, and simply connected. Then

$$\mathcal{D} = \mathcal{D}_0 \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_i, \quad \mathcal{D}_j \subseteq i(n_M \cap m_j)^*$$

$$\pi^0_{\lambda_0 + h_M} = e^{h_0} \otimes \pi_{\lambda_1; 0 + h_1} \otimes \cdots \otimes \pi_{\lambda_i; 0 + h_i}$$

$$\sigma^0_{\rho; h_M} = e^{h_0} \otimes \sigma_{\rho; h_1} \otimes \cdots \otimes \sigma_{\rho; h_i}$$
where $h_j$ is the projection of $h_M$ to $(v_M \cap m_j)^*$. Let $\mathcal{H}(\pi^0, \sigma^0): h_M \to C$ be defined by $f(h_0, x_0) = e^{h_0}(x_0)$. Then $f_0$ satisfies (4.3) for $M_0$. Let $\mathcal{F}_0(\pi: \sigma_1: \sigma_2)$ consist of the multiples of $f_0$. Theorems 4.4 and 4.5 then are trivial for $M_0$.

Fix $i > 0$. Suppose that $M_i$ either is compact or is not of hermitian type. Then $v_M \cap m_i = \{0\}$ and $\mathcal{O}_i = \{0\}$. So $h_i = 0$ and we define $\mathcal{F}_i(\pi: \sigma_1: \sigma_2)$ to be the linear span in $C^\infty(M_i)$ of the coefficients $x_i \mapsto \langle \pi_{z_0}(x_i), w_{z, i} \rangle$, $w_{z, i} \in \mathcal{H}(\pi_{z_0}, \sigma_i)$. Theorems 4.4 and 4.5 are then trivial for $M_i$.

Suppose that $M_i$ is non-compact and of hermitian type. Then $v_M \cap m_i$ is the center of $L_M$, one-dimensional, and $\mathcal{O}_i$ is an open finite interval or an open half-line in $i(v_M \cap m_i)^*$. If $\mathcal{O}_i$ is an open finite interval we define $\mathcal{F}_i(\pi: \sigma_1: \sigma_2)$ to be the restrictions to $i(v_M \cap m_i)^* \times M_i$ of the holomorphic functions defined in [9, Proposition 5.3]. They satisfy (4.3), [9, Theorem 5.14] says that Theorem 4.4 holds for $M_i$, and [9, Theorem 5.4] and its proof show that Theorem 4.5 holds for $M_i$.

If $\mathcal{O}_i$ is a half-line, we define $\mathcal{F}_i(\pi: \sigma_1: \sigma_2)$ to be the linear span of the functions defined in [9, Theorem 6.9]. They satisfy (4.3). Here Theorem 4.4 is obvious for $M_i$, and Theorem 4.5 is an easy calculation in $\mathcal{H}(m_i)$. Define $\mathcal{F}^0(\pi^0, \sigma_1^0: \sigma_2^0)$ to be the set of all finite linear combinations of the functions $f(h_M: x) = f_0(h_0: x_0) f_1(h_1: x_1) \cdots f_i(h_i: x_i)$, where $x = (x_0, x_1, \ldots, x_i) \in M_0 \times M_1 \times \cdots \times M_i = M^0$. Those functions satisfy (4.3), and Theorems 4.4 and 4.5 hold for $M^0$ because they hold for the $M_i$.

4.7. Case. General $M^0$. Let $p: M' \to M^0$ be the universal covering group and $\Gamma = \text{Kernel}(p)$. Then $\mathfrak{t}_M = [\mathfrak{t}_M, \mathfrak{t}_M] \oplus u_M \oplus v_M$ and $K_M = [K_M', K_M] \times U_M \times V_M$, where $\Gamma \subset [K_M', K_M] \times U_M$. $\Gamma \cap U_M$ is a lattice in $U_M'$, and $p$ maps $V_M'$ isomorphically onto $V_M$. Note that $\mathcal{Q}_M \subset \mathfrak{v}_M^*$ is just $\{h' \in \mathcal{Q}_M: h'(u_M) = 0\}$, where $\mathcal{Q}_M' = i(u_M \oplus v_M)^*$ is the topological component of $\{h' \in i(u_M \oplus v_M)^*: \beta(h') \neq -\beta(h_0)\}$ for $\beta \in \Phi^+_M \setminus \Phi^+_M_{K_M}$ that contains $\mathcal{Q}_M$.

Let $\mathcal{F}'(\pi': \sigma_1': \sigma_2')$ be the set of holomorphic families of relative discrete series coefficients for $M'$ constructed in (4.6). If $f' \in \mathcal{F}'(\pi': \sigma_1': \sigma_2')$, so $f'(u_M \oplus v_M)^* \times M' \to C$, define $f(h_M: x) = f'(h_M: x')$ whenever $x \in M$ and $x' \in p^{-1}(x)$. Define $\mathcal{F}^0(\pi^0: \sigma_1^0: \sigma_2^0)$ to consist of all such $f$. Then $f$ inherits (4.3) from $f'$, and $\mathcal{F}^0$ inherits the conclusions of Theorems 4.4 and 4.5 from $\mathcal{F}'$. 

SCHWARTZ SPACE OF A GENERAL SIMPLER LIE GROUP, I
4.8. Case. General $M^\dagger$. Define $\pi^0 = \{\pi_{\gamma_0 + h_M(h')}; \pi_{\gamma_1 + h_M(h)} \otimes \chi(h) \in \pi\}$. Write $\sigma_{\gamma, 0} = \sigma_{\gamma, 0}^0 \otimes \chi(0)$ as in (3.10) and set

$$\sigma_{\gamma, 0}^0 = \{\sigma_{\gamma, 0}^0 \otimes e^{h_M(h')}\}.$$

Let $\mathcal{F}^0(\pi^0; \sigma_{\gamma, 0}^0; \sigma_{\gamma, 0}^0)$ be the set of holomorphic families of discrete series coefficients for $M_0$ constructed in (4.7). Let $\psi(0)$ be any matrix coefficient of $\chi(h) = \chi(0) \otimes e^h$, so we can define a holomorphic family of matrix coefficients of $x(O)$ by $\psi(h:z) = \psi(0; z) e^h(z), z \in Z_M(M^0)$. Since $Z_{M_0} = Z_M(M_0) \cap M_0$, and both $\chi(h)$ and $\pi_{\gamma_0 + h_M(h)}$ have $Z_{M_0}$-character $e^{h_0 + h_M(h)}$ for any $f_0 \in \mathcal{F}^0(\pi^0; \sigma_{\gamma, 0}^0; \sigma_{\gamma, 0}^0)$ we have a well-defined $f: M^\dagger \times \Omega^* \to C$ given by $f(h; z_0) = \psi(h; z) f_0(h; z_0)$ for $z \in Z_M(M^0), x_0 \in M_0$. Define $\mathcal{F}(\pi; \sigma_1; \sigma_2)$ to be the space spanned by all such $f$ as $\psi(0)$ ranges over matrix coefficients of $x(O)$ and $f_0$ ranges over $\mathcal{F}^0(\pi^0; \sigma_{\gamma, 0}^0; \sigma_{\gamma, 0}^0)$. Clearly each $f \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ satisfies (4.3), and Theorem 4.4 holds. Finally Theorem 4.5 is satisfied since $f_0$ satisfies (4.5) and $f(h; D_1; x_0; D_2) = \psi(h; z) f_0(h; D_1; x_0; D_2)$ for any $D_1, D_2 \in \mathcal{H}(m)$.

We now have constructed a family $\mathcal{F}(\pi; \sigma_1; \sigma_2)$ of functions $f$ that satisfy (4.3), and for which Theorems 4.4 and 4.5 hold. Next, we consider some consequences of the construction.

**Lemma 4.9.** Dim $\mathcal{H}(\pi; \sigma; h)$ is constant for $h \in \text{cl} \mathcal{L}$.

*Proof.* This is obvious if $M$ is a vector group. Let $M$ be simple, connected and simply connected. If $M$ is compact or not of hermitian type, it again is trivial. If $M$ is non-compact and of hermitian type, the assertion is [9, Theorem 4.11. Now, as in (4.6), the lemma follows for $M_0$ simply connected. The result for general $M^0$ follows as in (4.7), and the result for $M^\dagger$ is immediate as in (4.8) since deg $\chi(h)$ is independent of $h$. Q.E.D.

**Proposition 4.10.** Fix $h' \in \text{cl}(\mathcal{L})$. Then there is a neighborhood $J$ of $h'$ in $\text{cl}(\mathcal{L})$, and a finite subset \{\(f_1, \ldots, f_r\)\} $\subset \mathcal{F}(\pi; \sigma_1; \sigma_2)$, such that \(\{f_1(h)\}\) is a basis of \(\mathcal{J}(\pi; \sigma_1; \sigma_2; h')\) for every $h \in J$.

*Proof.* Let \(\{w_{p, h}\}\) and \(\{v_q\}\) be bases of $\mathcal{H}(\pi; \sigma_1; h')$ and $\mathcal{H}(\pi; \sigma_1; h')$. Theorem 4.4 gives us \(\{f_{ap}\} \subset \mathcal{F}(\pi; \sigma_1; \sigma_2)\) such that $f_{ap}(h'; x) = \langle \pi(h')(x) w_p, v_q \rangle$ for $x \in M^\dagger$. Now \(\{f_{ap}(h')\}\) is a basis of \(\mathcal{F}(\pi; \sigma_1; \sigma_2; h')\). As in [9, Theorem 5.14 (3)], let $P$ be the complex projective space based on dim $\mathcal{H}(\pi; \sigma_1; h')$ by dim $\mathcal{H}(\pi; \sigma_2; h')$ matrices. Then

$$W = \left\{([a_{ap}], h) \in P \times \Omega^* : \sum a_{ap} f_{ap}(h) = 0 \right\}$$

is a holomorphic subvariety of $P \times \Omega^*$, and projection to $\Omega^*$ is a proper map whose image omits $h'$, thus omits a neighborhood $J$ of $h'$. Let
\[ J = \mathcal{J} \cap \text{cl}(\mathcal{G}). \] In view of Lemma 4.9, the linearly independent subset \( \{ f_{\eta}(h) \}, h \in J \), is a basis of \( \tau^*(\pi: \sigma_1: \sigma_2: h') \). Q.E.D.

**Proposition 4.11.** There is a finite open cover \( \{ J_1, \ldots, J_r \} \) of \( \text{cl}(\mathcal{G}) \) such that if \( h' \in J_j \) then we may take \( J = J_j \) in Proposition 4.10.

**Proof.** Let \( \psi_1(0), \ldots, \psi_s(0) \) be a basis for the space of matrix coefficients of \( \chi(0) \). Then \( \psi_1(h), \ldots, \psi_s(h) \) is a basis for the space of matrix coefficients of \( \chi(h) \) for all \( h \in \mathfrak{g}_x \). Suppose \( \{ f_1, \ldots, f_s \} \subseteq \mathcal{F}(\pi^0: \sigma_1^0: \sigma_2^0) \) give a basis for \( \tau^*(\pi^0: \sigma_1^0: \sigma_2^0) \) for every \( h \in J \), and \( \{ f_{ij} \} \subseteq \mathcal{F}(\pi: \sigma_1: \sigma_2) \) are defined by \( f_{ij}(h \cdot x_0) = \psi_j(h \cdot z) f_i(h \cdot x_0), 1 \leq i \leq r, 1 \leq j \leq s \). Then clearly \( \{ f_{ij}(h) \} \) gives a basis for \( \tau^*(\pi: \sigma_1: \sigma_2: h) \) for all \( h \in J \). Thus it is enough to prove the proposition for \( M^0 \), and as in (4.7) we can assume that \( M^0 \) is simply connected. Then we can decompose \( m = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \mathfrak{v}_M = \mathfrak{v}_{M,0} \oplus \mathfrak{v}_{M,1} \oplus \mathfrak{v}_{M,2} \), and \( \mathcal{G}_M = \mathfrak{v}_{M,0} \times \mathcal{G}_{M,1} \times \mathcal{G}_{M,2} \) with \( \mathcal{G}_{M,1} \subseteq \mathfrak{v}_{\mathfrak{g}_{J,j}} \). In such a way that \( \text{cl}(\mathcal{G}_{M,1}) \) is compact and \( \mathcal{G}_{M,2} \) is the product of open half-lines of the form \( P_1 \cap \mathfrak{m}', \mathfrak{m} \) runs over the simple ideals of \( \mathfrak{m}_2 \). Along those local factors of \( M^0, \mathcal{F}(\pi^0: \sigma_1^0: \sigma_2^0) \) was defined [9, Theorem 6.9] to be the span of a set of holomorphic families \( f^0 \) that, for every \( h_M \), give a basis of \( \tau^*(\pi^0: \sigma_1^0: \sigma_2^0: h_M) \). Thus the result follows from (4.10) using compactness of \( \text{cl}(\mathcal{G}_{M,1}) \). Q.E.D.

### 5. Spherical Functions

We reformulate the results of Section 4 as results on holomorphic families of spherical functions and prove some inequalities. We need these results for construction of Eisenstein integrals and for certain growth estimates. As in Section 4, we work with \( M^\tau \) rather than \( M \).

**Lemma 5.1.** Let \( \sigma = \{ \sigma_h \} \) be a holomorphic family of irreducible representations of \( K^*_M \) as in (3.11). Then there is a holomorphic family \( \tau = \{ \tau_h \} \) of irreducible representations of \( K \) such that each \( \sigma_h \) is a subrepresentation of \( \tau_h|_{K^*_M} \).

**Proof.** Recall (3.11) that \( \sigma_h = \sigma_0 \otimes e^h \) for all \( h \). Let \( \tau_0 \) be any irreducible summand of \( \text{Ind}_{K^*_M}^{K^*_M} (\sigma_0) \) such that \( \tau_0|_{K^*_M} \) contains \( \sigma_0 \). Then the representation \( \tau_h = \tau_0 \otimes e^h \) of \( K = K_1 \times V \) satisfies \( \tau_h|_{K^*_M} \supset \sigma \otimes e^h|_{K^*_M} = \sigma_h \). Q.E.D.

Fix a continuous family \( \pi \) of relative discrete series representations of \( M^\tau \), two holomorphic families \( \sigma_1 \) and \( \sigma_2 \) of irreducible representations of \( K^*_M \), and holomorphic families \( \tau_1 \) and \( \tau_2 \) of irreducible representations of \( K \) as in (5.1).
Denote
\[ \zeta(\tau_j:0) : V\text{-character of } \tau_{j,0}. \] (5.2)

Then it is clear that
\[ \zeta(\tau_j:h) = \zeta(\tau_j:0)e^h \]

is the \( V \)-character of \( \tau_{j,h} \) for all \( h \) (5.2b)

and
\[ \zeta(\sigma_j:h) = \zeta(\tau_j:h)|_{K_M^+ \cap V}. \]

is the \( V \cap K_M^+ \)-character of \( \sigma_{j,h} \) for all \( h \). (5.2c)

For every \( f \in \mathcal{F}(\pi:\sigma_1:\sigma_2) \),
\[ f(h:z_1z_2) = \zeta(\sigma_1:h:z_1) \zeta(\sigma_2:h:z_2) f(h:x) \]

for all \( h \in \mathfrak{u}_m^*, x \in M^+, \) and \( z_1, z_2 \in K_M^+ \cap V. \) (5.3)

Define \( K_M^V = K_M^+ \cdot V \) and \( K_{M,1}^V = K_M^+ \cap K_1 \) as in (2.2). Given \( f \in \mathcal{F}(\pi:\sigma_1:\sigma_2) \) we define
\[ F = F(f) : \mathfrak{u}_m^* \times M^+ \to L^2(K_{M,1}^V \times K_{M,1}^V) \] (5.4a)

by the formula
\[ F(h:x)(k_1,k_2) = \zeta(\tau_1:h:v_1) \zeta(\tau_2:h:v_2) f(h:k_1^{-1}k_2^{-1}xk_1k_2^{-1}). \] (5.4b)

where we decompose \( k_j \in K_{1,M}^V \subseteq K_M^+ \cdot V \) by \( k_j = k_{j,M} \cdot v_j, k_{j,M} \in K_M^+, v_j \in V. \)

This is well defined because of (5.3).

Extend \( \sigma_{j,h} \) to an irreducible representation \( \tilde{\sigma}_{j,h} \) of \( K_M^V \) by
\[ \tilde{\sigma}_{j,h} = \sigma_{j,h} \otimes \zeta(\tau_j:h). \] (5.5)

This is well defined because of (5.2c). Define an irreducible representation \( \sigma_j \) of \( K_{M,1}^V \) by
\[ \sigma_j = \tilde{\sigma}_{j,h}|_{K_{M,1}^V}. \] (5.6)

**Lemma 5.7.** Let \( E_{\sigma_j} \) denote the \( \sigma_j \)-isotypic subspace of \( L^2(K_{M,1}^V) \). Note that \( E_{\sigma_2} \) also is the right \( \sigma^*_2 \)-isotypic subspace. Then, for every \( h \in \mathfrak{u}_m^* \) and every \( f \in \mathcal{F}(\pi:\sigma_1:\sigma_2) \), \( F \) takes values in the finite dimensional space \( E_{\sigma_1} \otimes E_{\sigma_2}. \)

**Proof.** For \( h \in \mathfrak{g} \) this is obvious since \( f(h) \in \mathcal{F}(\pi:\sigma_1:\sigma_2:h) \) and \( \sigma_j = \tilde{\sigma}_{j,h}|_{K_{M,1}^V} \) for all \( h \). But both sides of the equation are holomorphic in \( h \), so the equation is valid for all \( h \in \mathfrak{u}_m^* \). \( \Box \)
Interpret the action of $K_{M,1}^I$ on $W(\sigma_1 : \sigma_2) = E_{\sigma_1} \otimes E_{\sigma_2}$ as a double representation $\sigma = (\sigma_1, \sigma_2)$.

\[
[\sigma(m_1 : m_2) \cdot \phi](k_1 : k_2) = [\sigma_1(m_1) \cdot \phi \cdot \sigma_2(m_2)](k_1 : k_2) = \phi(m_1^{-1}k_1 : k_2m_2^{-1}).
\]  

(5.8)

We obtain a holomorphic family $\sigma = (\sigma_1, \sigma_2)$ of double representations of $K_M^I$ on $W(\sigma_1 : \sigma_2)$ by

\[
\left[\sigma_h(z_1m_1 : z_2m_2) \cdot \phi\right](k_1 : k_2) = \left[\sigma_1_h(w_1m_1) \cdot \phi \cdot \sigma_2_h(w_2m_2)\right](k_1 : k_2)
= \zeta(\tau_1 : h : z_1) \cdot \zeta(\tau_2 : h : z_2) \cdot \phi(m_1^{-1}k_1 : k_2m_2^{-1}),
\]

\[
z_1, z_2 \in V, m_1, m_2 \in K^I_{M,1}. \tag{5.9}
\]

Now (5.4) defines a holomorphic family of $\sigma$-spherical functions.

**Theorem 5.10.** Let $f \in \mathcal{F}(\pi : \sigma_1 : \sigma_2)$. Then $F : \mathfrak{v}^* \times M^I \to W(\sigma_1 : \sigma_2)$ is smooth in $(h, x)$ and holomorphic in $h$. Given $h \in \mathfrak{v}^*$, $F(h)$ is $\sigma_h$-spherical,

\[
F(h : m_1xm_2) = \sigma_1_h(m_1) \cdot F(h : x) \cdot \sigma_2_h(m_2) \tag{5.11}
\]

for $x \in M^I, m_i \in K^I_{M,1}$.

**Proof.** Smoothness and holomorphicity are clear from (4.3) and (5.4). Write $m_i = z_iy_i$, where $z_i \in V$, $y_i \in K^I_{M,1}$, and $k_i = v_i k_{1, M}$ as in (5.4). Then using (5.4) and (5.9)

\[
\sigma_1_h(m_1) \cdot F(h : x) \cdot \sigma_2_h(m_2)(k_1 : k_2)
= \zeta(\tau_1 : h : z_1) \cdot \zeta(\tau_2 : h : z_2) F(h : x)(y_1 k_1 : k_2 y_2^{-1})
= \zeta(\tau_1 : h : z_1) \cdot \zeta(\tau_2 : h : z_2) \zeta(\tau_1 : h : z_1^{-1}v_1^{-1})
\times \zeta(\tau_2 : h : z_2^{-1}v_2^{-1}) f(h : k_{1, M}m_1xm_2 k_2^{-1})
\]

since $y_1 k_1 = (z_1 v_1)(m_1 k_{1, M})$ and $k_2 y_2^{-1} = (v_2 z_2)(k_{2, M}m_2^{-1})$ in $V \cdot K^I_{M,1}$. But this last expression equals $F(h : m_1xm_2)(k_1 : k_2)$. Q.E.D.

For the rest of this section we examine the growth estimates for a holomorphic family $F(h : x)$.

Fix a complex neighborhood $\mathcal{U}_\omega = \mathcal{U} + i\omega$ of $\mathcal{U}$ in $\mathfrak{v}_\omega^*$, where $\omega$ is an open neighborhood of 0 in $\mathfrak{v}^*$ with compact closure.

Let $\mathfrak{a} = \mathfrak{a}_0 \cap \mathfrak{m}$ and choose a positive restricted root system $\Phi_M^+ - \Phi^+(\mathfrak{m}, \mathfrak{a}_M)$. Let $\rho_{\mathfrak{a} \mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Phi_M^+} \alpha$ and $A_M^* = \{ a \in A_M : x(\log a) > 0 \text{ for all } a \in \Phi_M^+ \}$. Because of the Cartan decomposition $M^I = K_M^I \cdot \text{cl}(A_M^+) K_M^I$, any $\sigma$-spherical function on $M^I$ is determined by its restriction to $\text{cl}(A_M^+)$. 

Theorem 5.12. Let $f \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ and $F = F(f)$. Let $D_1, D_2 \in \mathcal{U}(m)$. Then there are constants $c, m \geq 0$ such that $\|F(h; D_1 : a; D_2)\| \leq c(1 + \|h\|)^m (1 + \sigma(a)^m) |\lambda e^{\omega(h)}(a)|$ for $h \in \text{cl}(\mathcal{D}_1)$, and $a \in \text{cl}(A^+_M)$, where $\omega(h) \in (A_M)^\ast$, $|\lambda e^{\omega(h)}(a)| < e^{-\varepsilon m}(a)$ for all $a \in \text{cl}(A^+_M) \setminus \frac{1}{1}$, and $h \mapsto \omega(h)$ is piecewise linear on $\mathcal{D}_1$.

Proof. Case 1. We will first prove the theorem in the case that $D_1 = D_2 = 1$. As in (4.8) we write $f(h; z_0) = \psi(h; z) f_0(h; z_0)$ for some holomorphic family $\psi(h)$ of matrix coefficients for $\chi(h)$ and some $f_0 \in \mathcal{F}(\pi; \sigma_0; \sigma_0)$. Now $\|F(h; a)\| = \sup_{k_1, k_2 \in K_{M,1}} |f(h; k_1^{-1} a k_2^{-1})|$. Since $K_{M,1}^\ast$ is compact, $[K_{M,1}^\ast / K_{M,1}^0]$ is finite. Let $\gamma_1, \ldots, \gamma_t \in Z(M^0) \cap K_{M,1}^\ast$ be coset representatives. Then we can write $\|F(h; a)\| = \sup_{\gamma_1, \ldots, \gamma_t} |\psi(h; \gamma_1 \gamma_2 \cdots \gamma_t)| \times \sup_{k_1, k_2 \in K_{M,1}} |f_0(h; k_1^{-1} a k_2^{-1})|$. Now since $\gamma_i \gamma_j^{-1} \in K_{M,1}^\ast$, $\psi(h; \gamma_i \gamma_j^{-1}) = \psi(0; \gamma_i \gamma_j^{-1})$ for all $h$. Thus there is a constant $C$ with $\sup_{\gamma_1, \ldots, \gamma_t} |\psi(h; \gamma_1 \gamma_2 \cdots \gamma_t)| \leq C$ for all $h$. Let $F^0$ be the spherical function with values in $L^2(K_{M,1}^\ast \times K_{M,1}^0)$ given by $F^0(h; x) = f_0(h; k_1^{-1} x k_2^{-1})$. Then we have shown that $\|F(h; a)\| \leq C \|F^0(h; a)\|$. Thus it is enough to prove the result for $F^0$ on $M^0$. But for general $M^0$, our estimate can be obtained as in (4.7) by restricting the parameters in the corresponding estimate on the universal covering group of $M^0$. Thus we can assume that $M^0$ is simply connected.

As in (4.6) we decompose $M^0 = M_0 \times \cdots \times M_t$. We can assume $f_0$ is a product $f_0 = f_0 \cdots f_t$, as in (4.6) with $f_i \in \mathcal{F}(\pi; \sigma_i; \sigma_i)$. Now $K_{M,t}^0 = K_{M,1}^0 \times \cdots \times K_{M,1}^0$ so $\|F^0(h; a)\| = \|F_0(h_0; a_0)\| \cdots \|F_t(h_t; a_t)\|$, where $F_i(h_i; a_i) \in L^2(K_{M,1}^\ast \times K_{M,1}^0)$ is the spherical function corresponding to $f_i$. Thus it is enough to prove the theorem when $M^0 = M_i$ for some $0 \leq i \leq t$.

For $i = 0$, $M_0$ is a real vector group so $K_{M,1} = \{1\}$ and $F_0(h_0; a_0) = e^{h_0 a_0}$ is given by $F_0(h_0; x_0) = e^{h_0 x_0}$. Since $A_{M,1} = \{1\}$, $F_0(h_0; a_0) \equiv 1$.

For $1 \leq i \leq t$, $M_i$ is simple, connected, and simply connected. If $M_i$ is compact, then $A_{M_i} = \{1\}$ and $v_i = \{0\}$ so the result is trivial.

If $M_i$ is non-compact and not of hermitian type, then $v_i = 0$, and the assertion reduces to a standard estimate for discrete series coefficients [1].

If $M_i$ is non-compact and of hermitian type, then $v_i$ is a line, and $\mathcal{D}_i$ is an open interval in $iu_i^\ast$. If $\text{cl}(\mathcal{D}_i)$ is compact, then for $h_i \in \mathcal{D}_i$ the assertion is just [9, Theorem 8.1].

If $h_i = h_R + h_I$ with $h_R \in \text{cl}(\mathcal{D}_i)$ and $h_I \in \text{cl}(i\omega)$, then the proof of [9, Theorem 8.1] extends the assertions from $F(h_R; a)$ to $F(h_I; a)$, for the absolute values of the exponential terms involved depend only on $h_R$, while the coefficient functions are bounded on compact sets, so that $\|F(h_R + h_I; a)\|$ satisfies the same type of estimate as $\|F(h_R; a)\|$ over the compact set $\text{cl}(i\omega)$.

If $\text{cl}(\mathcal{D}_i)$ is non-compact, then $\mathcal{D}_i$ has the form $(h_0, \infty)$, and the above argument holds for $h_R$ in an initial segment $[h_0, h_0 + \varepsilon]$ of $\text{cl}(\mathcal{D}_i)$. For $h_R \in$
we combine [9, Proposition 6.16] with the explicit formula [8, Theorem 5.11]. In the latter, both sides are holomorphic when $h_R \in (h_0, \infty)$, the absolute values of the exponential terms depend only on $h_R$, and the remaining dependence on $h \in [h_0 + \epsilon, \infty) + \text{cl}(i\omega)$ is polynomial, so the estimates of [8, Corollary 5.2] extend to $h \in \text{cl}(\mathcal{D}_{it})$, and our assertions follow.

Case II. Now let $D_1, D_2$ be arbitrary elements of $\mathcal{H}(m)$. Then for all $x \in M^+$, $F(h; D_1; x; D_2)(k_1; k_2) = f(h; \text{Ad} k_1^{-1} D_1; \text{Ad} k_1^{-1} x k_2^{-1}; \text{Ad} k_2 D_2)$. Now there are $D'_1, D''_2 \in \mathcal{H}(m)$ and $a'_i, a''_j \in C^\infty(K^+_{M,1})$ so that $\text{Ad} k_1^{-1} D_1 - \sum_i a'_i(k_1) D'_i$ and $\text{Ad} k_2 D_2 = \sum_j a''_j(k_2) D''_j$ for all $k_1, k_2 \in K^+_{M,1}$. Let $C = \max_{i,j,k_1,k_2} |a'_i(k_1) a''_j(k_2)| < \infty$. Then $F(h; D_1; x; D_2)(k_1; k_2) = \sum_{i,j} a'_i(k_1) a''_j(k_2) f(h; D'_i; k_1^{-1} x k_2^{-1}; D''_j)$. Now by Theorem 4.5 there are holomorphic families $f_{ij}$ and polynomials $p_{ij}$ so that $f(h; D'_i; x; D''_j) = \sum_i p_{ij}(h) f_{ij}(h; x)$ for all $x \in M^+$. Let $F_{ij} = F(f_{ij})$. Then $F(h; D_1; x; D_2)(k_1; k_2) = \sum_{i,j} a'_i(k_1) a''_j(k_2) p_{ij}(h) F_{ij}(h; x)(k_1; k_2)$. Thus $\|F(h; D_1; x; D_2)\| \leq C \sum_{i,j} \|p_{ij}(h) F_{ij}(h; x)\|$, and so we have a bound of the desired form by Case I applied to the functions $F_{ij}$.

Q.E.D.

6. Eisenstein Integrals

We define holomorphic families of Eisenstein integrals, show that they are spherical functions in the appropriate setting, and check that they satisfy systems of differential equations corresponding to the appropriate infinitesimal characters.

Fix a continuous family $\pi$ of relative discrete series representations of $M^+$ and holomorphic families $\sigma_\tau$ for $K^+_{M}$ and $\tau_j$ for $K$ such that $\tau_{f,h}$ contains $\sigma_{f,h}$. Denote subspaces $E_{\sigma_\tau}^M, E_{\tau_j}^M \subset L^2(K_{M,1}^+)$ as in Lemma 5.7 and let $E_{\tau_{1,1}}^G, E_{\tau_{2,1}}^G \subset L^2(K_1)$ be defined similarly. Denote

$$E_{\psi}^G, \text{kernel of restriction } E_{\psi}^G \rightarrow E_{\sigma_\tau}^M \text{ of functions}$$

from $K_1$ to $K_{M,1}^+$ followed by projection onto $E_{\sigma_\tau}^M$ (6.1a)

$$E_{\psi_{\tau_j}}^G : \text{orthocomplement of } E_{\psi}^G \text{ in } E_{\tau_j}^G$$

so that we have $K_{M,1}^+$-equivariant isomorphisms

$$i_1 : E_{\sigma_\tau}^M \cong E_{\psi}^G \quad \text{and} \quad i_2 : E_{\tau_j}^G \cong E_{\psi}^G \oplus E_{\tau_j}^G.$$ (6.1c)

Now let $f_M \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ and view the associated family of spherical functions

$$F_M = F(f_M) : \psi_2^* \times M^+ \rightarrow E_{\sigma_1}^M \otimes E_{\sigma_2}^M.$$ (6.2a)
as having values in
\[ E_{\sigma_1}^M \otimes E_{\sigma_2}^M \cong E_{\tau_1}^G \otimes E_{\tau_2}^G \subset E_{\tau_1}^G \otimes E_{\tau_2}^G \subset L^2(K_1 \times K_1). \] (6.2b)

Denote this by
\[ F_M: \mathcal{V}_*^M \times M^+ \to L^2(K_1 \times K_1). \] (6.2c)

As in (5.9), interpret the action of \( K_1 \) on \( W(\tau_1: \tau_2) = E_{\tau_1}^G \otimes E_{\tau_2}^G \) as a double representation \( \tau = (\tau_1, \tau_2) \):
\[ [\tau(g_1: g_2)\psi](k_1:k_2) = [\tau_1(g_1)\cdot \psi \cdot \tau_2(g_2)](k_1:k_2) \]
\[ = \psi(g_1^{-1}k_1:k_2 g_2^{-1}). \] (6.3a)

We define a holomorphic family \( \tau = (\tau_1: \tau_2) \) of double representations of \( K \) on \( W(\tau_1: \tau_2) \) by
\[ [\tau_h(g_1 z_1: g_2 z_2)\psi](k_1:k_2) \]
\[ = [\tau_1(h)(g_1 z_1)\cdot \psi \cdot \tau_2(h)(g_2 z_2)](k_1:k_2) \]
\[ = \zeta(\tau_1: h: z_1) \zeta(\tau_2: h: z_2) \psi(g_1^{-1}k_1:k_2 g_2^{-1}) \]
for \( z, \in V, g, \in K_1. \) (6.3b)

Then
\[ F_M(h: m_1, x m_2) = \tau_{1,h}(m_1) \cdot F_M(h: x) \cdot \tau_{2,h}(m_2) \] (6.3c)
for all \( m_1, m_2 \in K_1^+ \) since the embedding \( i_1 \otimes i_2: E_{\sigma_1}^M \otimes E_{\sigma_2}^M \to E_{\tau_1}^G \otimes E_{\tau_2}^G = W(\tau_1: \tau_2) \) defined in (6.1) is equivariant for the action of \( K_1^+. \)

The embedding \( i_1 \otimes i_2 \) is not in general the only possible one. In order to generate a larger class of spherical functions we denote
\[ \text{End}_{K_1^+} \left( W(\tau_1: \tau_2) \right) \]
\[ = \{ S \in \text{End}(W(\tau_1: \tau_2)) : S(\tau_1(k_1) \cdot \psi \cdot \tau_2(k_2)) \]
\[ = \tau_1(k_1) \cdot S\psi \cdot \tau_2(k_2) \text{ for all } k_1, k_2 \in K_1^+, \psi \in W(\tau_1: \tau_2) \}. \] (6.4a)

Note for all \( S \in \text{End}_{K_1^+} \left( W(\tau_1: \tau_2) \right), h \in \mathcal{V}_*^M, m_1, m_2 \in K_1^+, \psi \in W(\tau_1: \tau_2), \)
\[ S(\tau_{1,h}(m_1) \cdot \psi \cdot \tau_{2,h}(m_2)) = \tau_{1,h}(m_1) \cdot S\psi \cdot \tau_{2,h}(m_2) \] (6.4b)
since for \( m_1 = z_1/k_1, \ z_1 \in V, k_1 \in K_1^+, \ \tau_{1,h}(m_1) = \zeta(\tau_1: h: z_1) \tau_1(k_1). \) Now for any \( S \in \text{End}_{K_1^+} \left( W(\tau_1: \tau_2) \right) \) we define
\[ F_{M,S}(h: x) = S \cdot F_M(h: x) \quad \text{for all } h \in \mathcal{V}_*^M, x \in M^+. \] (6.4c)
Using (6.3c) and (6.4b) we have
\[ F_{M,S}(h;m_1,xm_2) = \tau_{1,h}(m_1) \cdot F_{M,S}(h;x) \cdot \tau_{2,h}(m_2) \quad (6.4d) \]
for all \( m_1, m_2 \in K_M^+ \).

\( F_{M,S} \) is called the holomorphic family of \( K_M^+ \)-spherical functions corresponding to \( f_M \in \mathcal{F}(\pi;\sigma_1;\sigma_3) \) and \( S \in \text{End}_{\alpha_{M-1}^+}(W(\tau_1;\tau_2)) \). Now, if \( x \in G \) use \( G = KM^+AN \) to express
\[ x = k(x) \cdot m(x) \cdot \exp H_p(x) \cdot n(x), \quad H_p(x) \in a. \quad (6.5a) \]
Then the function \( F_{M,S} \) of (6.4) extends to
\[ F_G: \mathfrak{g}^*_c \times G \rightarrow W(\tau_1;\tau_2) \subset L^2(K_1 \times K_1) \quad (6.5b) \]
by the formula
\[ F_G(h;x) = \tau_{1,h}(k(x)) \cdot F_{M,S}(h;m(x)). \quad (6.5c) \]
\( F_G \) is well defined: if \( l \in K_M^+ \) then, dropping \( h, \tau_{1}(kl) \cdot F_M(m) = \tau_{1}(k) \cdot \tau_{1}(l) \cdot F_M(m) = \tau_{1}(k) \cdot F_M(lm) \).

We can now define the Eisenstein integral
\[ E(P:F_{M,S}:h:y;x) \]
by
\[ E(P:F_{M,S}:h:y;x) = \int_{K_1 \times K_1} F_G(h:xk) \cdot \tau_{2,h}(k^{-1}) e^{\text{tr} \cdot \rho_p \cdot H_p(k)} d(kZ), \quad (6.6b) \]
where \( F_G \) and \( H_p \) are defined in (6.5), and \( \rho_p \) is \( \frac{1}{2} \) the trace of \( \text{ad}(a) \) on \( \mathfrak{n} \).

It is well defined because, dropping \( h \), if \( z \in Z \) then
\[ F_G(xkz) \cdot \tau_z(z^{-1}k^{-1}) = \tau_1(k(xkz)) \cdot F_{M,S}(m(xkz)) \cdot \tau_2(z^{-1}k^{-1}) = \tau_1(k(xk)) \cdot F_{M,S}(m(xk)) \cdot \tau_2(z^{-1}k^{-1}) \]
(because \( xk = k_1m_1a_1n_1 \) gives \( xkz = k_1 \cdot m_1z \cdot a_1 \cdot n_1 \))
\[ = \tau_1(k(xk)) \cdot F_{M,S}(m(xk)) \cdot \sigma_2(z) \cdot \tau_3(z^{-1}) \cdot \tau_2(k^{-1}) \]
\[ = F_G(xk) \cdot \tau_2(k^{-1}) \]
and \( H_p(xkz) = H_p(xk) \).

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**Theorem 6.7.** Let $F_{M,S}$ be the holomorphic family of $K^s$-spherical functions corresponding to $f_M \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ and $S \in \text{End}_{K_{M_1}}(W(\tau_1; \tau_2))$. Then the Eisenstein integral $E(P:F_{M,S})$ is jointly smooth, holomorphic in $h, v$, and if $(h, v, x) \in \mathfrak{t}_G^* \times \mathfrak{a}_E^* \times G$ then $E(P:F_{M,S}; h; v; x)$ is a $\tau$-spherical function.

**Proof.** To see that it is a spherical function, drop most of the variables and compute

$$E(k_1 x k_2)$$

$$= \int F_G(k_1 x k_2 k) \cdot \tau_2(k^{-1}) \cdot e^{i(v - \rho) H_p(k_1 x k_2)} d(kZ)$$

$$= \int F_G(k_1 x k) \cdot \tau_2(k^{-1} k_2) \cdot e^{i(v - \rho) H_p(k_1 x k)} d(kZ)$$

$$= \int \tau_1(k_1) \cdot F_G(x k) \cdot \tau_2(k^{-1}) \cdot \tau_2(k_2) \cdot e^{i(v - \rho) H_p(k)} d(kZ)$$

$$= \tau_1(k_1) \cdot E(x) \cdot \tau_2(k_2).$$

We check that $E(P:F_{M,S})$ is smooth in $(h, v, x)$ and holomorphic in $(h, v)$. Theorem 5.8 implies that $F_M$ is holomorphic in $h$ and smooth in $(h, m)$ as a map to $W(\sigma_1; \sigma_2)$. The same follows for $F_{M,S}$ as a map to $W(\tau_1; \tau_2)$. Now $F_G$ is holomorphic in $h$ and smooth in $(h, x)$. Since $H_p$ is real analytic, now the integrand in (6.6b) is holomorphic in $(h, v)$ and smooth in $(h, v, x, k)$. As $K/Z$ is compact, now $E(P:F_{M,S})$ is holomorphic in $(h, v)$, and smooth in $(h, v, x)$.

Let $\mathcal{Z}(g)$ denote the center of the enveloping algebra $\mathcal{U}(g)$. If $\beta \in \mathfrak{h}_G^*$ then $\chi_\beta: \mathcal{Z}(g) \to \mathbb{C}$ denotes the infinitesimal character with Harish-Chandra parameter $\beta$. Every $f_M \in \mathcal{F}(\pi; \sigma_1; \sigma_2)$ satisfies

$$f_M(h; m; u) = \chi_{\xi_{\delta_0 + h_M(h)}}(u) f_M(h; m) \quad \text{for} \quad u \in \mathcal{Z}(m)$$

because $\pi(h)$ has infinitesimal character $\xi_{\delta_0 + h_M(h)}$. Now, from its definition (5.4), $F_M = F(f_M)$ satisfies

$$F_M(h; m; u) = \chi_{\xi_{\delta_0 + h_M(h)}}(u) F_M(h; m) \quad \text{for} \quad u \in \mathcal{Z}(m).$$

This carries through trivially to $F_{M,S}$ and now

$$F_G(h; x; z) = \chi_{\xi_{\delta_0 + h_M(h)}}(z) F_G(h; x) \quad \text{for} \quad z \in \mathcal{Z}(g).$$

Differentiating under the integral we have, exactly as in [3, Lemma 19.1],

$$E(P:F_{M,S}; h; v; x; z)$$

$$= \chi_{\xi_{\delta_0 + h_M(h)} + n(z)} E(P:F_{M,S}; h; v; x) \quad \text{for} \quad z \in \mathcal{Z}(g). \quad (6.8)$$
Here note that $z_{i_0} + h(t_{i_1} + i)$ is the infinitesimal character of the $H$-series representation $\pi_{h_1}$ of (3.9).

7. Growth Estimates

In this section we will show how to use the differential equations (6.8) satisfied by the Eisenstein integrals to sharpen the a priori estimates on their growth which will be proved in Section 9. The main results are Theorems 6.31 and 6.33 which will be used in Sections 8 and 9 to show that wave packets of Eisenstein integrals are Schwartz functions on $G$. In order to carry out the induction required for the proof of Theorem 8.5, it is necessary to study more general classes of functions. The functions of type $I(H, L_p)$ defined in (7.5) generalize holomorphic families of Eisenstein integrals, and the functions of type $I(G, L_p)$ defined in (7.7) generalize the product of an Eisenstein integral with a Schwartz function in the parameter variables.

The results of this section extend the construction and estimates of Harish-Chandra [4] to include dependence on the extra continuous parameters in the Eisenstein integrals which come from continuous families of relative discrete series representations. The organization of this section closely follows Trombi’s account of Harish-Chandra’s work in [10].

We first review some standard results in invariant theory (see [10]). Fix $P_0 = M_0 A_0 N_0$ a minimal parabolic subgroup of $G$ and let $h_0 = t_0 + a_0$ be a Cartan subalgebra of $g$ with $t_0 \subseteq n_0$. For any parabolic subgroup $P = M_p A_p N_p$ of $G$, write $L_P = M_P A_P$ and $K_P = K \cap M_P$. Now if $A_P \subseteq A_0$, then $h_0$ is a Cartan subalgebra of $L_P$, and we write $W_P$ for the Weyl group of the pair $(L_P, h_0)$, $S(h_0)$ for the $W_P$-invariants in the symmetric algebra of $h_0$, $G_P$ for the center of $\mathbb{Z}(L_P)$ and $\mu_P: G_P \rightarrow S(h_0)$ for the canonical isomorphism.

Now suppose $Q \subseteq P$ are two parabolic subgroups of $G$ with $A_P \subseteq A_Q \subseteq A_0$, so that $*Q = Q \cap L_P = L_Q * N_Q$ is a parabolic subgroup of $L_P$. Then we can consider $W_Q \subseteq W_P$ so that $S(h_Q) = S(h_0)^{W_Q}$ for all $z \in G_P$. Let $\mu_{\rho_Q}: G_P \rightarrow G_Q$ be the algebra injection such that $\mu_{\rho_Q}(z) = \mu_Q(\mu_{\rho_Q}(z))$ for all $z \in G_P$. Let $A(\rho_Q, A_Q)$ denote the roots of the pair $(\rho_Q, A_Q)$, $\rho = \frac{1}{2} \sum x, x \in A(\rho_Q, A_Q)$. Then $d_Q$ is defined on $L_Q$ by $d_Q(\rho_Q(\mu_Q) = e^{i\rho_Q(\mu_Q)}$, $m \in M_Q, \rho \in A_Q$. For $b \in G_Q$, we write $b' = d_Q^{-1} b d_Q$. The mapping $\mu_{\rho_Q}$ also has the property that $z - \mu_{\rho_Q}(z) \in \theta(\rho_Q, Q) \mathbb{Z}(L_Q)$ for all $z \in G_P$. Further, $G_Q$ is a finite free module over $\mu_{\rho_Q}(G_P)$ of rank $r = [W_P : W_Q]$. Let $1 = v_1, v_2, ..., v_r$ be a free basis, and for $v \in G_Q$, denote by $z_{v_i}, 1 \leq i, j \leq r$, the unique element of $G_P$ satisfying

\[ v_i = \sum_{1 \leq i < j \leq r} \mu_{\rho_Q}(z_{v_i}) v_j, \quad 1 \leq i \leq r. \] (7.1)
Fix a complex Hilbert space $T$ of dimension $r$ with orthonormal basis \{\(e_1, \ldots, e_r\)\}. For \(A \in \mathfrak{l}_{\text{OC}}\), \(v \in \mathcal{L}_Q\), let \(\Gamma(A; v)\) be the endomorphism of \(T\) with matrix \(\mu_p(z; \cdot A) = \mu_p(z; A)\) with respect to this basis. Let \(\Phi^+_p, \Phi^-_p\) be positive systems of roots for \((\mathfrak{l}_{PC}, \mathfrak{h}_{\text{OC}}), (\mathfrak{l}_{OC}, \mathfrak{h}_{OC})\) respectively, chosen so that \(\Phi^-_p = \{x \in \Phi^+_p : x|_{\mathfrak{a}_Q} = 0\}\) and if \(x \in \Phi^+_p\) with \(x|_{\mathfrak{a}_Q} \neq 0\), then \(x|_{\mathfrak{a}_Q}\) is a root of \(\{\ast q, a_q\}\). Define
\[
\pi_p = \prod_{x \in \Phi^+_p} H_x, \quad \pi_Q = \prod_{x \in \Phi^-_p} H_x, \quad \pi_{PQ} = \pi_p/\pi_Q. \tag{7.2}
\]

Fix coset representatives \(s_1 = 1, s_2, \ldots, s_r\) for \(W_p/\mathcal{W}_Q\). The following lemma appears in [10].

**Lemma 7.3.** For \(A \in \mathfrak{l}_{\text{OC}}\), let \(e_k(A) = \sum_{1 \leq j \leq r} \mu_p(v_j; s_k A) e_j\). Then if \(A \in (\mathfrak{l}_{\text{OC}}^*)' = \{A \in \mathfrak{l}_{\text{OC}} : \pi_p (A) \neq 0\}\), the \(e_k(A), 1 \leq k \leq r\), form a basis for \(T\) and \(\Gamma(A; v) e_k(A) = \mu_Q(v; s_k A) e_k(A)\) for all \(v \in \mathcal{L}_Q, 1 \leq k \leq r\). Moreover, there is an \(r \times r\) matrix \(B\) with entries in the quotient field of \(S(\mathfrak{l}_{\text{OC}})^{W_q}\) so that
\[
\text{(i)} \quad \pi_{PQ} B \text{ has entries in } S(\mathfrak{l}_{\text{OC}})^{W_q};
\]
\[
\text{(ii)} \quad \text{for any } A \in (\mathfrak{l}_{\text{OC}}^*)', \text{ the } B(s_k A) \text{ are projections } T \to C e_k(A) \text{ corresponding to the direct sum } T = \sum_{1 \leq k \leq r} C e_k(A).
\]

Fix \(\mathfrak{h} = \mathfrak{l}_H + a_H\) a \(\theta\)-stable Cartan subalgebra of \(\mathfrak{g}\). We may as well assume that \(a_H \subseteq \mathfrak{a}_H\). Let \(P_H = M_H A_H N_H\) be a cuspidal parabolic subgroup with split component \(A_H\). As in (3.3), (3.7), and (3.18), we introduce parameters for a continuous series of representations induced from \(P_H\). Thus for \(\lambda_0 \in \mathcal{A}_0\) and \(h \in \mathcal{D}\), \(\lambda(h) = \lambda_0 + h_M(h) \in \mathfrak{t}_H\) and \(\chi(h) \in \mathcal{Z}_{M_H}(M_H^\vee)\) are parameters for a continuous family of relative discrete series representations on \(M_H\). Let \(\omega\) be a relatively compact neighborhood of 0 in \(i v^\ast\). Define \(\mathcal{D}_c = \mathcal{D}_c(\omega) = \{h \in i v^\ast : h_R + i h_I, h_k \in \mathcal{D}_c, h_\ell \in \omega\}\). For \(h \in \mathcal{D}_c\) define \(d(h)\) to be the distance from \(h_R\) to the boundary of \(\mathcal{D}_c\). For \(h \in \mathcal{D}_c, v \in \mathcal{F} = a_H^\ast\), extend \(\lambda(h)\) trivially to \(a_H\) and \(\nu\) trivially to \(t_{HC}\) so that \(\lambda(h) + i \nu \in \mathfrak{h}_c^\ast\). Let \((\tau_1, h, \tau_2, h)\) be a family of double representations of \(K\) on a finite dimensional vector space \(W\) with norm \(\|\cdot\|\) as in (6.4).

Fix \(P\) a parabolic subgroup of \(G\) with \(A_P \subseteq A_H \subseteq A_0\). Then \(\mathfrak{h}\) and \(\mathfrak{h}_0\) are both Cartan subalgebras of \(\mathfrak{I}_P\). Pick \(v \in \text{Int}(\mathfrak{l}_{PC})\) such that \(\gamma(\mathfrak{h}_c) = \mathfrak{h}_{\text{OC}}\). For \(h \in \mathcal{D}_c, v \in \mathcal{F}\), write \(A_{h,v} = v(\lambda(h) + i \nu) \in \mathfrak{h}_{\text{OC}}^\ast\). Let \(\mathcal{D}\) denote the set of all differential operators on \(v_c^\ast \times \mathcal{F}\) with coefficients which are polynomials in \(h \in \mathfrak{h}_c^\ast\) and \(v \in \mathcal{F}\). Write \(\mathcal{D}_c = \mathcal{D}_c(\omega) = \{h \in \mathfrak{v}_c^\ast : h = h_R + i h_I, h_k \in \mathcal{D}_c, h_\ell \in \omega\}\). For \(D_1 \otimes l_1 \otimes l_2 \in \mathcal{D}_c\), \(\varphi \in C_c(\mathcal{C}_c \times \mathcal{F} \times L_P, \mathcal{W})\), define \((D_1 \otimes l_1 \otimes l_2 \varphi)(h,v):x = \varphi(h,v; D_1,l_1,x;l_2)\). For \(D \in \mathcal{D}_c, r \in \mathbb{R}\), set
\[
S_{D,r}(\varphi) = \sup_{\mathcal{D}_c \times \mathcal{F} \times L_P} \|D \varphi(h,v,x)\| \Xi^{-1}(\chi) \|\varphi(h,v,x)\|^{-r} e^{-|h|_{\mathfrak{a}_H} |x|_{\mathfrak{a}_L}}. \tag{7.4}
\]
where \(|(h,v,x)| = (1 + |h|)(1 + |v|)(1 + \delta(x))(1 + d(h)^{-1})\), \(\Xi_p\) is the function \(\Xi\) defined as in (2.4) for the group \(L_p\), and \(\delta(x)\) and \(\sigma_v(x)\) are defined as in (2.7). For \(F\) any finite subset of \(\mathcal{F}_p\), set \(S_{r,F}(\varphi) = \sum_{P \in F} S_{P,F}(\varphi)\).

**Definition 7.5.** We will write \(II(\mathcal{G}, L_p)\) for the set of all \(\varphi \in C^\infty(\mathcal{G} \times \mathcal{F} \times L_p, W)\) satisfying

(i) for all \((v, x) \in \mathcal{F} \times L_p, h \mapsto \varphi(h ; v : x)\) is a holomorphic function on \(\mathcal{G}_c\);

(ii) for all \((h, v) \in \mathcal{G} \times \mathcal{F}, \varphi(h ; v)\) is a \((\tau_1, h |_{\mathfrak{p}}, \tau_2, h |_{\mathfrak{p}})\)-spherical function on \(L_p\);

(iii) for all \((h, v) \in \mathcal{G} \times \mathcal{F}, z \varphi(h ; v) = \mu_p(z ; A_{h,v}) \varphi(h ; v)\) for all \(z \in \mathcal{G}_p\);

(iv) for all \(D \in \mathcal{F}_p\), there is \(r \geq 0\) so that \(S_{D,F}(\varphi) < \infty\).

**Remark.** The holomorphic families of Eisenstein integrals constructed in (6.6) will be shown in Section 9 to be elements of \(II(\mathcal{G}_c , G)\).

For \(\varphi \in C^\infty(\mathcal{G} \times \mathcal{F} \times L_p, \mathcal{W}), D \in \mathcal{F}_p, r, t, \in \mathbb{R}\), set

\[
0^0 S_{D,F}(\varphi) = \sup_{\mathcal{G} \times \mathcal{F} \times L_p} \| D\varphi(h ; v : x) \| \Xi_p^{-1}(x)(1 + \delta(x))^{-1} \eta(1 + d(h)^{-1})^t.
\]

(7.6)

For \(F\) any finite subset of \(\mathcal{F}_p\), set \(0^0 S_{D,F}(\varphi) = \sum_{D \in F} 0^0 S_{D,F}(\varphi)\).

**Definition 7.7.** We will write \(II(\mathcal{G}, L_p)\) for the set of all \(\varphi \in C^\infty(\mathcal{G} \times \mathcal{F} \times L_p, W)\) satisfying

(i) there is a complex neighborhood \(\mathcal{G}_c\) of \(\mathcal{G}\) and a finite set of functions \(\varphi_1, \ldots, \varphi_k \in II(\mathcal{G}_c , L_p)\) so that for each \((h,v) \in \mathcal{G} \times \mathcal{F}\) there exist \(a_j(h:v) \in \mathbb{C}, 1 \leq j \leq k\), such that \(\varphi(h ; v ; x) = \sum_{j=1}^{k} a_j(h ; v) \varphi_j(h ; v ; x)\) for all \(x \in \mathcal{G}_p\);

(ii) for all \(D \in \mathcal{F}_p\), there is \(r \geq 0\) so that \(0^0 S_{D,F}(\varphi) < \infty\) for all \(t \geq 0\).

**Remark.** If \(\varphi \in II(\mathcal{G}_c , L_p)\) and \(x \in C(\mathcal{G} \times \mathcal{F}) = \{ x \in C^\infty(\mathcal{G} \times \mathcal{F}) : \| x \|_{D,F} = \sup_{D \times X \in \mathcal{G} \times \mathcal{F}} |D\varphi(h ; v)| (1 + d(h)^{-1})^t < \infty\) for all \(D \in \mathcal{F}, t \geq 0\},\) then \(\varphi \cdot x \in I(\mathcal{G}, L_p)\). In fact, given \(D \in \mathcal{F}_p, r \geq 0\), there is a finite subset \(F\) of \(\mathcal{F}_p \times \mathcal{F}\) so that \(0^0 S_{D,F}(\varphi \cdot x) \leq \sum_{D' \in \mathcal{F}_p \times \mathcal{F}} S_{D',F}(\varphi) \| x \|_{D',F}^t\) for all \(t \geq 0\).

We are now ready to study the asymptotic behavior of functions of types \(I(\mathcal{G}, L_p)\) and \(II(\mathcal{G}_c , L_p)\). Both types will be treated simultaneously with the understanding that if \(f \in I(\mathcal{G}; L_p), h\) ranges over \(\mathcal{G}\) while if \(f \in II(\mathcal{G}_c; L_p), h\) ranges over \(\mathcal{G}_c\). We return to the notation in the first part of this section. Thus \(\mathcal{Q}\) is a parabolic subgroup of \(L_p\).

For \(f \in I(\mathcal{G}, L_p)\) or \(II(\mathcal{G}_c , L_p)\), define \(\Phi(f)\) and \(\Psi_{\epsilon}(f)\) taking values in \(\mathcal{W} = \mathcal{W} \otimes T\) by

\[
\Phi(f ; h : v : m) = \sum_{D \in \mathcal{D}} d_\Omega(m) f(h : v ; m, v') \otimes e_i, \quad m \in L_{\Omega} \quad (7.8a)
\]
where for $v \in \mathcal{L}_Q$, $u_i(v;h:v) = \sum_{i=1} f(m) f(h:v;m;u_i(v;h:v)) \otimes e_i, \quad m \in L_Q, \quad (7.8b)$

where for $v \in \mathcal{L}_Q, u_i(v;h:v) = \sum_{i=1} f(m) f(h:v;m;u_i(v;h:v)) \otimes e_i, \quad m \in L_Q, \quad (7.8b)$

**Lemma 7.9.** Let $b_1, b_2 \in \mathcal{U}(l_Q), v \in \mathcal{L}_Q.$ Then $\Phi(f;h:v;b_1;m; b_2) = \Gamma(A_{h,v};v) \Phi(f;h:v;b_1;m; b_2) + \Psi_f(f;h:v;b_1;m; b_2)$ for all $m \in L_Q.$ Here $\Gamma(A_{h,v};v)$ has been extended to an endomorphism of $\mathcal{W}$ by making it act trivially on $W.$

**Proof.** This follows easily from (7.1) since $z_{ij} - \mu_p(z_{ij};A_{h,v})$ kills every $f \in I(\mathcal{Q}, L_p) \cup \mathcal{H}(\mathcal{Q}, L_p)$ (see [10, pp. 280–281]).

**Corollary 7.10.** Let $b_1, b_2 \in \mathcal{U}(l_Q), H \in \mathcal{A}_Q.$ Then for all $T \in \mathbb{R}, m \in L_Q,$ we have $\Phi(f;h:v;b_1;m; \exp TH; b_2) = \exp(T\Gamma(A_{h,v};H)) \Phi(f;h:v;b_1;m; b_2) + \int_0^T \exp((T-t)\Gamma(A_{h,v};H)) \Psi_f(f;h:v;b_1;m; \exp tH; b_2) dt.$

**Lemma 7.11.** Fix $D \in \mathcal{Q}, l_1, l_2 \in \mathcal{U}(l_p), \text{ and } X \in n_Q = n_Q \cap 1_p.$ Then we can choose a finite subset $F \subseteq F_p$ and $r_0 \geq 0$ such that

$$d_Q(ma) \| f(h;v;D;l_1;X; ma; l_1) \parallel \| f(h;v;D;l_1; ma; \theta(X)(l_2)) \parallel$$

$$\leq \begin{cases} 0 \text{ for all } r,t \geq 0, f \in I(\mathcal{Q}, L_p) \\
S_{r,t}(f) \Xi_Q(m) e^{\beta_Q(\log a)(1 + \hat{\sigma}(ma))^{r+n} (1 + d(h)^{-1})} \\
\text{for all } r \geq 0, f \in \mathcal{H}(\mathcal{Q}, L_p) \\
S_{r,t}(f) \Xi_Q(m) e^{\beta_Q(\log a)(1 + \hat{\sigma}(v))^{r+n} (1 + \sigma(a))^{r+n} e^{\|h\|_1 \sigma_1(m)}} \\
\text{for all } r \geq 0, f \in \mathcal{H}(\mathcal{Q}, L_p) \\
\end{cases}$$

and

$$d_Q(ma) \parallel f(h;v;D;l_1; ma; l_2) \parallel$$

$$\leq \begin{cases} 0 \text{ for all } r,t \geq 0, f \in I(\mathcal{Q}, L_p) \\
S_{r,s}(f) \Xi_Q(m) (1 + \hat{\sigma}(ma))^{r+n} (1 + d(h)^{-1}) \\
\text{for all } r \geq 0, f \in \mathcal{H}(\mathcal{Q}, L_p) \\
S_{r,s}(f) \Xi_Q(m) (1 + \hat{\sigma}(v))^{r+n} (1 + \sigma(a))^{r+n} e^{\|h\|_1 \sigma_1(m)} \\
\text{for all } r \geq 0, f \in \mathcal{H}(\mathcal{Q}, L_p) \\
\end{cases}$$

for all $m \in L_Q, = K_Q \text{cl}(A_{\alpha}^+(P)) K_Q$ and $a \in A_{\alpha}^+ = \{a \in A_Q: \alpha(\log a) > 0 \text{ for all } a \in A(Q, A_Q)\}.$ Here $A_{\alpha}^+(P) = \{a \in A_Q: \alpha(\log a) > 0 \text{ for all } a \in \Phi^+ \}$ and for $H \in \mathcal{A}_Q, \beta_Q(H) = \inf \{\alpha(H): \alpha \in A(Q, A_Q)\}.$

**Proof.** Write $\rho_P = \frac{1}{2} \Sigma x, x \in \Phi^+, \rho_Q = \frac{1}{2} \Sigma x, x \in \Phi^*_Q,$ and $\rho_{PQ} = \frac{1}{2} \Sigma x, x \in \mathcal{A}(Q, A_Q).$ Then $\rho_P = \rho_Q \cup \rho_{PQ}.$ Using (2.16) for the group $L_p$ we have constants $c_0$ and $r_0$ so that $\Xi_\alpha(a) \leq c_0 (1 + \sigma(a))^{\rho_0} e^{\rho_\alpha(a)}$ for all $a \in A_{\alpha}^+(P).$
But \( e^{-\sigma(a)} = d_{O}^{-1}(a) e^{-\sigma(a)} \leq d_{O}^{-1}(a) \Xi_{O}(a) \) by (2.16) for the group \( L_{Q} \). Thus for all \( m \in L_{Q}^{+}, d_{O}(m) \leq c_{0} \Xi_{O}(m) \leq \Xi_{O}(m)(1 + \sigma(m))^{\varepsilon_{0}}. \) Then the second set of inequalities follows trivially from the definitions of \( S_{F,r}(f) \) and \( S_{F,c}(f) \) if we take \( F = \{ c_{0} D \otimes l_{1} \otimes l_{2} \}. \)

We will prove the first set of inequalities for \( D \otimes l_{1} \otimes \emptyset(\Lambda) l_{2} \) only. The argument for \( D \otimes l_{1} \otimes \emptyset(\Lambda) l_{2} \) is similar. Write \( f(h; v; D; l_{1} X; ma; l_{2}) = f(h; v; D; l_{1} : ma; u; \varepsilon_{1} X_{1}). \) Now \( m = k_{1} a_{0} l_{2} \) for some \( k_{1}, k_{2} \in K_{Q} \) and \( a_{0} \in \emptyset(A_{0}^{+}(P)). \) Thus \( u^{-1} a_{0}^{-1} k_{1}^{-1} X_{1} = k_{2}^{-1} b^{-1} a_{0}^{-1} k_{1}^{-1} X. \) Write \( k_{2}^{-1} X = \sum c_{z}(k_{2}) X_{z} \), where the summation is taken over \( \{ x \in \Phi_{p} : x|_{\Lambda_{0}} \in \Delta(A_{Q}, A_{Q}) \} \), each \( c_{z} \) is a smooth function on the compact group \( K_{Q} \), and each \( X_{z} \in *n_{Q} \) satisfies \( u^{-1} a_{0}^{-1} X_{z} = e^{-\xi|\log a_{0}|} e^{-\xi|\log a|} X_{z}. \) Finally, write each \( k_{2}^{-1} X_{z} = \sum d_{z}(k_{2}) X_{z} \), where each \( d_{z} \) is a smooth function on \( K_{Q} \). Now \( e^{-\xi|\log a|} \leq 1 \) for all \( a \in A_{Q}^{+}(P) \) and \( e^{-\xi|\log a|} \leq e^{-\xi|\log a|} \) for all \( a \in A_{Q}^{+}(P) \) so that

\[
\|f(h; v; D; l_{1} X; ma; l_{2})\| \\
\leq \sum_{z} \left| c_{z}(k_{1}) d_{z}(k_{2}) e^{-\xi|\log a_{0}|} e^{-\xi|\log a|} \right| \\
\times \left\| f(h; v; D; l_{1} : ma; X_{z} l_{2})\right\| \\
\leq e^{-\xi|\log a|} \sum_{z} c_{z} \left\| f(h; v; D; l_{1} : ma; X_{z} l_{2})\right\|,
\]

where each \( c_{z} = \sup_{k_{1}, k_{2}} \sum_{z} \left| c_{z}(k_{1}) d_{z}(k_{2})\right| < \infty. \) Now the result follows as above where \( F = \{ c_{0} D \otimes l_{1} \otimes X_{z} l_{2}\}. \) Q.E.D.

**Corollary 7.12.** Fix \( D \in \mathcal{D}, v \in \mathcal{L}_{Q}, \) and \( b_{1}, b_{2} \in \mathcal{U}(l_{Q}). \) Then we can choose a finite subset \( F \) of \( \mathcal{D}_{p}^{\prime} \) and \( r_{0} \geq 0 \) so that for all \( m \in L_{Q}^{+}, H \in \emptyset(\Lambda_{Q}) \)

\[
\|\Psi_{\varepsilon}(f; h; v; D; b_{1} ; m \exp H; b_{2})\| \\
\leq \left\{
\begin{array}{ll}
0 S_{F,r}(f) \Xi_{Q}(m) e^{-\rho|\exp H|}(1 + \sigma(m \exp H))^{r + \tau_{0}} (1 + d(h)^{-1})^{r} & \text{for all } r, t \geq 0, f \in I(\mathcal{D}, L_{p})
S_{F,r}(f) \Xi_{Q}(m) e^{-|\exp H|} |(h, v, m)|^{r + \tau_{0}} (1 + |H|)^{r + \rho_{0}} e^{h|\sigma(m)|} & \text{for all } r \geq 0, f \in II(\mathcal{D}, L_{p})
\end{array}
\right.
\]

and

\[
\|\Phi(f; h; v; D; b_{1}; m \exp H; b_{2})\| \\
\leq \left\{
\begin{array}{ll}
0 S_{F,r}(f) \Xi_{Q}(m)(1 + \sigma(m \exp H))^{r + \tau_{0}} (1 + d(h)^{-1})^{r} & \text{for all } r, t \geq 0, f \in I(\mathcal{D}, L_{p})
S_{F,r}(f) \Xi_{Q}(m) |(h, v, m)|^{r + \tau_{0}} (1 + |H|)^{r + \rho_{0}} e^{h|\sigma(m)|} & \text{for all } r \geq 0, f \in II(\mathcal{D}, L_{p})
\end{array}
\right.
\]
Proof. This follows from (7.11) because, by definition (7.8), \( \Psi_r(h:v:b_1; m \exp H; b_2) \) is a sum of terms of the form

\[
d_Q(m \exp H) f(h:v:b_1'; m \exp H; (\mu_p(z)' - \mu_p(z':A_{h,v}) v'_h b'_2) = d_Q(m \exp H) f(h:v:b_1'; m \exp H; (\mu_p(z)' - z) v'_h b'_2)
\]

for some \( z \in \mathcal{I}_p \). But \( \mu_p(z)' - z \in \mathcal{U}(\mathcal{Q}_p) \mathcal{U}(\mathcal{Q}_p) \). Q.E.D.

For any \( A \in h_{\text{loc}}^\bullet \), let \( B_1(A) \) be the endomorphism of \( W = W \otimes \mathcal{E} \) given by \( \mathcal{P} \circ \mathcal{R}(A) \mathcal{B}(A) \), where \( \mathcal{B} \) is defined as in Lemma 7.3. For \( 1 \leq i \leq r \) and \( v \in \mathcal{I}_Q \) set

\[
\Phi_i(f:h:v:m) = B_1(s_{i,A_{h,v}}) \Phi(f:h:v:m);
\]

\[
\Psi_{r,i}(f:h:v:m) = B_1(s_{i,A_{h,v}}) \Psi_i(f:h:v:m).
\]

Since \( B_1(s_{i,A_{h,v}}) \) depends polynomially on \( h \) and \( v \), there are constants \( c \geq 0 \) and \( h \geq 0 \) so that

\[
\|B_1(s_{i,A_{h,v}})\|_{DP} \leq c(1 + |h|)^h (1 + |v|)^h.
\]

Lemma 7.7. Let \( b_1, b_2 \in \mathcal{U}(\mathcal{Q}_p), m \in L_Q, H \in a_Q \). Then for all \( T \in \mathbb{R}, 1 \leq i \leq r \),

\[
\Phi_i(f:h:v:b_1; m \exp TH; b_2) = e^{T \frac{1}{1+h} \mathcal{H}} \Phi_i(f:h:v:b_1; m; b_2)
\]

\[
+ \int_0^T e^{t \frac{1}{1+h} \mathcal{H}} \Psi_{r,i}(f:h:v:b_1; m \exp t \mathcal{H}; b_2) dt.
\]

Proof. This is an immediate consequence of (7.10) together with (7.3). Q.E.D.

For \( 1 \leq i \leq r \), let \( \lambda_i(h) \) be restriction of the real part of \( s_{i,h} \) to \( a_Q \). Note \( \lambda_i(h) = \lambda_i(h_R) \) if \( h = h_R + i h_I \) with \( h_R \in \mathcal{D}, h_I \in \omega \). Set \( I = \{1, 2, ..., r\} \) and define

\[
I^0 = \{i \in I: \lambda_i(h:H) = 0 \text{ for all } h \in \mathcal{D}, H \in a_Q\};
\]

\[
I^+ = \{i \in I: \lambda_i(h:H) > 0 \text{ for some } h \in \mathcal{D} \text{ and some } H \in a^+_Q\};
\]

\[
I^- = \{i \in I: \lambda_i(h:H) < 0 \text{ for all } h \in \mathcal{D} \text{ and all } H \in a^+_Q\}.
\]

Note that \( I = I^0 \cup I^+ \cup I^- \), since if \( i \notin I^+ \), \( \lambda_i(h:H) \leq 0 \) for all \( h \in \mathcal{D}, H \in a^+_Q \). But \( a^+_Q \) and \( \mathcal{D} \) are open, and \( \lambda_i(h:H) \) is a linear function of \( H \) and an affine function of \( h \), so that either \( \lambda_i(h:H) \) is identically zero, or else \( \lambda_i(h:H) < 0 \) for all \( h \in \mathcal{D} \) and \( H \in a^+_Q \).
**Lemma 7.16.** Let $C$ be a compact subset of $L_Q$, $\Omega$ a compact subset of $a_Q^+$. Then we can choose $T_0 \geq 0$ so that $m \exp TH \in L_Q^+$ for $m \in C$ and $T \geq T_0$, $H \in \Omega$.

**Proof.** See [2, Lemma 54].

For $H \in a_Q^+$, $i \in I^0 \cup I^+$, define $\mathcal{Q}^i(H) = \{ h \in \mathcal{Q}_i : \lambda_i(h; H) + \beta_Q(H) > 0 \}$, $\mathcal{Q}'(H) = \mathcal{Q}^i(H) \cap \mathcal{Q}$. Note if $i \in I^0$, then $\mathcal{Q}_i'(H) = \mathcal{Q}_i$ for all $H \in a_Q^+$.

**Lemma 7.17.** Let $D \in \mathcal{D}$, $b_1, b_2 \in \mathcal{H}(l_0)$, $i \in I^0 \cup I^+$, $H \in a_Q^+$. Then $\int_0^\infty \| \Phi_H, i(f; h; v ; D \cdot e^{t \lambda_i(h; H)} ; b_1 ; m \exp tH ; b_2)\| dt$ converges uniformly for $v$ and $m$ in compact subsets of $\mathcal{D}$ and $L_Q$, respectively, and for $h$ in compact subsets of

$$\mathcal{Q}'(H) \quad \text{if} \quad f \in I(\mathcal{Q}, L_R);$$

$$\mathcal{Q}_i'(H) \quad \text{if} \quad f \in II(\mathcal{Q}_i, L_R).$$

**Proof.** This follows from (7.12), (7.13c), and (7.16). Q.E.D.

**Lemma 7.18.** Let $i \in I^0 \cup I^+$, $H \in a_Q^+$. Then $\Phi_{i, \nu}(f; h; v ; m ; H) = \lim_{\tau \to -\infty} \Phi_i(f; h; v ; m \exp \tau H) e^{-\int_{-\infty}^\tau \lambda_i(h; H) dt}$ exists and is $C^r$ on

$$\mathcal{Q}'(H) \times \mathcal{D} \times L_Q$$

if $f \in I(\mathcal{Q}, L_R)$;

$$\mathcal{Q}_i'(H) \times \mathcal{D} \times L_Q$$

and holomorphic for $h \in \mathcal{Q}_i'(H)$ if $f \in II(\mathcal{Q}_i, L_R)$.

Further, for all $D \in \mathcal{D}$, $b_1, b_2 \in \mathcal{H}(l_0)$,

$$\Phi_{i, \nu}(f; h; v ; D; b_1; m; b_2; H)$$

$$= \Phi_i(f; h; v ; D; b_1; m; b_2)$$

$$+ \int_0^{\infty} \Psi_{n,i}(f; h; v ; D \cdot e^{t \lambda_i(h; H)} ; b_1 ; m \exp tH ; b_2) dt.$$

**Proof.** Combine Lemmas (7.14) and (7.17). Q.E.D.

Let $H_1, H_2 \in a_Q^+$. For $i \in I^0 \cup I^+$ and

$$h \in \begin{cases} \mathcal{Q}'(H_1) \cap \mathcal{Q}'(H_2) & \text{if} \quad f \in I(\mathcal{Q}, L_R), \\ \mathcal{Q}_i'(H_1) \cap \mathcal{Q}_i'(H_2) & \text{if} \quad f \in II(\mathcal{Q}_i, L_R), \end{cases}$$

the argument in [10, p. 285] shows that $\Phi_{i, \nu}(f; h; v ; m; H_1) = \Phi_{i, \nu}(f; h; v ; m; H_2)$. Thus whenever there is an $H \in a_Q^+$ such that $\lambda_i(h; H) + \beta_Q(H) > 0$, we can define $\Phi_{i, \nu}(f; h; v ; m) = \Phi_{i, \nu}(f; h; v ; m; H)$, and the definition does not depend on the choice of $H$. 

LEMMA 7.19. Suppose for \( i \in I^+ \) and \( h \in \{ \phi \} \), there is an \( H \in \mathcal{A} \) such that \( \lambda_i(h;H) > 0 \). Then \( \Phi_{i,x}(f;h;v;m) = 0 \) for all \((v,m) \in \mathcal{F} \times L_Q \) and \( f \in \{ \mathcal{A}(L_p) \} \).

Proof. Since \( \lambda_i(h;H) > 0 \), \( h \in \{ \phi \} \) so that \( \Phi_{i,x}(f;h;v;m) = \Phi_{i,x}(f;h;v;m;H) = \lim_{T \to \infty} \Phi_i(f;h;v;m \exp TH) e^{-T \phi(A_{h,v},s_i)} \). But using (7.12) and (7.13c), this limit is zero since \( \Phi_i(f;h;v;m \exp TH) \) grows polynomially in \( T \) while \( |e^{-T \phi(A_{h,v},s_i)}| = e^{-T \phi(A_{h,v},s_i)} \) decays exponentially. Q.E.D.

LEMMA 7.20. Let \( i \in I^0 \). Then

(i) \( \Phi_{i,x}(f;h;v;m;\nu) = \mu_0(v;s_iA_{h,v}) \Phi_{i,x}(f;h;v;m) \) for all \( v \in \mathcal{A}_0 \);

(ii) given \( h_1, h_2 \in \mathcal{A}_0 \) and \( D \in \mathcal{D} \) there exists a finite subset \( F \subseteq \mathcal{D}_p \) such that for all \( r, t \geq 0 \) there is a \( C > 0 \) so that

\[
\| \Phi_{i,x}(f;h;v;D;h_1;m;h_2) \| = \begin{cases} 0 & \text{if } \nu \in \{ \mathcal{A}(L_p) \} \\
C^0 S_{F,r}(f) \cdot \mathcal{A}(m)(1 + \phi(m))^{r + \phi(h_1)}(1 + d(h)^{-1})^{-1} & \text{if } f \in \{ \mathcal{A}(L_p) \} \\
C S_{F,r}(f) \cdot \mathcal{A}(m) \cdot \phi(h_1, v, m)^{r + \phi(h_1) + \phi(m)} & \text{if } f \in \{ \mathcal{A}(L_p) \}. 
\end{cases}
\]

Here \( r_0 \) and \( h \) are the constants given in (7.12) and (7.13c).

Proof. (i) From (7.3) and (7.9), we have

\[
\Phi_i(f;h;v;m \exp TH;v) = \mu_0(v;s_iA_{h,v}) \Phi_i(f;h;v;m \exp TH) + \Psi_{i,A}(f;h;v;m \exp TH).
\]

But by using the estimate in (7.12) we see that

\[
\lim_{T \to \infty} e^{-T \phi(A_{h,v},s_i)} \Psi_{i,A}(f;h;v;m \exp TH) = 0 \text{ for } i \in I^0.
\]

(ii) Combine the formula for \( \Phi_{i,x} \) in the second part of (7.18) with the estimates of (7.12) and (7.13c) and use (7.16). Q.E.D.

LEMMA 7.21. There exists a continuous, piecewise affine function \( \delta \) on \( \mathcal{D} \) satisfying \( 0 < \delta(h) \leq \frac{1}{2} \) for all \( h \in \mathcal{D} \) so that given \( D \in \mathcal{D} \) there exists a finite subset \( F \subseteq \mathcal{D}_p \) and \( C, r_1 > 0 \) so that, for all \( f \in \{ \mathcal{D}(L_p) \} \cup \{ \mathcal{D}_p \}, i \in I \).

\[
\| \Phi_i(f;h;v;D;h_1;m \exp TH;h_2) \| = \begin{cases} 0 & \text{if } i \in I^+ \cup I^- \\
\Phi_{i,x}(f;h;v;D;h_1;m \exp TH;h_2) & \text{if } i \in I^0 \end{cases}
\]
We will need some preparation before we can prove this lemma. This is the first result in this section where the continuous relative discrete series parameter plays a significant role. The point is that \( i \in I^+ \) if there is \( H \in a_Q^+ \) with \( \lambda_i(h: H) > 0 \) for some \( h \in \mathcal{G} \), rather than for all \( h \in \mathcal{G} \). In order to obtain the estimate required in the case that \( i \in I^+ \), we need to use the holomorphicity in \( \lambda \) of functions in \( \mathcal{H}(Q', L_p) \). After we have the result in the case that \( f \in \mathcal{H}(Q', L_p) \), we use the fact that each \( f \in \mathcal{H}(Q', L_p) \) is a linear combination of functions in \( \mathcal{H}(Q', L_p) \) to obtain the result in case \( f \in \mathcal{H}(Q', L_p) \).

Suppose \( \lambda \) is any real-valued linear function on \( a_Q \). Write \( \lambda = \sum_{j} c_j \alpha_j \), where \( \alpha_1, \ldots, \alpha_l \) are the simple roots of \( a_Q \) giving \( (1; \alpha) \) as positive chamber. Recall \( \beta(H) = \min_{s \leq j \leq l} \{ \alpha_j(H) \} \) for \( H \in a_Q^+ \). The following lemma is elementary.

**Lemma 7.22.** Let \( \lambda \in a_Q^+ \) and define \( c_1, \ldots, c_l \) as above. Then

(i) \( \lambda(H) = 0 \) for all \( H \in a_Q \) if and only if \( c_1 = \cdots = c_l = 0 \);

(ii) \( \lambda(H) < 0 \) for all \( H \in a_Q^+ \) if and only if \( c_j \leq 0 \) for all \( 1 \leq j \leq l \), and \( \sum_{j=1}^{l} c_j < 0 \):

(iii) \( \lambda(H) > 0 \) for some \( H \in a_Q^+ \) if and only if \( c_j > 0 \) for some \( 1 \leq j \leq l \).

(iv) \( \lambda(H) + \beta(H) > 0 \) for some \( H \in a_Q^+ \) if and only if \( c_j > 0 \) for some \( 1 \leq j \leq l \) or \( \sum_{j=1}^{l} c_j > 0 \).

Now for each \( i \in I \), we write \( \lambda_i(h) = \sum_{j=1}^{l} c_i(h) \alpha_j \) and let \( d_i(h) = -\sum_{j=1}^{l} c_i(h) \). Define

\[ \mathcal{G}_i^+ = \{ h \in \mathcal{G} : c_i(h) > 0 \} \quad \text{for some } 1 \leq j \leq l \}; \quad \text{(7.23a)} \]

\[ \mathcal{G}_i = \{ h \in \mathcal{G} : c_i(h) \leq 0 \} \text{ for all } 1 \leq j \leq l \text{ and } d_i(h) > 0 \}; \quad \text{(7.23b)} \]

\[ \mathcal{G}_i^0 = \{ h \in \mathcal{G} : c_i(h) = 0 \} \text{ for all } 1 \leq j \leq l \}; \quad \text{(7.23c)} \]

\[ \mathcal{G}_i^1 = \{ h \in \mathcal{G} : d_i(h) < 1 \}. \quad \text{(7.23d)} \]
Lemma 7.24. Suppose $\mathcal{D}_i^+ \neq \emptyset$ and $\mathcal{D}_i^- \neq \emptyset$. Then either

(i) $\mathcal{D}_i^+ \cap \mathcal{D}_i^- \neq \emptyset$ or

(ii) $\mathcal{D} = \mathcal{D}_i^+ \cup \mathcal{D}_i^-$, where $\mathcal{D}_i^+ \neq \emptyset$ and $\inf_{h \in \mathcal{D}_i} d_i(h) > 0$.

Proof. Fix $i \in I$ and drop $i$ from the notation for simplicity. Suppose (ii) does not hold. Thus either $\mathcal{D}^0 \neq \emptyset$, $\mathcal{D}^- \neq \emptyset$, or $\inf_{h \in \mathcal{D}^0} d(h) = 0$. Suppose $\mathcal{D} = \mathcal{D}^0 \cup \mathcal{D}^+$. Since $\mathcal{D}^+ \neq \emptyset$ and $\mathcal{D}^- \neq \emptyset$ by assumption, $\mathcal{D}^+$ is a dense open subset of $\mathcal{D}$ and $\mathcal{D}^-$ is a non-empty open subset of $\mathcal{D}$. Thus $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$. Thus we may assume $\mathcal{D}^0 \neq \emptyset$. Suppose $\mathcal{D}^0 \neq \emptyset$. Pick $h_0 \in \mathcal{D}^0$ and $h_1 \in \mathcal{D}^-$. Since $\mathcal{D}$ is convex, $h_t = th_1 + (1-t)h_0 \in \mathcal{D}$ for all $0 \leq t \leq 1$. But for all $j, c_j(h_t) = tc_j(h_0) \leq 0$ for $0 \leq t \leq 1$ and $d(h_t) = td(h_1) > 0$ for $0 < t < 1$. Thus $h_t \in \mathcal{D}^-$ for $0 < t < 1$ and $d(h_t) \to 0$ as $t \to 0$. Thus $\inf_{h \in \mathcal{D}} d(h) = 0$. Thus it is enough to show that $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$ and $\inf_{h \in \mathcal{D}^+} d(h) = 0$ imply that $\mathcal{D} \neq \emptyset$.

Pick $\{h_n\} \in \mathcal{D}^-$ so that $d(h_n) \to 0$ as $n \to \infty$. Fix $h^+ \in \mathcal{D}^+$. Then for some $1 \leq j \leq l, c_j(h^+) > 0$. If $h^+ \in \mathcal{D}^1$ we are done, so we can assume that $d(h^+) > 1$. For each $n, h_n = th^+ + (1-t)h_n \in \mathcal{D}$ for $0 \leq t \leq 1$. Let $T = 1/2d(h^+)$. Then $0 < T \leq \frac{1}{2}$. Pick $N$ large enough that $d(h_N) < 1/2(1-T)$ and $c_j(h_N) > -Tc_j(h^+)/(1-T)$. Then it is easy to check that $h_{N,T} \in \mathcal{D}^+ \cap \mathcal{D}^-$. Q.E.D.

We are now ready to define the function $\delta$ which is required by (7.21). Let $\tilde{T}^+ = \{i \in I^+ : \mathcal{D}_i^0 = \emptyset, \mathcal{D}_i^- \neq \emptyset, \text{ and } d_i = \inf_{h \in \mathcal{D}_i} d_i(h) > 0\}$. Then for each $h \in \mathcal{D}$ we define

$$
\delta(h) = \min\{\{d_i(h) : i \in I^\prime\} ; \{\delta_i, i \in \tilde{T}^+\} ; \frac{1}{2}\}.
$$

Then it is easy to check that $\delta$ is continuous piecewise affine function on $\mathcal{D}$ satisfying

$$
0 < \delta(h) \leq \frac{1}{2} \quad \text{for all } h \in \mathcal{D}
$$

and

$$
\lambda_i(h; H) \leq -\delta(h) \beta_\mathcal{Q}(H) \quad \text{for all } H \in \mathcal{D}_i^+ \\
\text{if } i \in I^\prime \text{ or if } i \in \tilde{T}^+ \text{ and } h \in \mathcal{D}_i^-.
$$

Lemma 7.26. Let $i \in I^+$. Suppose $h_0$ satisfies $\lambda_i(h_0; H) + \beta_\mathcal{Q}(H) > 0$ for some $H \in \mathcal{D}_i^\dagger$. Then either

(i) $i \in \tilde{T}^+$ and $\Re h_0 \in \mathcal{D}_i^\dagger$, or

(ii) there is a neighborhood $U(h_0)$ of $h_0$ so that $\Phi_{i,x}(f; h; v; m) = 0$ for all $(h, v, m) \in U(h_0) \times \mathcal{F} \times \mathcal{L}_Q$. 


Proof. Using (7.22), Re $h_0 \in \mathcal{S}^+ \cup \mathcal{S}^1$. By (7.19), $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m) = 0$ for all $(v, m) \in \mathcal{S} \times L_\mathcal{S}^1$ if Re $h \in \mathcal{S}^+$. Thus when Re $h_0 \in \mathcal{S}^+$, (ii) is satisfied with $U(h_0) = \{ h \colon$ Re $h \in \mathcal{S}^+ \}$. Thus we can assume that Re $h \notin \mathcal{S}^+$. Now if $i \in \mathcal{T}^+$, $\mathcal{S}^0 = \emptyset$ so Re $h \notin \mathcal{S}^+$ implies that Re $h \notin \mathcal{T}^+$ so that (i) is satisfied. Thus we can assume that $i \notin \mathcal{T}^+$, so that by (7.24), $\mathcal{S}^+ \cap \mathcal{S}^1 \neq \emptyset$.

Suppose first that $f \in \mathcal{S}(\mathcal{S}_0, L_\mathcal{S})$. Then by (7.18) $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m)$ is holomorphic on $\{ h \in \mathcal{S} \colon$ Re $h \in \mathcal{S}^+ \cup \mathcal{S}^1 \}$, as above. $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m) = 0$ for $h \in \mathcal{S}^+(\mathcal{S}) = \{ h \in \mathcal{S} :$ Re $h \in \mathcal{S}^+ \}$ is an open convex set, hence connected, and by hypothesis $\mathcal{S}^+(\mathcal{C}) \cap \mathcal{S}^1(\mathcal{C})$ is a non-empty open subset of $\mathcal{S}^1(\mathcal{C})$ on which $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m) = 0$. Thus $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m) = 0$ on all of $\mathcal{S}^1(\mathcal{C})$, so that (ii) is satisfied.

Now suppose $f \in \mathcal{S}(\mathcal{L}, L_\mathcal{L})$. Then $f$ is a finite linear combination of functions $f_i \in \mathcal{S}((\mathcal{S}_i, L_\mathcal{S}))$. Thus $\Phi_{h,\mathcal{S}}(f)$ is a finite linear combination of the corresponding $\Phi_{h,\mathcal{S}}(f_i)$. Now $\Phi_{h,\mathcal{S}}(f_i\colon h\colon v\colon m) = 0$ for $h \in \mathcal{S}^+(\mathcal{C}) \cup \mathcal{S}^1(\mathcal{C})$ implies that $\Phi_{h,\mathcal{S}}(f\colon h\colon v\colon m) = 0$ for $h \in \mathcal{S}^+ \cup \mathcal{S}^1$. Thus (ii) is satisfied.

Q.E.D.

Proof of Lemma 7.21. For $D \in \mathcal{S}$, $i \in I$, there is a finite subset $F_i$ of $\mathcal{S}$ and for each $D' \in F_i$ a polynomial $P_i(D')$ on $a_\mathcal{S}$ so that $D \cdot e^{v_i, \mathcal{A}(H)} = \sum_{D' \in F_i} P_i(D') \cdot D'$ for all $H \in a_\mathcal{S}$.

Pick $r_1 \geq 0$ and $c > 0$ so that $|P_i(D' \colon H)| \leq C(1 + \| H \|^T)$ for all $i \in I$. $D' \in F_i$. Let $F = \bigcup_{i \in I} F_i$.

Case I. Suppose $i \in I^0$. Then $|e^{v_i, \mathcal{A}(H)}| = 1$ for all $H \in a_\mathcal{S}$. and using (7.18),

$$
\| \Phi(f\colon h\colon v\colon D\colon b_1\colon m \exp TH\colon b_2) - \Phi_{h,\mathcal{S}}(f\colon h\colon v\colon D\colon b_1\colon m \exp TH\colon b_2) \|
\leq \sum_{D' \in F_i} \int_T^T \| \Psi_H, i(f\colon h\colon v\colon D \cdot e^{v_i, \mathcal{A}(H)} \colon b_1\colon m \exp(t + T) \cdot H\colon b_2) \| dt
\leq C \sum_{D' \in F_i} \int_T^T (1 + (t - T) \| H \|^T)
\times \| \Psi_H, i(f\colon h\colon v\colon D'\colon b_1\colon m \exp tH\colon b_2) \| dt
\leq Ce^{-T \delta(h) \| H \|^2} \int_T^T \| \Psi_H, i(f\colon h\colon v\colon D'\colon b_1\colon m \exp tH\colon b_2) \|
\times e^{T \delta(h) \| H \|^2}(1 + t \| H \|^T) dt
\text{since } 0 < \delta(h) \leq \frac{1}{2}.
$$
Case II. Suppose \( i \in I^+ \). Using (7.14),

\[
\| \Phi_i(f;h;v;D:b_1;m \exp TH; b_2) \|
\leq \| \Phi_i(f;h;v;D:e^{T\lambda_i(h,H)};b_1;m; b_2) \|
\]

\[
+ \int_0^T \| \Psi_{\nu,i}(f;h;v;D:e^{(T-t)\lambda_i(h,H)};b_1;m \exp tH; b_2) \| \, dt
\]

\[
\leq C(1 + T \| H \|)^{\gamma_1} e^{T\lambda_i(h,H)} \sum_{D' \in F_i} \| \Phi_i(f;h;v;D';b_1;m; b_2) \|
\]

\[
+ C \sum_{D' \in F_i} \int_0^T (1 + (T-t) \| H \|)^{\gamma_1} e^{(T-t)\lambda_i(h,H)}
\]

\[
\times \| \Psi_{\nu,i}(f;h;v;D';b_1;m \exp tH; b_2) \| \, dt
\]

\[
\leq C(1 + T \| H \|)^{\gamma_1} e^{-T\delta(h)\beta_H(H)} \sum_{D' \in F_i} \left\{ \| \Phi_i(f;h;v;D';b_1;m; b_2) \| \right. \\

\left. + \int_0^T e^{T\beta_H(H)/2} \| \Psi_{\nu,i}(f;h;v;D';b_1;m \exp tH; b_2) \| \, dt \right\}
\]

since \( 0 < \delta(h) \leq \frac{1}{2} \) and \( \lambda_i(h; H) \leq -\delta(h) \beta_H(H) \).

Case III. Suppose \( i \in I^+ \) and \( h \) satisfies \( \lambda_i(h; H) + \frac{1}{2} \beta_H(H) \leq 0 \). Then using (7.14) as above and \( \lambda_i(h; H) \leq -\beta_H(H)/2 \) we have

\[
\| \Phi_i(f;h;v D;b_1;m \exp TH; b_2) \|
\]

\[
\sum C(1 + T \| H \|)^{\gamma_1} e^{-\frac{T}{2} \beta_H(H)} \sum_{D' \in F_i} \| \Phi_i(f;h;v;D';b_1;m; b_2) \|
\]

\[
+ C \sum_{D' \in F_i} \int_0^T (1 + (T-t) \| H \|)^{\gamma_1} e^{-(T-t)\beta_H(H)/2}
\]

\[
\times \| \Psi_{\nu,i}(f;h;v;D';b_1;m \exp tH; b_2) \| \, dt
\]

\[
\leq C(1 + T \| H \|)^{\gamma_1} e^{-T\delta(h)\beta_H(H)} \sum_{D' \in F_i} \left\{ \| \Phi_i(f;h;v;D';b_1;m; b_2) \| \\

\left. + \int_0^T e^{\frac{T}{2} \beta_H(H)/2} \| \Psi_{\nu,i}(f;h;v;D';b_1;m \exp tH; b_2) \| \, dt \right\}
\]

since \( \delta(h) \leq \frac{1}{2} \).

Case IV. Suppose \( i \in I^+ \) and \( h \) satisfies \( \lambda_i(h; H) + \frac{1}{2} \beta_H(H) > 0 \). Then \( \lambda_i(h; H) + \beta_H(H) > 0 \) also so that (7.26) can be used. In case (ii) of (7.26) we have \( \Phi_{i,A}(f;h;v;D;m) = 0 \) for all \( (v, m) \in \mathcal{F} \times \mathcal{L}_Q \). Then using (7.18) as in Case I,
\[ \| \Phi_i(f; h: v; D: h_1; m \exp TH; h_2) \| \]
\[ \leq C \sum_{D'' \in F_i} \int_0^t (1 + t - T) \| H \| \alpha \exp tH \| H \| dt \]
\[ \times \| \Psi_{H', i}(f; h: v; D': h_1; m \exp tH; h_2) \| \]
\[ \leq C \sum_{D'' \in F_i} \int_0^t (1 + t \| H \| \alpha \exp tH \| H \| dt \]
\[ \times \| \Psi_{H', i}(f; h: v; D': h_1; m \exp tH; h_2) \| \]
\[ \leq Ce^{-\frac{1}{2}(h: H)} \sum_{D'' \in F_i} \int_0^t (1 + t \| H \| \alpha \exp tH \| H \| dt \]
\[ \times \| \Psi_{H', i}(f; h: v; D': h_1; m \exp tH; h_2) \| \]

since \( \lambda_i(h: H) > -\frac{1}{2} \beta_Q(H) \) and \( \delta(h) \leq \frac{1}{2} \). In case (i) of (7.26), we have \( \lambda_i(h: H) \leq -\delta(h) \beta_Q(H) \), and the same estimate as that used in Case II works. Q.E.D.

From now on, we define (or redefine) \( \Phi_{i, \gamma}(f) = 0 \) if \( i \in I \cup I' \).

**Lemma 7.27.** Let \( i \in I \). Then \( \Phi_{i, \gamma}(f; h: v; m \exp H) = e^{\gamma_i - \gamma_i(h: H)} \Phi_{i, \gamma}(f; h: v; m) \) for all \( H \in a_Q, m \in L_Q \).

**Proof.** This follows from Lemma 7.20, part (i). Q.E.D.

**Lemma 7.28.** Fix \( i \in I \) and suppose that \( \Phi_{i, \gamma}(f) \) is not identically zero on \( \Omega \times \mathcal{F} \times L_Q \). Then \( s_i^{-1}a_Q \subseteq a_H \).

**Proof.** We know that \( \Phi_{i, \gamma}(f) \) is \( C^\infty \) on \( \Omega \times \mathcal{F} \times L_Q \). Thus if it is not identically zero, it must be not identically zero on the dense set of points for which \( f \) factors through a quotient of Harish-Chandra class. Thus by [4, Lemma 6.3], \( s_i^{-1}a_Q \subseteq a_H \). Q.E.D.

Let \( W(a_H, a_Q) \) denote the set of linear maps \( s \) of \( a_Q \) into \( a_H \) such that there exists \( k \in K_F \) with \( s(H) = \text{Ad} k(H) \) for all \( H \in a_Q \). Note \( s \) determines the coset \( kK_Q \). If \( B \) is any subgroup of \( L_F \) normalized by \( K_Q \) we write \( B' = kBk^{-1} \) for \( s \) and \( k \) as above. In particular, \( Q' = M_Q' A_Q' N_Q' \) is a parabolic subgroup of \( G \) with \( A_Q' \subseteq A_H \). For \( \varphi \) a \( (\tau_1, \tau_2) \)-spherical function on \( L_Q \), we define \( \varphi^* \) on \( L_Q' \) by \( \varphi^*(kmk^{-1}) = \tau_i(k) \varphi(m) \tau_2(k^{-1}) \) for \( m \in L_Q \).

**Lemma 7.29.** Given \( s \in W(a_H, a_Q) \), there is a unique \( i = i(s) \in I \) such that \( s(H) = s_i^{-1}(H) \) for all \( H \in a_Q \).

**Proof.** See [4, Lemma 6.3]. Q.E.D.
Recall that \( \Phi(f) \) takes values in \( W = W \otimes \epsilon T \), where \( T \) has a distinguished basis \( e_1, \ldots, e_r \). Thus we can write, for each \( i \in I \), \( \Phi_{t,z}(f; h; v; m) = \sum_{j \in I} \varphi_{ij}(f; h; v; m) \otimes e_j \), where the \( \varphi_{ij}(f) \) take values in \( W \). Define
\[
\psi_j(h; v; m) = \pi_p(A_{h,v}) f(h; v; m); \\
\psi_{t,s}(h; v; m) = \varphi_{dts,1}(f; h; v; m), \quad s \in W(a_H, a_Q) \\
\psi_{t,s}(h; v; m) = \pi_q(s_{dts} A_{h,v})^{-1} \psi_{j,s}(h; v; m).
\]

Let \( \Omega \) be a compact subset of \( a_+ \). Choose \( \epsilon_0 > 0 \) so that \( \beta_0(H) \geq 2\epsilon_0 \) for all \( H \in \Omega \). Put \( \epsilon(h) = \delta(h) \epsilon_0 \), where \( \delta(h) \) is defined as in (7.25). For \( s \in W(a_H, a_Q) \), define \( s = \pm 1 \) by \( \pi_p(sA) = \pm \pi_p(A) \) for all \( A \in a_+^* \).

**Theorem 7.31.** Given \( b_1, b_2 \in \mathcal{U}(l_Q) \) and \( D \in \mathcal{P} \), there exist a finite subset \( F \subseteq \mathcal{D}_p \) and an \( r_1 > 0 \) so that for all \( r, t \geq 0 \) there is a \( c > 0 \) so that for all \( m \in L_+^Q, H \in \Omega, T \geq 0 \),
\[
\| d_q(m \exp TH) \psi_j(h; v; D; b_1; m \exp TH; b_2) \|
\leq C \sum_{s \in W(a_H, a_Q)} \det s_{\psi_{j,s}}(h; v; D; b_1; m \exp TH; b_2)
\leq \begin{cases} 
C^0 \xi_{F,t}(f) e^{-\epsilon(h)^T} \xi_{Q}(m)(1 + \delta(m) \exp TH)^{r + r_1} \\
\times (1 + d(h)^{-1})^t & \text{for } f \in I(D, L_p) \\
C S_{F,t}(f) e^{-\epsilon(h)^T} \xi_{Q}(m) \|h, v, m, TH\|^{r + r_1} e^{[h]| \sigma(m)} & \text{for } f \in I(D_C, L_p).
\end{cases}
\]

**Proof.** This follows from combining (7.21) with (7.12) and (7.13c).

**Q.E.D.**

**Lemma 7.32.** For \( s \in W(a_H, a_Q) \), \( \psi_{j,s} \) extends to a smooth function on
\[
\mathcal{D} \times \mathcal{F} \times L_Q \quad \text{if } f \in I(D, L_p); \\
\mathcal{D} \times \mathcal{F} \times L_Q \quad \text{if } f \in I(D_C, L_p).
\]

Further, given \( b_1, b_2 \in \mathcal{U}(l_Q), D \in \mathcal{P} \), there is a finite subset \( F \) of \( \mathcal{D}_p \) and an \( r_1 > 0 \) such that for all \( r, t \geq 0 \) there is a \( c > 0 \) with
\[
\| \psi_{j,s}(h; v; D; b_1; m; b_2) \|
\leq \begin{cases} 
C^0 \xi_{F,t}(f) \xi_{Q}(m)(1 + \sigma(m))^{r + r_1} (1 + d(h)^{-1})^t & \text{if } f \in I(D, L_p) \\
C S_{F,t}(f) \xi_{Q}(m) \|h, v, m\|^{r + r_1} e^{[h]| \sigma(m)} & \text{if } f \in I(D_C, L_p).
\end{cases}
\]

Finally, for all \( v \in L_Q \), \( \psi_{j,s}(h; v; m; v) = \mu_Q(v; s_{dts} A_{h,v}) \psi_{j,s}(h; v; m) \).
Proof. For \( i \in I \), \( \pi_{Q}(s_{i}A_{h,v}) = B_{i}(s_{i}A_{h,v}) = 1 \otimes \pi_{P_{Q}}(s_{i}A_{h,v}) B(s_{i}A_{h,v}) \). But using (7.3), \( \pi_{P_{Q}}(s_{i}A_{h,v}) B(s_{i}A_{h,v}) \) will have entries polynomial in \( h \) and \( v \). Thus it is clear that \( \psi_{L,t}^{\pm} \) extends to be smooth, and that the inequality can be proved the same way as in (ii) of (7.20). The final claim follows from (i) of (7.20). Q.E.D.

8. SCHWARTZ WAVE PACKETS

In this section we will prove that certain wave packets are Schwartz functions on \( G \). The main result is Theorem 8.4. In Section 9 we will see that this class of wave packets includes wave packets of Eisenstein integrals. We use the notation of Section 7. Thus \( H \) is a fixed \( \theta \)-stable Cartan subgroup of \( G \), and \( P \) is a parabolic subgroup of \( G \) with \( A_{P} \subseteq A_{H} \). Let \( \Phi_{R} \) be the set of real roots of \( (g, h) \), \( \Phi_{R}^{+} \) a choice of positive roots. For \( \Lambda \in \mathfrak{h}^{\ast}_{L} \), write \( \pi_{\Lambda}(\Lambda) = \prod \langle x, \Lambda \rangle, x \in \Phi_{R}^{+} \). Define \( \mathcal{F}' = \{ v \in \mathcal{F} = a_{h}^{\mathbb{R}} : \pi_{\Lambda}(v) \neq 0 \} \).

DEFINITION 8.1. We say \( f \in I'(\mathcal{G}, \mathcal{L}_{p}) \) if \( f \in I(\mathcal{G}, \mathcal{L}_{p}) \) and if for every parabolic subgroup \( \mathcal{Q}' = \mathcal{Q} \cap \mathcal{L}_{p} \) \( f \) factors through a group of Harish-Chandra class. Let \( \pi_{\Lambda}^{-1}(v) \psi_{L,s}^{\pm}(h; v; m) \) has a smooth extension from \( \mathcal{F}' \) to \( \mathcal{F} \) for all \( (h, m) \in \mathcal{G} \times L_{Q} \).

LEMMA 8.2. Suppose \( f \in I'(\mathcal{G}, \mathcal{L}_{p}) \). Then for all \( \Lambda, s \) as above, \( (\psi_{L,s}^{\pm})' \in I'(\mathcal{G}, \mathcal{L}_{Q}^{\ast}) \).

Proof. By (7.33), \( (\psi_{L,s}^{\pm})' \in I(\mathcal{G}, \mathcal{L}_{Q}^{\ast}) \). Let \( \mathcal{Q}' = \mathcal{Q} \cap L_{Q} \) be a parabolic subgroup of \( L_{Q} \) and let \( t \in W(\mathfrak{a}_{H}, \Lambda^{\prime}) \). Write \( g = (\psi_{L,s}^{\prime})' \). Then by [4, Lemma 7.4], there is \( t \in W(\mathfrak{a}_{H}, \Lambda^{\prime}) \) so that \( (\psi_{L,s}^{\prime})' = (\psi_{L,s}^{\prime})' \) for the dense set of points for which \( f \) factors through a group of Harish-Chandra class. But both sides are smooth, so that the equality persists for all values of \( (h, v) \). But \( f \in I'(\mathcal{G}, \mathcal{L}_{p}) \) implies that \( v \mapsto \pi_{\Lambda}^{-1}(v) \psi_{L,s}^{\prime}(h; v; m) \) has a smooth extension from \( \mathcal{F}' \) to \( \mathcal{F} \). Thus \( v \mapsto \pi_{\Lambda}^{-1}(v) \psi_{L,s}(h; v; m) \) does also. Q.E.D.
For \( \varphi \in I'(\mathcal{D}, G) \), define

\[
I_\varphi(x) = \int_{\mathcal{D} \times \mathcal{D}} \psi_\varphi(h: v: x) \pi_R^{-1}(v) \, dh \, dv
\]

\[
= \int_{\mathcal{D} \times \mathcal{D}} \varphi(h: v: x) \pi_G(A_{h,v}) \pi_R^{-1}(v) \, dh \, dv. \tag{8.3}
\]

**Theorem 8.4.** Let \( \varphi \in I'(\mathcal{D}, G) \). Then \( I_\varphi \in C(G, W) \). Given any \( r \geq 0 \) and \( g_1, g_2 \in \mathcal{U}(g) \), there is a finite subset \( F \) of \( \mathcal{L} \) so that given any \( r' \geq 0 \) there are \( C > 0 \) and \( t \geq 0 \) so that \( g_1 \| I_\varphi \|_{r, g_2} \leq C^0 S_{F, r', t}(\varphi) \).

**Proof.** Note that the first claim follows from the second. This is because, by Definition 7.7, given \( F \), we can choose \( r' \) so that \( S_{F, r', t}(\varphi) < 1 \) for all \( t \geq 0 \). We will reduce the second claim to a theorem which can be proved by induction using the machinery of Section 7.

Write \( \tilde{\varphi}(h: v: x) = \varphi(h: v: x) \pi_G(A_{h,v}) \pi_R^{-1}(v) \). Then for \( r \geq 0 \) and \( g_1, g_2 \in \mathcal{U}(g) \),

\[
g_1 \| I_\varphi \|_{r, g_2} = \sup_{x \in G} (1 + \sigma(a)) \varphi \Xi^{-1}(x) \int_{\mathcal{D} \times \mathcal{D}} \tilde{\varphi}(h: v: g_1: x; g_2) \, dh \, dv
\]

\[
\leq C \sup_{k_1, k_2 \in K} \sup_{a \in \text{cl}(A_{g_k})} (1 + \sigma(a)) \Xi^{-1}(a)(1 + \delta(k_1, k_2))
\]

\[
\times \left| \int_{\mathcal{D} \times \mathcal{D}} \tau_{1,h}(k_1) \tilde{\varphi}(h: v: k_1: g_1; a; k_2: g_2) \tau_{2,h}(k_2) \, dh \, dv \right|.
\]

Write \( k_1 = \sum_i f_i(k_1) g_i \), \( k_2 = \sum_i f_i''(k_2) g_i'' \), where both sums are finite, the \( g_i, g_i'' \in \mathcal{U}(g) \), and the \( f_i, f_i'' \in C^\infty_{\mathfrak{g}}(K/Z) \). Let \( C_1 = \sup_{i,j,k_1,k_2} |f_i(k_1)f_j''(k_2)| < \infty \). Write \( k_i = k_i'k_i'' \), \( i = 1, 2 \), where \( k_i' \in K, k_i'' \in V \). Then \( \tau_{i,h}(k_i) = \tau_i(h)(e^{h})(k_i'') \), and \( \delta(k_1, k_2) = \delta(k_1', k_2') \). Thus

\[
g_1 \| I_\varphi \|_{r, g_2} \leq C C_1 \sum_{i,j} \sup_{k_i \in V} (1 + \sigma(a)) \Xi^{-1}(a)(1 + \delta(k))
\]

\[
\times \left| \int_{\mathcal{D} \times \mathcal{D}} e^{h}(k) \tilde{\varphi}(h: v: g_i' \cdot a; g_i'') \, dh \, dv \right|.
\]

Thus the result follows from the special case \( L_p = G \) of Theorem 8.5 below.

Q.E.D.

**Theorem 8.5.** Let \( P \) be a parabolic subgroup of \( G \) with \( A_P \subseteq A_H \). Given
$l_1, l_2 \in \mathcal{H}(l_\rho), r \geq 0$, we can choose a finite subset $F \subseteq \mathcal{L}_\rho$ so that given any $r' \geq 0$ there are $C > 0$ and $t \geq 0$ so that
\[
\sup_{k \in F(a)\setminus (P)} (1 + \sigma(a)) \frac{\Xi^1_P(a)(1 + \delta(k))^{r'}}{a \in \text{cl}(A_0^+(P))} \\
\times \left[ \int_{\mathbb{R}_+} (e^h(k)) \psi_P(h; \nu; l_1; a; l_2) \pi^{-1}(v) \, dv \, dh \right] \\
\leq C^0 \mathcal{S}_{F, r', s} \varphi \quad \text{for all} \quad \varphi \in P'(\mathcal{D}, L_\rho).
\]

Here $A_0^+(P)$ is a positive chamber of $A_0$ with respect to $\Delta^+(L_\rho, A_0)$, a set of positive roots for $(L_\rho, A_0)$, and $\Xi_P$ is the spherical function $\Xi$ for $L_\rho$.

Before we start the proof of Theorem 8.5, we will need some lemmas.

Let $E = \mathbb{R}^n$. For a multi-index $x = (x_1, \ldots, x_n)$ put $D^x = (\partial/\partial x_1)^{x_1} \cdots (\partial/\partial x_n)^{x_n}$. Write $|x| = x_1 + \cdots + x_n$ and denote by $M$ the set of all multi-indices. For $W$ a finite dimensional vector space with norm $\| \cdot \|$, put $\mathcal{G}(E; W) = \{ f \in C^\infty(E; W) : s_{x_0}(f) = \sup f (1 + |x|)^r \|D^x f(x)\| < \infty \, \text{for all} \, r \geq 0, x \in M \}$.

**Lemma 8.6.** Fix $x \in M$ and let $F = \{ \beta \in M : |\beta| \leq |x| + N \}$. Then for every $r \geq 0$, we can choose $C_x \geq 1$ with the following property. Suppose $f \in \mathcal{G}(E; W)$ and $p \big| f$ is locally bounded on $E$. Then $f = pg$, where $g \in \mathcal{G}(E; W)$ and $s_{x_0}(g) \leq C \sup_{x \in F} \sup s_{x_0}(f)$.

**Q.E.D.**

**Proof.** See [4, Lemma 22.2].

**Lemma 8.7.** Let $\varepsilon(h)$ be a continuous, piecewise affine function on $\mathcal{D}$ so that $\varepsilon(h) > 0$ for all $h \in \mathcal{D}$. Then for all $r \geq 0$ there are a $C > 0$ and an $r_1 > 0$ so that
\[
\sup_{a \in \text{cl}(A_0^+(P)) \cap M_\rho} (1 + \sigma(a))^{r_1} \Xi_P^{\varepsilon(h)}(a) \\
\leq C' (1 + d(h))^{-r_1} \quad \text{for all} \quad h \in \mathcal{D}.
\]

**Proof.** There are constants $q \geq 0$ and $C \geq 0$ so that $\Xi_P(a) \leq C(1 + \sigma(a))^q e^{-\rho_P \log a}$ for all $a \in \text{cl}(A_0^+(P)) \cap M_\rho$. Let $z_1, \ldots, z_d$ be the set of simple roots of $(L_\rho, A_0)$ determining $A_0^+(P)$. Pick $H_1, \ldots, H_d \in (a_0 \cap m_\rho)$ so that $z_i(H_j) = \delta_{ij}, 1 \leq i, j \leq d$. Then $a_0^+(P) \cap m_\rho = \left[ \sum_{i=1}^d 1 \leq 0 \right.$ for $1 \leq i \leq d \}$ and $\rho_P = \sum_{i=1}^d n_i z_i$ for some $n_i > 0, 1 \leq i \leq d$. Thus $\sup_{a \in \text{cl}(A_0^+(P)) \cap M_\rho} (1 + \sigma(a))^{r_1} \Xi_P^{\varepsilon(h)}(a) \leq C \sum_{i=1}^d \left[ \sup_{a \geq 1} (1 + t_i)^{-r_1} e^{-n_i \varepsilon(h)} \right] \leq C \left[ \sup_{t \geq 1} (1 + t)^{-q} e^{-n \varepsilon(h)} \right]^d$, where $n = \min_{1 \leq i \leq d} n_i > 0$. Write $\mathcal{D} = \bigcup_{i=1}^k \mathcal{D}_i$, where $\varepsilon_i(h) = n \varepsilon(h)_{|\mathcal{D}_i}$ is affine for $1 \leq i \leq k, \varepsilon_i(h) > 0$ on $\mathcal{D}_i$. Then these are constants $C_i$ so that
\[ \sup_{t \geq 0} (1 + t)^{r+q} e^{-\omega_t(h)} \leq C_t (1 + 1/\epsilon_t(h))^{r+q} \text{ for all } h \in D_t. \] But since \( \{ h \in \mathfrak{u}^*: \epsilon_t(h) = 0 \} \) is outside of \( D \), there is a constant \( C_t \) so that for all \( h \in D_t, d(h) \leq C_t \epsilon_t(h) \). Q.E.D.

**Proof of Theorem 8.5.** Let \( \{ \alpha_1, \ldots, \alpha_d \} \) be the simple roots for the set of positive roots of \( (L_P, A_o) \) determining \( A_0^+(P) \). The proof will be by induction on \( d \), the number of simple roots.

**Case I.** Suppose \( d = 0 \). Then \( A_0 \) is central in \( L_P \) so that \( P = M_0 A_0 N_0 \) is a minimal parabolic. Since we assume that \( A_0 \subseteq A_H \subseteq A_0 \), this occurs only when \( A_0 = A_H \). Note then \( \Xi_p = 1, \text{cl}(A_0^+(P)) = A_0, \) and \( \psi \phi = \phi \).

Every element of \( \mathcal{H}(l_0) \) is a finite sum of terms of the form \( l_i = k_i u_i \), where \( k_i \in \mathcal{H}(m_0 \cap 1) \) and \( u_i \in S(\alpha_0) \). Since \( \phi \) is \( (K \cap M_0) \)-spherical, \( \phi(h: \cdot: k; u_1 \cdot a; u_2) = \phi(h: k_1; u_1; a; u_2) \phi(h: k_2; u_1; a; u_2) \), where \( \phi(h) \), \( i = 1, 2 \), depends polynomially on \( h \). Thus there are finitely many polynomials \( P_j(h) \) such that

\[
\| \int (e^h(k)) \phi(h: k_1; u_1; a; u_2) \pi^R_1(v) dh dv \| \leq \sum \| (e^h(k)) \phi(h: v; a; u_1 u_2) P_j(h) \pi^R_1(v) dh dv \|
\]

Further, since \( S(\alpha_0) \subseteq \mathcal{P} \) and \( \phi \) is an eigenfunction for \( \mathcal{L}_P, \phi(h: v; a) = e^{i\alpha_0(log a)} \phi(h; v: 1) \) for all \( a \in A_0 \). But \( a_0 = a_H \) so that \( A_0 A_0 \) (log \( a \)) = (iv(log \( a \))). Thus \( \phi(h: v; u_1 u_2) = u_1 u_2(iv) e^{iv(log a)} \phi(h; v: 1) \) for all \( a \in A_0 \). Note that \( u_1 u_2(iv) = Q(v) \) is a polynomial in \( v \). Now since \( \phi \in \mathcal{I}(\mathcal{L}_P, L_P), f_j(h: v) = P_j(h) Q(v) \phi(h; v: 1) \in C(i\nu^* \times \mathcal{F}, W) \) and \( \pi^R \) is a product of real linear forms on \( \mathcal{F} \) for which \( \pi^R f_j \) is locally bounded. Thus by (8.6), \( g_j = \pi^R f_j \in C(i\nu^* \times \mathcal{F}, W) \). Now \( \mathcal{F} \) is the dual of \( A_0 \) and \( \mathcal{D} \) is a subset of the dual of \( V \) so that \( g_j(h; v) \) is supported in \( \text{cl}(\mathcal{D} \times \mathcal{F}) \). Pick a polynomial \( R(h, v) \) such that \( C - \frac{1}{V} \| R(h, v) \|^{-1} dh dv < \infty \). Then by abelian Fourier analysis, there is \( D_j \in \mathcal{D} \) so that

\[
\sup_{k \in V} \sum_{a \in A_0} (1 + \sigma(a))^{r+q} (1 + \tilde{\sigma}(k))^{r+q} \| \int \int (e^h(k)) \phi(h: v; D_j) \pi^R_1(v) dh dv \|
\]

But, by (8.6) there is a finite subset \( F_j \) of \( \mathcal{D} \) so that \( \| g_j(h; v; D_j) \| \leq \sum_{D' \in F_j} \| f_j(h; v; D') \| \). Thus

\[
\sup_{k \in V} \sum_{a \in A_0} (1 + \sigma(a))^{r+q} (1 + \tilde{\sigma}(k))^{r+q} \| \int \int (e^h(k)) \phi(h: v; l_1; a; l_2) \pi^R_1(v) dh dv \|
\]

\[
\leq C \sum_{j} \sum_{D' \in F_j} \| f_j(h; v; D') \|
\]

\[
\leq C \sum_{D \in F} \| \phi(h; v; D: 1) \|
\]
for some finite subset $F$ of $\mathcal{P}$. But for any $r' \geq 0$, $\sup_{x \in F} \| \phi(h; v; D; 1) \| \leq oS_{D, r, 0}(\varphi)$. This finishes the case $d = 0$.

**Case II.** Pick $d \geq 1$ and assume inductively that the theorem is true when $d' < d$. For $1 \leq i \leq d$, let $a_i = \{ H \in a_i : \chi_i(H) = 0 \}$ and let $L_i$ be the centralizer of $a_i$ in $G$. Let $A_i = A(L_i, A_0) \cap A^+(L_i, A_0)$. Let $Q_i = L_iN_i$ be the maximal parabolic subgroup of $L_i$ for which $N_i$ corresponds to $A_i$. For $H \in a_i$, let $\rho(H) = \frac{1}{2} \sum m(\alpha) \alpha(H)$, $\alpha \in A_i^+$, $\rho(H) = \sum m(\alpha) \alpha(H)$, $\alpha \in A^+(L_i, A_0)$, $\rho(H) = \rho(H) - \rho_i(H)$. For $h > 0$, $1 \leq i \leq d$, let $A_i^+(h) = \{ a \in A_i^+(P) : \chi_i(\log(a)) > h \rho(H) \}$. Fix $h$ small enough that $A_i^+(h) \subseteq \bigcup_{1 \leq i \leq d} A_i^+(h)$.

Let $l_i, l_2 \in \mathcal{U}(l_p), r \geq 0$. Since $cl(A_i^+(P)) \subseteq \bigcup_{1 \leq i \leq d} cl(A_i^+(h))$, we must show for each $1 \leq i \leq d$ that there is $F_i \subseteq L_i$ so that given any $r' \geq 0$ there are $C > 0$ and $t \geq 0$ so that

$$
\sup_{a \in cl(A_i^+(h))} (1 + \sigma(a))^r \Xi_p^{-1}(a)(1 + \delta(k))^r $$

$$\times \left\| \int_{x \in F_i} \left( e^h(x) \psi_h(h; v; l_1; a; l_2) \pi_R^{-1}(v) dh dv \right) \right\| $$

$$\leq C oS_{F_i, r, i}(\psi).$$

Fix an $i$, and drop it from the notation so that $Q = LN = Q_i$.

Write $\mathcal{U}(l_p) = \mathcal{U}(m_p) S(a_p)$, where $S(a_p) \subseteq L_p$. Then if $l_i = m_i u_i, m_i \in \mathcal{U}(m_p), u_i \in S(a_p), i = 1, 2$, and if $a = a_1 a_2$, where $a_i \in M_p \cap A_0, a_2 \in A_p$ then, as in the case $d = 0$, recalling that $A_p \subseteq A_H$, $\psi(h; v; m_i u_1, a_1 a_2; m_2 u_2) = Q(v) e^{\epsilon(\log_{x_i})} \psi_h(h; v; m_1 a_1; m_2 a_2)$, where $u_1 u_2(\epsilon) = Q(v)$ is a polynomial in $v$.

Write $\mathcal{U}(m_p) = \mathcal{U}(l_p) \mathcal{U}(l \cap m_p) \mathcal{U}(n) = \mathcal{U}(\theta(n)) \mathcal{U}(l \cap m_p) \mathcal{U}(l_p)$. There exist $b_1 \in \mathcal{U}(l_p), b_2 \in \mathcal{U}(l \cap m_p), m_1 \in \mathcal{U}(l \cap m_p)$ and $m'_1 \in \mathcal{U}(m_p)n$ such that $m_i = b_i + m'_i, i = 1, 2$. Thus $\psi_h(h; v; m_i u_1, a_1 a_2; m_2 u_2) = \psi_h(h; v; m'_1 u_1, a_1 a_2; m_2 u_2) + \psi_h(h; v; b_i u_1, a_1 a_2; m'_2 u_2) + \psi_h(h; v; b_2 u_1; a_1 a_2; b_2 u_2)$. We will estimate each of these terms separately. First

$$
\sup_{k \in F} (1 + \sigma(a))^r \Xi_p^{-1}(a)(1 + \delta(k))^r $$

$$\times \left\| \int_{x \in F} \left( e^h(x) \psi_h(h; v; m_i u_1, a; m_2 u_2) \pi_R^{-1}(v) dh dv \right) \right\| $$


As in the case \( d = 0 \), we can use (8.6) and abelian Fourier analysis to find a finite subset \( F_1 \) of \( \mathcal{P} \) and \( c_1 > 0 \) so that this last expression is bounded by

\[
C_1 \sup_{a \in \text{cl}(A_1^+(b)) \cap M_P} (1 + \sigma(a_1))' \Xi_p^{-1}(a_1) \times \sum_{D \in F_1} \sup_{\mathcal{G} \times \mathcal{F}} \|\psi_{\phi}(h : v; m_1'; a_1; m_2)\|.
\]

But now using (7.11), there are a finite subset \( F \subseteq \mathcal{P} \) and \( r_0 > 0 \) so that for all \( r' \geq 0 \) this is bounded by

\[
0^{S_{F, r, r_0}}(f) \sup_{a \in \text{cl}(A_1^+(b)) \cap M_P} (1 + \sigma(a_1))' \Xi_p(a_1) e^{-\beta_{p}(\log a_1)} (1 + \sigma(a_1))^{r' + r_0} d_Q^{-1}(a_1).
\]

But there are constants \( D \geq 0 \) and \( q \geq 0 \) so that \( \Xi_p(a_1) \leq De^{-\beta_{p}(\log a_1)} (1 + \sigma(a_1))^q \). Further, \( d_Q^{-1}(a_1) = e^{-\beta_{q}(\log a_1)} \) and \( e^{-\beta_{p}(\log a_1)} \leq e^{-\beta_{q}(\log a_1)} \) since \( a_1 \in A_1^+(b) \). Thus \( \Xi_p(a_1) d_Q^{-1}(a_1) e^{-\beta_{q}(\log a_1)} \leq D e^{-\beta_{q}(\log a_1)} (1 + \sigma(a_1))^q \). Thus

\[
\sup_{a \in \text{cl}(A_1^+(b)) \cap M_P} (1 + \sigma(a_1))' e^{r' + r_0} d_Q^{-1}(a_1) \leq D \Xi_p(a_1)^{r' + r_0 + q} = C_r < \infty,
\]

since \( b > 0 \).

Thus the term involving \( \psi_{\phi}(h : v; m_1'; a_1; m_2 u_2) \) can be bounded by \( C_{r, 0^{S_{F, r, r_0}}}(f) \) for any \( r' \geq 0 \). The same argument also works for \( \psi_{\phi}(h : v; b_1 u_1; a_1; m_2 u_2) \). It remains to estimate the terms with \( \psi_{\phi}(h ; v; b_1 u_1; a_1; b_2 u_2) \), where \( b_1 \in \mathcal{U}(1), b_2 \in \mathcal{U}(m_1 \cap 1) \). Write \( b_1 = \kappa_1 \beta_1', b_2 = \beta_2', \) where \( \kappa_1 \in \mathcal{U}(1), \beta_i \in \mathcal{U}(m_1 \cap 1), i = 1, 2 \). Then \( \beta_i = d_Q^{-1} \beta_i' d_Q \), as in Section 7. As in the \( d = 0 \) case, since \( \phi \) is \( K_r \)-spherical, there are
polynomials \( P_j(h) \) so that \( \| \int_{\mathcal{J} \times \mathcal{F}} (e^h)(k) \pi_R^{-1}(v) \psi_{\phi}(h: v; \beta_1' u_1; a; u_2' \beta_2') \) \( dh \) \( dv \) \( \leq \sum_j \| \int_{\mathcal{J} \times \mathcal{F}} (e^h)(k) \pi_R^{-1}(v) P_j(h) \psi_{\phi}(h: v; \beta_1' u_1; a; u_2' \beta_2') \) \( dh \) \( dv \). Thus it suffices to estimate terms of the form

\[
(1 + \sigma(a))' \mathcal{E}_\rho^{-1}(a)(1 + \delta(k))'
\]

where \( P(h) \) is a polynomial. We will split this up further. Write

\[
d_{\phi}(h: v; \beta_1' u_1; a; \beta_2' u_2) = \psi_{\phi}(h: v; \beta_1' u_1; a; \beta_2' u_2) - \sum_{z \in u \langle a_{1z}, a_{2z} \rangle} \det s_{\phi}^{-1}(a) \times \psi_{\phi}(h: v; \beta_1' u_1; a; \beta_2' u_2).
\]

Then, as before,

\[
\text{sup}_{k \in \mathcal{J}} (1 + \sigma(a))' \mathcal{E}_\rho^{-1}(a)(1 + \delta(k))'
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

\[
C_2 \text{sup}_{k \in \mathcal{J}} \text{sup}_{a_2 \in A_{\rho} \ q \in \mathcal{J} \cap M_p} (1 + \sigma(a_2))' (1 + \delta(k))' \mathcal{E}_\rho^{-1}(a_1)
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

\[
C_2 \text{sup}_{a_1 \in \mathcal{J} \cap M_p} (1 + \sigma(a_1))' \mathcal{E}_\rho^{-1}(a_1)
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

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\[
C_2 \text{sup}_{a_1 \in \mathcal{J} \cap M_p} (1 + \sigma(a_1))' \mathcal{E}_\rho^{-1}(a_1)
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

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C_2 \text{sup}_{a_1 \in \mathcal{J} \cap M_p} (1 + \sigma(a_1))' \mathcal{E}_\rho^{-1}(a_1)
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

\[
C_2 \text{sup}_{a_1 \in \mathcal{J} \cap M_p} (1 + \sigma(a_1))' \mathcal{E}_\rho^{-1}(a_1)
\]

where \( Q(v) = (u_1 u_2) \). Again, since \( \phi \in \mathcal{I}(\mathcal{L}, L_p) \), we can use (8.6) and abelian Fourier analysis to find a subset \( F_2 \) of \( \mathcal{J} \) and \( C_2 > 0 \) so that this expression is bounded by

\[
C_2 \text{sup}_{a_1 \in \mathcal{J} \cap M_p} (1 + \sigma(a_1))' \mathcal{E}_\rho^{-1}(a_1)
\]
$\varepsilon(h) = \frac{1}{2} \delta(h)$. Then using Theorem 7.31 there exists a finite subset $F$ of $\mathcal{L}_P$ and an $r_1 > 0$ so that for all $r' \geq 0$, $t \geq 0$, there is a $C' > 0$ so that this is bounded by

$$C_2 C' \sup_{a_1 \in \text{cl}(A_{r_1}^+(h)) \cap M_P, h \in \mathcal{L}_P} (1 + \sigma(a_1))^{r'} \times \mathcal{E}_P^{-1}((a_1)^0) S_{F,r'_1}(f) e^{\varepsilon(h) T} \times \mathcal{E}_Q((a_1)(1 + \sigma(a_1))^{r'_1 + r_1}(1 + d(h)^{-1})^{-1}.$$ 

But as before, for $a_1 \in \text{cl}(A_{r_1}^+(h)) \cap M_P$, $\mathcal{E}_P^{-1}(a_1) d_{O_1}^{-1}(a_1) \mathcal{E}_Q(a_1) e^{-\varepsilon(h) T}$ is bounded by some constants $D > 0, q > 0$. But by (8.7), there exists $C_r$ and $t$ such that $\sup_{a_1 \in \text{cl}(A_{r_1}^+(h)) \cap M_P} (1 + \sigma(a_1))^{r'_1 + r_1 + q} \mathcal{E}_P((a_1)^{e(h)}) \leq C_r (1 + d(h)^{-1})^{r'}$. Thus for any $r' \geq 0$ we can find $C$ and $t$ so that we can bound our expression involving $d_{F, r'_1}(f)$.

Finally, for each $s \in \mathcal{W}(a_H, a_G)$, we look at

$$\sup_{a_1 \in \text{cl}(A_{r_1}^+(h))} (1 + \sigma(a))^{r'} \mathcal{E}_P^{-1}(a)(1 + \hat{\sigma}(k))^{r'} d_{O_1}^{-1}(a) \times \left\| \int_{\mathcal{P} \times \mathcal{P}} (e^h)(k) \psi_{\psi_{1 \psi}(h:v:u_1; a; \beta_2 u_2)} P(h) \pi_R^{-1}(v) dh dv \right\|.$$ 

Now $\mathcal{E}_P^{-1}(a) d_{O_1}^{-1}(a) \leq D \mathcal{E}_P^{-1}(a)(1 + \sigma(a))^q$, $\text{cl}(A_{r_1}^+(h)) \subseteq \text{cl}(A_{r_1}^+(Q))$, and $\psi_{\psi_{1 \psi}(h:v:u_1; a; \beta_2 u_2)} = \psi_{\psi_{1 \psi}(h:v:u_1; a; \beta_2 u_2)} \pi_Q(s(h)) \pi_R(u_1)$, so we can bound this by

$$D \sup_{a_1 \in \text{cl}(A_{r_1}^+(h))} (1 + \sigma(a))^{r'_1 + q} \mathcal{E}_P^{-1}(a)(1 + \hat{\sigma}(k))^{r'} \times \left\| \int_{\mathcal{P} \times \mathcal{P}} (e^h)(k) \pi_R^{-1}(v) P(h) \psi_{\psi_{1 \psi}(h:v:u_1; a; \beta_2 u_2)}^{\psi_{1 \psi}} dh dv \right\|,$$

where $Q'$ is a parabolic subgroup of $G$ with $A_{Q'} \subseteq A_H, d_1, d_2 \in \Psi(1_{Q'})$, and $P \cdot (\psi_{1 \psi}(h:v:u_1; a; \beta_2 u_2))^{\psi_{1 \psi}} \subseteq \mathcal{P}(\mathcal{P}, L_{Q'})$. Thus by the induction hypothesis, there is $F' \subseteq \mathcal{L}_Q$, such that for any $r', r_1 > 0$ there are $C'$ and $t$ so that the above is bounded by $C'' S_{F', r'_1} (P(\psi_{1 \psi}(h:v))^{\psi_{1 \psi}}) = C'' S_{F', r'_1} (P(\psi_{1 \psi}(h:v))^{\psi_{1 \psi}})$, where $F'' = \{ DP : D \in F' \}$. But now using Theorem 7.33, there is a finite subset $F$ of $\mathcal{L}_P$ and an $r_1 > 0$ and $C > 0$ so that $S_{F', r'_1} (P(\psi_{1 \psi}(h:v))^{\psi_{1 \psi}}) \leq C'' S_{F', r'_1} (f)$. \(Q.E.D.\)
9. Wave Packets of Eisenstein Integrals

In this section we relate the abstract families defined in Section 7 and used to form Schwartz wave packets in Section 8 to the holomorphic families of Eisenstein integrals defined in Section 6. The first main result is Theorem 9.10 which gives the a priori estimates which are needed to show that Eisenstein integrals are functions of type $II(\mathcal{Z}, G)$. The second main result is Theorem 9.14 which characterizes those wave packets of Eisenstein integrals which are Schwartz functions on $G$.

Finally, we show in Theorem 9.18 that the Schwartz wave packets of Eisenstein integrals corresponding to the $H$-series of representations are in $\mathcal{C}_H(G)$, the closed subspace of $\mathcal{C}(G)$ consisting of functions whose Plancherel formula expansions involve only the $H$-series of tempered representations.

Let $F: \mathfrak{u}_t^* \times G \rightarrow W = W(\tau_1; \tau_2)$ be a holomorphic family of $K_M$-spherical functions on $G$ coming from a holomorphic family of matrix coefficients on $M_1$ and a $K_{M,1}$-endomorphism of $W(\tau_1; \tau_2)$. Then, as in (6.6), for $(h, v, x) \in \mathfrak{u}_t^* \times \mathfrak{a}_t^* \times G$, we define the Eisenstein integral

$$E(P; F; h; v; x) = \int_{K, Z} F(h; Xk) \tau_{2, h}(k^{-1}) e^{i h \cdot \mu^H p(x) k} d(kZ). \quad (9.1)$$

**Lemma 9.2.** Let $\Phi: \mathfrak{u}_t^* \times G \rightarrow W$ be any smooth family of $\tau$-spherical functions. Then given $D_1, D_2 \in \mathcal{U}(\mathfrak{g})$, there are a finite subset $S$ of $\mathcal{U}(\mathfrak{g})$ and an $r \geq 0$ so that $\| \Phi(h; D_1; x; D_2) \| \leq (1 + |h|)^r \sum_{D \in S} \| \Phi(h; D; x) \|$ for all $(h, x) \in \mathfrak{u}_t^* \times G$.

**Proof.** For fixed $h \in \mathfrak{u}_t^*$, a similar estimate is proved in [2, Lemma 17]. The constants involved are independent of $h$ except for terms of the form $\|d\tau_{1,h}(k)\|$ for some $\kappa \in \mathcal{U}(1)$, depending on $D_1, D_2$. These grow polynomially in $h$.

**Q.E.D.**

**Corollary 9.3.** For any $D_1, D_2 \in \mathcal{U}(\mathfrak{g})$, there are a finite subset $S$ of $\mathcal{U}(\mathfrak{g})$ and an $r \geq 0$ so that

$$\| E(P; F; h; v; D_1; x; D_2) \| \leq (1 + |h|)^r \sum_{D \in S} \| E(P; F; h; v; D; x) \|.$$

Write $F_\kappa(h; x) = F(h; x) e^{i \kappa \mu^H p(x)}$. Then, since $\|w \tau_{2, h}(k)\| = |e^h(k)| \|w\|$ for all $k \in K$, $w \in W$, we have

$$\| E(P; F; h; v; D; x) \| \leq \int_{K, Z} \| F_\kappa(h; D; x) \| |e^h(k^{-1})| d(kZ) \quad (9.4)$$
for all \((h, v, x) \in v_c^* \times a_c^* \times G\). For \(x \in G\), write \(x = k(x) m(x) \exp H_p(x) n(x)\) as in (6.5a). For \(v \in a_c^*\), write \(v = v_R + iv_J\), where \(v_R, v_J \in a^*\).

**Lemma 9.5.** For any \(D \in \mathcal{U}(g)\), there are a finite subset \(S \subset \mathcal{U}(m)\) and constants \(C > 0, \ r \geq 0\) so that \(\|F_v(h; D; x)\| \leq C(1 + |h|)^r e^{|b(k(x))|} (1 + |v|)^r e^{|v| H_p(x)} e^{-\mu H_p(x)} \sum_x \|F_m(h; v; m(x))\|\) for all \((h, v, x) \in v_c^* \times a_c^* \times G\).

**Proof.** Let \(k \in K, \ m \in M^+, \ a \in A, \ n \in N\). Then using (6.5c),

\[
\|F_v(h; D; km)\| = \|\mathcal{H}_1(k) F_v(h; k^{-1} D; ma)\| \leq C(e^{|b(k)|} \sum_k \|F_v(h; D; ma)\|,
\]

where we express \(k^{-1} D = \sum_i a_i(k) D_i\) and let \(C = \sup_k \|a_i(k)\|\). Now fix \(i\) and write \(D_i = kub\), where \(k \in \mathcal{U}(1), \ u \in \mathcal{U}(m + a), \) and \(h \in \mathcal{U}(n)\). Then

\[
\|F_v(h; km; ma)\| = \|d_1(k) F_v(h; \mu; ma; u^{-1})\| \leq C(1 + |h|)^r \|F_v(h; \mu; ma; u^{-1})\|, \text{ since } d_1(k) \text{ is a polynomial in } h. \]

But \(F_v\) is right \(N\)-invariant and \(m^{-1} u^{-1} h \in \mathcal{U}(n)\), so \(F_v(h; \mu; ma; u^{-1}) = \mathcal{H}_1(k) F_v(h; \mu; ma)\), where \(\mathcal{H}_1\) is the constant term of \(b\). Finally, we write \(\mu \in \mathcal{U}(m + a)\) as \(\mu = \nu v', \) where \(v \in \mathcal{U}(m)\) and \(v' \in \mathcal{U}(a)\). Then differentiating with respect to \(v'\) gives a polynomial in \(v\). Thus \(\|F_v(h; \nu v'; ma)\| \leq C(1 + |v|)^r \|F_v(h; v; ma)\| = C(1 + |v|)^r |e^{(v - \rho H_p(x))}| \|F_v(h; m; v)\|\).

**Q.E.D.**

**Lemma 9.6.** There is a constant \(C_0\) so that

\[
\|E(P; F; h; v; D; x)\| \leq C(1 + |h|)^r (1 + |v|)^r \sum_k \|e^{|b(k(x))|} \Xi_m(m(x))^{-1} \sum_{i \in S} \|F(h; v; m(x))\|\}
\]

for all \((h, v, x) \in v_c^* \times a_c^* \times G\).

**Proof.** Combine (9.4) and (9.5) to obtain

\[
\|E(P; F; h; v; D; x)\| \leq C(1 + |h|)^r (1 + |v|)^r \sum_{i \in S} \int_{K/Z} |e^{|b(k)|} |e^{|b(k(x))|} |e^{-|v| H_p(x)} |e^{-\mu H_p(x)} \times \|F(h; v; m(x))\| d(kZ).
\]

But by [10, p. 275], there is \(C_0 > 0\) so that \(|v| H_p(x) \leq C_0 |v| \sigma(x)\) for all \(k \in K\). Also, by [10, p. 275], \(\int_{K/Z} e^{-\mu H_p(x)} \Xi_m(m(x)) d(kZ) = \Xi(x)\).

**Q.E.D.**

**Lemma 9.7.** There are constants \(C\) and \(c\) so that \(\sup_{k \in K} |e^{|b(k(x))|}| \leq C e^{(|b| \sigma(\sigma(x)) + c)}\).
Proof. Write \( x = k_1 ak_2 \) where \( k_1, k_2 \in K, a \in A_0 \). Then
\[
\sup_{k \in K} |e^h(k(ak^{-1}))| = |e^h(k_1 k_2)| \sup_{k \in K} |e^h(k(ak^{-1}))|.
\]
Now \( |e^h(k_1 k_2)| \leq e^{h|\sigma_1(x)|} \), so it suffices to show \( \sup_{k \in K} |e^h(k(ak^{-1}))| < \infty \). But
\[
|e^h(k(ak^{-1}))| \leq e^{h|\sigma_1(x)(k(ak^{-1}))|}
\]
and by (2.11), there is a constant \( c \) so that
\[
\sup_{k,a} |e^h(k(ak^{-1}))| \leq c.
\]
Q.E.D.

Let \( \mathcal{D} \) be the chamber in \( iv^* \) for which \( \Gamma_M(h) \) is a spherical function of matrix coefficients and \( \mathcal{D} = \mathcal{D} + i\omega \), where \( \omega \) is a relatively compact neighborhood of 0 in \( iv^* \). Let \( d(h) \) denote the distance from \( h_R \) to the boundary of \( \mathcal{D} \), where \( h = h_R + ih_j, h_R, h_j \in iv^* \).

Lemma 9.8. For any \( v \in \mathcal{U}(m) \), there are constants \( C > 0, r, t \geq 0 \) so that
\[
\sup_k |\Xi_{iv}(m(ak))| \leq C(1 + |h|)^r (1 + d(h)^{-1})^t
\]
for all \( (h, x) \in \mathcal{D}_c \times G \).

Proof. Write \( M^+ = K^+ c(A_{0,M}^+)K^+ \). When we decompose \( xk = k(ak) \) \( m(ak) \exp H(ak) m(ak) \), we can assume \( m(ak) \in c(A_{0,M}^+)K^+ \) as \( K^+ \subset K \). Also, since \( Z \) is central in \( M \), elements of \( Z \) can be commuted past \( \exp H(ak) \) into \( K^+ \), so we can assume that \( \sigma_1(m(ak)) \) is bounded.

Thus we can assume that \( m(ak) = ak_1 \), where \( a \in c(A_{0,M}^+) \) and \( \sigma(k_1) \) is bounded. Now \( \Xi_{iv}(m(ak)) = \Xi_{iv}(a) \) and
\[
|F(h, x; m(ak))| \leq C(1 + |h|^r (1 + d(h)^{-1})^t\Xi(x) e^{i\nu x} (1 + |h|^r (1 + d(h)^{-1})^t\Xi(x) e^{i\nu x} \]
for all \( (h, v, x) \in \mathcal{D}_c \times a^\times \times G \).

Lemma 9.9. Let \( D \in \mathcal{U}(g) \). Then there are constants \( C, c_0, r, t \geq 0 \) so that
\[
|E(P : F : h, v, D, x)| \leq C(1 + |h|^r (1 + |v|^r (1 + d(h)^{-1})^t\Xi(x) e^{i\nu x} |\sigma(x)| e^{ih|\sigma(x)|} \]
for all \( (h, v, x) \in \mathcal{D}_c \times a^\times \times G \).

Proof. This follows from combining (9.6), (9.7), and (9.8) since \( |h| \) is bounded in \( \mathcal{D}_c \).

Theorem 9.10. Let \( g_1, g_2 \in \mathcal{U}(g) \) and \( D \in \mathcal{P} \). Then there are constants \( C, c_0 \) so that
\[
|E(P : F : h, v, D, g_1, x; g_2)| \leq C(1 + |h|^r (1 + |v|^r (1 + d(h)^{-1})^t\Xi(x) e^{i\nu x} |\sigma(x)| e^{ih|\sigma(x)|} \]
for all \( (h, v, x) \in \mathcal{D}_c \times a^\times \times G \).

Proof. We know from Theorem 6.7 that \( E(P : F : h, v, x) \) is holomorphic as a function of \( (h, v) \in \mathcal{D}_c \times a^\times \). We use the same method to estimate derivatives in \( (h, v) \) that Harish-Chandra used to estimate derivatives in \( \nu \) of the ordinary Eisenstein integral. Namely, if \( f \) is a holomorphic function in a neighborhood of \( |z - w| < C \), then \( |(d^n/dz^n)f(z)| \leq \).
(n! / C^n) sup_{\omega \in \mathbb{C}} |f(w)|. For derivatives in \nu, we use radius \((1 + \sigma(x))^1\) and for derivatives in \h we use radius \(C < \min\{1 + \sigma_1(x), \frac{1}{2} d(h)\}\). Combined, this gives us the polynomial growth in \(\sigma(x) = \sigma(x) + \sigma_1(x)\) in addition to the terms needed to estimate \(E(P:F:h:\nu; g_1; x; g_2)\) coming from (9.3) and (9.9).

**Q.E.D.**

**Corollary 9.11.** \(E(P, F) \in \mathcal{I}(\mathcal{O}, G)\).

**Proof.** Combine (6.7), (6.8), and (9.10). **Q.E.D.**

We are now ready to discuss wave packets of Eisenstein integrals. Since \(E(P, F) \in \mathcal{I}(\mathcal{O}, G)\), \(E(P: F) \cdot \mathcal{z} \in \mathcal{I}(\mathcal{O}, G)\) for any \(\mathcal{z} \in \mathcal{G}(\mathcal{O} \times \mathcal{F})\). (See the remark following Definition 7.7.) However, there is no reason to expect that \(E(P, F) \cdot \mathcal{z} \in \mathcal{I}(\mathcal{O}, G)\). This, as in the finite center case, is because of "poles of the c-function." In order to eliminate these poles, we bring in the Plancherel measure. Recall that the Eisenstein integral is a spherical function of matrix coefficients for a series of induced representations. Thus for each \((h, \nu) \in (\mathcal{O} \times \mathcal{F})\), \(E(P:F:h: \nu)\) is associated to a representation \(\pi_{h, \nu}\), defined as in (3.8). By the results of [6], the Plancherel measure for this representation is, up to a constant factor independent of \((h, \nu)\), given by

\[
m(h: \nu) = \pi_G(A_{h, \nu}) \pi_R^{1/2}(\nu) \prod_{a \in \Phi^+_R} m_a(h: \nu),
\]

where \(m_a(h: \nu) = \nu_2 \sinh \pi \nu_2 / (\cosh \pi \nu_2 - \cos \pi h_2), \nu_2 = 2 \langle \alpha, \alpha \rangle / \langle \alpha, \alpha \rangle\), and \(h \to h_*\), is an affine linear functional on \(\mathcal{O}\) for which we do not need the exact formula (see [6]). Write \(m_R(h: \nu) = \prod_{a \in \Phi^+_R} m_a(h: \nu)\).

Multiplying \(E(P:F) \cdot \mathcal{z}\) by \(m_R\) will eliminate the problem of poles of the c-function. However, in our situation, it introduces new difficulties because \(m_a(h: \nu)\) is not jointly continuous at points \((h, \nu)\), where \(h_\alpha \in \mathbb{Z}\) and \(\nu_\alpha = 0\) for some \(a \in \Phi^+_R\) with \(h_2\) not constant. (These are points corresponding to principal series which are reducible, or which fail to be reducible because certain limits of discrete series are zero.) Thus we will need to assume that \(a\) is chosen so that \(E(P:F) \cdot \mathcal{z} \cdot m_R\) is jointly smooth. This will certainly be true if \(a \cdot m_R\) is jointly smooth. We will need the following lemma.

Let \(E = \mathbb{R}^{n+2}, n \geq 0\), and denote the coordinates by \((x, y, z)\), where \(x, y \in \mathbb{R}, z \in \mathbb{R}^{n}\). Define \(\mathcal{C}(E, W)\) as in (8.6).

**Lemma 9.13.** Suppose \(f \in \mathcal{C}(E, W)\) satisfies \(g(x, y, z) = x \sinh \pi x / (\cosh \pi x - \cos \pi y) f(x, y, z)\) is jointly smooth on \(E\). Then \(g \in \mathcal{C}(E, W)\), and given \(a \in M, r \geq 0\), there exist constants \(C > 0, \tau \geq 0\) and a finite subset \(F_{x, y, z}\) of \(M\) so that \(s_{x, y, z}(g) \leq C^r \sum_{p \in F_{x, y, z}} g_p, f\).

**Proof.** Fix \(r \geq 0\) and \(x \in M\). Then \(s_{x, y, z}(g) = \sup_{m \in \mathbb{Z}} \sup_{1 < x < t} \sup_{1 < z < w} (1 + ||(x, y + 2m, x)||) Y |D^2g(x, y + 2m, z)|\). For \(m \in \mathbb{Z}, \) write \(g_m(x, y, z) = \)
where \( h(x, y) = x \sinh \pi x (x^2 + y^2) / (\cosh \pi x - \cos \pi y) \) is jointly smooth on \( \mathbb{R} \times [-1, 1] \) and satisfies the condition that for all \( \beta \in M \) there exist constants \( C_\mu \geq 0, t_\mu \geq 0 \) so that \( |D^\mu h(x, y)| \leq C_\mu (1 + |x|)^{t_\mu} \) for all \( x \in \mathbb{R}, -1 \leq y \leq 1 \). Then \( g(x, y + 2m, z) = 1/(x^2 + y^2) g_m(x, y, z) \). Write \( k = |\alpha_1| + |\alpha_2| \) for the total degree of \( D^x \) in \( x \) and \( y \).

Then there are finite subsets \( F_1, F_2 \) of \( M \), for each \( \beta \in F_1 \) a polynomial \( P_\beta(x, y) \), and for each \( \beta \in F_2 \) a constant \( C_\beta \) so that

\[
D^2 g(x, y + 2m, z) = \begin{cases} 
\sum_{\beta \in F_1} P_\beta(x, y) D^\mu g_m(x, y, z)(x^2 + y^2)^k & \text{if } (x, y) \neq (0, 0), \\
\sum_{\beta \in F_2} C_\beta D^\mu g_m(0, 0, z) & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Further, by Taylor's theorem there is a finite subset \( F_3 \) of \( M \) and for each \( \beta \in F_3 \) a polynomial \( P_\beta'(x, y) \) so that for all \( m \in \mathbb{Z} \),

\[
|D^x g(x, y + 2m, z) - D^x g(0, 2m, z)| \leq \sqrt{x^2 + y^2} \sup_{(x_1, y_1) \in \mathbb{R}^2} \sum_{\beta \in F_3} |P_\beta'(x_1, y_1) D^\mu g_m(x_1, y_1, z)|,
\]

where the sup is taken over \( (x_1, y_1) \in \mathbb{R}^2 \) such that \( |x_1| \leq |x|, |y_1| \leq |y| \).

Now for each \( m \in \mathbb{Z} \),

\[
\sup_{1 \leq |y| \leq 1} \left| (1 + |(x, y + 2m, z)|)^r \right| D^x g(x, y + 2m, z) \right| \\
\leq \sup_{x^2 + y^2 \leq 1, \ |y| \leq 1} \left( 1 + |(x, y + 2m, z)| \right)^r |D^x g(x, y + 2m, z)| \\
+ \sup_{x^2 + y^2 \geq 1, \ |y| \leq 1} \left( 1 + |(x, y + 2m, z)| \right)^r |D^x g(x, y + 2m, z)|.
\]

But

\[
\sup_{x^2 + y^2 \geq 1, \ |y| \leq 1} (1 + |(x, y + 2m, z)|)^r |D^x g(x, y + 2m, z)| \\
\leq \sup_{x^2 + y^2 \geq 1, \ |y| \leq 1} (1 + |(x, y + 2m, z)|)^r \\
\times \sum_{\beta \in F_1} |P_\beta(x, y)| |D^\mu g_m(x, y, z)| \\
\leq \sup_{|y| \leq 1, x, z} (1 + |(x, y + 2m, z)|)^r \\
\times \sum_{\beta \in F_1} |P_\beta(x, y)| \sum_{(\nu, \zeta) \in F_4(\beta)} |D^{\nu} h(x, y)| \\
\times |D^\zeta f(x, y + 2m, z)|,
\]

where for each \( \beta \in F_1, F_4(\beta) \) is a finite subset of \( M \times M \). Pick \( C_1, t_1 \geq 0 \) so
that $|P_{\beta}(x, y)| \leq C_1(1 + |x|)^{\nu}$ for all $\beta \in F_1$, $x \in \mathbb{R}$, $|y| \leq 1$. Then this last expression is bounded by

$$C_1 \sum_{\beta \in F_1} \sum_{(\gamma, \gamma') \in F_0(\beta)} C_7 \times \sup_{|y| \leq 1, x, z} (1 + |(x, y + 2m, z)|)^{\nu + n + \nu} |D^\gamma f(x, y + 2m, z)|$$

$$\leq C_1 C_2 \sum_{\beta \in F_1} \sum_{(\gamma, \gamma') \in F_0(\beta)} S_{\gamma, \gamma'}(f).$$

Now

$$\sup_{x^2 + y^2 \leq 1, z} |(1 + |(x, y + 2m, z)|)^{\nu} |D^2 g(x, y + 2m, z)|$$

$$\leq \sup_{x^2 + y^2 \leq 1, z} |(1 + |(x, y + 2m, z)|)^{\nu} |D^2 g(0, 2m, z)|$$

$$+ \sup_{x^2 + y^2 \leq 1, z} |(1 + |(x, y + 2m, z)|)^{\nu} \times |D^2 g(x, y + 2m, z) - D^2 g(0, 2m, z)|.$$

But there is a constant $C_3$ so that the first of these is bounded by

$$C_3 \sup_{x^2 + y^2 \leq 1, z} |(1 + |(0, 2m, z)|)^{\nu} \sum_{\beta \in F_2} C_\beta |D^\beta g_m(0, 0, z)|$$

$$\leq C_3 \sup_{x^2 + y^2 \leq 1, z} |(1 + |(0, 2m, z)|)^{\nu} \sum_{\beta \in F_2} C_\beta \sum_{(\gamma, \gamma') \in F_0(\beta)} |(1 + |(0, 2m, z)|)^{\nu} \times |D^\gamma h(0, 0)| |D^\gamma f(0, 2m, z)|,$$

where for each $\beta \in F_2$, $F_0(\beta)$ is a finite subset of $M \times M$. Let $C_4 = \max_{\beta \in F_2, (\gamma, \gamma') \in F_0(\beta)} C_3 C_\beta |D^\gamma h(0, 0)|$. Then this expression is bounded by

$$C_4 \sum_{\beta \in F_2} C_\beta \sum_{(\gamma, \gamma') \in F_0(\beta)} S_{\gamma, \gamma'}(f).$$

Finally

$$\sup_{x^2 + y^2 \leq 1, z} \times |D^2 g(x, y + 2m, z) - D^2 g(0, 2m, z)|$$

$$\leq \sup_{x^2 + y^2 \leq 1, z} |(1 + |(x, y + 2m, z)|)^{\nu} \times \sum_{\beta \in F_2} \sum_{x_1^2 + y_1^2 \leq 1} |P_{\beta}(x_1, y_1)| |D^\beta g_m(x_1, y_1, z)|$$
where for each $\beta \in F_\alpha$, $C^{m}_\beta = \sup_{x^2 + y^2 \leq 1, z} |P^{m}_\beta(x_1, y_1)| < \infty$ and $F_\alpha(\beta)$ is a finite subset of $M \times M$. Write

$$C_s = \max_{\beta \in F_\alpha} C^{m}_\beta \sup_{x^2 + y^2 \leq 1} |D^k h(x_1, y_1)|.$$

Then the above is bounded by

$$C_s \sup_{x^2 + y^2 \leq 1, z} (1 + |(x, y + 2m, z)|)^r \sum_{\beta \in F_\alpha} \|D^k h(x_1, y_1)| \times (1 + (2|m| - 1)^2 + |z|^2)^{1/2} r'.

But there is a constant $C_r$ so that for all $m \in \mathbb{Z}$, $\sup_{x^2 + y^2 \leq 1, z} (1 + |(x, y + 2m, z)|)^r (1 + (2|m| - 1)^2 + |z|^2)^{1/2} r' \leq C_r$. Thus we have a bound of the desired form. Q.E.D.

**Theorem 9.14.** Suppose $E(P; F)$ is a holomorphic family of Eisenstein integrals defined as in (6.6) and $\varphi \in \mathcal{C}(\mathcal{G} \times \mathcal{F})$ such that $\varphi \cdot m_\mathcal{G}$ is jointly smooth as a function on $\mathcal{G} \times \mathcal{F}$. Then $E(P; F) \cdot \varphi \cdot m_\mathcal{G} \in I(\mathcal{G}, \mathcal{G})$. Given any $D \in \mathcal{L}_\mathcal{G}$ there is $r \geq 0$ so that given any $t \geq 0$ there is a continuous seminorm $\mu_\mathcal{G}$ on $\mathcal{C}(\mathcal{G} \times \mathcal{F})$ so that $0 S_{r, \mathcal{G}}(E(P; F) \cdot \varphi \cdot m_\mathcal{G}) \leq \mu_\mathcal{G}$ for all $\varphi$ as above.

**Corollary 9.15.** Suppose $E(P; F)$ and $\varphi$ are as above. Then $F_\alpha(x) = \int_{\mathcal{G} \times \mathcal{F}} E(P; F; h \cdot v \cdot x) \varphi(h \cdot v) m(h \cdot v) dh \, dv$ is in $\mathcal{C}(G, W)$. Given any $t \geq 0$, there is a continuous seminorm $\mu$ on $\mathcal{C}(\mathcal{G} \times \mathcal{F})$ so that $S_{r, \mathcal{G}}(E(P; F) \cdot \varphi \cdot m_\mathcal{G}) \leq \mu_\mathcal{G}$ for all $\varphi$ as above.

**Proof.** By Theorem 9.14, $\varphi = E(P; F) \cdot \varphi \cdot m_\mathcal{G} \in I(\mathcal{G}, \mathcal{G})$. Then by Theorem 8.2, $I_\varphi$ is a Schwartz function on $\mathcal{G}$ and there is $F \subseteq \mathcal{L}_\mathcal{G}$, so that given any $r' \geq 0$, there are $C, t \geq 0$ so that $R \|I_\varphi\| \leq C 0 S_{r, \mathcal{G}}(\varphi)$. But $I_\varphi(x) = \int_{\mathcal{G} \times \mathcal{F}} E(P; F; h \cdot v \cdot x) \varphi(h \cdot v) m(h \cdot v) \pi_\mathcal{G}(A_{h, v}) \pi_R(v) dh \, dv$. But by (9.12), $m_R(h \cdot v) \pi_\mathcal{G}(A_{h, v}) \pi_R(v) \leq 1 = m(h \cdot v)$ so that $I_\varphi = F_\alpha$. Further, there is $r' \geq 0$ so that given any $t \geq 0$ there is $\mu$ so that $0 S_{r, \mathcal{G}}(\varphi) \leq \mu_\mathcal{G}$. Q.E.D.

Rather than prove Theorem 9.14 as stated, we will prove a slightly generalized version.

**Theorem 9.16.** Suppose $E(P; F_1), \ldots, E(P; F_k)$ are holomorphic families of Eisenstein integrals, $x_1, \ldots, x_k \in \mathcal{C}(\mathcal{G} \times \mathcal{F})$, and $\varphi = m_\mathcal{G} \sum_{i=1}^k E(P; F_i) x_i$ is
jointly smooth on $\mathcal{D} \times \mathcal{F} \times G$. Then $\varphi \in I'(\mathcal{D}, G)$, and given $D \in \mathcal{L}_G$, there is $r > 0$ so that given any $t \geq 0$, there are continuous seminorms $\mu_1, ..., \mu_k$ on $C(\mathcal{D} \times \mathcal{F})$ so that $0S_{D,r,t}(\varphi) \leq \sum_{i=1}^k \mu_i(\varphi_i)$.

Proof. We know that $\varphi_1 = \sum_{i=1}^k E(P:F_i) \varphi_i \in I(\mathcal{D}, G)$. But using Lemma 9.13, we see that $\varphi \in I(\mathcal{D}, G)$ also, and that for any $D \in \mathcal{L}_G, r, t > 0$, there are a finite subset $F$ of $\mathcal{L}_G$ and $r_1 > 0$ so that $S_{D,r,t}(\varphi) \leq \sum_{i=1}^k S_{F,r_1,t}(E(P:F_i) \varphi_i)$. But since each $E(P:F_i) \in II(\mathcal{D}, G)$, by making $r_1$ sufficiently large, given $t > 0$ there are continuous seminorms $\mu_i$ on $C(\mathcal{D} \times \mathcal{F})$ so that $S_{F,r_1,t}(E(P:F_i) \varphi_i) \leq \mu_i(\varphi_i)$.

Now let $Q$ be any parabolic subgroup of $G, s \in W(a, a_Q)$. We must show that $v \mapsto \pi_R^{-1}(v) \psi_{\varphi,s}(h:x)$ has a smooth extension from $\mathcal{F}'$ to $\mathcal{F}$ for all $(h, x) \in \mathcal{D} \times L_Q$. By [4, Lemma 22.11], it is enough to show that $\psi_{\varphi,s}(h:v_0:x) = 0$ for all $(h, x) \in \mathcal{D} \times L_Q$ if $\pi_R(v_0) = 0$. But $\varphi \in I(\mathcal{D}, G)$ so that $\varphi_{\mathcal{D}}(h:v) = 0$ is jointly smooth on $\mathcal{D} \times \mathcal{F} \times L_Q$. Thus it is enough to show that for all $h_0 \in \mathcal{D}$ such that $\varphi$ factors through a group of Harish-Chandra class, and all $v_0$ such that $\pi_R(v_0) = 0$, $\lim_{v \to v_0} \psi_{\varphi,s}(h_0:v:x) = 0$ for all $x \in L_Q$.

Now let $Q$ be any parabolic subgroup of $G, s \in W(a, a_Q)$. We must show that $v \mapsto \pi_R^{-1}(v) \psi_{\varphi,s}(h:x)$ has a smooth extension from $\mathcal{F}'$ to $\mathcal{F}$ for all $(h, x) \in \mathcal{D} \times L_Q$. By [4, Lemma 22.11], it is enough to show that $\psi_{\varphi,s}(h:v_0:x) = 0$ for all $(h, x) \in \mathcal{D} \times L_Q$ if $\pi_R(v_0) = 0$. But $\varphi \in I(\mathcal{D}, G)$ so that $\varphi_{\mathcal{D}}(h:v) = 0$ is jointly smooth on $\mathcal{D} \times \mathcal{F} \times L_Q$. Thus it is enough to show that for all $h_0 \in \mathcal{D}$ such that $\varphi$ factors through a group of Harish-Chandra class, and all $v_0$ such that $\pi_R(v_0) = 0$, $\lim_{v \to v_0} \psi_{\varphi,s}(h_0:v:x) = 0$ for all $x \in L_Q$.

Now suppose $Q$ is any parabolic subgroup of $G$ with $W(a, a_Q) \neq \emptyset$. Then for $s \in W(a, a_Q)$, let $g = (\psi_{\varphi,s})^s$. Let $P' \in \mathcal{P}(A)$, $*P' = P' \cap L_Q$. As in (8.2), for $t \in W(a, a')$ there is $t' \in W(a, a)$ so that $(\psi_{\varphi,s})^{t'} = (\psi_{\varphi,s'})^{t'}$. But $\psi_{\varphi,s}(h_0,v_0) = 0$ as above. Thus $(\psi_{\varphi,s'})(h_0,v_0) = 0$ for all $P' \in \mathcal{P}(A), t \in W(a, a)$. Thus, again using [4, Lemma 11.1], $\pi_R(s_{h,v}(h_0,v_0)) \psi_{\varphi,s}(h_0,v_0) = \psi_{\varphi,s}(h_0,v_0) = 0$ on $L_Q$.

For $s \in W(a, a), h, h' \in \mathcal{D}$, write $h' = sh$ if $\lambda(h') = s\lambda(h)$. Let $W_0(h) = \{s \in W(a, a): sh = h\}$ and $W_1(h) = \{s \in W(a, a): sh = h' \text{ for some } h' \in \mathcal{D}\}$. Note that $W_1 = W_1(h)$ is independent of $h \in \mathcal{D}$. Let $\Theta(H:h:v)$ denote the distribution character of the representation $\pi_{h,v}$.

LEMMA 9.17. There is a constant $c$ so that for $F_2$ as above, and any $h, v \in \mathcal{D} \times \mathcal{F}, \Theta(H:h:v:R(x)F_2) = c \sum_{s \in W_1} \alpha(sh:sv) E(P:F_M:sh:sv:x) m(h:v) dv$, and define $\zeta(h) \in \hat{Z}$ by $\zeta(h) = e^{h|_Z}$. Let $\mathcal{D}_h = \{h' \in \mathcal{D} : e^{h|_Z} = 0 \}$.
Then it follows from a Poisson summation argument similar to [8, 7.12] that for all \( h \in \mathcal{G} \), \( \tilde{F}_s(x; \zeta(h)) = \int_{\mathcal{G}} F_s(xz) \zeta(h)(z) \, dz = c \sum h' \in H \, F_s(h' \cdot x) \). Now \( \Theta(H; h: v; R(x)F_s) = \int_{\mathcal{G}} \Theta(H; h: v; y) \tilde{F}_s(yx; \zeta(h)) \, dy \). Fix \( h \) rational, that is, for which \( (\tau_{1, h}, \tau_{2, h}) \) factors through a group of Harish-Chandra class. Then \( h' \) is rational for all \( h' \in \mathcal{G} \), and using [5, Theorems 20.1 and 27.1]

Thus in this case \( \Theta(H; h: v; y) F_s(h' \cdot xy) \, dy \). But both sides are smooth functions of \( h \), so equality persists for all \( h \in \mathcal{G} \).

Q.E.D.

**Theorem 9.18.** \( F_s \in \mathcal{C}_H(G, W) \). In fact there is a constant \( c \) so that

\[
F_s(x) = c \int_{\mathcal{G} \times \mathcal{F}} \Theta(H; h: v)(R(x)F_s) \, m(h: v) \, dh \, dv.
\]

**Proof.** Using (9.17) \( \int_{\mathcal{G} \times \mathcal{F}} \Theta(H; h: v)(R(x)F_s) \, m(h: v) \, dh \, dv = c \sum_{s \in W_v} \alpha(h; sv) E(P; F: M; h; sv; x) \).

We have constructed wave packets

\[
F_s(x) = \int_{\mathcal{G} \times \mathcal{F}} E(P; F; h; v; x) \alpha(h; v) \, m(h; v) \, dh \, dv
\]

and shown they are elements of \( \mathcal{C}_H(G; W) \). If we want scalar-valued wave packets, we need only take

\[
f_s(x) = F_s(x)(1:1)
\]

Since \( \psi \to \psi(1:1) \) is a linear functional on the finite dimensional vector space \( W \), we will have \( x_1 \| F_s \|_{r, r_2} \leq C x_1 \| F_s \|_{r, r_2} \) for all \( g_1, g_2 \in W(\psi) \), \( r \geq 0 \). Thus \( f_s \in \mathcal{C}(G) \) whenever \( F_s \in \mathcal{C}(G; W) \). We can also evaluate both sides of (9.18) at \( (1:1) \in K_1 \times K_1 \) to obtain:
**Theorem 9.20.** Suppose $E(P:F)$ is a holomorphic family of Eisenstein integrals defined as in (6.6) and $x \in \mathcal{C}(X \times F)$ such that $x \cdot m_R$ is jointly smooth on $X \times F$. Define the wave packet $f_x$ as in (9.19b). Then $f_x \in \mathcal{C}_H(G)$. More precisely:

(i) Given any $g_1, g_2 \in \mathfrak{U}(g), r \geq 0$, there is a continuous seminorm $\mu$ on $\mathcal{C}(X \times F)$ so that $g_1 \|f_x\|_{r, g_2} \leq \mu(x)$ for all $x$ as above.

(ii) There is a constant $c$ so that for all $x \in G$,

$$f_x(x) = c \int_{\mathcal{C}} \Theta(H: h:v)(R(x)f_x)m(h:v) dh dv.$$

### 10. Extension to Disconnected Groups

Finally we extend our results for connected reductive Lie groups to the class [11, 6, 7, 8, 9] of real reductive Lie groups $G$ such that

- $G$ has a closed normal abelian subgroup $Z$
- such that $Z \subset Z_G(G^0)$ and $|G/ZG^0| < \infty$, \hspace{1cm} (10.1a)
- if $x \in G$ then $\text{Ad}(x) \in \text{Int}(g_1)$, \hspace{1cm} (10.1b)

and

$G/G^0$ is finitely generated. \hspace{1cm} (10.1c)

The Harish-Chandra class consists of the groups (10.1) such that $[G^0, G^0]$ has finite center and $G/G^0$ is finite.

The first step is to show that there is a particularly good choice of $Z$.

Fix a Cartan involution $\theta$ of $G$ as in [11]. The fixed point set $K = G^\theta$ is the inverse image of a maximal compact subgroup of the linear semisimple group $G/Z_G(G^0)$. $K$ meets, and has connected intersection with, every component of $G$. As in the connected case every Cartan subgroup of $G$ is $G^0$-conjugate to a $\theta$-stable one. So every cuspidal parabolic subgroup of $G$ is $G^0$-conjugate to one of the form $MAN$, where $M$ and $A$ are $G^0$-stable, $MA = M \times A = Z_G(A)$, and $M$ and $MA$ satisfy (10.1).

Proposition 2.1 says, here, that $K^0$ has a unique maximal compact subgroup $K^0$ and a closed normal vector subgroup $V$ such that $K^0 = K^0 \times V$ and $Z_G \cap V$ is co-compact in both $V$ and $Z_G$. Since $\text{Ad}_G(K)$ is compact we may assume that it stabilizes the Lie algebra of $V$, and thus that $V$ is normal in $K$. 
**Proposition 10.2.** The group $Z$ of (10.1a) can be chosen so that $Z = (Z \cap G^0) \times E$, where

(a) $E$ is a finitely generated free abelian group,

(b) $E$ is a closed normal subgroup of $G$, and

(c) $Z \cap G^0 = Z_{G^0} \cap V$.

Then $ZG^0 = G^0 \times E$ and $ZK^0 = K^0 \times E = K_1^0 \times V \times E$.

**Proof.** Start with $Z_1$ that satisfies (10.1a). Then $Z_2 = Z_1Z_{G^0}$ does also, and $Z_2/Z_2^0$ is finitely generated by (10.1c). Now [7, Lemma 6.3] $Z_2 = \{Z_{G^0}, F\} \times E'$, finite abelian, $E'$ finitely generated free abelian.

$Z_3 = \{Z_{G^0} \cap V\} \times E'$ has finite index in $Z_2$. $Z_{G^0} \cap V$ is normal in $G = KG^0$ because it is normal in $K$ and centralized by $G^0$, and the finite index subgroup $Z_2G^0 \subset G$ centralizes $Z_3$. So $Z_3$ has a finite index subgroup $Z_4 = \{Z_{G^0} \cap V\} \times E''$ that is normal in $G$ and thus satisfies (10.1a).

Split $V = V' \times V''$, where $V' = V \cap [G^0, G^0]$ and $V'' \subset Z_{G^0}$ is normal in $K$. Then $Z_{G^0} \cap V = L' \times V''$, where $L'$ is a lattice in $V'$, normal in $G = KG^0$ because it is normal in $K$ and central in $G^0$. By (10.1b), $V''$ is central in $G$. So is any lattice $L'' \subset V''$. Now $L = L' \times L''$ is a lattice in $V$, normal in $G$. So $Z'' = L \times E''$ is a finitely generated free abelian group that satisfies (10.1a).

The action of $G$ on $Z''$ by conjugation defines a linear representation $\varphi$ of $G$ on the rational vector space $Z''_Q = Z'' \otimes Q$. As $\varphi(G)$ is finite the invariant subspace $L_\varphi$ has an invariant complement $B$. Let $E = Z'' \cap B$ and $Z' = L \times E$. Then $L$ and $E$ are normal in $G$ so $Z'$ satisfies (10.1a). Proposition 10.2 follows with $Z = (Z_{G^0} \cap V) \times E$.

Q.E.D.

From now on, we choose $Z$ as in Proposition 10.2. For convenience we write $G''$ for $ZG^0$ and use $^0$ and $''$ to indicate items pertaining to $G^0$ and $G''$.

Recall that the Schwartz spaces for $G''$ and $G$ were defined in [7, Sect. 6] as follows. For $x \in G^0$, define $\Xi$ and $\hat{\sigma}$ as in (2.4) and (2.7). Since $V$ is normal in $K$ we can assume that $\sigma_V$ is $K$-invariant. Let $\sigma_E$ be a norm on $E$ coming from an $Ad_G(K)$-invariant positive definite inner product on $E_{12} = E \otimes_{\sigma} \mathbb{R}$. Now we extend $\hat{\sigma}$ to $G'' = G^0 \times E$ by

$$\hat{\sigma}(xe) = \hat{\sigma}(x) + \sigma_E(e), \quad x \in G^0, e \in E.$$ (10.3a)

This is equivalent to the definition of $\hat{\sigma}$ in [7, Sect. 6]. Using (2.9c), we see that

$$\hat{\sigma}(x''y'') \leq 3(\hat{\sigma}(x'') + \hat{\sigma}(y'')) \quad \text{for all} \quad x'', y'' \in G''$$ (10.3b)
and since $\sigma_1$ and $\sigma_2$ are chosen to be $K$-invariant we have

$$\tilde{\sigma}(kx^{-1}k^{-1}) = \tilde{\sigma}(x^{-1}) \quad \text{for all } k \in K. \quad (10.3c)$$

Extend $\Xi$ to $G''$ by

$$\Xi(xe) = \Xi(x), \quad x \in G^0, e \in E. \quad (10.3d)$$

Then clearly

$$\Xi(k''x'') = \Xi(x''k'') = \Xi(x'') \quad \text{for all } x'' \in G'', k'' \in K''. \quad (10.3e)$$

Finally, because of $(10.1b)$, every element of $N_{G}(A_0)/Z_{G}(A_0)$ can be represented by an element of $K^0$. Now since coset representatives of $K/K^0$ can be chosen to normalize $A_0$, we have $\Xi(kak^{-1}) = \Xi(a)$ for all $k \in K$, $a \in A_0$. Thus

$$\Xi(kxk^{-1}) = \Xi(x') \quad \text{for all } k \in K, x'' \in G''. \quad (10.3f)$$

For $f \in C^\infty(G'')$, $g_1, g_2 \in \mathcal{U}(g)$, $r \geq 0$, we define

$$\gamma_1(f)_{r, g_2} = \sup_{x \in G''} |f(g_1 : x ; g_2)| \Xi(x)^{-1}(1 + \tilde{\sigma}(x))'. \quad (10.3g)$$

Then

$$\mathcal{C}(G'') = \{ f \in C^\infty(G'') : \gamma_1(f)_{r, g_2} < \infty \text{ for all } g_1, g_2 \in \mathcal{U}(g), r \geq 0 \}, \quad (10.3h)$$

and

$$\mathcal{C}(G) = \{ f \in C^\infty(G) : (L(x)f)|_{G'} \in \mathcal{C}(G'') \quad \text{for all } x \in G \}. \quad (10.3i)$$

Let $\{b_1, \ldots, b_n\}$ be coset representatives for $G/G''$. For $f \in C^\infty(G)$ and $1 \leq t \leq n$, define $f_t = (L(b_t^{-1})f)|_{G'}$. Then $\mathcal{C}(G) = \{ f \in C^\infty(G) : f_t \in \mathcal{C}(G''), 1 \leq t \leq n \}$, and it can be topologized by the seminorms

$$\gamma_1(f)_{r, g_2} = \gamma_1(f_t)_{r, g_2}, \quad g_1, g_2 \in \mathcal{U}(g), r \geq 0, 1 \leq t \leq n. \quad (10.3j)$$

The next step is to extend Theorem 9.20 from $G^0$ to $G'' = G^0 \times E$.

Fix a $\theta$-stable Cartan subgroup $H'' \subset G''$ and let $P'' = M''AN$ be an associated cuspidal parabolic. Let

$$\pi^0_G = \{ \pi^0_{h, v} : h \in \mathcal{D} \text{ and } v \in a^* \} \quad (10.4a)$$

be a continuous family of $H'' \cap G^0$-series representations of $G^0$ as in (3.8). The corresponding continuous family of $H''$-series representations of $G''$ is

$$\pi_G'' = \{ \pi''_{h, v, \eta} = \pi^0_{h, v} \otimes \eta : h \in \mathcal{D}, v \in a^*, \eta \in \mathcal{E} \}. \quad (10.4b)$$
Let $E^0(P'' \cap G^0 : F : h : v : x)$ be a family of Eisenstein integrals on $G^0$ as in (6.6) corresponding to the family $\pi^0_G$ and a family $F$ of $\tau$-spherical functions on $G^0$ coming from a holomorphic family of matrix coefficients on $M'' \cap G^0$ as in (6.5). Now

$$E^0(P'' \cap G^0 : F : h : v : x) = E^0(P'' \cap G^0 : F : h : v : x)(1:1) \quad (10.5a)$$

is a smooth family of matrix coefficients of $\pi^0_G$. Since every matrix coefficient of $\pi^0_{h,v}$ is of the form $f(x:e) = f^0(x) \eta(e)$, where $f^0$ is a matrix coefficient of $\pi^0_{h,v}$ we obtain a smooth family of coefficients of $\pi^0_{h,v}$ by defining

$$E^0(P'' : F : h : v : \eta : x : e) = \eta(e) E^0(P'' \cap G^0 : F : h : v : x). \quad (10.5b)$$

Let $\mathcal{D}^0$ and $\mathcal{H}''$ denote the respective algebras of differential operators on $\mathcal{G} \times \mathcal{F}$ and $\mathcal{G} \times \mathcal{F} \times \hat{E}$ whose coefficients are polynomials on $\mathcal{G} \times \mathcal{F}$ and constant on $\hat{E}$. If $h \in \mathcal{G}$ then, as before, $d(h)$ is the distance from $h$ to $bd(\mathcal{G})$. The seminorms on $C^{-t}(\mathcal{G} \times \mathcal{F})$

$$\|x\|_{p, t}^0 = \sup_{x \in \mathcal{G} \times \mathcal{F}} |Dx(h : v)| (1 + d(h)^{-1})^t \quad (10.6a)$$

and on $C^{-t}(\mathcal{G} \times \mathcal{F} \times \hat{E})$

$$\|\beta\|_{D, t}^0 = \sup_{\beta \in \mathcal{G} \times \mathcal{F} \times \hat{E}} |D\beta(h : v : \eta)| (1 + d(h)^{-1})^t \quad (10.6b)$$

define Schwartz spaces

$$\mathcal{S}(\mathcal{G} \times \mathcal{F}) = \{x \in C^{-t}(\mathcal{G} \times \mathcal{F}) : \|x\|_{p, t}^0 < \infty \text{ for } D \in \mathcal{D}^0, t \geq 0\} \quad (10.7a)$$

and

$$\mathcal{S}(\mathcal{G} \times \mathcal{F} \times \hat{E}) = \{\beta \in C^{-t}(\mathcal{G} \times \mathcal{F} \times \hat{E}) : \|\beta\|_{D, t}^0 < \infty \text{ for } D \in \mathcal{D}'' t > 0\}. \quad (10.7b)$$

The space (10.7a) was used to form the wave packets for $G^0$ in Theorem 9.20. We will use (10.7b) to form the analogous wave packets for $G''$.

Two remarks:

**Lemma 10.8.** Let $d$ be a constant coefficient differential operator on $\hat{E}$ and $\mu$ a continuous seminorm on $\mathcal{S}(\mathcal{G} \times \mathcal{F})$. If $\beta \in \mathcal{S}(\mathcal{G} \times \mathcal{F} \times \hat{E})$ define $\gamma_{d, \eta}(h : v) = \beta(h : v : \eta ; d)$. Then $\gamma_{d, \eta} \in \mathcal{S}(\mathcal{G} \times \mathcal{F})$ and $\beta \mapsto \sup_{\eta} \mu(\gamma_{d, \eta})$ is a continuous seminorm on $\mathcal{S}(\mathcal{G} \times \mathcal{F} \times \hat{E})$. 

LEMMA 10.9. Let $\beta \in \mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E})$ and let $\alpha_{d,\eta}$ be as in Lemma 10.8. If $m_r(h:v) \beta(h:v;\eta)$ extends to be $C^\infty$ on $\mathcal{D} \times \mathcal{F} \times \mathcal{E}$ then each $m_r(h:v) \alpha_{d,\eta}(h:v)$ extends to be $C^\infty$ on $\mathcal{D} \times \mathcal{F}$.

Now we extend Theorem 9.20 to $G''$. Let $\beta \in \mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E})$ such that $m_r(h:v) \beta(h:v;\eta)$ extends $C^\infty$ on $\mathcal{D} \times \mathcal{F} \times \mathcal{E}$. Form the wave packets

$$
\phi''_\beta(x:e) = \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \varepsilon''(P^\prime:F:h:v;\eta:x:e) \beta(h:v;\eta) \, dh \, dv \, d\eta. \tag{10.10}
$$

THEOREM 10.11. Let $\varepsilon''(P^\prime:F)$ be a smooth family of matrix coefficients on $G'' = G^0 \times E$ defined as in (10.5b). Let $\beta \in \mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E})$ such that $m_r(h:v) \beta(h:v;\eta)$ extends $C^\infty$ on $\mathcal{D} \times \mathcal{F} \times \mathcal{E}$. Then $\phi''_\beta \in \mathcal{C}_H(G'')$. More precisely:

(i) Let $g_1, g_2 \in \mathcal{U}(g), r \geq 0$. Then there is a continuous seminorm $\mu$ on $\mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E})$, independent of $\beta \in \mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E})$, such that $g_1, \|\phi''_\beta\|_{r, g_2} \leq \mu(\beta)$.

(ii) There is a constant $c$ so that for all $x \in G''$,

$$
\phi''_\beta(x) = c \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \Theta(H'';h:v;\eta)(R(x)\phi''_\beta) m(h:v) \, dh \, dv \, d\eta.
$$

Proof. For (i) split $\tilde{\sigma}(xe) = \tilde{\sigma}(x) + \sigma_{k}(e)$, where $x \in G^0$ and $e \in E$. Then the integral defining $g_1, \|\phi''_\beta\|_{r, g_2}$ is bounded by

$$
\sup_{x \in G^0} (1 + \tilde{\sigma}(x))' \Xi(x)^{-1} \sup_{e \in E} (1 + \sigma_{k}(e))' \\
\times \left| \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \varepsilon''(P^\prime:F:h:v;\eta; g_1; x; g_2; e) \beta(h:v;\eta) m(h:v) \, dh \, dv \, d\eta \right|
$$

$$
\leq \sup_{x \in G^0} (1 + \tilde{\sigma}(x))' \Xi(x)^{-1} \sup_{e \in E} (1 + \sigma_{k}(e))' \left| \int_{\mathcal{E}} \psi_{\beta}(x;\eta) \, d\eta \right|,
$$

where

$$
\psi_{\beta}(x;\eta) = \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \varepsilon'(P^\prime \cap G^0:F:h:v; g_1; x; g_2) \beta(h:v;\eta) m(h:v) \, dh.
$$

Now $\psi_{\beta}(x) \in C^\infty(\mathcal{E})$. For given $r \geq 0$ we have a constant coefficient operator $d$ on $\mathcal{E}$ such that

$$
\sup_{e \in E} (1 + \sigma_{k}(e))' \left| \int_{\mathcal{E}} \eta(e) \psi_{\beta}(x;\eta) \, d\eta \right| \leq \sup_{n \in \mathcal{E}} |d\psi_{\beta}(x;\eta)|$$
independent of \( x \). So the integral defining \( \| \phi_p \|_{r, r_2} \) is bounded by

\[
\sup_{x \in G^0} (1 + \delta(x))^{-1} \int_{\overline{\mathcal{E}}} e^0(P'' \cap G^0; F; h: v; g_1; x; g_2) \cdot \beta(h; v; \eta; d) m(h; v) \, dh \, dv.
\]

Since \( \varepsilon_{d, \eta}(h; v) = \beta(h; v; \eta; d) \in \mathcal{C}(\mathcal{D} \times \mathcal{F}) \), Theorem 9.20(i) gives us a continuous seminorm \( \mu^0 \) on \( \mathcal{C}(\mathcal{D} \times \mathcal{F}) \), depending only on \( g_1, g_2, \) and \( r \), such that this is bounded by \( \sup_{\eta \in \mathcal{E}} \mu^0(\varepsilon_{d, \eta}) \). Thus we have a continuous seminorm \( \mu(\beta) - \sup_{\eta \in \mathcal{E}} \mu^0(\varepsilon_{d, \eta}) \) on \( \mathcal{C}(\mathcal{D} \times \mathcal{F} \times \mathcal{E}) \) such that the integral defining \( \| \phi_p \|_{r, r_2} \) is bounded by \( \mu(\beta) \).

For (ii) note \( \Theta(H''; h; v; \eta; x; e) = \eta(e) \Theta(H'' \cap G^0; h; v; x) \) and note that the Plancherel density functions for \( G'' \) and \( G^0 \) are related by \( m''(h; v; \eta) = m(h; v) \). Now, using Theorem 9.20(ii),

\[
\phi_p^0(x) = \int_{\mathcal{E}} \eta(e) \left\{ \int_{\mathcal{D} \times \mathcal{F}} e^0(P'' \cap G^0; F; h; v; x) \beta(h; v; \eta) m(h; v) \, dh \, dv \right\} \, d\eta
\]

where \( \varepsilon_{d, \eta}(h; v) = \beta(h; v; \eta; d) \in \mathcal{C}(\mathcal{D} \times \mathcal{F}) \) and

\[
\phi^0_{\varepsilon_{d, \eta}}(x) = \int_{\mathcal{D} \times \mathcal{F}} e^0(P'' \cap G^0; F; h; v; x) \varepsilon_{d, \eta}(h; v) m(h; v) \, dh \, dv.
\]

So

\[
\phi^0_{\varepsilon_{d, \eta}}(x) = c \int_{\mathcal{D} \times \mathcal{F}} \Theta(H'' \cap G^0; h; v)(R(x) \phi^0_{\varepsilon_{d, \eta}}) m(h; v) \, dh \, dv
\]

and thus

\[
\phi^0_p(x) = c \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \Theta(H'' \cap G^0; h; v)(R(x) \phi^0_{\varepsilon_{d, \eta}}) \eta(e) m(h; v) \, dh \, dv \, d\eta
\]

\[
= c \int_{\mathcal{D} \times \mathcal{F} \times \mathcal{E}} \Theta(H''; h; v; \eta)(R(xe) \phi^0_p) m(h; v) \, dh \, dv \, d\eta
\]

using the Fourier inversion formula on \( \mathcal{E} \).

Q.E.D.

Our final step is to extend Theorem 10.11 from \( G'' = ZG^0 \) to \( G \). First note that \( G'' \) is normal in \( G \) and of finite index.
Our continuous family of $H$-series representations, as in (10.4), will be

$$\pi_G = \{ \pi_{h,v} = \text{Ind}_G^G(\pi_{h,v}^n): h \in H, v \in a^*, \eta \in E \}. \quad (10.12)$$

Note that in general $G'' \subset Z_G(G^0)G^0$. Thus the representations $\pi_{h,v}$ are in general finite sums of irreducible $H$-series representations.

Choose coset representatives $\{b_1, ..., b_n\}$ for $G/G''$. Since $K$ meets every component of $G$, we can assume that $b_i \in K$, $1 \leq i \leq n$.

Now $\pi_{h,v} = \sum_{i=1}^n \pi_i^v \cdot \text{Ad}(b_i)^{-1}$. In fact, given $v \in \mathcal{H}(\pi_{h,v})$ we construct $\{v_1, ..., v_n\} \subset \mathcal{H}(\pi_{h,v})$ as follows:

$$v_k : G \to \mathcal{H}(\pi_{h,v}) \text{ is supported in } b_k G'', \quad (10.13a)$$

$$v_k(b_k x) = \pi_i^v(x)^{-1} \cdot v \quad \text{for } x \in G''. \quad (10.13b)$$

Similarly, if $w \in \mathcal{H}(\pi_{h,v})$ we have $\{w_1, ..., w_n\}$ in $\mathcal{H}(\pi_{h,v})$.

**Lemma 10.14**. The coefficients $x \mapsto \langle \pi_{h,v}(x) w_i, v_k \rangle$ of $\pi_{h,v}$ is supported in the coset $b_j G''$ for which $b_j \equiv b_i b_j \mod G''$. On $b_j G''$ it is given by $b_j x'' \mapsto \langle \pi_i^v(a_{sl}) \cdot \pi_{h,v}(b_j^{-1} x'' b_j) \cdot w, v \rangle_{\mathcal{H}(\pi_{h,v})}$, where $a_{sl} = b_k^{-1} b_s b_j \in K''$.

**Proof.** Drop the subscripts on $\pi_{h,v}$ and $\pi_{h,v}$ and compute

$$\langle \pi(x) w_i, v_k \rangle_{\mathcal{H}(\pi)} = \sum_{1 \leq i \leq n} \langle w_i(x^{-1} b_i), v_k(b_i) \rangle_{\mathcal{H}(\pi)}. \quad (10.14a)$$

Let $x = b_s x'', x'' \in G''$, so $w_i(x^{-1} b_i) = 0$ unless $b_s^{-1} b_i \in b_j G''$. Note $v_k(b_i) = 0$ unless $k = t$. So $x \mapsto \langle \pi(x) w_i, v_k \rangle$ is supported in the coset $b_j G''$ with $b_k \equiv b_s b_i$. Given that, $b_s b_i = b_k a_{sl}$ and we compute

$$\langle \pi(x) w_i, v_k \rangle_{\mathcal{H}(\pi)} = \langle w_i(x^{-1} b_k), v_k(b_k) \rangle_{\mathcal{H}(\pi)}$$

$$= \langle w_i(b_i \cdot (b_i^{-1} x'' b_i)^{-1} a_{sl}^{-1}), v \rangle_{\mathcal{H}(\pi)}$$

$$= \langle \pi''(a_{sl}) \cdot \pi''(b_i^{-1} x'' b_i) \cdot w, v \rangle_{\mathcal{H}(\pi)}$$

as asserted. Q.E.D.

Lemma 10.14 tells us how the family (10.5b) of matrix coefficients of $G''$ defines families for $G$. Let $\varepsilon''(P''; F; h; \eta; v; x'')$ be as in (10.5b) with $x''$ in place of $(x; e)$. For $1 \leq i, k \leq n$ define

$$\varepsilon_{ik}(P; F; h; \eta; v) : G \to \mathbb{C} \text{ supported in the coset } b_k G'' \text{ for which } b_k \equiv b_s b_i \mod G'' \quad (10.15a)$$

by the formula

$$\varepsilon_{ik}(P; F; h; \eta; v; b_s x'') = \varepsilon''(P''; F; h; \eta; v; a_{sl} b_i^{-1} x'' b_i) \text{ for } x'' \in G'', b_s \text{ as specified in (10.15a)} \text{ and } a_{sl} = b_k^{-1} b_s b_i. \quad (10.15b)$$
The $\varepsilon_{ik}(P:F:h;\eta;v)$ are matrix coefficients of $\pi_{k,\eta,v}$. Now, as in (10.10), let $\beta \in \mathcal{C}(\mathfrak{D} \times \mathfrak{F} \times \hat{E})$ such that $m_{ik}(h;v) \beta(h;v;\eta)$ extends to $C^\prime$ on $\mathfrak{D} \times \mathfrak{F} \times \hat{E}$, and form the wave packets

$$\varphi_{ik\beta}(x) = \int_{\mathfrak{D} \times \mathfrak{F} \times \hat{E}} \varepsilon_{ik}(P:F:h;\eta;v) \beta(h;v;\eta) m(h;v) \, dh \, dv \, d\eta. \quad (10.16)$$

**Theorem 10.17.** Let $E^0(P \cap G^0:F)$ be a family of Eisenstein integrals on $G^0$ as in (6.6) and let $\beta \in \mathcal{C}(\mathfrak{D} \times \mathfrak{F} \times \hat{E})$ such that $m_{ik}(h;v) \beta(h;v;\eta)$ extends to be $C^\prime$ on $\mathfrak{D} \times \mathfrak{F} \times \hat{E}$. Then the $\varphi_{ik\beta}$ of (10.16) belong to $\mathcal{C}_H(G)$. More precisely:

(i) Let $g_1, g_2 \in \mathfrak{H}(g), r \geq 0, 0 \leq t \leq n$. Define $s$ as in (10.15). Then if $t \neq s$, $g_1 \| \varphi_{ik\beta} \|_{r,g_2} = 0$. For $t = s$, there is a continuous seminorm $\mu$ on $\mathcal{C}(\mathfrak{D} \times \mathfrak{F} \times \hat{E})$, independent of $\beta$, so that $g_1 \| \varphi_{ik\beta} \|_{r,g_2} \leq \mu(\beta)$.

(ii) There is a constant $c$ so that for all $x \in G$,

$$\varphi_{ik\beta}(x) = c \int_{\mathfrak{D} \times \mathfrak{F} \times \hat{E}} \Theta(H:h;v;\eta;R(x)\varphi_{ik\beta}) m(h;v) \, dh \, dv \, d\eta.$$  

**Proof.** Using (10.15) we see that $\varphi_{ik\beta}$ is supported in the coset $b_r G''$ and $\varphi_{ik\beta}(h,x\eta') = \varphi_{ik\beta}(a_1 b_i \ldots x b_i)$. Now using (10.3b, c, e, f) it is easy to see that there are a constant $C'$ and $\gamma, \delta \in \mathfrak{H}(g)$ depending on $r, g_1, g_2, a_i, \xi$ but not on $\beta$, so that $g_1 \| \varphi_{ik\beta} \|_{r,g_2} \leq C' \| \gamma \|_{r,g_2}$. Part (i) now follows from Theorem 10.11.

For (ii), we first fix $h, \eta, v$ and write $\Theta$ and $\Theta''$ for the characters $\Theta(H:h;\eta;v)$ and $\Theta(H'';h;\eta;v)$, respectively. Since $\Theta$ is supported on $G''$, we have $\Theta(R(x)\varphi_{ik\beta}) = 0$ unless $x \in b_r G''$. Thus it suffices to prove (ii) for $x \in b_r G''$. On $G''$ we have $\Theta = \sum_{1 \leq i \leq n} \Theta''(b_i)$, where $\Theta'' = \Theta''(b_i \ldots x b_i)$. Thus we have $\Theta(R(h,x)\varphi_{ik\beta}) = \sum_{1 \leq i \leq n} \Theta''(R(a_i b_i \ldots x b_i))$. We can assume as in [6, 6.5] that $b_1 = 1, b_2, \ldots, b_n$ normalize $H$. Let $w_1 \in W(G, H)$ be the Weyl group element represented by $b_1$. Then $(\Theta(H'';h;\eta;v))'' = \chi(R(h)) = \Theta(H'';h;\eta;v)$ is the character of the $H''$-series representation with parameters $w_1(\lambda_0 + \lambda_M(h)) \in i^* w_1(\lambda_0) \in Z_M(M^0), \eta \in i^* a^*$, and $w_1 \eta \in \hat{E}$. By part (ii) of Theorem 10.11 and the uniqueness of the Plancherel formula representation of $\varphi_{ik\beta}, \Theta(H'';h;\eta;v) \varphi_{ik\beta} \equiv 0$ unless

$$w_1(\chi(h)) = s\chi(h') \text{ and } w_1(\lambda_0 + \lambda_M(h)) = s(\lambda_0 + \lambda_M(h')). \quad (10.18)$$

Let $S = \{ 1 \leq t \leq n \}$ satisfies 10.18. Note $[S] \geq 1$ since $1 \in S$. Fix $t \in S$ and write $w = w_t$. Since $w$ and $1^w$ represent the same coset of $G/G''$, we may as well assume that $w_2 = \xi_0, w\lambda_0 = \lambda_0, \text{ and } w\xi = \xi', \text{ so that } \Theta(H'';h;\eta;v)'' = \Theta(H'';wh;\eta;wv)$ as in [6, 6.8], $m(wh;wv) = m(h;v)$ so
we can change variables in the integration and apply part (ii) of Theorem 10.11 to write

\[ c \int_{\mathcal{H} \times \mathcal{F} \times \mathcal{E}} \Theta(H':h:v;\eta)^n(R(a, b_i^{-1}x'' b_i) \varphi_{\beta}') m(h:v) \, dh \, dv \, d\eta \]

\[ = c \int_{\mathcal{H} \times \mathcal{F} \times \mathcal{E}} \Theta(H'';h:v;\eta)(R(a, b_i^{-1}x'' b_i) \varphi_{\beta}') m(h:v) \, dh \, dv \, d\eta \]

\[ = \varphi_{\beta}'(a, b_i^{-1}x'' b_i) = \varphi_{\beta}(b, x''). \]

Thus

\[ \varphi_{\beta}(b, x'') = c' \int_{\mathcal{H} \times \mathcal{F} \times \mathcal{E}} \Theta(H:h:v;\eta)(R(b, x'') \varphi_{\beta}) m(h:v) \, dh \, dv \, d\eta, \]

where \( c' = c/[\mathcal{S}] \).

Q.E.D.

REFERENCES