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The Schwartz Space of a General Semisimple Lie Group. I. Wave Packets of Eisenstein Integrals

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Contents.

1. INTRODUCTION

Suppose G is a semisimple Lie group with infinite center. As in the finite center case, the tempered spectrum of G consists of families of representations induced from cuspidal parabolic subgroups $P = MAN$. However, in the infinite center case, the representations of M to be induced are not discrete series, but are relative discrete series which occur in continuous families. In two previous papers [8, 9] we studied matrix coefficients of relative discrete series representations for a connected simple group with

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infinite center and showed how to combine them into wave packets along the continuous parameter to construct Schwartz functions on the group.

In this paper we construct Schwartz class wave packets of matrix coefficients corresponding to the various induced series of tempered representations for any connected reductive Lie group G. In the special case that $G = P = M$, these are exactly the wave packets of relative discrete series matrix coefficients constructed in [8, 9]. To study induced representations on G we must be able to define continuous families of relative discrete series representations on the Levi components M of arbitrary cuspidal parabolic subgroups. These groups M need not be connected, but lie in the class of reductive groups studied in [11]. (See (10.1) for the precise definition.) Since we do not prove our wave packets are Schwartz by induction on G, but rather by induction on parabolic subgroups of G, we do not need to assume G is a general group in the class (10.1). This simplifies the constructions of the machinery needed to define wave packets. In Section 10, we indicate how the results for connected groups can be extended to arbitrary groups in the class (10.1).

To construct wave packets we proceed as follows. We start with continuous families of relative discrete series matrix coefficients on M of the type constructed in [9]. We use these to form Eisenstein integrals similar to those defined by Harish-Chandra in [3]. Thus our Eisenstein integrals have two types of continuous parameters, those corresponding to unitary characters of A which occur in Harish-Chandra's Eisenstein integrals, plus additional continuous parameters coming from families of relative discrete series representations on M. Wave packets must be taken along both types of continuous parameters to obtain Schwartz class functions on the group G. Roughly speaking, the wave packets considered will be integrals of the form

$$F_\alpha(x) = \iint E(P:h:\nu:x)\,\alpha(h:\nu)\,m(h:\nu)\,dh\,d\nu, \qquad x \in G, \tag{1.1}$$

where $E(P:h:\nu:x)$ is an Eisenstein integral corresponding to $\nu \in \hat{A}$ and to a continuous family of matrix coefficients for relative discrete series representations π_h of M; $m(h:\nu)\,dh\,d\nu$ is the Plancherel measure corresponding to the associated family of induced representations $\pi_{h,\nu} = \mathrm{Ind}_P^G(\pi_h \otimes \nu \otimes 1)$ of G; and $\alpha(h:\nu)$ is a suitable Schwartz function in the parameter variables.

The space $\hat{A}$ is a Euclidean space, and Harish-Chandra proved in the finite center case that a necessary and sufficient condition for the wave packet F_α to be a Schwartz function on G is that α be an ordinary Schwartz function on $\hat{A}$ [5]. The infinite center situation is more complicated. First, α must be a jointly smooth function of h and ν which decays rapidly at infinity as for an ordinary Schwartz function. However, as a function of h, α must also decay rapidly, in the sense of having a zero of infinite order, as h approaches values on walls where π_h is a limit of discrete series representation. Finally, there are conditions on α at points (h_0, ν_0) for which the induced representation π_{h_0, ν_0} is reducible. One way to phrase this condition is to require that the product $\alpha(h:\nu)\, m(h:\nu)$ be jointly smooth in h and ν. This is a restriction only at points (h_0, ν_0) as above where the Plancherel function $m(h:\nu)$, which is separately smooth in each variable, fails to be jointly continuous.

In this paper we make a sightly stronger assumption on α, namely, that $\alpha(h:\nu)\, m_R(h:\nu)$ is jointly smooth in h and ν, where $m_R(h:\nu)$ is part of the Plancherel function. (See (9.12) for the definition.) Points (h_0, ν_0) at which $m_R(h:\nu)$ is not smooth, but $m(h:\nu)$ is, correspond to induced representations π_{h_0, ν_0} which only fail to be reducible because certain limits of discrete series are zero. For such α, we are able to prove that F_α is a Schwartz function using Harish-Chandra's theory of the c-function, in particular the result which says that for a fixed discrete series representation of M, the Plancherel measure cancels the poles of the c-function considered as a meromorphic function of ν. In order to remove this extra assumption on α, we will need to know more about the c-function as a meromorphic function of h and ν jointly. These results will also be needed to study the "mixed wave packets" described in the next paragraph, and so are deferred to the paper in which we will study the mixed wave packets.

In the finite center case, Harish-Chandra proved that every K-finite Schwartz function on G is a finite sum of Schwartz wave packets of Eisenstein integrals coming from the various series of tempered representations [5]. (Of course, discrete series representations have no continuous parameters so the degenerate wave packets in this case are just single matrix coefficients.) In the infinite center case, the analogue of K, the maximal compact subgroup, is non-compact, and there are no K-finite functions in the Schwartz space of G. However, there is a dense subspace of the Schwartz space consisting of "K-compact" functions, that is, ones for which the K-types are restricted to lie in a compact subset of $\hat{K}$. (See [9].) However, it is not true in the infinite center case that a K-compact Schwartz function will be a finite sum of Schwartz wave packets of the type described above. The problem comes from the fact that the different series of tempered representations interfere where a reducible principal series representation breaks up as a sum of limits of relative discrete series representations. As a result of this interference between series, not all Schwartz functions on the group decompose as sums of Schwartz wave packets. The typical K-compact Schwartz function on G breaks up as a sum of pieces from different series of representations which individually are not Schwartz functions, but which "patch together" at reducible principle series and limits of discrete series to form a Schwartz function. The wave packets studied in this paper are the ones which patch together with the zero wave packet from all other series. In another paper we will study the mixed wave packets which have non-trivial patches.

The organization of this paper is as follows. In Section 2 we develop some structural information about G and reductive components of its cuspidal parabolic subgroups. We also recall the basic definitions of the Schwartz space.

In Section 3 we discuss the parameterization of relative discrete series representations on M and the corresponding continuous families of induced representations on G.

In Section 4 we extend results on holomorphic families of relative discrete series matrix coefficients which were proved in [9] for the case that M is a simple, simply connected group of hermitian type, to all Levi components M of cuspidal parabolic subgroups.

In Section 5 we reformulate the results of Section 4 as results on holomorphic families of spherical functions, and extend the growth estimates proved in [8, 9] to our general class of groups M.

In Section 6 we define holomorphic families of Eisenstein integrals and check that they are eigenfunctions of the center of the enveloping algebra.

In Sections 7 and 8 we extend the machinery developed by Harish-Chandra to study growth properties of abstract families of functions generalizing Eisenstein integrals to include dependence on the continuous parameters coming from the relative discrete series. Specifically, in Section 7 we use differential equations to sharpen a priori estimates, and in Section 8 we use these estimates to show that wave packets formed from a certain class of functions are Schwartz.

In Section 9 we show that the Eisenstein integrals defined in Section 6 are members of the abstract family studied in Section 7 and use the results of Section 8 to show that wave packets of Eisenstein integrals of the type described in this section are Schwartz functions. Further, we show that when these Schwartz functions are written in terms of tempered characters using the Plancherel formula, only the series of representations used to form the Eisenstein integrals occurs in the expansion.

In Section 10 we show how to extend the results of Section 9 from the case of connected groups to arbitrary groups in the class (10.1).

2. GROUP STRUCTURE

Throughout the first nine sections of this paper, G is a connected reductive Lie group. Fix a Cartan involution θ of G as in [11] and let K denote the fixed point set of θ. It is the full inverse image of a maximal compact subgroup of the linear group G/Z_G, but is compact only when the center Z_G of G is compact.

PROPOSITION 2.1. *K has a unique maximal compact subgroup K_1 and has a closed normal vector subgroup V such that*

(a) $K = K_1 \times V$,

(b) *$Z = Z_G \cap V$ is co-compact in both V and Z_G.*

Proof. Let p: $'G \to G$ be the universal covering and $'K = p^{-1}(K)$. Then $'K$ is direct product of the compact semisimple group $['K, 'K]$ and a vector group $'W$. Let $'Z = Z_{'G} \cap 'W$; it has finite index in $Z_{'G}$ and is co-compact in $'W$. Let $'U$ be the subspace of $'W$ spanned by $\mathrm{Ker}(p) \cap 'W$ and define $K_1 = p(['K, 'K] \times 'U)$. Then K_1 is the unique maximal compact subgroup of K.

Decompose $'W = 'U \oplus 'V$ such that $Z_{'G} \cap 'V$ is co-compact in $'V$. Then $p|_{'V}$ is one to one, so $V = p('V)$ is a closed vector subgroup of K, and $K = K_1 \times V$.

For (b), $(\mathrm{Ker}(p) \cap 'W) \times (Z_{'G} \cap 'V)$ is co-compact in $'U \oplus 'V = 'W$, so $p(Z_{'G} \cap 'V) = Z_G \cap V$ is co-compact in V. Also, $Z_{'G} \cap 'W$ has finite index in $Z_{'G}$, and $(\mathrm{Ker}(p) \cap 'W) \times (Z_{'G} \cap 'V)$ has finite index in $Z_{'G} \cap 'W$, so $Z_G \cap V = p(Z_{'G} \cap 'V)$ has finite index in $p(Z_{'G}) = Z_G$. Q.E.D.

Let P be a cuspidal parabolic subgroup of G. In other words, up to G-conjugacy, P is given as follows. Start with a θ-stable Cartan subgroup $H \subset G$. Then $H = T \times A$, where $T = H \cap K$ and $A = \exp \mathfrak{a}$, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under θ. Then $Z_G(A) = M \times A$, where $\theta M = M$, we can choose a positive system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{a})$ of restricted roots, and $N = \exp(\mathfrak{n})$, where $\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Now $P = MAN$. Note that $Z \subset M$ and that although M need not be connected, it is in the class of reductive groups studied in [11, 6, 7]. (See (10.1) for the precise definition.)

2.2 *Remark.* Write $K_M^0 = M^0 \cap K$. Our structural results (2.1) for K can be applied to K_M^0 to write $K_M^0 = K_{1,M}^0 \times V_M$ where $K_{1,M}^0 = K_1 \cap M^0$ is maximal compact in K_M^0. Note that $V_M \not\subset V$ in general. Write $M^\dagger = Z_M(M^0) M^0$ and $K_M^\dagger = K \cap M^\dagger$.

In order to define spherical functions we will need to consider the larger group $K_M^V = K_M^\dagger \cdot V$. Note this is a group since V centralizes K. We can write $K_M^V = K_{M,1}^V \times V$, where $K_{M,1}^V = K_M^V \cap K_1$ is the unique maximal compact subgroup of K_M^V.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, ± 1 eigenspace decomposition under θ. Choose a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{p}$ and a positive restricted root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{a}_0)$. As usual, $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} m(\alpha)\alpha$, where $m(\alpha) = \dim \mathfrak{g}_\alpha$. The Iwasawa decomposition

$$G = N_0 A_0 K, \qquad x = n(x) \cdot \exp H(x) \cdot \kappa(x) \tag{2.3}$$

specifies the zonal spherical function on G for $0 \in \mathfrak{a}_0^*$,

$$\Xi(x) = \int_{K/Z} e^{-\rho(H(kx))} \, d(kZ). \tag{2.4}$$

It is the lift of the corresponding function on the linear semisimple group G/Z_G, and thus if σ is defined as in (2.7) there are constants $C, q \geqslant 0$ so that

$$e^{-\rho}(a) \leqslant \Xi(a) \leqslant C(1+\sigma(a))^q \, e^{-\rho}(a) \tag{2.5}$$

for all $a \in A_0^+ = \{a \in A_0 : \alpha(\log a) > 0 \text{ for all } \alpha \in \Phi^+\}$.

Growth in G is determined by a function $\tilde{\sigma}: G \to \mathbb{R}^+$ which is defined as follows. Choose an $\mathrm{Ad}_G(K)$-invariant positive definite inner product on V. If $x \in G$ we decompose

$$x = v(x) \cdot k_1(x) \cdot \exp \xi(x) \in VK_1 \cdot \exp(\mathfrak{p}) \tag{2.6}$$

and then we set

$$\sigma_V(x) = \|v(x)\|, \quad \sigma(x) = \|\xi(x)\|, \qquad \text{and} \qquad \tilde{\sigma}(x) = \sigma_V(x) + \sigma(x). \tag{2.7}$$

The main properties of σ are

$$\sigma(k_1 x k_2) = \sigma(x) \qquad \text{for all} \quad x \in G, \quad k_1, k_2 \in K; \tag{2.8a}$$

and

$$\sigma(xy) \leqslant \sigma(x) + \sigma(y) \qquad \text{for all} \quad x, y \in G. \tag{2.8b}$$

The corresponding properties of $\tilde{\sigma}$ are

$$\tilde{\sigma}(kxk^{-1}) = \tilde{\sigma}(x) \qquad \text{for all} \quad x \in G, k \in K; \tag{2.9a}$$

$$\tilde{\sigma}(k_1 x k_2) = \tilde{\sigma}(x) \qquad \text{for all} \quad x \in G, k_1, k_2 \in K_1; \tag{2.9b}$$

$$\tilde{\sigma}(xy) \leqslant 3(\tilde{\sigma}(x) + \tilde{\sigma}(y)) \qquad \text{for all} \quad x, y \in G. \tag{2.9c}$$

Let W be a Banach space and $f \in C^\infty(G:W)$. If $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$ and $r \in \mathbb{R}$ we define

$$_{D_1}\|f\|_{r,D_2} = \sup_{x \in G} \, (1+\tilde{\sigma}(x))^r \, \Xi(x)^{-1} \, \|f(D_1 ; x ; D_2)\|_W. \tag{2.10a}$$

The Schwartz space is

$$\mathscr{C}(G:W) = \{ f \in C^\infty(G:W) : {}_{D_1}\|f\|_{r,D_2} < \infty$$
$$\text{for all } D_1, D_2 \in \mathscr{U}(\mathfrak{g}) \text{ and all } r \in \mathbb{R} \}. \tag{2.10b}$$

It is a complete locally convex topological vector space with the topology defined by the seminorms (2.10a). And of course the most important case is

$$\mathscr{C}(G) = \mathscr{C}(G:\mathbb{C}). \tag{2.10c}$$

PROPOSITION 2.11. *Define* $\kappa_V = G \to V$ by $x \in \kappa_V(x)\, K_1 A_0 N_0$, *where* $G = KA_0N_0 = VK_1A_0N_0$ *is the Iwasawa decomposition. Then* $\kappa_V(\exp \mathfrak{p})$ *is bounded.*

It suffices to prove Proposition 2.11 in the case where G is simply connected, non-compact, and of hermitian type.

LEMMA 2.12. *There exists* $C > 0$ *such that* $\sigma_V(xy) \leqslant C + \sigma_V(x) + \sigma_V(y)$ *for all* $x, y \in G$.

LEMMA 2.13. *There exists* $C' > 0$ *such that* $\sigma_V(n) \leqslant C'$ *for all* $n \in N_0$.

Proof of Proposition from Lemmas. Let $\xi \in \mathfrak{p}$, $\exp \xi = \kappa(\exp \xi) \cdot a \cdot n$ with $\kappa(\exp \xi) \in K$, $a \in A_0$, and $n \in N_0$. Using Lemma 2.12 $\|\kappa_V(\exp \xi)\| = \sigma_V(\kappa_V(\exp \xi)) = \sigma_V(\kappa(\exp \xi)) = \sigma_V((\exp \xi)\, n^{-1} a^{-1}) \leqslant C + \sigma_V(\exp \xi) + \sigma_V(n^{-1}a^{-1}) = C + \sigma_V(n^{-1}a^{-1}) \leqslant 2C + \sigma_V(n^{-1}) + \sigma_V(a^{-1}) = 2C + \sigma_V(n^{-1})$. Now $\|\kappa_V(\exp \xi)\| \leqslant 2C + C'$ by Lemma 2.13. Q.E.D.

2.14 *Proof of* (2.12). Write $x = vk \exp \xi$ and $y = v'k' \exp \xi'$, where $v, v' \in V$, $k, k' \in K_1$, and $\xi, \xi' \in \mathfrak{p}$. Then

$$xy = (vv')(kk')((\exp \xi'')(\exp \xi')) \qquad \text{where} \quad \xi'' = \operatorname{Ad}(v'k')^{-1}(\xi).$$

Write $(\exp \xi'')(\exp \xi') = wh \exp \eta$, where $w \in V$, $h \in K_1$, and $\eta \in \mathfrak{p}$. Then $xy = (vv'w)(kk'h)(\exp \eta)$, so

$$\sigma_V(xy) = \|vv'w\| \leqslant \|v\| + \|v'\| + \|w\| = \|w\| + \sigma_V(x) + \sigma_V(y).$$

Use $G \subset P_+ \tilde{K}_{\mathbb{C}} P_-$ from [8, Theorem 2.17], $b = p_+(b) \cdot \tilde{\kappa}(b) \cdot p_-(b)$, and let $\tilde{\kappa}_V(b)$ denote the $V_{\mathbb{C}}$ projection of $\tilde{K}_{\mathbb{C}} = V_{\mathbb{C}} \times (K_1)_{\mathbb{C}}$. Then

$$\|w\| = \sigma_V((\exp \xi'')(\exp \xi')) = \|\operatorname{Im} \tilde{\kappa}_V((\exp \xi'')(\exp \xi'))\|$$

by [9, Lemma 10.6], and that is bounded by some constant C according to [9, Lemma 10.7]. Q.E.D.

2.15 *Proof of* (2.13). Let $\lambda \in \Phi^+_{\mathfrak{a}_0}$, $\xi_\lambda \in \mathfrak{g}_\lambda$, and $\eta_\lambda = \theta \xi_\lambda \in \mathfrak{g}_{-\lambda}$. Assume $\xi_\lambda \neq 0$ so that $\xi_\lambda, \eta_\lambda$, and $h_\lambda = [\xi_\lambda, \eta_\lambda] \in \mathfrak{a}_0$ span a three-dimensional simple algebra $\mathfrak{g}[\lambda]$. Then $\zeta_\lambda = \xi_\lambda + \eta_\lambda$ spans $\mathfrak{k}[\lambda] = \mathfrak{k} \cap \mathfrak{g}[\lambda]$, h_λ spans $\mathfrak{a}[\lambda] = \mathfrak{a}_0 \cap \mathfrak{g}[\lambda]$, ξ_λ spans $\mathfrak{n}[\lambda] = \mathfrak{n}_0 \cap \mathfrak{g}[\lambda]$, and $\{h_\lambda, \xi_\lambda - \eta_\lambda\}$ spans $\mathfrak{p}[\lambda] = \mathfrak{p} \cap \mathfrak{g}[\lambda]$. The analytic subgroup $G[\lambda]$ for $\mathfrak{g}[\lambda]$ has form $G[\lambda] =$

$K[\lambda]\cdot\exp\mathfrak{p}[\lambda]$, so $\exp\xi_\lambda=\exp(\zeta)\exp(\xi)$, where $\zeta\in\mathfrak{k}[\lambda]$ and $\xi\in\mathfrak{p}[\lambda]$. If $\zeta_\lambda\in[\mathfrak{k},\mathfrak{k}]=\mathfrak{k}_1$ then $\zeta\in\mathfrak{k}_1$ and $\sigma_{I'}(\exp\xi_\lambda)=0$. Now write $\zeta=\zeta_0+\zeta_1$, $\zeta_0\in\mathfrak{v}$, and $\zeta_1\in\mathfrak{k}_1$, so $\sigma_{I'}(\exp\xi_\lambda)=\|\zeta_0\|\leqslant\|\zeta\|$. An $\widetilde{SL}(2;\mathbb{R})$ calculation shows that $\cos(r\,\|\zeta\|)\neq 0$, where r is a constant that depends only on $\mathfrak{g}$ and $\mathfrak{g}[\lambda]$, so $\|\zeta\|<\pi/2r$.

Summary: given $\lambda\in\Phi^+_{\mathfrak{a}_0}$ we have $C_\lambda>0$ such that $\sigma_{I'}(\exp\xi_\lambda)\leqslant C_\lambda$ for all $\xi_\lambda\in\mathfrak{g}_\lambda$.

Let $n\in N_0$. Then we can write $n=\prod_{\lambda\in\Phi^+_{\mathfrak{a}_0}}\exp(\xi_\lambda)$, where $\xi_\lambda\in\mathfrak{g}_\lambda$. Using Lemma 2.12, $\sigma_{I'}(n)\leqslant|\Phi^+_{\mathfrak{a}_0}|C+\sum_{\lambda\in\Phi^+_{\mathfrak{a}_0}}\sigma_{I'}(\exp\xi_\lambda)\leqslant|\Phi^+_{\mathfrak{a}_0}|C+\sum C_\lambda$.

Q.E.D.

3. Discrete Series Parameters

Fix a θ-stable Cartan subgroup $H\subset G$ and an associated cuspidal parabolic subgroup $P=MAN$. Here $H=T\times A$, where $T=H\cap K$ and $A=\exp(\mathfrak{a})$, $\mathfrak{a}=\mathfrak{h}\cap\mathfrak{p}$, and $Z_G(A)=M\times A$ with $\theta M=M$. In this section we discuss holomorphic families of coefficients of relative discrete series representations of M.

Fix a positive root system $\Phi^+_M=\Phi^+(\mathfrak{m},\mathfrak{t})$ such that

$$\text{if } \mathfrak{m}_1 \text{ is a non-compact simple ideal of hermitian type in } \mathfrak{m} \text{ then } \Phi^+(\mathfrak{m}_1,\mathfrak{t}\cap\mathfrak{m}_1) \text{ contains a unique non-compact simple root.} \tag{3.1}$$

We denote $\Phi^+_{M,K}=\Phi^+(\mathfrak{k}_M,\mathfrak{t})$ and write ρ_M and $\rho_{M,K}$ for the half sums over Φ^+_M and $\Phi^+_{M,K}$. Define

Λ_M: all $\lambda\in i\mathfrak{t}^*$ such that

$$\begin{aligned}&\lambda-\rho_M \text{ is integral, i.e., } e^{\lambda-\rho_M} \text{ is defined on } T^0\\&\lambda \text{ is } \Phi_M\text{-non-singular, i.e., } \langle\lambda,\alpha\rangle\neq 0 \text{ for } \alpha\in\Phi_M\\&\lambda \text{ is } \Phi^+_{M,K} \text{ dominant, i.e., } \langle\lambda,\alpha\rangle\geqslant 0 \text{ for } \alpha\in\Phi^+_{M,K}.\end{aligned} \tag{3.2a}$$

Write $\mathfrak{t}=\mathfrak{t}_1\oplus\mathfrak{v}_M$ where $\mathfrak{t}_1=\mathfrak{t}\cap\mathfrak{k}_1$, and set $i\mathfrak{v}^*_M=\{\lambda\in i\mathfrak{t}^*:\lambda(\mathfrak{t}_1)=0\}$. Then Λ_M is the disjoint union of subsets

$$\{\lambda_h=\lambda_0+h_M: h_M\in i\mathfrak{v}^*_M \text{ and } \beta(\lambda_h)\neq 0 \text{ for } \beta\in\Phi^+_M\backslash\Phi^+_{M,K}\}, \tag{3.2b}$$

where λ_0 belongs to the discrete set

$$\begin{aligned}\Lambda_{M,0}\text{: all } \lambda_0\in i\mathfrak{t}^* \text{ such that } &\lambda_0-\rho_M \text{ is integral,}\\&\lambda_0 \text{ is } \Phi^+_{M,K}\text{-dominant, and } \Phi^+_{M,K} \text{ non-singular,}\\&\text{and } \lambda_0(\mathfrak{v}_M)=0.\end{aligned} \tag{3.3}$$

If $\lambda \in \Lambda_M$ then π_λ^0 denotes the corresponding relative discrete series representation of M^0. Thus the relative discrete series of M^0 is the disjoint union of *continuous families*

$$\{\lambda_0 + h_M : h_M \in \mathscr{D}_M = \mathscr{D}_M(\lambda_0)\}, \tag{3.4a}$$

where $\lambda_0 \in \Lambda_{M,0}$ and $\mathscr{D}_M$ is a topological component of

$$\{h_M \in i\mathfrak{v}_M^* : \beta(h_M) \neq -\beta(\lambda_0) \text{ for } \beta \in \Phi_M^+ \backslash \Phi_{M,K}^+\}. \tag{3.4b}$$

To define continuous families of relative discrete series representations on $M^\dagger$ and M we proceed as follows. Recall the decomposition $K = K_1 \times V$ of (2.1). Since V is central in K, each $h \in i\mathfrak{v}^*$ gives a unitary character e^h of K which is trivial on K_1. Consider $i\mathfrak{v}^* \subseteq i\mathfrak{k}^*$ by $i\mathfrak{v}^* = \{h \in i\mathfrak{k}^* : h(\mathfrak{k}_1) = 0\}$. Then for each $h \in i\mathfrak{v}^*$, we can define $h_M(h) = h|_\mathfrak{t}$. Since $\mathfrak{t}_1 \subseteq \mathfrak{k}_1$, $h_M(h) \in i\mathfrak{v}_M^*$. Further, any $h_M \in i\mathfrak{v}_M^*$ can be extended to $\mathfrak{v}_M \oplus \mathfrak{k}_1$ by making it zero on $\mathfrak{k}_1$, and then extended arbitrarily from $\mathfrak{v}_M \oplus \mathfrak{k}_1$ to $\mathfrak{k}$ to give a linear functional $h \in i\mathfrak{v}^*$ with $h_M(h) = h_M$. Thus

$$i\mathfrak{v}_M^* = \{h_M(h) = h|_\mathfrak{t} : h \in i\mathfrak{v}^*\}. \tag{3.5}$$

We can now reparametrize our continuous families on M^0 as follows. Fix $\lambda_0 \in \Lambda_{M,0}$ and let $\mathscr{D} = \{h \in i\mathfrak{v}^* : h_M(h) \in \mathscr{D}_M\}$. Then $\{\lambda_0 + h_M : h_M \in \mathscr{D}_M\} = \{\lambda_0 + h_M(h) : h \in \mathscr{D}\}$. Further, the representation $\pi^0_{\lambda_0 + h_M(h)}$ has Z_{M^0}-character $e^{\lambda_0 - \rho_M + h_M(h)}|_{Z_{M^0}} = e^{\lambda_0 - \rho_M}|_{Z_{M^0}} \otimes e^h|_{Z_{M^0}}$. Let $\chi(0)$ be any element of $Z_M(M^0)^\wedge$ with Z_{M^0}-character $e^{\lambda_0 - \rho_M}$. Since $Z_M(M^0) \subseteq K$ and e^h is a character of all of K, $e^h|_{Z_M(M^0)}$ is a character, which we will also denote by e^h, of $Z_M(M^0)$. Set

$$\chi(h) = \chi(0) \otimes e^h, \qquad h \in i\mathfrak{v}^*. \tag{3.6}$$

Then $\chi = \{\chi(h) : h \in i\mathfrak{v}^*\}$ is a continuous family of irreducible unitary representations of $Z_M(M^0)$, and $\chi(h)$ has the same Z_{M^0}-character $e^{\lambda_0 - \rho_M + h_M(h)}$ as $\pi^0_{\lambda_0 + h_M(h)}$. Thus we can define *continuous families of relative discrete series representations* of $M^\dagger = Z_M(M^0)M^0$ by

$$\pi = \{\pi(h) = \chi(h) \otimes \pi^0_{\lambda_0 + h_M(h)} : h \in \mathscr{D}\} \tag{3.7a}$$

and *continuous families of relative discrete representations* of M by

$$\pi^M = \{\pi^M(h) = \operatorname{Ind}_{M^\dagger}^M \pi(h) : h \in \mathscr{D}\}. \tag{3.7b}$$

The parametrization (3.7) of the relative discrete series of M extends in the obvious way to a parametrization of the tempered series of representations of G associated to H:

$$\pi_{h,\nu} = \operatorname{Ind}_{MAN}^G (\pi^M(h) \otimes e^{i\nu} \otimes 1_N). \tag{3.8a}$$

Here we use normalized induction, so $\pi_{h,\nu}$ is unitary just when $\nu \in \mathfrak{a}^*$. The corresponding *continuous families of H-series representations* of G are the sets

$$\boldsymbol{\pi}_G = \{\pi_{h,\nu} : h \in \mathcal{S}, \nu \in \mathfrak{a}^*\}. \tag{3.8b}$$

Since $\pi^M(h)$ is obtained via induction from $M^\dagger$ to M, we can obtain $\pi_{h,\nu}$ directly as

$$\pi_{h,\nu} = \mathrm{Ind}_{M^\dagger AN}^G (\pi(h) \otimes e^{i\nu} \otimes 1_N). \tag{3.9}$$

We will use this fact to avoid extending results from $M^\dagger$ to M.

We will also use $i\mathfrak{v}^*$ to parametrize families of K_M^0-types of $\pi^0_{\lambda_0 + h_M(h)}$ and $K_M^\dagger$-types of $\pi(h)$. Let

$$\sigma_0^0: \text{irreducible unitary representation of } K_M^0$$
$$\text{with } Z_{M^0}\text{-character } e^{\lambda_0 - \rho_M}; \tag{3.10a}$$
$$\sigma_{h_M}^0 = \sigma_0^0 \otimes e^{h_M}: \text{representation of } K_M^0$$
$$\text{with } Z_{M^0}\text{-character } e^{\lambda_0 - \rho_M + h_M}; \tag{3.10b}$$
$$\sigma_h = \sigma^0_{h_M(h)} \otimes \chi(h): \text{representation of } K_M^\dagger. \tag{3.10c}$$

Note that

$$\sigma_h = \sigma_0 \otimes e^h \qquad \text{for all } h. \tag{3.10d}$$

The representations σ_h are well defined for $h \in \mathfrak{v}_{\mathbb{C}}^*$. We will denote

$$\boldsymbol{\sigma} = \{\sigma_h : h \in \mathfrak{v}_{\mathbb{C}}^*\} \tag{3.11}$$

and will refer to $\boldsymbol{\sigma}$ as a holomorphic family of irreducible representations of $K_M^\dagger$.

Let τ_0 be an irreducible unitary representation of K. Set

$$\tau_h = \tau_0 \otimes e^h \tag{3.12a}$$
$$\boldsymbol{\tau} = \{\tau_h : h \in \mathfrak{v}_{\mathbb{C}}^*\}. \tag{3.12b}$$

Then $\boldsymbol{\tau}$ is called a holomorphic family of irreducible representations of K.

4. Discrete Series Coefficients

Because our continuous families of H-series representations can be induced directly from $M^\dagger AN$, we will be able to define Eisenstein integrals of matrix coefficients for the family $\boldsymbol{\pi}_G$ of induced representations directly from continuous families of matrix coefficients for the family $\boldsymbol{\pi}$ of relative discrete series representations on $M^\dagger$.

Fix a continuous family $\boldsymbol{\pi}$ of relative discrete series representations of $M^\dagger$ as in (3.7), and two holomorphic families $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of irreducible representations of $K_M^\dagger$ as in (3.10). If $h \in \operatorname{bd} \mathscr{D}$ we understand $\pi(h)$ to be the limit, by coherent continuation, of representations $\pi(h')$, $h' \in \mathscr{D}$. Denote

$$\mathscr{H}(\boldsymbol{\pi}:\boldsymbol{\sigma}_j:h) = \mathscr{H}(\pi(h):\sigma_{j,h})\text{: the } \sigma_{j,h}\text{-isotypic subspace of the representation space } \mathscr{H}(\boldsymbol{\pi}:h) = \mathscr{H}(\pi(h)). \tag{4.1}$$

$$\mathscr{V}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2:h)\text{: The linear span in } C^\infty(M^\dagger) \text{ of the } x \to \langle \pi(h)(x)\, w_2, w_1\rangle,\ w_i \in \mathscr{H}(\boldsymbol{\pi}:\boldsymbol{\sigma}_i:h). \tag{4.2}$$

It is an easy consequence of [9, Theorem 4.1] that $\dim \mathscr{H}(\boldsymbol{\pi}:\boldsymbol{\sigma}_j:h)$ is constant for $h \in \operatorname{cl}(\mathscr{D})$; see Lemma 4.9 below.

First we will construct a family $\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ of functions $f \in C^\infty(\mathfrak{v}_{\mathbb{C}}^* \times M^\dagger)$ such that

$$h \to f(h:x) \text{ is holomorphic on } \mathfrak{v}_{\mathbb{C}}^* \text{ for all } x \in M^\dagger \tag{4.3a}$$

and

$$f(h) \in \mathscr{V}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2:h) \qquad \text{when} \quad h \in \operatorname{cl} \mathscr{D}. \tag{4.3b}$$

Functions in the family $\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ will be called *holomorphic families of discrete series coefficients*. In the course of the construction we will prove

THEOREM 4.4. *Fix $h' \in \operatorname{cl} \mathscr{D}$ and $w_j \in \mathscr{H}(\boldsymbol{\pi}:\boldsymbol{\sigma}_j:h)$. Then there exists $f \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ such that $f(h':x) = \langle \pi(h')(x)\, w_2, w_1\rangle$ for all $x \in M^\dagger$.*

THEOREM 4.5. *Let $f \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ and $D_1, D_2 \in \mathscr{U}(\mathfrak{m})$. Then $f(h:D_2;x;D_1) = \sum_{i=1}^n p_i(h) f_i(h:x)$, where the the p_i are polynomials and $f_i \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_{i_1}:\boldsymbol{\sigma}_{i_2})$ for appropriate holomorphic families $\boldsymbol{\sigma}_{i_j}$ of irreducible representations of $K_M^\dagger$.*

Second, we will work out a number of consequences of the construction and of Theorems 4.4 and 4.5.

We construct $\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ by reducing to the case [9, Sects. 5 and 6] of a non-compact simply connected simple group of hermitian type.

4.6. *Case. M^0 is connected and simply connected.* Then $M^0 = M_0 \times M_1 \times \cdots \times M_t$, where M_0 is a vector group and the other M_j are simple, connected, and simply connected. Then

$$\mathscr{D}_M = \mathscr{D}_0 \times \mathscr{D}_1 \times \cdots \times \mathscr{D}_t, \qquad \mathscr{D}_j \subseteq i(\mathfrak{v}_M \cap \mathfrak{m}_j)^*$$

$$\pi^0_{\lambda_0 + h_M} = e^{h_0} \otimes \pi_{\lambda_{1,0} + h_1} \otimes \cdots \otimes \pi_{\lambda_{t,0} + h_t}$$

$$\sigma^0_{j,h_M} = e^{h_0} \otimes \sigma_{j,h_1} \otimes \cdots \otimes \sigma_{j,h_t},$$

where h_i is the projection of h_M to $(\mathfrak{v}_M \cap \mathfrak{m}_i)^*_{\mathbb{C}}$, and where σ_{j,h_i} is the K_{M_i}-factor of σ_{j,h_M}. Note that $\mathscr{H}(\pi^0 : \sigma_j^0 : h_M)$ is finite dimensional, hence is the algebraic tensor product of $\mathbb{C}_{h_0}$ with the $\mathscr{H}(\pi_{\lambda_{i,0}+h_i} : \sigma_{j,h_i})$.

Let $f_0 : (\mathfrak{v}_M \cap \mathfrak{m}_0)^*_{\mathbb{C}} \times M_0 \to \mathbb{C}$ be defined by $f_0(h_0 : x_0) = e^{h_0}(x_0)$. Then f_0 satisfies (4.3) for M_0. Let $\mathscr{F}_0(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ consist of the multiples of f_0. Theorems 4.4 and 4.5 then are trivial for M_0.

Fix $i > 0$. Suppose that M_i either is compact or is not of hermitian type. Then $\mathfrak{v}_M \cap \mathfrak{m}_i = \{0\}$ and $\mathscr{D}_i = \{0\}$. So $h_i = 0$ and we define $\mathscr{F}_i(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ to be the linear span in $C^\infty(M_i)$ of the coefficients $x_i \mapsto \langle \pi_{\lambda_{0,i}}(x_i) w_{2,i}, w_{1,i} \rangle$, $w_{j,i} \in \mathscr{H}(\pi_{\lambda_{0,i}} : \sigma_{j,0})$. Theorems 4.4 and 4.5 are then trivial for M_i.

Suppose that M_i is non-compact and of hermitian type. Then $\mathfrak{v}_M \cap \mathfrak{m}_i$ is the center of $\mathfrak{k}_{M_i}$, one-dimensional, and $\mathscr{D}_i$ is an open finite interval or an open half-line in $i(\mathfrak{v}_M \cap \mathfrak{m}_i)^*$. If $\mathscr{D}_i$ is an open finite interval we define $\mathscr{F}_i(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ to be the restrictions to $i(\mathfrak{v}_M \cap \mathfrak{m}_i)^*_{\mathbb{C}} \times M_i$ of the holomorphic functions defined in [9, Proposition 5.3]. They satisfy (4.3), [9, Theorem 5.14] says that Theorem 4.4 holds for M_i, and [9, Theorem 5.4] and its proof show that Theorem 4.5 holds for M_i.

If $\mathscr{D}_i$ is a half-line, we define $\mathscr{F}_i(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ to be the linear span of the functions defined in [9, Theorem 6.9]. They satisfy (4.3). Here Theorem 4.4 is obvious for M_i, and Theorem 4.5 is an easy calculation in $\mathscr{U}(\mathfrak{m}_i)$.

Define $\mathscr{F}^0(\boldsymbol{\pi}^0 : \boldsymbol{\sigma}_1^0 : \boldsymbol{\sigma}_2^0)$ to be the set of all finite linear combinations of the

$$f(h_M : x) = f_0(h_0 : x_0)\, f_1(h_1 : x_1) \cdot \cdots \cdot f_t(h_t : x_t),$$

where $x = (x_0, x_1, ..., x_t) \in M_0 \times M_1 \times \cdots \times M_t = M^0$. Those functions satisfy (4.3), and Theorems 4.4 and 4.5 hold for M^0 because they hold for the M_i.

4.7. *Case. General* M^0. Let $p : M' \to M^0$ be the universal covering group and $\Gamma = \mathrm{Kernel}(p)$. Then $\mathfrak{k}_M = [\mathfrak{k}_M, \mathfrak{k}_M] \oplus \mathfrak{u}_M \oplus \mathfrak{v}_M$ and $K_{M'} = [K_{M'}, K_{M'}] \times U'_M \times V'_M$, where $\Gamma \subset [K_{M'}, K_{M'}] \times U'_M$, $\Gamma \cap U'_M$ is a lattice in U'_M, and p maps V'_M isomorphically onto V_M. Note that $\mathscr{D}_M \subset i\mathfrak{v}_M^*$ is just $\{h' \in \mathscr{D}_{M'} : h'(\mathfrak{u}_M) = 0\}$, where $\mathscr{D}_{M'} \subset i(\mathfrak{u}_M \oplus \mathfrak{v}_M)^*$ is the topological component of $\{h' \in i(\mathfrak{u}_M \oplus \mathfrak{v}_M)^* : \beta(h') \neq -\beta(\lambda_0)$ for $\beta \in \Phi_M^+ \setminus \Phi_{M,K}^+\}$ that contains $\mathscr{D}_M$.

Let $\boldsymbol{\pi}' = \{\pi'_{\lambda_0 + h'} : h' \in \mathscr{D}_{M'}\}$, where $\pi'_{\lambda_0 + h_M} = \pi^0_{\lambda_0 + h_M} \cdot p$ for $h_M \in \mathscr{D}_M$. Similarly, let $\boldsymbol{\sigma}'_j = \{\sigma'_{j,h'} : h' \in (\mathfrak{u}_M \oplus \mathfrak{v}_M)^*_{\mathbb{C}}\}$, where $\sigma'_{j,h_M} = \sigma^0_{j,h_M} \cdot (p|_{K_{M'}})$ for $h_M \in (\mathfrak{v}_M)^*_{\mathbb{C}}$. Let $\mathscr{F}'(\boldsymbol{\pi}' : \boldsymbol{\sigma}'_1 : \boldsymbol{\sigma}'_2)$ be the set of holomorphic families of relative discrete series coefficients for M' constructed in (4.6). If $f' \in \mathscr{F}'(\boldsymbol{\pi}' : \boldsymbol{\sigma}'_1 : \boldsymbol{\sigma}'_2)$, so $f' : (\mathfrak{u}_M \oplus \mathfrak{v}_M)^*_{\mathbb{C}} \times M' \to \mathbb{C}$, define $f(h_M : x) = f'(h_M : x')$ whenever $x \in M$ and $x' \in p^{-1}(x)$. Define $\mathscr{F}^0(\boldsymbol{\pi}^0 : \boldsymbol{\sigma}_1^0 : \boldsymbol{\sigma}_2^0)$ to consist of all such f. Then f inherits (4.3) from f', and $\mathscr{F}^0$ inherits the conclusions of Theorems 4.4 and 4.5 from $\mathscr{F}'$.

4.8. *Case.* *General* $M^\dagger$. Define $\boldsymbol{\pi}^0 = \{\pi^0_{\lambda_0 + h_M(h)} : \pi^0_{\lambda_0 + h_M(h)} \otimes \chi(h) \in \boldsymbol{\pi}\}$. Write $\sigma_{j,0} = \sigma^0_{j,0} \otimes \chi(0)$ as in (3.10) and set

$$\boldsymbol{\sigma}^0_j = \{\sigma^0_{j,0} \otimes e^{h_M(h)}\}.$$

Let $\mathscr{F}^0(\boldsymbol{\pi}^0 : \boldsymbol{\sigma}^0_1 : \boldsymbol{\sigma}^0_2)$ be the set of holomorphic families of discrete series coefficients for M^0 constructed in (4.7). Let $\psi(0)$ be any matrix coefficient of $\chi(0) \in Z_M(M^0)^\wedge$. Now $\chi(h) = \chi(0) \otimes e^h$, so we can define a holomorphic family of matrix coefficients of $\chi(h)$ by $\psi(h:z) = \psi(0:z)\, e^h(z)$, $z \in Z_M(M^0)$. Since $Z_{M^0} = Z_M(M^0) \cap M^0$, and both $\chi(h)$ and $\pi^0_{\lambda_0 + h_M(h)}$ have Z_{M^0}-character $e^{\lambda_0 + h_M(h) - \rho_M}$, for any $f_0 \in \mathscr{F}^0(\boldsymbol{\pi}^0 : \boldsymbol{\sigma}^0_1 : \boldsymbol{\sigma}^0_2)$ we have a well-defined $f : M^\dagger \times \mathfrak{v}^*_{\mathbb{C}} \to \mathbb{C}$ given by $f(h : zx_0) = \psi(h:z)\, f_0(h:x_0)$ for $z \in Z_M(M^0)$, $x_0 \in M^0$. Define $\mathscr{F}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ to be the space spanned by all such f as $\psi(0)$ ranges over matrix coefficients of $\chi(0)$ and f_0 ranges over $\mathscr{F}^0(\boldsymbol{\pi}^0 : \boldsymbol{\sigma}^0_1 : \boldsymbol{\sigma}^0_2)$. Clearly each $f \in \mathscr{F}(\boldsymbol{\pi}_1 : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ satisfies 4.3, and Theorem 4.4 holds. Finally Theorem 4.5 is satisfied since f_0 satisfies (4.5) and $f(h : D_1; zx_0; D_2) = \psi(h:z)\, f_0(h : D_1; x_0; D_2)$ for any $D_1, D_2 \in \mathscr{U}(\mathfrak{m})$.

We now have constructed a family $\mathscr{F}(\boldsymbol{\pi}_1 : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ of functions f that satisfy (4.3), and for which Theorems 4.4 and 4.5 hold. Next, we consider some consequences of the construction.

LEMMA 4.9. Dim $\mathscr{H}(\boldsymbol{\pi} : \boldsymbol{\sigma}_j : h)$ *is constant for* $h \in \operatorname{cl} \mathscr{D}$.

Proof. This is obvious if M is a vector group. Let M be simple, connected and simply connected. If M is compact or not of hermitian type, it again is trivial. If M is non-compact and of hermitian type, the assertion is [9, Theorem 4.1]. Now, as in (4.6), the lemma follows for M^0 simply connected. The result for general M^0 follows as in (4.7), and the result for $M^\dagger$ is immediate as in (4.8) since deg $\chi(h)$ is independent of h. Q.E.D.

PROPOSITION 4.10. *Fix* $h' \in \operatorname{cl}(\mathscr{D})$. *Then there is a neighborhood* J *of* h' *in* $\operatorname{cl}(\mathscr{D})$, *and a finite subset* $\{f_1, ..., f_r\} \subset \mathscr{F}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$, *such that* $\{f_i(h)\}$ *is a basis of* $\mathscr{V}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2 : h)$ *for every* $h \in J$.

Proof. Let $\{w_p\}$ and $\{v_q\}$ be bases of $\mathscr{H}(\boldsymbol{\pi} : \boldsymbol{\sigma}_2 : h')$ and $\mathscr{H}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : h')$. Theorem 4.4 gives us $\{f_{qp}\} \subset \mathscr{F}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2)$ such that $f_{qp}(h' : x) = \langle \pi(h')(x)\, w_p, v_q \rangle$ for $x \in M^\dagger$. Now $\{f_{qp}(h')\}$ is a basis of $\mathscr{V}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2 : h')$. As in [9, Theorem 5.14 (3)], let P be the complex projective space based on dim $\mathscr{H}(\boldsymbol{\pi} : \boldsymbol{\sigma}_1 : h')$ by dim $\mathscr{H}(\boldsymbol{\pi} : \boldsymbol{\sigma}_2 : h')$ matrices. Then

$$W = \left\{([a_{qp}], h) \in P \times \mathfrak{v}^*_{\mathbb{C}} : \sum a_{qp} f_{qp}(h) = 0\right\}$$

is a holomorphic subvariety of $P \times \mathfrak{v}^*_{\mathbb{C}}$, and projection to $\mathfrak{v}^*_{\mathbb{C}}$ is a proper map whose image omits h', thus omits a neighborhood $\tilde{J}$ of h'. Let

$J = \tilde{J} \cap \mathrm{cl}(\mathscr{S})$. In view of Lemma 4.9, the linearly independent subset $\{f_{qp}(h)\}$, $h \in J$, is a basis of $\mathscr{V}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2:h')$. Q.E.D.

PROPOSITION 4.11. *There is a finite open cover $\{J_1, ..., J_s\}$ of $\mathrm{cl}(\mathscr{S})$ such that if $h' \in J_i$ then we may take $J = J_i$ in Proposition* 4.10.

Proof. Let $\psi_1(0), ..., \psi_s(0)$ be a basis for the space of matrix coefficients of $\chi(0)$. Then $\psi_1(h), ..., \psi_s(h)$ is a basis for the space of matrix coefficients of $\chi(h)$ for all $h \in \mathfrak{v}_{\mathbb{C}}^*$. Suppose $\{f_1, ..., f_r\} \subseteq \mathscr{F}^0(\boldsymbol{\pi}^0:\boldsymbol{\sigma}_1^0:\boldsymbol{\sigma}_2^0)$ give a basis for $\mathscr{V}(\boldsymbol{\pi}^0:\boldsymbol{\sigma}_1^0:\boldsymbol{\sigma}_2^0)$ for every $h \in J$, and $\{f_{ij}\} \subseteq \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ are defined by $f_{ij}(h:zx_0) = \psi_j(h:z) f_i(h:x_0)$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Then clearly $\{f_{ij}(h)\}$ gives a basis for $\mathscr{V}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2:h)$ for all $h \in J$. Thus it is enough to prove the proposition for M^0, and as in (4.7) we can assume that M^0 is simply connected. Then we can decompose $\mathfrak{m} = \mathfrak{v}_{M,0} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\mathfrak{v}_M = \mathfrak{v}_{M,0} \oplus \mathfrak{v}_{M,1} \oplus \mathfrak{v}_{M,2}$, and $\mathscr{S}_M = i\mathfrak{v}_{M,0}^* \times \mathscr{S}_{M,1} \times \mathscr{S}_{M,2}$ with $\mathscr{S}_{M,j} \subseteq i\mathfrak{v}_{M,j}^*$, in such a way that $\mathrm{cl}(\mathscr{S}_{M,1})$ is compact and $\mathscr{S}_{M,2}$ is the product of open half-lines of the form $\mathscr{S}_M \cap i(\mathfrak{v}_M \cap \mathfrak{m}')^*$, where $\mathfrak{m}'$ runs over the simple ideals of $\mathfrak{m}_2$. Along those local factors of M^0, $\mathscr{F}^0(\boldsymbol{\pi}^0:\boldsymbol{\sigma}_1^0:\boldsymbol{\sigma}_2^0)$ was defined [9, Theorem 6.9] to be the span of a set of holomorphic families f^0 that, for every h_M, give a basis of $\mathscr{V}(\boldsymbol{\pi}^0:\boldsymbol{\sigma}_1^0:\boldsymbol{\sigma}_2^0:h_M)$. Thus the result follows from (4.10) using compactness of $\mathrm{cl}(\mathscr{S}_{M,1})$. Q.E.D.

5. SPHERICAL FUNCTIONS

We reformulate the results of Section 4 as results on holomorphic families of spherical functions and prove some inequalities. We need these results for construction of Eisenstein integrals and for certain growth estimates. As in Section 4, we work with $M^\dagger$ rather than M.

LEMMA 5.1. *Let $\boldsymbol{\sigma} = \{\sigma_h\}$ be a holomorphic family of irreducible representations of $K_M^\dagger$ as in* (3.11). *Then there is a holomorphic family $\boldsymbol{\tau} = \{\tau_h\}$ of irreducible representations of K such that each σ_h is a subrepresentation of $\tau_h|_{K_M^\dagger}$.*

Proof. Recall (3.11) that $\sigma_h = \sigma_0 \otimes e^h$ for all h. Let τ_0 be any irreducible summand of $\mathrm{Ind}_{K_M^\dagger}^{K_1}(\sigma_0)$ such that $\tau_0|_{K_M^\dagger}$ contains σ_0. Then the representation $\tau_h = \tau_0 \otimes e^h$ of $K = K_1 \times V$ satisfies $\tau_h|_{K_M^\dagger} \supset \sigma \otimes e^h|_{K_M^\dagger} = \sigma_h$. Q.E.D.

Fix a continuous family $\boldsymbol{\pi}$ of relative discrite series representations of $M^\dagger$, two holomorphic families $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of irreducible representations of $K_M^\dagger$, and holomorphic families $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ of irreducible representations of K as in (5.1).

Denote

$$\zeta(\boldsymbol{\tau}_j:0): V\text{-character of } \tau_{j,0}. \tag{5.2}$$

Then it is clear that

$$\zeta(\boldsymbol{\tau}_j:h)=\zeta(\boldsymbol{\tau}_j:0)e^h$$
$$\text{is the } V\text{-character of } \tau_{j,h} \text{ for all } h \tag{5.2b}$$

and

$$\zeta(\boldsymbol{\sigma}_j:h)=\zeta(\boldsymbol{\tau}_j:h)|_{K_M^\dagger\cap V}$$
$$\text{is the } V\cap K_M^\dagger\text{-character of } \sigma_{j,h} \text{ for all } h. \tag{5.2c}$$

For every $f\in\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$,

$$f(h:z_1xz_2)=\zeta(\boldsymbol{\sigma}_1:h:z_1)\,\zeta(\boldsymbol{\sigma}_2:h:z_2)\,f(h:x)$$
$$\text{for all } h\in\mathfrak{v}_{\mathbb{C}}^*,\ x\in M^\dagger, \text{ and } z_1, z_2\in K_M^\dagger\cap V. \tag{5.3}$$

Define $K_M^V=K_M^\dagger\cdot V$ and $K_{M,1}^V=K_M^V\cap K_1$ as in (2.2). Given $f\in\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ we define

$$F=F(f):\mathfrak{v}_{\mathbb{C}}^*\times M^\dagger\to L^2(K_{M,1}^V\times K_{M,1}^V) \tag{5.4a}$$

by the formula

$$F(h:x)(k_1:k_2)=\zeta(\boldsymbol{\tau}_1:h:v_1^{-1})\,\zeta(\boldsymbol{\tau}_2:h:v_2^{-1})\,f(h:k_{1,M}^{-1}xk_{2,M}^{-1}), \tag{5.4b}$$

where we decompose $k_j\in K_{1,M}^V\subseteq K_M^\dagger\cdot V$ by $k_j=k_{j,M}\cdot v_j$, $k_{j,M}\in K_M^\dagger$, $v_j\in V$. This is well defined because of (5.3).

Extend $\sigma_{j,h}$ to an irreducible representation $\tilde{\sigma}_{j,h}$ of K_M^V by

$$\tilde{\sigma}_{j,h}=\sigma_{j,h}\otimes\zeta(\boldsymbol{\tau}_j:h). \tag{5.5}$$

This is well defined because of (5.2c). Define an irreducible representation σ_j of $K_{M,1}^V$ by

$$\sigma_j=\tilde{\sigma}_{j,h}|_{K_{M,1}^V}. \tag{5.6}$$

LEMMA 5.7. *Let E_{σ_j} denote the σ_j-isotypic subspace of $L^2(K_{M,1}^V)$. Note that E_{σ_2} also is the right σ_2^*-isotypic subspace. Then, for every $h\in\mathfrak{v}_{\mathbb{C}}^*$ and every $f\in\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$, F takes values in the finite dimensional space $E_{\sigma_1}\otimes E_{\sigma_2}$.*

Proof. For $h\in\mathscr{D}$ this is obvious since $f(h)\in\mathscr{V}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2:h)$ and $\sigma_j=\tilde{\sigma}_{j,h}|K_{M,1}^V$ for all h. But both sides of the equation are holomorphic in h, so the equation is valid for all $h\in\mathfrak{v}_{\mathbb{C}}^*$. Q.E.D.

Interpret the action of $K_{M,1}^{V}$ on $W(\sigma_1:\sigma_2)=E_{\sigma_1}\otimes E_{\sigma_2}$ as a double representation $'\sigma=('\sigma_1,'\sigma_2)$,

$$[{}'\sigma(m_1:m_2)\phi](k_1:k_2)=[{}'\sigma_1(m_1)\cdot\phi\cdot{}'\sigma_2(m_2)](k_1:k_2)$$
$$=\phi(m_1^{-1}k_1:k_2m_2^{-1}). \tag{5.8}$$

We obtain a *holomorphic family* $'\boldsymbol{\sigma}=('\boldsymbol{\sigma}_1,'\boldsymbol{\sigma}_2)$ *of double representations* of K_M^V on $W(\sigma_1:\sigma_2)$ by

$$[{}'\sigma_h(z_1m_1:z_2m_2)\phi](k_1:k_1)=[{}'\sigma_{1,h}(z_1m_1)\cdot\phi\cdot{}'\sigma_{2,h}(z_2m_2)](k_1:k_2)$$
$$=\zeta(\boldsymbol{\tau}_1:h:z_1)\,\zeta(\boldsymbol{\tau}_2:h:z_2)\,\phi(m_1^{-1}k_1:k_2m_2^{-1}),$$
$$z_1,z_2\in V,\ m_1,m_2\in K_{M,1}^{V}. \tag{5.9}$$

Now (5.4) defines a *holomorphic family* of $'\boldsymbol{\sigma}$*-spherical functions.*

THEOREM 5.10. *Let $f\in\mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$. Then $F\colon\mathfrak{v}_{\mathbb{C}}^*\times M^\dagger\to W(\sigma_1:\sigma_2)$ is smooth in (h,x) and holomorphic in h. Given $h\in\mathfrak{v}_{\mathbb{C}}^*$, $F(h)$ is $'\sigma_h$-spherical,*

$$F(h:m_1xm_2)={}'\sigma_{1,h}(m_1)\cdot F(h:x)\cdot{}'\sigma_{2,h}(m_2) \tag{5.11}$$

for $x\in M^\dagger$, $m_i\in K_M^\dagger$.

Proof. Smoothness and holomorphicity are clear from (4.3) and (5.4). Write $m_i=z_iy_i$, where $z_i\in V$, $y_i\in K_{M,1}^{V}$, and $k_i=v_ik_{i,M}$ as in (5.4). Then using (5.4) and (5.9),

$$\begin{aligned}
&{}'\sigma_{1,h}(m_1)\cdot F(h:x)\cdot{}'\sigma_{2,h}(m_2)(k_1:k_2)\\
&=\zeta(\boldsymbol{\tau}_1:h:z_1)\,\zeta(\boldsymbol{\tau}_2:h:z_2)\,F(h:x)(y_1^{-1}k_1:k_2y_2^{-1})\\
&=\zeta(\boldsymbol{\tau}_1:h:z_1)\,\zeta(\boldsymbol{\tau}_2:h:z_2)\,\zeta(\boldsymbol{\tau}_1:h:z_1^{-1}v_1^{-1})\\
&\quad\times\zeta(\boldsymbol{\tau}_2:h:z_2^{-1}v_2^{-1})\,f(h:k_{1,M}^{-1}m_1xm_2k_{2,M}^{-1})
\end{aligned}$$

since $y_1^{-1}k_1=(z_1v_1)(m_1^{-1}k_{1,M})$ and $k_2y_2^{-1}=(v_2z_2)(k_{2,M}m_2^{-1})$ in $V\cdot K_M^\dagger$. But this last expression equals $F(h:m_1xm_2)(k_1:k_2)$. Q.E.D.

For the rest of this section we examine the growth estimates for a holomorphic family $F(h:x)$.

Fix a complex neighborhood $\mathscr{D}_{\mathbb{C}}=\mathscr{D}+i\omega$ of $\mathscr{D}$ in $\mathfrak{v}_{\mathbb{C}}^*$, where ω is an open neighborhood of 0 in $i\mathfrak{v}^*$ with compact closure.

Let $\mathfrak{a}_M=\mathfrak{a}_0\cap\mathfrak{m}$ and choose a positive restricted root system $\Phi_M^+=\Phi^+(\mathfrak{m},\mathfrak{a}_M)$. Let $\rho_{\mathfrak{a}_M}=\frac{1}{2}\sum_{\alpha\in\Phi_M^+}\alpha$ and $A_M^+=\{a\in A_M:\alpha(\log a)>0$ for all $a\in\Phi_M^+\}$. Because of the Cartan decomposition $M^\dagger=K_M^\dagger\,\mathrm{cl}(A_M^+)\,K_M^\dagger$, any $'\boldsymbol{\sigma}$-spherical function on $M^\dagger$ is determined by its restriction to $\mathrm{cl}(A_M^+)$.

THEOREM 5.12. *Let* $f \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ *and* $F = F(f)$. *Let* $D_1, D_2 \in \mathscr{U}(\mathfrak{m})$. *Then there are constants* $c, m \geqslant 0$ *such that* $\|F(h:D_1;a;D_2)\| \leqslant c(1+\|h\|)^m (1+\sigma(a))^m |\lambda e^{\omega(h)}(a)|$ *for* $h \in \mathrm{cl}(\mathscr{D}_{\mathbb{C}})$, *and* $a \in \mathrm{cl}(A_M^+)$, *where* $\omega(h) \in (\mathfrak{a}_M)^*_{\mathbb{C}}$, $|e^{\omega(h)}(a)| < e^{-\rho_{\mathfrak{a}_M}}(a)$ *for all* $a \in \mathrm{cl}(A_M^+) \backslash \{1\}$, *and* $h \mapsto \omega(h)$ *is piecewise linear on* $\mathscr{D}_{\mathbb{C}}$.

Proof. *Case* I. We will first prove the theorem in the case that $D_1 = D_2 = 1$. As in (4.8) we write $f(h:zx_0) = \psi(h:z) f^0(h:x_0)$ for some holomorphic family $\psi(h)$ of matrix coefficients for $\chi(h)$ and some $f^0 \in \mathscr{F}^0(\boldsymbol{\pi}^0:\boldsymbol{\sigma}_1^0:\boldsymbol{\sigma}_2^0)$. Now $\|F(h:a)\| = \sup_{k_1 k_2 \in K^\dagger_{M,1}} |f(h:k_1^{-1} a k_2^{-1})|$. Since $K^\dagger_{M,1}$ is compact, $[K^\dagger_{M,1}/K^0_{M,1}]$ is finite. Let $\gamma_1, \ldots, \gamma_r \in Z_M(M^0) \cap K^\dagger_{M,1}$ be coset representatives. Then we can write $\|F(h:a)\| = \sup_{i,j} |\psi(h:\gamma_i^{-1}\gamma_j^{-1})| \times \sup_{k_1 k_2 \in K^0_{M,1}} |f^0(h:k_1^{-1} a k_2^{-1})|$. Now since $\gamma_i^{-1}\gamma_j^{-1} \in K^\dagger_{M,1}$, $\psi(h:\gamma_i^{-1}\gamma_j^{-1}) = \psi(0:\gamma_i^{-1}\gamma_j^{-1})$ for all h. Thus there is a constant C with $\sup_{i,j} |\psi(h:\gamma_i^{-1}\gamma_j^{-1})| \leqslant C$ for all h. Let F^0 be the spherical function with values in $L^2(K^0_{M,1} \times K^0_{M,1})$ given by $F^0(h:x)(k_1:k_2) = f^0(h:k_1^{-1} x k_2^{-1})$. Then we have shown that $\|F(h:a)\| \leqslant C\, \|F^0(h:a)\|$. Thus it is enough to prove the result for F^0 on M^0. But for general M^0, our estimate can be obtained as in (4.7) by restricting the parameters in the corresponding estimate on the universal covering group of M^0. Thus we can assume that M^0 is simply connected.

As in (4.6) we decompose $M^0 = M_0 \times \cdots \times M_t$. We can assume f^0 is a product $f^0 = f_0 \cdots f_t$ as in (4.6) with $f_i \in \mathscr{F}_i(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$. Now $K^0_{M,1} = K_{M_0,1} \times \cdots \times K_{M_t,1}$ so $\|F^0(h:a)\| = \|F_0(h_0:a_0)\| \cdots \|F_t(h_t:a_t)\|$, where $F_i(h_i:a_i) \in L^2(K_{M_i,1} \times K_{M_i,1})$ is the spherical function corresponding to f_i. Thus it is enough to prove the theorem when $M^0 = M_i$ for some $0 \leqslant i \leqslant t$.

For $i = 0$, M_0 is a real vector group so $K_{M_0,1} = \{1\}$ and $F_0 : \mathfrak{v}^*_{0,\mathbb{C}} \times M_0 \to \mathbb{C}$ is given by $F_0(h_0:x_0) = e^{h_0}(x_0)$. Since $A_{M_0} = \{1\}$, $F_0(h_0:a_0) \equiv 1$.

For $1 \leqslant i \leqslant t$, M_i is simple, connected, and simply connected. If M_i is compact, then $A_{M_i} = \{1\}$ and $\mathfrak{v}_i = \{0\}$ so the result is trivial.

If M_i is non-compact and not of hermitian type, then $\mathfrak{v}_i = 0$, and the assertion reduces to a standard estimate for discrete series coefficients [1].

If M_i is non-compact and of hermitian type, then $\mathfrak{v}_i$ is a line, and $\mathscr{D}_i$ is an open interval in $i\mathfrak{v}_i^*$. If $\mathrm{cl}(\mathscr{D}_i)$ is compact, then for $h_i \in \mathscr{D}_i$ the assertion is just [9, Theorem 8.1].

If $h_i = h_R + h_I$ with $h_R \in \mathrm{cl}(\mathscr{D}_i)$ and $h_I \in \mathrm{cl}(i\omega)$, then the proof of [9, Theorem 8.1] extends the assertions from $F(h_R:a)$ to $F(h_i:a)$, for the absolute values of the exponential terms involved depend only on h_R, while the coefficient functions are bounded on compact sets, so that $\|F(h_R + h_I:a)\|$ satisfies the same type of estimate as $\|F(h_R:a)\|$ over the compact set $\mathrm{cl}(i\omega)$.

If $\mathrm{cl}(\mathscr{D}_i)$ is non-compact, then $\mathscr{D}_i$ has the form (h_0, ∞), and the above argument holds for h_R in an initial segment $[h_0, h_0 + \varepsilon]$ of $\mathrm{cl}(\mathscr{D}_i)$. For $h_R \in$

$[h_0+\varepsilon, \infty)$, we combine [9, Proposition 6.16] with the explicit formula [8, Theorem 5.1]. In the latter, both sides are holomorphic when $h_R \in (h_0, \infty)$, the absolute values of the exponential terms depend only on h_R, and the remaining dependence on $h_i \in [h_0+\varepsilon, \infty)+\mathrm{cl}(i\omega)$ is polynomial, so the estimates of [8, Corollary 5.2] extend to $h_i \in \mathrm{cl}(\mathscr{D}_{i\lambda})$, and our assertions follow.

Case II. Now let D_1, D_2 be arbitrary elements of $\mathscr{U}(\mathfrak{m})$. Then for all $x \in M^\dagger$, $F(h:D_1:x;D_2)(k_1:k_2) = f(h:\mathrm{Ad}\,k_1^{-1}D_1:k_1^{-1}xk_2^{-1};\mathrm{Ad}\,k_2D_2)$. Now there are $D_i', D_j'' \in \mathscr{U}(\mathfrak{m})$ and $a_i', a_j'' \in C^\infty(K_{M,1}^\dagger)$ so that $\mathrm{Ad}\,k_1^{-1}D_1 = \sum_i a_i'(k_1)D_i'$ and $\mathrm{Ad}\,k_2D_2 = \sum a_j''(k_2)D_j''$ for all $k_1, k_2 \in K_{M,1}^\dagger$. Let $C = \max_{i,j,k_1,k_2}|a_i'(k_1)a_j''(k_2)| < \infty$. Then $F(h:D_1:x;D_2)(k_1:k_2) = \sum_{i,j} a_i'(k_1) a_j''(k_2) f(h:D_i';k_1^{-1}xk_2^{-1};D_j'')$. Now by Theorem 4.5 there are holomorphic families f_{ijl} and polynomials p_{ijl} so that $f(h:D_i';x;D_j'') = \sum_l p_{ijl}(h) f_{ijl}(h:x)$ for all $x \in M^\dagger$. Let $F_{ijl} = F(f_{ijl})$. Then $F(h:D_1:x;D_2)(k_1:k_2) = \sum_{i,j,l} a_i'(k_1) a_j''(k_2) p_{ijl}(h) F_{ijl}(h:x)(k_1:k_2)$. Thus $\|F(h:D_1;a;D_2)\| \leqslant C\sum_{i,j,l}\|p_{ijl}(h) F_{ijl}(h:a)\|$, and so we have a bound of the desired form by Case I applied to the functions F_{ijl}. Q.E.D.

6. Eisenstein Integrals

We define holomorphic families of Eisenstein integrals, show that they are spherical functions in the appropriate setting, and check that they satisfy systems of differential equations corresponding to the appropriate infinitesimal characters.

Fix a continuous family $\boldsymbol{\pi}$ of relative discrete series representations of $M^\dagger$ and holomorphic families $\boldsymbol{\sigma}_j$ for $K_M^\dagger$ and $\boldsymbol{\tau}_j$ for K such that $\tau_{j,h}$ contains $\sigma_{j,h}$. Denote subspaces $E_{\sigma_1}^M, E_{\sigma_2}^M \subset L^2(K_{M,1}^\dagger)$ as in Lemma 5.7 and let $E_{\tau_1}^G, E_{\tau_2}^G \subset L^2(K_1)$ be defined similarly. Denote

$$''E_{\tau_j}^G\text{: kernel of restriction } E_{\tau_j}^G \to E_{\sigma_j}^M \text{ of functions from } K_1 \text{ to } K_{M,1}^\dagger \text{ followed by projection onto } E_{\sigma_j}^M \tag{6.1a}$$

$$'E_{\tau_j}^G\text{: orthocomplement of } ''E_{\tau_j}^G \text{ in } E_{\tau_j}^G \tag{6.1b}$$

so that we have $K_{M,1}^\dagger$-equivariant isomorphisms

$$i_1: E_{\sigma_j}^M \cong {}'E_{\tau_j}^G \qquad \text{and} \qquad i_2: E_{\tau_j}^G \cong {}'E_{\tau_j}^G \oplus {}''E_{\tau_j}^G. \tag{6.1c}$$

Now let $f_M \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ and view the associated family of spherical functions

$$F_M = F(f_M): \mathfrak{v}_{\mathbb{C}}^* \times M^\dagger \to E_{\sigma_1}^M \otimes E_{\sigma_2}^M \tag{6.2a}$$

as having values in

$$E^M_{\sigma_1} \otimes E^M_{\sigma_2} \cong 'E^G_{\tau_1} \otimes 'E^G_{\tau_2} \subset E^G_{\tau_1} \otimes E^G_{\tau_2} \subset L^2(K_1 \times K_1). \tag{6.2b}$$

Denote this by

$$'F_M : \mathfrak{v}^*_{\mathbb{C}} \times M^\dagger \to L^2(K_1 \times K_1). \tag{6.2c}$$

As in (5.9), interpret the action of K_1 on $W(\tau_1 : \tau_2) = E^G_{\tau_1} \otimes E^G_{\tau_2}$ as a double representation $'\tau = ('\tau_1, '\tau_2)$:

$$\begin{aligned}[]['\tau(g_1 : g_2)\psi](k_1 : k_2) &= ['\tau_1(g_1) \cdot \psi \cdot '\tau_2(g_2)](k_1 : k_2) \\ &= \psi(g_1^{-1} k_1 : k_2 g_2^{-1}).\end{aligned} \tag{6.3a}$$

We define a holomorphic family $'\boldsymbol{\tau} = ('\boldsymbol{\tau}_1 : '\boldsymbol{\tau}_2)$ of double representations of K on $W(\tau_1 : \tau_2)$ by

$$\begin{aligned}&['\tau_h(g_1 z_1 : g_2 z_2)\psi](k_1 : k_2) \\ &\quad = ['\tau_{1,h}(g_1 z_1) \cdot \psi \cdot '\tau_{2,h}(g_2 z_2)](k_1 : k_2) \\ &\quad = \zeta(\boldsymbol{\tau}_1 : h : z_1)\, \zeta(\boldsymbol{\tau}_2 : h : z_2)\, \psi(g_1^{-1} k_1 : k_2 g_2^{-1}) \\ &\quad\quad \text{for} \quad z_i \in V,\ g_i \in K_1.\end{aligned} \tag{6.3b}$$

Then

$$'F_M(h : m_1 x m_2) = '\tau_{1,h}(m_1) \cdot 'F_M(h : x) \cdot '\tau_{2,h}(m_2) \tag{6.3c}$$

for all $m_1, m_2 \in K^\dagger_M$ since the embedding $i_1 \otimes i_2 : E^M_{\sigma_1} \otimes E^M_{\sigma_2} \to E^G_{\tau_1} \otimes E^G_{\tau_2} = W(\tau_1 : \tau_2)$ defined in (6.1) is equivariant for the action of $K^\dagger_M$.

The embedding $i_1 \otimes i_2$ is not in general the only possible one. In order to generate a larger class of spherical functions we denote

$$\begin{aligned}&\operatorname{End}_{K^V_{M,1}}(W(\tau_1 : \tau_2)) \\ &\quad = \{S \in \operatorname{End}(W(\tau_1 : \tau_2)) : S('\tau_1(k_1) \cdot \psi \cdot '\tau_2(k_2)) \\ &\quad = '\tau_1(k_1) \cdot S\psi \cdot '\tau_2(k_2) \text{ for all } k_1, k_2 \in K^V_{M,1}, \psi \in W(\tau_1 : \tau_2)\}.\end{aligned} \tag{6.4a}$$

Note for all $S \in \operatorname{End}_{K^V_{M,1}}(W(\tau_1 : \tau_2))$, $h \in \mathfrak{v}^*_{\mathbb{C}}$, $m_1, m_2 \in K^\dagger_M$, $\psi \in W(\tau_1 : \tau_2)$,

$$S('\tau_{1,h}(m_1) \cdot \psi \cdot '\tau_{2,h}(m_2)) = '\tau_{1,h}(m_1) \cdot S\psi \cdot '\tau_{2,h}(m_2) \tag{6.4b}$$

since for $m_i = z_i k_i$, $z_i \in V$, $k_i \in K^V_{1,M}$, $'\tau_{i,h}(m_i) = \zeta(\boldsymbol{\tau}_i : h : z_i)\, '\tau_i(k_i)$. Now for any $S \in \operatorname{End}_{K^V_{M,1}}(W(\tau_1 : \tau_2))$ we define

$$'F_{M,S}(h : x) = S \cdot 'F_M(h : x) \qquad \text{for all} \quad h \in \mathfrak{v}^*_{\mathbb{C}},\ x \in M^\dagger. \tag{6.4c}$$

Using (6.3c) and (6.4b) we have

$$'F_{M,S}(h:m_1 x m_2) = {}'\tau_{1,h}(m_1)\cdot {}'F_{M,S}(h:x)\cdot {}'\tau_{2,h}(m_2) \tag{6.4d}$$

for all $m_1, m_2 \in K_M^\dagger$.

$'F_{M,S}$ is called the *holomorphic family of $K_M^\dagger$-spherical functions* corresponding to $f_M \in \mathscr{F}(\pi:\sigma_1:\sigma_2)$ and $S \in \operatorname{End}_{K_{M,1}^\dagger}(W(\tau_1:\tau_2))$. Now, if $x \in G$ use $G = KM^\dagger AN$ to express

$$x = \mathbf{k}(x)\cdot \mathbf{m}(x)\cdot \exp H_P(x)\cdot \mathbf{n}(x),\ H_P(x) \in \mathfrak{a}. \tag{6.5a}$$

Then the function $'F_{M,S}$ of (6.4) extends to

$$F_G: \mathfrak{v}_{\mathbb{C}}^* \times G \to W(\tau_1:\tau_2) \subset L^2(K_1 \times K_1) \tag{6.5b}$$

by the formula

$$F_G(h:x) = {}'\tau_{1,h}(\mathbf{k}(x))\cdot {}'F_{M,S}(h:\mathbf{m}(x)). \tag{6.5c}$$

F_G is well defined: if $l \in K_M^\dagger$ then, dropping h, $'\tau_1(kl)\cdot {}'F_M(m) = {}'\tau_1(k)\cdot {}'\tau_1(l)\cdot {}'F_M(m) = {}'\tau_1(k)\cdot {}'F_M(lm)$.

We can now define the *Eisenstein integral*

$$E(P:{}'F_{M,S}): \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G \to W(\tau_1:\tau_2) \tag{6.6a}$$

by

$$E(P:{}'F_{M,S}:h:\nu:x)$$
$$= \int_{K/Z} F_G(h:xk)\cdot {}'\tau_{2,h}(k^{-1})\, e^{(i\nu-\rho_p)\cdot H_p(xk)}\, d(kZ), \tag{6.6b}$$

where F_G and H_p are defined in (6.5), and ρ_p is $\frac{1}{2}$ the trace of $\operatorname{ad}(\mathfrak{a})$ on $\mathfrak{n}$. It is well defined because, dropping h, if $z \in Z$ then

$$\begin{aligned}
&F_G(xkz)\cdot {}'\tau_2(z^{-1}k^{-1})\\
&\quad = {}'\tau_1(\mathbf{k}(xkz))\cdot {}'F_{M,S}(\mathbf{m}(xkz))\cdot {}'\tau_2(z^{-1}k^{-1})\\
&\quad = {}'\tau_1(\mathbf{k}(xk))\cdot {}'F_{M,S}(\mathbf{m}(xk)z)\cdot {}'\tau_2(z^{-1}k^{-1})\\
&\qquad (\text{because } xk = k_1 m_1 a_1 n_1 \text{ gives } xkz = k_1\cdot m_1 z\cdot a_1\cdot n_1)\\
&\quad = {}'\tau_1(\mathbf{k}(xk))\cdot {}'F_{M,S}(\mathbf{m}(xk))\cdot {}'\sigma_2(z)\cdot {}'\tau_2(z^{-1})\cdot {}'\tau_2(k^{-1})\\
&\quad = F_G(xk)\cdot {}'\tau_2(k^{-1})
\end{aligned}$$

and $H_p(xkz) = H_p(xk)$.

THEOREM 6.7. *Let $'F_{M,S}$ be the holomorphic family of $K_M^\dagger$-spherical functions corresponding to $f_M \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ and $S \in \operatorname{End}_{K_{M,1}^\dagger}(W(\tau_1:\tau_2))$. Then the Eisenstein integral $E(P:'F_{M,S}))$ is jointly smooth, holomorphic in h, v, and if $(h, v, x) \in \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G$ then $E(P:'F_{M,S}:h:v:x)$ is a $'\tau$-spherical function.*

Proof. To see that it is a spherical function, drop most of the variables and compute

$$
\begin{aligned}
&E(k_1 x k_2)\\
&\quad = \int F_G(k_1 x k_2 k) \cdot {'\tau_2}(k^{-1}) \cdot e^{(iv-\rho_P) H_P(k_1 x k_2 k)}\, d(kZ)\\
&\quad = \int F_G(k_1 x k) \cdot {'\tau_2}(k^{-1} k_2) \cdot e^{(iv-\rho_P) H_P(k_1 x k)}\, d(kZ)\\
&\quad = \int {'\tau_1}(k_1) \cdot F_G(xk) \cdot {'\tau_2}(k^{-1}) \cdot {'\tau_2}(k_2) \cdot e^{(iv-\rho_P) H_P(xk)}\, d(kZ)\\
&\quad = {'\tau_1}(k_1) \cdot E(x) \cdot {'\tau_2}(k_2).
\end{aligned}
$$

We check that $E(P:'F_{M,S})$ is smooth in (h, v, x) and holomorphic in (h, v). Theorem 5.8 implies that F_M is holomorphic in h and smooth in (h, m) as a map to $W(\sigma_1:\sigma_2)$. The same follows for $'F_{M,S}$ as a map to $W(\tau_1:\tau_2)$. Now F_G is holomorphic in h and smooth in (h, x). Since H_P is real analytic, now the integrand in (6.6b) is holomorphic in (h, v) and smooth in (h, v, x, k). As K/Z is compact, now $E(P:'F_{M,S})$ is holomorphic in (h, v), and smooth in (h, v, x).

Let $\mathscr{Z}(\mathfrak{g})$ denote the center of the enveloping algebra $\mathscr{U}(\mathfrak{g})$. If $\beta \in \mathfrak{h}_{\mathbb{C}}^*$ then $\chi_\beta: \mathscr{Z}(\mathfrak{g}) \to \mathbb{C}$ denotes the infinitesimal character with Harish-Chandra parameter β. Every $f_M \in \mathscr{F}(\boldsymbol{\pi}:\boldsymbol{\sigma}_1:\boldsymbol{\sigma}_2)$ satisfies

$$f_M(h:m;u) = \chi^M_{\lambda_0 + h_M(h)}(u)\, f_M(h:m) \qquad \text{for} \quad u \in \mathscr{Z}(\mathfrak{m})$$

because $\pi(h)$ has infinitesimal character $\chi^M_{\lambda_0 + h_M(h)}$. Now, from its definition (5.4), $F_M = F(f_M)$ satisfies

$$F_M(h:m;u) = \chi^M_{\lambda_0 + h_M(h)}(u)\, F_M(h:m) \qquad \text{for} \quad u \in \mathscr{Z}(\mathfrak{m}).$$

This carries through trivially to $'F_{M,S}$ and now

$$F_G(h:x;z) = \chi_{\lambda_0 + h_M(h)}(z)\, F_G(h:x) \qquad \text{for} \quad z \in \mathscr{Z}(\mathfrak{g}).$$

Differentiating under the integral we have, exactly as in [3, Lemma 19.1],

$$
\begin{aligned}
&E(P:'F_{M,S}:h:v:x;z)\\
&\qquad = \chi_{\lambda_0 + h_M(h) + iv}(z)\, E(P:'F_{M,S}:h:v:x) \qquad \text{for} \quad z \in \mathscr{Z}(\mathfrak{g}). \qquad (6.8)
\end{aligned}
$$

Here note that $\chi_{\lambda_0+h_M(h)+i\nu}$ is the infinitesimal character of the H-series representation $\pi_{h,\nu}$ of (3.9).

7. Growth Estimates

In this section we will show how to use the differential equations (6.8) satisfied by the Eisenstein integrals to sharpen the a priori estimates on their growth which will be proved in Section 9. The main results are Theorems 6.31 and 6.33 which will be used in Sections 8 and 9 to show that wave packets of Eisentein integrals are Schwartz functions on G. In order to carry out the induction required for the proof of Theorem 8.5, it is necessary to study more general classes of functions. The functions of type $H(\mathscr{S}_{\mathfrak{a}}, L_P)$ defined in (7.5) generalize holomorphic families of Eisenstein integrals, and the functions of type $I(\mathscr{S}, L_P)$ defined in (7.7) generalize the product of an Eisenstein integral with a Schwartz function in the parameter variables.

The results of this section extend the construction and estimates of Harish-Chandra [4] to include dependence on the extra continuous parameters in the Eisenstein integrals which come from continuous families of relative discrete series representations. The organization of this section closely follows Trombi's account of Harish-Chandra's work in [10].

We first review some standard results in invariant theory (see [10]). Fix $P_0=M_0A_0N_0$ a minimal parabolic subgroup of G and let $\mathfrak{h}_0=\mathfrak{t}_0+\mathfrak{a}_0$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{t}_0\subseteq\mathfrak{m}_0$. For any parabolic subgroup $P=M_PA_PN_P$ of G, write $L_P=M_PA_P$ and $K_P=K\cap M_P$. Now if $A_P\subseteq A_0$, then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{l}_P$, and we write W_P for the Weyl group of the pair $(\mathfrak{l}_{P\mathbb{C}}, \mathfrak{h}_{0\mathbb{C}})$, $S(\mathfrak{h}_{0\mathbb{C}})^{W_P}$ for the W_P-invariants in the symmetric algebra of $\mathfrak{h}_0$, $\mathscr{Z}_P$ for the center of $\mathscr{U}(\mathfrak{l}_P)$ and $\mu_P:\mathscr{Z}_P\to S(\mathfrak{h}_{0\mathbb{C}})^{W_P}$ for the canonical isomorphism.

Now suppose $Q\subseteq P$ are two parabolic subgroups of G with $A_P\subseteq A_Q\subseteq A_0$, so that $*Q=Q\cap L_P=L_Q{*N_Q}$ is a parabolic subgroup of L_P. Then we can consider $W_Q\subseteq W_P$ so that $S(\mathfrak{h}_{0\mathbb{C}})^{W_P}\subseteq S(\mathfrak{h}_{0\mathbb{C}})^{W_Q}$. Let $\mu_{PQ}:\mathscr{Z}_P\to\mathscr{Z}_Q$ be the algebra injection such that $\mu_P(z)=\mu_Q(\mu_{PQ}(z))$ for all $z\in\mathscr{Z}_P$. Let $\Delta(*Q, A_Q)$ denote the roots of the pair $(*\mathfrak{q}, \mathfrak{a}_Q)$, $\rho=\frac{1}{2}\sum\alpha$, $\alpha\in\Delta(*Q, A_Q)$. Then d_Q is defined on L_Q by $d_Q(ma)=e^{\rho(\log a)}$, $m\in M_Q$, $a\in A_Q$. For $b\in\mathscr{U}(\mathfrak{l}_Q)$, we write $b'=d_Q^{-1}\cdot b\cdot d_Q$. The mapping μ_{PQ} also has the property that $z-\mu_{PQ}(z)'\in\theta(*\mathfrak{n}_Q)\,\mathscr{U}(\mathfrak{l}_P)*\mathfrak{n}_Q$ for all $z\in\mathscr{Z}_P$. Further, $\mathscr{Z}_Q$ is a finite free module over $\mu_{PQ}(\mathscr{Z}_P)$ of rank $r=[W_P:W_Q]$. Let $1=v_1, v_2, ..., v_r$ be a free basis, and for $v\in\mathscr{Z}_Q$, denote by z_{vij}, $1\leqslant i, j\leqslant r$, the unique element of $\mathscr{Z}_P$ satisfying

$$vv_i=\sum_{1\leqslant j\leqslant r}\mu_{PQ}(z_{vij})v_j, \qquad 1\leqslant i\leqslant r. \tag{7.1}$$

Fix a complex Hilbert space T of dimension r with orthonormal basis $\{e_1, ..., e_r\}$. For $\Lambda \in \mathfrak{h}_{0\mathbb{C}}^*$, $v \in \mathscr{Z}_Q$, let $\Gamma(\Lambda : v)$ be the endomorphism of T with matrix $\mu_P(z_{vij} : \Lambda) = \mu_P(z_{vij})(\Lambda)$ with respect to this basis. Let Φ_P^+, Φ_Q^+ be positive systems of roots for $(\mathfrak{l}_{P\mathbb{C}}, \mathfrak{h}_{0\mathbb{C}})$, $(\mathfrak{l}_{Q\mathbb{C}}, \mathfrak{h}_{0\mathbb{C}})$, respectively, chosen so that $\Phi_Q^+ = \{\alpha \in \Phi_P^+ : \alpha|_{\mathfrak{a}_Q} = 0\}$ and if $\alpha \in \Phi_P^+$ with $\alpha|_{\mathfrak{a}_Q} \neq 0$, then $\alpha|_{\mathfrak{a}_Q}$ is a root of $({}^*\mathfrak{q}, \mathfrak{a}_Q)$. Define

$$\pi_P = \prod_{\alpha \in \Phi_P^+} H_\alpha, \qquad \pi_Q = \prod_{\alpha \in \Phi_Q^+} H_\alpha, \qquad \pi_{PQ} = \pi_P / \pi_Q. \tag{7.2}$$

Fix coset representatives $s_1 = 1, s_2, ..., s_r$ for W_P / W_Q. The following lemma appears in [10].

LEMMA 7.3. *For* $\Lambda \in \mathfrak{h}_{0\mathbb{C}}^*$, *let* $e_k(\Lambda) = \sum_{1 \leqslant j \leqslant r} \mu_Q(v_j : s_k \Lambda) e_j$. *Then if* $\Lambda \in (\mathfrak{h}_{0\mathbb{C}}^*)' = \{\Lambda \in \mathfrak{h}_{0\mathbb{C}}^* : \pi_P(\Lambda) \neq 0\}$, *the* $e_k(\Lambda)$, $1 \leqslant k \leqslant r$, *form a basis for* T *and* $\Gamma(\Lambda : v)\, e_k(\Lambda) = \mu_Q(v : s_k \Lambda)\, e_k(\Lambda)$ *for all* $v \in \mathscr{Z}_Q$, $1 \leqslant k \leqslant r$. *Moreover, there is an* $r \times r$ *matrix* B *with entries in the quotient field of* $S(\mathfrak{h}_{0\mathbb{C}})^{W_Q}$ *so that*

(i) $\pi_{PQ} B$ *has entries in* $S(\mathfrak{h}_{0\mathbb{C}})^{W_Q}$;

(ii) *for any* $\Lambda \in (\mathfrak{h}_{0\mathbb{C}}^*)'$, *the* $B(s_k \Lambda)$ *are projections* $T \to \mathbb{C} e_k(\Lambda)$ *corresponding to the direct sum* $T = \sum_{1 \leqslant k \leqslant r} \mathbb{C} e_k(\Lambda)$.

Fix $\mathfrak{h} = \mathfrak{t}_H + \mathfrak{a}_H$ a θ-stable Cartan subalgebra of $\mathfrak{g}$. We may as well assume that $\mathfrak{a}_H \subseteq \mathfrak{a}_0$. Let $P_H = M_H A_H N_H$ be a cuspidal parabolic subgroup with split component A_H. As in (3.3), (3.7), and (3.18), we introduce parameters for a continuous series of representations induced from P_H. Thus for $\lambda_0 \in \Lambda_{0,M}$ and $h \in \mathscr{D} = \{h \in i\mathfrak{v}^* : h_M(h) \in \mathscr{D}_M\}$, $\lambda(h) = \lambda_0 + h_M(h) \in i\mathfrak{t}_H^*$ and $\chi(h) \in Z_{M_H}(M_H^0)^\wedge$ are parameters for a continuous family of relative discrete series representations on M_H. Let ω be a relatively compact neighborhood of 0 in $i\mathfrak{v}^*$. Define $\mathscr{D}_\mathbb{C} = \mathscr{D}_\mathbb{C}(\omega) = \{h \in \mathfrak{v}_\mathbb{C}^* : h = h_R + ih_I,$ $h_R \in \mathscr{D},\ h_I \in \omega\}$. For $h \in \mathscr{D}_\mathbb{C}$ define $d(h)$ to be the distance from h_R to the boundary of $\mathscr{D}$. For $h \in \mathscr{D}_\mathbb{C}$, $v \in \mathscr{F} = \mathfrak{a}_H^*$, extend $\lambda(h)$ trivially to $\mathfrak{a}_{H\mathbb{C}}$ and v trivially to $\mathfrak{t}_{H\mathbb{C}}$ so that $\lambda(h) + iv \in \mathfrak{h}_\mathbb{C}^*$. Let $(\tau_{1,h}, \tau_{2,h})$ be a family of double representations of K on a finite dimensional vector space W with norm $\|\cdot\|$ as in (6.4).

Fix P a parabolic subgroup of G with $A_P \subseteq A_H \subseteq A_0$. Then $\mathfrak{h}$ and $\mathfrak{h}_0$ are both Cartan subalgebras of $\mathfrak{l}_P$. Pick $y \in \mathrm{Int}(\mathfrak{l}_{P\mathbb{C}})$ such that $y(\mathfrak{h}_\mathbb{C}) = \mathfrak{h}_{0\mathbb{C}}$. For $h \in \mathscr{D}_\mathbb{C}$, $v \in \mathscr{F}$, write $\Lambda_{h,v} = y(\lambda(h) + iv) \in \mathfrak{h}_{0\mathbb{C}}^*$. Let $\mathscr{P}$ denote the set of all differential operators on $\mathfrak{v}_\mathbb{C}^* \times \mathscr{F}$ with coefficients which are polynomials in $h \in \mathfrak{v}_\mathbb{C}^*$ and $v \in \mathscr{F}$. Write $\tilde{\mathscr{D}}_P = \mathscr{P} \otimes \mathscr{U}(\mathfrak{l}_P)^{(2)}$. For $D_1 \otimes l_1 \otimes l_2 \in \tilde{\mathscr{D}}_P$, $\varphi \in C^\infty(\mathscr{D}_\mathbb{C} \times \mathscr{F} \times L_P, W)$, define $(D_1 \otimes l_1 \otimes l_2 \varphi)(h : v : x) = \varphi(h : v; D_1 : l_1 ; x; l_2)$. For $D \in \tilde{\mathscr{D}}_P$, $r \in \mathbb{R}$, set

$$S_{D,r}(\varphi) = \sup_{\mathscr{D}_\mathbb{C} \times \mathscr{F} \times L_P} \|D\varphi(h : v : x)\|\, \Xi_P^{-1}(x)\, |(h, v, x)|^{-r} e^{-|h_I|\, \sigma_P(x)}, \tag{7.4}$$

where $|(h, \nu, x)| = (1 + |h|)(1 + |\nu|)(1 + \tilde{\sigma}(x))(1 + d(h)^{-1})$, Ξ_P is the function Ξ defined as in (2.4) for the group L_P, and $\tilde{\sigma}(x)$ and $\sigma_{V'}(x)$ are defined as in (2.7). For F any finite subset of $\tilde{\mathscr{D}}_P$, set $S_{F,r}(\varphi) = \sum_{D \in F} S_{D,r}(\varphi)$.

DEFINITION 7.5. We will write $II(\mathscr{D}_{\mathbb{C}}, L_P)$ for the set of all $\varphi \in C^\infty(\mathscr{D}_{\mathbb{C}} \times \mathscr{F} \times L_P, W)$ satisfying

(i) for all $(\nu, x) \in \mathscr{F} \times L_P$, $h \mapsto \varphi(h:\nu:x)$ is a holomorphic function on $\mathscr{D}_{\mathbb{C}}$;

(ii) for all $(h, \nu) \in \mathscr{D}_{\mathbb{C}} \times \mathscr{F}$, $\varphi(h:\nu)$ is a $(\tau_{1,h}|_{K_P}, \tau_{2,h}|_{K_P})$-spherical function on L_P;

(iii) for all $(h, \nu) \in \mathscr{D}_{\mathbb{C}} \times \mathscr{F}$, $z\varphi(h:\nu) = \mu_P(z:\Lambda_{h,\nu})\, \varphi(h:\nu)$ for all $z \in \mathscr{Z}_P$;

(iv) for all $D \in \tilde{\mathscr{D}}_P$, there is $r \geqslant 0$ so that $S_{D,r}(\varphi) < \infty$.

Remark. The holomorphic families of Eisenstein integrals constructed in (6.6) will be shown in Section 9 to be elements of $II(\mathscr{D}_{\mathbb{C}}, G)$.

For $\varphi \in C^\infty(\mathscr{D} \times \mathscr{F} \times L_P, W)$, $D \in \tilde{\mathscr{D}}_P$, $r, t, \in \mathbb{R}$, set

$$ {}^0S_{D,r,t}(\varphi) = \sup_{\mathscr{D} \times \mathscr{F} \times L_P} \|D\varphi(h:\nu:x)\|\, \Xi_P^{-1}(x)(1 + \tilde{\sigma}(x))^{-r}(1 + d(h)^{-1})^t. \tag{7.6} $$

For F any finite subset of $\tilde{\mathscr{D}}_P$, set ${}^0S_{F,r,t}(\varphi) = \sum_{D \in F} {}^0S_{D,r,t}(\varphi)$.

DEFINITION 7.7. We will write $I(\mathscr{D}, L_P)$ for the set of all $\varphi \in C^\infty(\mathscr{D} \times \mathscr{F} \times L_P, W)$ satisfying

(i) there are a complex neighborhood $\mathscr{D}_{\mathbb{C}}$ of $\mathscr{D}$ and a finite set of functions $\varphi_1, ..., \varphi_k \in II(\mathscr{D}_{\mathbb{C}}, L_P)$ so that for each $(h, \nu) \in \mathscr{D} \times \mathscr{F}$ there exist $a_j(h:\nu) \in \mathbb{C}$, $1 \leqslant j \leqslant k$, such that $\varphi(h:\nu:x) = \sum_{j=1}^k a_j(h:\nu)\, \varphi_j(h:\nu:x)$ for all $x \in L_P$;

(ii) for all $D \in \tilde{\mathscr{D}}_P$, there is $r \geqslant 0$ so that ${}^0S_{D,r,t}(\varphi) < \infty$ for all $t \geqslant 0$.

Remark. If $\varphi \in II(\mathscr{D}_{\mathbb{C}}, L_P)$ and $\alpha \in \mathscr{C}(\mathscr{D} \times \mathscr{F}) = \{\alpha \in C^\infty(\mathscr{D} \times \mathscr{F}) : \|\alpha\|_{D,t} = \sup_{\mathscr{D} \times \mathscr{F}} |D\alpha(h:\nu)|\, (1 + d(h)^{-1})^t < \infty$ for all $D \in \mathscr{P}, t \geqslant 0\}$, then $\varphi \cdot \alpha \in I(\mathscr{D}, L_P)$. In fact, given $D \in \tilde{\mathscr{D}}_P$, $r \geqslant 0$, there is a finite subset F of $\tilde{\mathscr{D}}_P \times \mathscr{P}$ so that ${}^0S_{D,r,t}(\varphi \cdot \alpha) \leqslant \sum_{(D', D'') \in F} S_{D',r}(\varphi)\, \|\alpha\|_{D'', r+t}$ for all $t \geqslant 0$.

We are now ready to study the asymptotic behavior of functions of types $I(\mathscr{D}, L_P)$ and $II(\mathscr{D}_{\mathbb{C}}, L_P)$. Both types will be treated simultaneously with the understanding that if $f \in I(\mathscr{D}, L_P)$, h ranges over $\mathscr{D}$ while if $f \in II(\mathscr{D}_{\mathbb{C}}, L_P)$, h ranges over $\mathscr{D}_{\mathbb{C}}$. We return to the notation in the first part of this section. Thus ${}^*Q$ is a parabolic subgroup of L_P.

For $f \in I(\mathscr{D}, L_P) \cup II(\mathscr{D}_{\mathbb{C}}, L_P)$, define $\Phi(f)$ and $\Psi_r(f)$ taking values in $\mathbf{W} = W \otimes_{\mathbb{C}} T$ by

$$ \Phi(f:h:\nu:m) = \sum^{r} d_Q(m)\, f(h:\nu:m; v_i')\otimes e_i, \qquad m \in L_Q \tag{7.8a} $$

and

$$\Psi_v(f:h:v:m) = \sum_{i-1}^{r} d_Q(m)\, f(h:v:m; u_i(v:h:v)')\otimes e_i, \qquad m \in L_Q, \qquad (7.8b)$$

where for $v \in \mathscr{Z}_Q$, $u_i(v:h:v) = \sum_{j=1}^{r} \mu_{PQ}(z_{vij} - \mu_P(z_{vij}:\Lambda_{h,v}))v_j$.

LEMMA 7.9. *Let* $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$, $v \in \mathscr{Z}_Q$. *Then* $\Phi(f:h:v:b_1; m; b_2 v) = \Gamma(\Lambda_{h,v}:v)\,\Phi(f:h:v:b_1; m; b_2) + \Psi_v(f:h:v:b_1; m; b_2)$ *for all* $m \in L_Q$. *Here* $\Gamma(\Lambda_{h,v}:v)$ *has been extended to an endomorphism of* **W** *by making it act trivially on* W.

Proof. This follows easily from (7.1) since $z_{vij} - \mu_P(z_{vij}:\Lambda_{h,v})$ kills every $f \in I(\mathscr{D}, L_P) \cup II(\mathscr{D}_{\mathbb{C}}, L_P)$ (see [10, pp. 280–281]).

COROLLARY 7.10. *Let* $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$, $H \in \mathfrak{a}_Q$. *Then for all* $T \in \mathbb{R}$, $m \in L_Q$, *we have* $\Phi(f:h:v:b_1; m \exp TH; b_2) = \exp(T\Gamma(\Lambda_{h,v}:H))\,\Phi(f:h:v:b_1; m; b_2) + \int_0^T \exp\{(T-t)\,\Gamma(\Lambda_{h,v}:H)\}\,\Psi_H(f:h:v:b_1; m \exp tH; b_2)\,dt$.

LEMMA 7.11. *Fix* $D \in \mathscr{P}$, $l_1, l_2 \in \mathscr{U}(\mathfrak{l}_P)$, *and* $X \in {}^*\mathfrak{n}_Q = \mathfrak{n}_Q \cap \mathfrak{l}_P$. *Then we can choose a finite subset* $F \subseteq \tilde{\mathscr{L}}_P$ *and* $r_0 \geqslant 0$ *such that*

$$d_Q(ma)\{\|f(h:v; D:l_1X; ma; l_1)\| + \|f(h:v; D:l_1; ma; \theta(X)l_2)\|\}$$
$$\leqslant \begin{cases} {}^0S_{F,r,t}(f)\,\Xi_Q(m)\,e^{-\beta_Q(\log a)}(1+\tilde{\sigma}(ma))^{r+r_0}(1+d(h)^{-1})^{t} \\ \qquad \text{for all} \quad r, t \geqslant 0,\ f \in I(\mathscr{D}, L_P) \\ S_{F,r}(f)\,\Xi_Q(m)\,e^{-\beta_Q(\log a)}\,|(h, v, m)|^{r+r_0}(1+\sigma(a))^{r+r_0}\,e^{|h_I|\,\sigma_I(m)} \\ \qquad \text{for all} \quad r \geqslant 0,\ f \in II(\mathscr{D}_{\mathbb{C}}, L_P) \end{cases}$$

and

$$d_Q(ma)\,\|f(h:v; D:l_1; ma; l_2)\|$$
$$\leqslant \begin{cases} {}^0S_{F,r,t}(f)\,\Xi_Q(m)(1+\tilde{\sigma}(ma))^{r+r_0}(1+d(h)^{-1})^{t} \\ \qquad \text{for all} \quad r, t \geqslant 0,\ f \in I(\mathscr{D}, L_P) \\ S_{F,r}(f)\,\Xi_Q(m)\,|(h, v, m)|^{r+r_0}(1+\sigma(a))^{r+r_0}\,e^{|h_I|\,\sigma_I(m)} \\ \qquad \text{for all} \quad r \geqslant 0,\ f \in II(\mathscr{D}_{\mathbb{C}}, L_P) \end{cases}$$

for all $m \in L_Q^+ = K_Q\,\mathrm{cl}(A_0^+(P))K_Q$ *and* $a \in A_Q^+ = \{a \in A_Q : \alpha(\log a) > 0$ *for all* $\alpha \in \Delta(^*Q, A_Q)\}$. *Here* $A_0^+(P) = \{a \in A_0 : \alpha(\log a) > 0$ *for all* $\alpha \in \Phi_P^+\}$ *and for* $H \in \mathfrak{a}_Q$, $\beta_Q(H) = \inf\{\alpha(H) : \alpha \in \Delta(^*Q, A_Q)\}$.

Proof. Write $\rho_P = \frac{1}{2}\sum \alpha$, $\alpha \in \Phi_P^+$, $\rho_Q = \frac{1}{2}\sum \alpha$, $\alpha \in \Phi_Q^+$, and $\rho_{PQ} = \frac{1}{2}\sum \alpha$, $\alpha \in \Delta(^*Q, A_Q)$. Then $\rho_P = \rho_Q + \rho_{PQ}$. Using (2.16) for the group L_P we have constants c_0 and r_0 so that $\Xi_P(a) \leqslant c_0(1+\sigma(a))^{r_0}\,e^{-\rho_P}(a)$ for all $a \in A_0^+(P)$.

But $e^{-\rho_P}(a) = d_Q^{-1}(a)\, e^{-\rho_Q}(a) \leqslant d_Q^{-1}(a)\, \Xi_Q(a)$ by (2.16) for the group L_Q. Thus for all $m \in L_Q^+$, $d_Q(m) \leqslant c_0 \Xi_P^{-1}(m)\, \Xi_Q(m)(1+\sigma(m))^{r_0}$. Then the second set of inequalities follows trivially from the definitions of ${}^0S_{F,r,t}(f)$ and $S_{F,r}(f)$ if we take $F = \{c_0 D \otimes l_1 \otimes l_2\}$.

We will prove the first set of inequalities for $D \otimes l_1 X \otimes l_2$ only. The argument for $D \otimes l_1 \otimes \theta(X) l_2$ is similar. Write $f(h:v; D:l_1X; ma; l_2) = f(h:v; D:l_1; ma; {}^{a^{-1}m^{-1}}Xl_2)$. Now $m = k_1 a_0 k_2$ for some $k_1, k_2 \in K_Q$ and $a_0 \in \mathrm{cl}(A_0^+(P))$. Thus ${}^{a^{-1}m^{-1}}X = {}^{k_2^{-1}a^{-1}a_0^{-1}k_1^{-1}}X$. Write ${}^{k_1^{-1}}X = \sum c_\alpha(k_1) X_\alpha$, where the summation is taken over $\{\alpha \in \Phi_P^+ : \alpha|_{\mathfrak{a}_Q} \in \Delta({}^*Q, A_Q)\}$, each c_α is a smooth function on the compact group $\mathrm{Ad}(K_Q)$, and each $X_\alpha \in {}^*\mathfrak{n}_Q$ satisfies ${}^{a^{-1}a_0^{-1}}X_\alpha = e^{-\alpha(\log a_0)}\, e^{-\alpha(\log a)} X_\alpha$. Finally, write each ${}^{k_2^{-1}}X_\alpha = \sum d_{\alpha\beta}(k_2) X_\beta$, where each $d_{\alpha\beta}$ is a smooth function on $\mathrm{Ad}(K_Q)$. Now $e^{-\alpha(\log a_0)} \leqslant 1$ for all $a_0 \in A_0^+(P)$ and $e^{-\alpha(\log a)} \leqslant e^{-\beta_Q(\log a)}$ for all $a \in A_Q^+$ so that

$$
\begin{aligned}
&\|f(h:v; D:l_1X; ma; l_2)\| \\
&\quad \leqslant \sum_{\alpha,\beta} |c_\alpha(k_1)\, d_{\alpha\beta}(k_2)\, e^{-\alpha(\log a_0)}\, e^{-\alpha(\log a)}| \\
&\qquad \times \|f(h:v; D:l_1; ma; X_\beta l_2)\| \\
&\quad \leqslant e^{-\beta_Q(\log a)} \sum_\beta c_\beta \|f(h:v; D:l_1; ma; X_\beta l_2)\|,
\end{aligned}
$$

where each $c_\beta = \sup_{k_1,k_2} \sum_\alpha |c_\alpha(k_1)\, d_{\alpha\beta}(k_2)| < \infty$. Now the result follows as above where $F = \{c_0 c_\beta D \otimes l_1 \otimes X_\beta l_2\}$. Q.E.D.

COROLLARY 7.12. *Fix $D \in \mathscr{P}$, $v \in \mathscr{Z}_Q$, and $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$. Then we can choose a finite subset F of $\tilde{\mathscr{S}}_P$ and $r_0 \geqslant 0$ so that for all $m \in L_Q^+$, $H \in \mathrm{cl}(\mathfrak{a}_Q^+)$*

$$
\|\Psi_P(f:h:v; D:b_1; m\exp H; b_2)\|
\leqslant \begin{cases} {}^0S_{F,r,t}(f)\, \Xi_Q(m)\, e^{-\beta_Q(H)}(1+\tilde\sigma(m\exp H))^{r+r_0}(1+d(h)^{-1})^{-t} \\ \qquad \text{for all} \quad r, t \geqslant 0,\ f \in I(\mathscr{S}, L_P) \\ S_{F,r}(f)\, \Xi_Q(m)\, e^{-\beta_Q(H)}\, |(h,v,m)|^{r+r_0}(1+|H|)^{r+r_0}\, e^{|h_I|\sigma_V(m)} \\ \qquad \text{for all} \quad r \geqslant 0,\ f \in II(\mathscr{S}_{\mathbb{C}}, L_P) \end{cases}
$$

and

$$
\|\Phi(f:h:v; D:b_1; m\exp H; b_2)\|
\leqslant \begin{cases} {}^0S_{F,r,t}(f)\, \Xi_Q(m)(1+\tilde\sigma(m\exp H))^{r+r_0}(1+d(h)^{-1})^{-t} \\ \qquad \text{for all} \quad r, t \geqslant 0,\ f \in I(\mathscr{S}, L_P) \\ S_{F,r}(f)\, \Xi_Q(m)\, |(h,v,m)|^{r+r_0}(1+|H|)^{r+r_0}\, e^{|h_I|\sigma_V(m)} \\ \qquad \text{for all} \quad r \geqslant 0,\ f \in II(\mathscr{S}_{\mathbb{C}}, L_P). \end{cases}
$$

Proof. This follows from (7.11) because, by definition (7.8), $\Psi_v(h:v:b_1; m\exp H; b_2)$ is a sum of terms of the form $d_Q(m\exp H)\, f(h:v:b_1'; m\exp H; (\mu_{PQ}(z)' - \mu_P(z:\Lambda_{h,v}))\, v_j' b_2') = d_Q(m\exp H)\, f(h:v:b_1'; m\exp H; (\mu_{PQ}(z)' - z)\, v_j' b_2')$ for some $z \in \mathcal{Z}_P$. But $\mu_{PQ}(z)' - z \in \theta(^*\mathfrak{n}_Q)\, \mathcal{U}(\mathfrak{l}_P)\, ^*\mathfrak{n}_Q$. Q.E.D.

For any $\Lambda \in \mathfrak{h}_{0\mathbb{C}}^*$, let $B_1(\Lambda)$ be the endomorphism of $\mathbf{W} = W \otimes_{\mathbb{C}} T$ given by $1 \otimes \pi_P(\Lambda)\, B(\Lambda)$, where B is defined as in Lemma 7.3. For $1 \leqslant i \leqslant r$ and $v \in \mathcal{Z}_Q$ set

$$\Phi_i(f:h:v:m) = B_1(s_i \Lambda_{h,v})\, \Phi(f:h:v:m); \tag{7.13a}$$

$$\Psi_{v,i}(f:h:v:m) = B_1(s_i \Lambda_{h,v})\, \Psi_v(f:h:v:m). \tag{7.13b}$$

Since $B_1(s_i \Lambda_{h,v})$ depends polynomially on h and v, there are constants $c \geqslant 0$ and $b \geqslant 0$ so that

$$\|B_1(s_i \Lambda_{h,v})\|_{OP} \leqslant c(1 + |h|)^b\, (1 + |v|)^b. \tag{7.13c}$$

LEMMA 7.14. *Let $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_Q)$, $m \in L_Q$, $H \in \mathfrak{a}_Q$. Then for all $T \in \mathbb{R}$, $1 \leqslant i \leqslant r$,*

$$\begin{aligned} \Phi_i(f:h:v:b_1; m\exp TH; b_2) &= e^{T s_i \Lambda_{h,v}(H)} \Phi_i(f:h:v:b_1; m; b_2) \\ &\quad + \int_0^T e^{(T-t) s_i \Lambda_{h,v}(H)} \Psi_{H,i}(f:h:v:b_1; m\exp tH; b_2)\, dt. \end{aligned}$$

Proof. This is an immediate consequence of (7.10) together with (7.3). Q.E.D.

For $1 \leqslant i \leqslant r$, let $\lambda_i(h)$ be restriction of the real part of $s_i \lambda_h^v$ to $\mathfrak{a}_Q$. Note $\lambda_i(h) = \lambda_i(h_R)$ if $h = h_R + ih_I$ with $h_R \in \mathcal{D}$, $h_I \in \omega$. Set $I = \{1, 2, ..., r\}$ and define

$$\begin{aligned} I^0 &= \{i \in I : \lambda_i(h:H) = 0 \text{ for all } h \in \mathcal{D},\ H \in \mathfrak{a}_Q\}; \\ I^+ &= \{i \in I : \lambda_i(h:H) > 0 \text{ for some } h \in \mathcal{D} \text{ and some } H \in \mathfrak{a}_Q^+\}; \\ I^- &= \{i \in I : \lambda_i(h:H) < 0 \text{ for all } h \in \mathcal{D} \text{ and all } H \in \mathfrak{a}_Q^+\}. \end{aligned} \tag{7.15}$$

Note that $I = I^0 \cup I^- \cup I^+$, since if $i \notin I^+$, $\lambda_i(h:H) \leqslant 0$ for all $h \in \mathcal{D}$, $H \in \mathfrak{a}_Q^+$. But $\mathfrak{a}_Q^+$ and $\mathcal{D}$ are open, and $\lambda_i(h:H)$ is a linear function of H and an affine function of h, so that either $\lambda_i(h:H)$ is identically zero, or else $\lambda_i(h:H) < 0$ for all $h \in \mathcal{D}$ and $H \in \mathfrak{a}_Q^+$.

LEMMA 7.16. *Let C be a compact subset of L_Q, Ω a compact subset of $\mathfrak{a}_Q^+$. Then we can choose $T_0 \geqslant 0$ so that $m \exp TH \in L_Q^+$ for $m \in C$ and $T \geqslant T_0$, $H \in \Omega$.*

Proof. See [2, Lemma 54].

For $H \in \mathfrak{a}_Q^+$, $i \in I^0 \cup I^+$, define $\mathscr{D}_{\mathbb{C}}^i(H) = \{h \in \mathscr{D}_{\mathbb{C}} : \lambda_i(h:H) + \beta_Q(H) > 0\}$, $\mathscr{D}^i(H) = \mathscr{D}_{\mathbb{C}}^i(H) \cap \mathscr{D}$. Note if $i \in I^0$, then $\mathscr{D}_{\mathbb{C}}^i(H) = \mathscr{D}_{\mathbb{C}}$ for all $H \in \mathfrak{a}_Q^+$.

LEMMA 7.17. *Let $D \in \mathscr{P}$, $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$, $i \in I^0 \cup I^+$, $H \in \mathfrak{a}_Q^+$. Then $\int_0^\infty \|\Psi_{H,i}(f:h:v; D \circ e^{-t s_i \Lambda_{h,v}(H)}:b_1; m \exp tH; b_2)\|\, dt$ converges uniformly for v and m in compact subsets of $\mathscr{F}$ and L_Q, respectively, and for h in compact subsets of*

$$\begin{aligned} &\mathscr{D}^i(H) && \text{if } f \in I(\mathscr{D}, L_P); \\ &\mathscr{D}_{\mathbb{C}}^i(H) && \text{if } f \in II(\mathscr{D}_{\mathbb{C}}, L_P). \end{aligned}$$

Proof. This follows from (7.12), (7.13c), and (7.16). Q.E.D.

LEMMA 7.18. *Let $i \in I^0 \cup I^+$, $H \in \mathfrak{a}_Q^+$. Then $\Phi_{i,\infty}(f:h:v:m:H) = \lim_{T \to \infty} \Phi_i(f:h:v:m \exp TH)\, e^{-T s_i \Lambda_{h,v}(H)}$ exists and is C^∞ on*

$$\begin{aligned} &\mathscr{D}^i(H) \times \mathscr{F} \times L_Q && \text{if } f \in I(\mathscr{D}, L_P); \\ &\mathscr{D}_{\mathbb{C}}^i(H) \times \mathscr{F} \times L_Q \text{ and holomorphic for } h \in \mathscr{D}_{\mathbb{C}}^i(H) && \text{if } f \in II(\mathscr{D}_{\mathbb{C}}, L_P). \end{aligned}$$

Further, for all $D \in \mathscr{P}$, $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_P)$,

$$\begin{aligned} &\Phi_{i,\infty}(f:h:v; D:b_1; m; b_2:H) \\ &\quad = \Phi_i(f:h:v; D:b_1; m; b_2) \\ &\qquad + \int_0^\infty \Psi_{H,i}(f:h:v; D \circ e^{-t s_i \Lambda_{h,v}(H)}:b_1; m \exp tH; b_2)\, dt. \end{aligned}$$

Proof. Combine Lemmas (7.14) and (7.17). Q.E.D.

Let $H_1, H_2 \in \mathfrak{a}_Q^+$. For $i \in I^0 \cup I^+$ and

$$h \in \begin{cases} \mathscr{D}^i(H_1) \cap \mathscr{D}^i(H_2) & \text{if } f \in I(\mathscr{D}, L_P), \\ \mathscr{D}_{\mathbb{C}}^i(H_1) \cap \mathscr{D}_{\mathbb{C}}^i(H_2) & \text{if } f \in II(\mathscr{D}_{\mathbb{C}}, L_P), \end{cases}$$

the argument in [10, p. 285] shows that $\Phi_{i,\infty}(f:h:v:m:H_1) = \Phi_{i,\infty}(f:h:v:m:H_2)$. Thus whenever there is an $H \in \mathfrak{a}_Q^+$ such that $\lambda_i(h:H) + \beta_Q(H) > 0$, we can define $\Phi_{i,\infty}(f:h:v:m) = \Phi_{i,\infty}(f:h:v:m:H)$, and the definition does not depend on the choice of H.

LEMMA 7.19. *Suppose for* $i \in I^+$ *and* $h \in \left\{ \begin{smallmatrix} \mathscr{D} \\ \mathscr{D}_1 \end{smallmatrix} \right\}$, *there is an* $H \in \mathfrak{a}_Q^+$ *such that* $\lambda_i(h:H) > 0$. *Then* $\Phi_{i,\infty}(f:h:v:m) = 0$ *for all* $(v, m) \in \mathscr{F} \times L_Q$ *and* $f \in \left\{ \begin{smallmatrix} I(\mathscr{D}, L_P) \\ II(\mathscr{D}_1, L_P) \end{smallmatrix} \right\}$.

Proof. Since $\lambda_i(h:H) > 0$, $h \in \left\{ \begin{smallmatrix} \mathscr{D}'(H) \\ \mathscr{D}_1'(H) \end{smallmatrix} \right\}$ so that $\Phi_{i,\infty}(f:h:v:m) = \Phi_{i,\infty}(f:h:v:m:H) = \lim_{T \to \infty} \Phi_i(f:h:v:m \exp TH)\, e^{-Ts_i\Lambda_{h,v}(H)}$. But using (7.12) and (7.13c), this limit is zero since $\Phi_i(f:h:v:m \exp TH)$ grows polynomially in T while $|e^{-Ts_i\Lambda_{h,v}(H)}| = e^{-T\lambda_i(h:H)}$ decays exponentially. Q.E.D.

LEMMA 7.20. *Let* $i \in I^0$. *Then*

(i) $\Phi_{i,\infty}(f:h:v:m; v) = \mu_Q(v:s_i\Lambda_{h,v})\, \Phi_{i,\infty}(f:h:v:m)$ *for all* $v \in \mathscr{Z}_Q$;

(ii) *given* $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$ *and* $D \in \mathscr{P}$ *there exists a finite subset* $F \subseteq \tilde{\mathscr{Z}}_P$ *such that for all* $r, t \geq 0$ *there is a* $C > 0$ *so that*

$$\|\Phi_{i,\infty}(f:h:v; D:b_1; m; b_2)\|$$
$$\leq \begin{cases} C\,{}^0S_{F,r,t}(f)\, \Xi_Q(m)(1 + \tilde{\sigma}(m))^{r + r_0 + b}(1 + d(h)^{-1})^{-t} \\ \qquad \text{if } f \in I(\mathscr{D}, L_P) \\ CS_{F,r}(f)\, \Xi_Q(m)\, |(h, v, m)|^{r + r_0 + b} e^{|h_I|\, \sigma_V(m)} \\ \qquad \text{if } f \in II(\mathscr{D}_{\mathbb{C}}, L_P). \end{cases}$$

Here r_0 *and* b *are the constants given in* (7.12) *and* (7.13c).

Proof. (i) From (7.3) and (7.9), we have

$$\begin{aligned} &\Phi_i(f:h:v:m \exp TH; v) \\ &\quad = \mu_Q(v:s_i\Lambda_{hv})\, \Phi_i(f:h:v:m \exp TH) \\ &\qquad + \Psi_{v,i}(f:h:v:m \exp TH). \end{aligned}$$

But by using the estimate in (7.12) we see that

$$\lim_{T \to \infty} e^{-Ts_i\Lambda_{h,v}(H)} \Psi_{v,i}(f:h:v:m \exp TH) = 0 \text{ for } i \in I^0.$$

(ii) Combine the formula for $\Phi_{i,\infty}$ in the second part of (7.18) with the estimates of (7.12) and (7.13c) and use (7.16). Q.E.D.

LEMMA 7.21. *There exists a continuous, piecewise affine function* δ *on* $\mathscr{D}$ *satisfying* $0 < \delta(h) \leq \frac{1}{2}$ *for all* $h \in \mathscr{D}$ *so that given* $D \in \mathscr{P}$ *there exists a finite subset* $F \subseteq \mathscr{P}$ *and* $C, r_1 > 0$ *so that, for all* $f \in I(\mathscr{D}, L_P) \cup II(\mathscr{D}_{\mathbb{C}}, L_P)$, $i \in I$,

$$\Bigg\| \Phi_i(f:h:v; D:b_1; m \exp TH; b_2) - \begin{cases} 0 & \text{if } i \in I^+ \cup I^- \\ \Phi_{i,\infty}(f:h:v; D:b_1; m \exp TH; b_2) & \text{if } i \in I^0 \end{cases} \Bigg\|$$

$$\leqslant Ce^{-T\delta(h)\beta_Q(H)}(1+T\|H\|)^{r_1}$$

$$\times \sum_{D'\in F}\left\{\|\Phi_i(f:h:v;D':b_1;m;b_2)\|\right.$$

$$+\int_0^\infty \|\Psi_{H,i}(f:h:v;D':b_1;m\exp tH;b_2)\|$$

$$\left.\times e^{t\beta_Q(H)/2}(1+t\|H\|)^{r_1}\,dt\right\}$$

for all $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_P)$, $m \in L_Q$, $H \in \mathfrak{a}_Q^+$, *and* $T \geqslant 0$.

We will need some preparation before we can prove this lemma. This is the first result in this section where the continuous relative discrete series parameter plays a significant role. The point is that $i \in I^+$ if there is $H \in \mathfrak{a}_Q^+$ with $\lambda_i(h:H) > 0$ for some $h \in \mathscr{D}$, rather than for all $h \in \mathscr{D}$. In order to obtain the estimate required in the case that $i \in I^+$, we need to use the holomorphicity in h of functions in $H(\mathscr{D}_{\mathbb{C}}, L_P)$. After we have the result in the case that $f \in H(\mathscr{D}_{\mathbb{C}}, L_P)$, we use the fact that each $f \in I(\mathscr{D}, L_P)$ is a linear combination of functions in $H(\mathscr{D}_{\mathbb{C}}, L_P)$ to obtain the result in case $f \in I(\mathscr{D}, L_P)$.

Suppose λ is any real-valued linear function on $\mathfrak{a}_Q$. Write $\lambda = \sum_{j=1}^{l} c_j \alpha_j$, where $\alpha_1, ..., \alpha_l$ are the simple roots of $\mathfrak{a}_Q$ giving $\mathfrak{a}_Q^+$ as positive chamber. Recall $\beta_Q(H) = \min_{1 \leqslant j \leqslant l} \{\alpha_j(H)\}$ for $H \in \mathfrak{a}_Q^+$. The following lemma is elementary.

LEMMA 7.22. *Let* $\lambda \in \mathfrak{a}_Q^*$ *and define* $c_1, ..., c_l$ *as above. Then*

(i) $\lambda(H) = 0$ *for all* $H \in \mathfrak{a}_Q$ *if and only if* $c_1 = \cdots = c_l = 0$;

(ii) $\lambda(H) < 0$ *for all* $H \in \mathfrak{a}_Q^+$ *if and only if* $c_j \leqslant 0$ *for all* $1 \leqslant j \leqslant l$, *and* $\sum_{j=1}^{l} c_j < 0$;

(iii) $\lambda(H) > 0$ *for some* $H \in \mathfrak{a}_Q^+$ *if and only if* $c_j > 0$ *for some* $1 \leqslant j \leqslant l$.

(iv) $\lambda(H) + \beta_Q(H) > 0$ *for some* $H \in \mathfrak{a}_Q^+$ *if and only if* $c_j > 0$ *for some* $1 \leqslant j \leqslant l$ *or* $\sum_{j=1}^{l} c_j > -1$.

Now for each $i \in I$, we write $\lambda_i(h) = \sum_{j=1}^{l} c_{ij}(h)\alpha_j$ and let $d_i(h) = -\sum_{j=1}^{l} c_{ij}(h)$. Define

$$\mathscr{D}_i^+ = \{h \in \mathscr{D} : c_{ij}(h) > 0 \text{ for some } 1 \leqslant j \leqslant l\}; \tag{7.23a}$$

$$\mathscr{D}_i^- = \{h \in \mathscr{D} : c_{ij}(h) \leqslant 0 \text{ for all } 1 \leqslant j \leqslant l \text{ and } d_i(h) > 0\}; \tag{7.23b}$$

$$\mathscr{D}_i^0 = \{h \in \mathscr{D} : c_{ij}(h) = 0 \text{ for all } 1 \leqslant j \leqslant l\}; \tag{7.23c}$$

$$\mathscr{D}_1^1 = \{h \in \mathscr{D} : d_i(h) < 1\}. \tag{7.23d}$$

LEMMA 7.24. *Suppose* $\mathscr{D}_i^+ \neq \varnothing$ *and* $\mathscr{D}_1^1 \neq \varnothing$. *Then either*

(i) $\mathscr{D}_i^+ \cap \mathscr{D}_1^1 \neq \varnothing$ *or*

(ii) $\mathscr{D} = \mathscr{D}_i^+ \cup \mathscr{D}_i$, *where* $\mathscr{D}_i^- \neq \varnothing$ *and* $\inf_{h \in \mathscr{D}_i} d_i(h) > 0$.

Proof. Fix $i \in I$ and drop i from the notation for simplicity. Suppose (ii) does not hold. Thus either $\mathscr{D}^0 \neq \varnothing$, $\mathscr{D} \ = \varnothing$, or $\inf_{h \in \mathscr{D}^-} d(h) = 0$. Suppose $\mathscr{D} \ = \varnothing$. Then $\mathscr{D} = \mathscr{D}^0 \cup \mathscr{D}^+$. Since $\mathscr{D}^+ \neq \varnothing$ and $\mathscr{D}^1 \neq \varnothing$ by assumption, $\mathscr{D}^+$ is a dense open subset of $\mathscr{D}$ and $\mathscr{D}^1$ is a non-empty open subset of $\mathscr{D}$. Thus $\mathscr{D}^+ \cap \mathscr{D}^1 \neq \varnothing$. Thus we may assume $\mathscr{D} \ \neq \varnothing$. Suppose $\mathscr{D}^0 \neq \varnothing$. Pick $h_0 \in \mathscr{D}^0$ and $h_1 \in \mathscr{D}^-$. Since $\mathscr{D}$ is convex $h_t = th_1 + (1-t)h_0 \in \mathscr{D}$ for all $0 \leqslant t \leqslant 1$. But for all j, $c_j(h_t) = tc_1(h_0) \leqslant 0$ for $0 \leqslant t \leqslant 1$ and $d(h_t) = td(h_1) > 0$ for $0 < t \leqslant 1$. Thus $h_t \in \mathscr{D}^-$ for $0 < t \leqslant 1$ and $d(h_t) \to 0$ as $t \to 0$. Thus $\inf_{h \in \mathscr{D}} d(h) = 0$. Thus it is enough to show that $\mathscr{D}^- \neq \varnothing$ and $\inf_{h \in \mathscr{D}^-} d(h) = 0$ imply that $\mathscr{D}^1 \cap \mathscr{D}^+ \neq \varnothing$.

Pick $\{h_n\} \in \mathscr{D}$ so that $d(h_n) \to 0$ as $n \to \infty$. Fix $h^+ \in \mathscr{D}^+$. Then for some $1 \leqslant j \leqslant l$, $c_j(h^+) > 0$. If $h^+ \in \mathscr{D}^1$ we are done, so we can assume that $d(h^+) \geqslant 1$. For each n, $h_{n,t} = th^+ + (1-t)h_n \in \mathscr{D}$ for $0 \leqslant t \leqslant 1$. Let $T = 1/2d(h^+)$. Then $0 < T \leqslant \frac{1}{2}$. Pick N large enough that $d(h_N) < 1/2(1-T)$ and $c_j(h_N) > -Tc_j(h^+)/(1-T)$. Then it is easy to check that $h_{N,T} \in \mathscr{D}^+ \cap \mathscr{D}^1$. Q.E.D.

We are now ready to define the function δ which is required by (7.21). Let $\tilde{I}^+ = \{i \in I^+ : \mathscr{D}_i^0 = \varnothing, \mathscr{D}_i \ \neq \varnothing$, and $d_i = \inf_{h \in \mathscr{D}_i} d_i(h) > 0\}$. Then for each $h \in \mathscr{D}$ we define

$$\delta(h) = \min[\{d_i(h) : i \in I^-\}; \{d_i, i \in \tilde{I}^+\}; \{\tfrac{1}{2}\}]. \tag{7.25a}$$

Then it is easy to check that δ is continuous piecewise affine function on $\mathscr{D}$ satisfying

$$0 < \delta(h) \leqslant \tfrac{1}{2} \qquad \text{for all} \quad h \in \mathscr{D} \tag{7.25b}$$

and

$$\lambda_i(h : H) \leqslant -\delta(h)\, \beta_Q(H) \qquad \text{for all} \quad H \in a_Q^+$$
$$\text{if } i \in I \ \text{ or if } i \in \tilde{I}^+ \text{ and } h \in \mathscr{D}_i^-. \tag{7.25c}$$

LEMMA 7.26. *Let* $i \in I^+$. *Suppose* h_0 *satisfies* $\lambda_i(h_0 : H) + \beta_Q(H) > 0$ *for some* $H \in \mathfrak{a}_Q^+$. *Then either*

(i) $i \in \tilde{I}^+$ *and* $\operatorname{Re} h_0 \in \mathscr{D}_i$ *or*

(ii) *there is a neighborhood* $U(h_0)$ *of* h_0 *so that* $\Phi_{i,\mathscr{K}}(f : h : v : m) = 0$ *for all* $(h, v, m) \in U(h_0) \times \mathscr{F} \times L_Q$.

Proof. Using (7.22), $\operatorname{Re} h_0 \in \mathscr{D}_i^+ \cup \mathscr{D}_i^1$. By (7.19), $\Phi_{i,\infty}(f:h:v:m)=0$ for all $(v,m) \in \mathscr{F} \times L_Q$ if $\operatorname{Re} h \in \mathscr{D}_i^+$. Thus when $\operatorname{Re} h_0 \in \mathscr{D}_i^+$, (ii) is satisfied with $U(h_0)=\{h:\operatorname{Re} h \in \mathscr{D}_i^+\}$. Thus we can assume that $\operatorname{Re} h \notin \mathscr{D}_i^+$. Now if $i \in \tilde{I}^+$, $\mathscr{D}_i^0=\varnothing$ so $\operatorname{Re} h \notin \mathscr{D}_i^+$ implies that $\operatorname{Re} h \in \mathscr{D}_i^-$ so that (i) is satisfied. Thus we can assume that $i \notin \tilde{I}^+$, so that by (7.24), $\mathscr{D}_i^+ \cap \mathscr{D}_i^1 \neq \varnothing$.

Suppose first that $f \in II(\mathscr{D}_{\mathbb{C}}, L_P)$. Then by (7.18) $\Phi_{i,\infty}(f:h:v:m)$ is holomorphic on $\{h \in \mathscr{D}_{\mathbb{C}}:\operatorname{Re} h \in \mathscr{D}_i^+ \cup \mathscr{D}_i^1\}$. As above, $\Phi_{i,\infty}(f:h:v:m)=0$ for $h \in \mathscr{D}_i^+(\mathbb{C})=\{h \in \mathscr{D}_{\mathbb{C}}:\operatorname{Re} h \in \mathscr{D}_i^+\}$. Now $\mathscr{D}_i^1(\mathbb{C})=\{h \in \mathscr{D}_{\mathbb{C}}:\operatorname{Re} h \in \mathscr{D}_i^1\}$ is an open convex set, hence connected, and by hypothesis $\mathscr{D}_i^+(\mathbb{C}) \cap \mathscr{D}_i^1(\mathbb{C})$ is a non-empty open subset of $\mathscr{D}_i^1(\mathbb{C})$ on which $\Phi_{i,\infty}(f:h:v:m)=0$. Thus $\Phi_{i,\infty}(f:h:v:m)=0$ on all of $\mathscr{D}_i^1(\mathbb{C})$, so that (ii) is satisfied.

Now suppose $f \in I(\mathscr{D}, L_P)$. Then f is a finite linear combination of functions $f_j \in II(\mathscr{D}_{\mathbb{C}}, L_P)$. Thus $\Phi_{i,\infty}(f)$ is a finite linear combination of the corresponding $\Phi_{i,\infty}(f_j)$. Now $\Phi_{i,\infty}(f_j:h:v:m)=0$ for $h \in \mathscr{D}_i^+(\mathbb{C}) \cup \mathscr{D}_i^1(\mathbb{C})$ implies that $\Phi_{i,\infty}(f:h:v:m)=0$ for $h \in \mathscr{D}_i^+ \cup \mathscr{D}_i^1$. Thus (ii) is satisfied.

Q.E.D.

Proof of Lemma 7.21. For $D \in \mathscr{P}$, $i \in I$, there is a finite subset F_i of $\mathscr{P}$ and for each $D' \in F_i$ a polynomial $P_i(D')$ on $\mathfrak{a}_Q$ so that $D \circ e^{s_i \cdot \Lambda_{h,v}(H)} = e^{s_i \cdot \Lambda_{h,v}(H)} \sum_{D' \in F_i} P_i(D':H) \circ D'$ for all $H \in \mathfrak{a}_Q$. Pick $r_1 \geqslant 0$ and $c>0$ so that $|P_i(D':H)| \leqslant C(1+\|H\|)^{r_1}$ for all $i \in I$, $D' \in F_i$. Let $F=\bigcup_{i \in I} F_i$.

Case I. Suppose $i \in I^0$. Then $|e^{s_i \cdot \Lambda_{h,v}(H)}|=1$ for all $H \in \mathfrak{a}_Q$, and using (7.18),

$$\begin{aligned}
&\|\Phi_i(f:h:v;D:b_1;m \exp TH;b_2) \\
&\qquad - \Phi_{i,\infty}(f:h:v;D:b_1;m \exp TH;b_2\| \\
&\leqslant \int_0^\infty \|\Psi_{H,i}(f:h:v;D \circ e^{(s_i \cdot \Lambda_{h,v}(H)}:b_1;m \exp(t+T)H;b_2)\|\,dt \\
&= \int_T^\infty \|\Psi_{H,i}(f:h:v;D \circ e^{-(t-T)s_i \cdot \Lambda_{h,v}(H)}:b_1;m \exp tH;b_2)\|\,dt \\
&\leqslant C \sum_{D' \in F_i} \int_T^\infty (1+(t-T)\|H\|)^{r_1} \\
&\qquad \times \|\Psi_{H,i}(f:h:v;D':b_1;m \exp tH;b_2)\|\,dt \\
&\leqslant Ce^{-T\delta(h)\beta_Q(H)} \sum_{D' \in F_i} \int_0^\infty \|\Psi_{H,i}(f:h:v;D':b_1;m \exp tH;b_2)\| \\
&\qquad \times e^{t\beta_Q(H)/2}(1+t\|H\|)^{r_1}\,dt
\end{aligned}$$

since $0<\delta(h) \leqslant \frac{1}{2}$.

Case II. Suppose $i \in I^-$. Using (7.14),

$$\|\Phi_i(f:h:v; D:b_1; m \exp TH; b_2)\|$$
$$\leqslant \|\Phi_i(f:h:v; D \circ e^{Ts_i A_{h,v}(H)}:b_1; m; b_2)\|$$
$$+\int_0^T \|\Psi_{H,i}(f:h:v; D \circ e^{(T-t)s_i A_{h,v}(H)}:b_1; m \exp tH; b_2)\| \, dt$$
$$\leqslant C(1+T\|H\|)^{r_1} e^{T\lambda_i(h:H)} \sum_{D' \in F_i} \|\Phi_i(f:h:v; D':b_1; m; b_2)\|$$
$$+C \sum_{D' \in F_i} \int_0^T (1+(T-t)\|H\|)^{r_1} e^{(T-t)\lambda_i(h:H)}$$
$$\times \|\Psi_{H,i}(f:h:v; D':b_1; m \exp tH; b_2)\| \, dt$$
$$\leqslant C(1+T\|H\|)^{r_1} e^{-T\delta(h)\beta_Q(H)} \sum_{D' \in F_i} \left\{ \|\Phi_i(f:h:v; D':b_1; m; b_2)\| \right.$$
$$\left. + \int_0^\infty e^{t\beta_Q(H)/2} \|\Psi_{H,i}(f:h:v; D':b_1; m \exp tH; b_2)\| \, dt \right\}$$

since $0 < \delta(h) \leqslant \frac{1}{2}$ and $\lambda_i(h:H) \leqslant -\delta(h)\,\beta_Q(H)$.

Case III. Suppose $i \in I^+$ and h satisfies $\lambda_i(h:H) + \frac{1}{2}\beta_Q(H) \leqslant 0$. Then using (7.14) as above and $\lambda_i(h:H) \leqslant -\beta_Q(H)/2$ we have

$$\|\Phi_i(f:h:v\, D:b_1; m \exp TH; b_2)U$$
$$\sum C(1+T\|H\|)^{r_1} e^{-T\beta_Q(H)/2} \sum_{D' \in F_i} \|\Phi_i(f:h:v; D':b_1; m; b_2)\|$$
$$+C \sum_{D' \in F_i} \int_0^T (1+(T-t)\|H\|)^{r_1} e^{-(T-t)\beta_Q(H)/2}$$
$$\times \|\Psi_{H,i}(f:h:v; D':b_1; m \exp tH; b_2)\| \, dt$$
$$\leqslant C(1+T\|H\|)^{r_1} e^{\quad T\delta(h)\beta_Q(H)} \sum_{D' \in F_i} \left\{ \|\Phi_i(f:h:v; D':b_1; m; b_2)\| \right.$$
$$\left. + \int_0^\infty e^{t\beta_Q(H)/2} \|\Psi_{H,i}(f:h:v; D':b_1; m \exp tH; b_2)\| \, dt \right\}$$

since $\delta(h) \leqslant \frac{1}{2}$.

Case IV. Suppose $i \in I^+$ and h satisfies $\lambda_i(h:H) + \frac{1}{2}\beta_Q(H) > 0$. Then $\lambda_i(h:H) + \beta_Q(H) > 0$ also so that (7.26) can be used. In case (ii) of (7.26) we have $\Phi_{i,\infty}(f:h:v; D:m) = 0$ for all $(v, m) \in \mathscr{F} \times L_Q$. Then using (7.18) as in Case I,

$$\begin{aligned}
&\|\Phi_i(f:h:v; D:h_1; m \exp TH; h_2)\| \\
&\quad \leqslant C \sum_{D' \in F_i} \int_T^\infty (1+(t-T)\,\|H\|)^{r_1}\, e^{-(t-T)\lambda_i(h:H)} \\
&\qquad \times \|\Psi_{H,i}(f:h:v; D':h_1; m \exp tH; h_2)\|\, dt \\
&\quad \leqslant C \sum_{D' \in F_i} \int_T^\infty (1+t\,\|H\|)^{r_1}\, e^{(t-T)\beta_Q(H)/2} \\
&\qquad \times \|\Psi_{H,i}(f:h:v; D':h_1; m \exp tH; h_2)\|\, dt \\
&\quad \leqslant C e^{-T\delta(h)\beta_Q(H)} \sum_{D' \in F_i} \int_0^\infty (1+t\,\|H\|)^{r_1}\, e^{t\beta_Q(H)/2} \\
&\qquad \times \|\Psi_{H,i}(f:h:v; D':h_1; m \exp tH; h_2)\|\, dt
\end{aligned}$$

since $\lambda_i(h:H) > -\frac{1}{2}\beta_Q(H)$ and $\delta(h) \leqslant \frac{1}{2}$. In case (i) of (7.26), we have $\lambda_i(h:H) \leqslant -\delta(h)\,\beta_Q(H)$, and the same estimate as that used in Case II works. Q.E.D.

From now on, we define (or redefine) $\Phi_{i,\infty}(f) = 0$ if $i \in I^+ \cup I^-$.

LEMMA 7.27. *Let $i \in I$. Then $\Phi_{i,\infty}(f:h:v:m \exp H) = e^{s_i^{-1}\nu_{h,\infty}(H)}\Phi_{i,\infty}(f:h:v:m)$ for all $H \in \mathfrak{a}_Q$, $m \in L_Q$.*

Proof. This follows from Lemma 7.20, part (i). Q.E.D.

LEMMA 7.28. *Fix $i \in I$ and suppose that $\Phi_{i,\infty}(f)$ is not identically zero on $\mathscr{D} \times \mathscr{F} \times L_Q$. Then $s_i^{-1}\mathfrak{a}_Q \subseteq \mathfrak{a}_H$.*

Proof. We know that $\Phi_{i,\infty}(f)$ is C^∞ on $\mathscr{D} \times \mathscr{F} \times L_Q$. Thus if it is not identically zero, it must be not identically zero on the dense set of points for which f factors through a quotient of Harish-Chandra class. Thus by [4, Lemma 6.3], $s_i^{-1}\mathfrak{a}_Q \subseteq \mathfrak{a}_H$. Q.E.D.

Let $W(\mathfrak{a}_H, \mathfrak{a}_Q)$ denote the set of linear maps s of $\mathfrak{a}_Q$ into $\mathfrak{a}_H$ such that there exists $k \in K_P$ with $s(H) = \mathrm{Ad}\, k(H)$ for all $H \in \mathfrak{a}_Q$. Note s determines the coset kK_Q. If B is any subgroup of L_P normalized by K_Q we write $B^s = kBk^{-1}$ for s and k as above. In particular, $Q^s = M_Q^s A_Q^s N_Q^s$ is a parabolic subgroup of G with $A_Q^s \subseteq A_H$. For φ a (τ_1, τ_2)-spherical function on L_Q, we define φ^s on L_Q^s by $\varphi^s(kmk^{-1}) = \tau_1(k)\,\varphi(m)\,\tau_2(k^{-1})$ for $m \in L_Q$.

LEMMA 7.29. *Given $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$, there is a unique $i = i(s) \in I$ such that $s(H) = s_i^{-1}(H)$ for all $H \in \mathfrak{a}_Q$.*

Proof. See [4, Lemma 6.3]. Q.E.D.

Recall that $\Phi(f)$ takes values in $\mathbf{W} = W \otimes_{\mathbb{C}} T$, where T has a distinguished basis $e_1, \ldots, e_r$. Thus we can write, for each $i \in I$, $\Phi_{i,\chi}(f:h:v:m) = \sum_{1 \leqslant j \leqslant r} \varphi_{ij}(f:h:v:m) \otimes e_j$, where the $\varphi_{ij}(f)$ take values in W. Define

$$\psi_f(h:v:m) = \pi_P(\Lambda_{h,v})\, f(h:v:m); \tag{7.30a}$$

$$\psi_{f,s}(h:v:m) = \varphi_{i(s),1}(f:h:v:m), \qquad s \in W(\mathfrak{a}_H, \mathfrak{a}_Q) \tag{7.30b}$$

$$\psi^1_{f,s}(h:v:m) = \pi_Q(s_{i(s)}\Lambda_{h,v})^{-1}\, \psi_{f,s}(h:v:m). \tag{7.30c}$$

Let Ω be a compact subset of $\mathfrak{a}_Q^+$. Choose $\varepsilon_0 > 0$ so that $\beta_Q(H) \geqslant 2\varepsilon_0$ for all $H \in \Omega$. Put $\varepsilon(h) = \delta(h)\varepsilon_0$, where $\delta(h)$ is defined as in (7.25). For $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$, define $\det s = \pm 1$ by $\pi_P(s\Lambda) = \det s\, \pi_P(\Lambda)$ for all $\Lambda \in \mathfrak{a}_H^*$.

THEOREM 7.31. *Given $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$ and $D \in \mathscr{P}$, there exist a finite subset $F \in \tilde{\mathscr{L}}_P$ and an $r_1 > 0$ so that for all $r, t \geqslant 0$ there is a $c > 0$ so that for all $m \in L_Q^+$, $H \in \Omega$, $T \geqslant 0$,*

$$\begin{aligned} &\| d_Q(m \exp TH)\, \psi_f(h:v; D:b_1'; m \exp TH; b_2') \\ &\quad - \sum_{s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)} \det s\, \psi_{f,s}(h:v; D:b_1; m \exp TH; b_2) \| \\ &\leqslant \begin{cases} C\, {}^0S_{F,r,t}(f)\, e^{-\varepsilon(h)T}\, \Xi_Q(m)(1 + \tilde{\sigma}(m \exp TH))^{r+r_1} \\ \qquad \times (1 + d(h)^{-1})^{-t} \qquad \text{for} \quad f \in I(\mathscr{D}, L_P) \\ C S_{F,r}(f)\, e^{-\varepsilon(h)T}\, \Xi_Q(m)\, |(h, v, m, TH)|^{r+r_1}\, e^{|h_I|\, \sigma_V(m)} \\ \qquad \text{for} \quad f \in II(\mathscr{D}_{\mathbb{C}}, L_P). \end{cases} \end{aligned}$$

Proof. This follows from combining (7.21) with (7.12) and (7.13c). Q.E.D.

LEMMA 7.32. *For $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$, $\psi^1_{f,s}$ extends to a smooth function on*

$$\mathscr{D} \times \mathscr{F} \times L_Q \qquad \text{if} \quad f \in I(\mathscr{D}, L_P);$$

$$\mathscr{D}_{\mathbb{C}} \times \mathscr{F} \times L_Q \qquad \text{if} \quad f \in II(\mathscr{D}_{\mathbb{C}}, L_P).$$

Further, given $b_1, b_2 \in \mathscr{U}(\mathfrak{l}_Q)$, $D \in \mathscr{P}$, there is a finite subset F of $\tilde{\mathscr{L}}_P$ and an $r_1 > 0$ such that for all $r, t \geqslant 0$ there is a $C > 0$ with

$$\begin{aligned} &\| \psi^1_{f,s}(h:v; D:b_1; m; b_2) \| \\ &\leqslant \begin{cases} C\, {}^0S_{F,r,t}(f)\, \Xi_Q(m)(1 + \tilde{\sigma}(m))^{r+r_1}\, (1 + d(h)^{-1})^{-t} \\ \qquad \text{if} \quad f \in I(\mathscr{D}, L_P) \\ C S_{F,r}(f)\, \Xi_Q(m)\, |(h, v, m)|^{r+r_1}\, e^{|h_I|\, \sigma_V(m)} \\ \qquad \text{if} \quad f \in II(\mathscr{D}_{\mathbb{C}}, L_P). \end{cases} \end{aligned}$$

Finally, for all $v \in \mathscr{Z}_Q$ $\psi^1_{f,s}(h:v:m; v) = \mu_Q(v; s_{i(s)}\Lambda_{h,v})\, \psi^1_{f,s}(h:v:m)$.

Proof. For $i \in I$, $\pi_Q(s_i \Lambda_{h,v})^{-1} B_1(s_i \Lambda_{h,v}) = 1 \otimes \pi_{PQ}(s_i \Lambda_{h,v}) B(s_i \Lambda_{h,v})$. But using (7.3), $\pi_{PQ}(s_i \Lambda_{h,v}) B(s_i \Lambda_{h,v})$ will have entries polynomial in h and v. Thus it is clear that $\psi^1_{f,s}$ extends to be smooth, and that the inequality can be proved the same way as in (ii) of (7.20). The final claim follows from (i) of (7.20). Q.E.D.

THEOREM 7.33. *For all* $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$,

$$(\psi^1_{f,s})^s \in \begin{cases} I(\mathcal{S}, L^s_Q) & \text{if } f \in I(\mathcal{S}, L_P) \\ II(\mathcal{S}_{\mathbb{C}}, L^s_Q) & \text{if } f \in II(\mathcal{S}_{\mathbb{C}}, L_P). \end{cases}$$

Given $D \in \tilde{\mathcal{L}}_{Q^s}$, *there are a finite subset* $F \subset \tilde{\mathcal{L}}_P$ *and an* $r_1 > 0$ *such that for all* $r, t \geqslant 0$ *there is a* $C > 0$ *so that*

$$\begin{aligned} {}^0S_{D,r+r_1,t}((\psi^1_{f,s})^s) &\leqslant C\, {}^0S_{F,r,t}(f) && \text{if } f \in I(\mathcal{S}, L_P) \\ S_{D,r+r_1}((\psi^1_{f,s})^s) &\leqslant C S_{F,r}(f) && \text{if } f \in II(\mathcal{S}_{\mathbb{C}}, L_P). \end{aligned}$$

8. SCHWARTZ WAVE PACKETS

In this section we will prove that certain wave packets are Schwartz functions on G. The main result is Theorem 8.4. In Section 9 we will see that this class of wave packets includes wave packets of Eisentein integrals. We use the notation of Section 7. Thus H is a fixed θ-stable Cartan subgroup of G, and P is a parabolic subgroup of G with $A_P \subseteq A_H$. Let Φ_R be the set of real roots of $(\mathfrak{g}, \mathfrak{h})$, Φ^+_R a choice of positive roots. For $\Lambda \in \mathfrak{h}^*_{\mathbb{C}}$, write $\pi_R(\Lambda) = \prod \langle \alpha, \Lambda \rangle$, $\alpha \in \Phi^+_R$. Define $\mathcal{F}' = \{ v \in \mathcal{F} = \mathfrak{a}^*_H : \pi_R(v) \neq 0 \}$.

DEFINITION 8.1. We say $f \in I'(\mathcal{S}, L_P)$ if $f \in I(\mathcal{S}, L_P)$ and if for every parabolic subgroup $*Q = Q \cap L_P$ of L_P and $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$, $v \mapsto \pi_R^{-1}(v)\, \psi_{f,s}(h:v:m)$ has a smooth extension from $\mathcal{F}'$ to $\mathcal{F}$ for all $(h, m) \in \mathcal{S} \times L_Q$.

LEMMA 8.2. *Suppose* $f \in I'(\mathcal{S}, L_P)$. *Then for all* Q, s *as above*, $(\psi^1_{f,s})^s \in I'(\mathcal{S}, L^s_Q)$.

Proof. By (7.33), $(\psi^1_{f,s})^s \in I(\mathcal{S}, L^s_Q)$. Let $*Q' = Q \cap L_Q$ be a parabolic subgroup of L_Q and let $t \in W(\mathfrak{a}_H, \mathfrak{a}^s_{Q'})$. Write $g = (\psi^1_{f,s})^s$. Then by [4, Lemma 7.4], there is $t \in W(\mathfrak{a}_H, \mathfrak{a}_{Q'})$ so that $(\psi_{g,t})^t = (\psi^{Q'}_{f,t'})^{t'}$ for the dense set of points for which f factors through a group of Harish-Chandra class. But both sides are smooth, so that the equality persists for all values of (h, v). But $f \in I'(\mathcal{S}, L_P)$ implies that $v \mapsto \pi_R^{-1}(v)\, \psi^{Q'}_{f,t'}(h:v:m)$ has a smooth extension from $\mathcal{F}'$ to $\mathcal{F}$. Thus $v \mapsto \pi_R^{-1}(v)\, \psi_{g,t}(h:v:m)$ does also. Q.E.D.

For $\varphi \in I'(\mathcal{D}, G)$, define

$$I_\varphi(x) = \int_{\mathcal{S} \times \mathcal{F}} \psi_\varphi(h:v:x)\, \pi_R^{-1}(v)\, dh\, dv$$

$$= \int_{\mathcal{S} \times \mathcal{F}} \varphi(h:v:x)\, \pi_G(A_{h,v})\, \pi_R^{-1}(v)\, dh\, dv. \tag{8.3}$$

THEOREM 8.4. *Let $\varphi \in I'(\mathcal{D}, G)$. Then $I_\varphi \in \mathscr{C}(G, W)$. Given any $r \geqslant 0$ and $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$, there is a finite subset F of $\mathscr{L}_G$ so that given any $r' \geqslant 0$ there are $C > 0$ and $t \geqslant 0$ so that ${}_{g_1}\|I_\varphi\|_{r, g_2} \leqslant C\, {}^0S_{F, r', t}(\varphi)$.*

Proof. Note that the first claim follows from the second. This is because, by Definition 7.7, given F, we can choose r' so that ${}^0S_{F,r',t}(\varphi) < \infty$ for all $t \geqslant 0$. We will reduce the second claim to a theorem which can be proved by induction using the machinery of Section 7.

Write $\tilde{\varphi}(h:v:x) = \varphi(h:v:x)\, \pi_G(A_{h,v})\, \pi_R^{-1}(v)$. Then for $r \geqslant 0$ and $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$,

$${}_{g_1}\|I_\varphi\|_{r, g_2} = \sup_{x \in G} (1 + \tilde{\sigma}(x))^r\, \Xi^{-1}(x) \left\| \int_{\mathcal{S} \times \mathcal{F}} \tilde{\varphi}(h:v:g_1;x;g_2)\, dh\, dv \right\|$$

$$\leqslant C \sup_{\substack{k_1, k_2 \in K \\ a \in \mathrm{cl}(A_0^+)}} (1 + \sigma(a))^r\, \Xi^{-1}(a)(1 + \tilde{\sigma}(k_1 k_2))^r$$

$$\times \left\| \int_{\mathcal{S} \times \mathcal{F}} \tau_{1,h}(k_1)\, \tilde{\varphi}(h:v:{}^{k_1^{-1}}g_1; a; {}^{k_2}g_2)\, \tau_{2,h}(k_2)\, dh\, dv \right\|.$$

Write ${}^{k_1^{-1}}g_1 = \sum_i f_i'(k_1)\, g_i'$, ${}^{k_2}g_2 = \sum_j f_j''(k_2)\, g_j''$, where both sums are finite, the $g_i', g_j'' \in \mathscr{U}(\mathfrak{g})$, and the $f_i', f_j'' \in C^\infty(K/Z)$. Let $C_1 = \sup_{i,j,k_1,k_2} |f_i'(k_1)\, f_j''(k_2)| < \infty$. Write $k_i = k_i' k_i''$, $i = 1, 2$, where $k_i' \in K_1$, $k_i'' \in V$. Then $\tau_{i,h}(k_i) = \tau_i(k_i)(e^h)(k_i'')$, and $\tilde{\sigma}(k_1 k_2) = \tilde{\sigma}(k_1'' k_2'')$. Thus

$${}_{g_1}\|I_\varphi\|_{r, g_2} \leqslant C C_1 \sum_{i,j} \sup_{\substack{k \in V \\ a \in \mathrm{cl}(A_0^+)}} (1 + \sigma(a))^r\, \Xi^{-1}(a)(1 + \tilde{\sigma}(k))^r$$

$$\times \left\| \int_{\mathcal{S} \times \mathcal{F}} e^h(k)\, \tilde{\varphi}(h:v:g_i'; a; g_j'')\, dh\, dv \right\|.$$

Thus the result follows from the special case $L_P = G$ of Theorem 8.5 below. Q.E.D.

THEOREM 8.5. *Let P be a parabolic subgroup of G with $A_P \subseteq A_H$. Given*

$l_1, l_2 \in \mathcal{U}(\mathfrak{l}_P)$, $r \geqslant 0$, we can choose a finite subset $F \subseteq \mathcal{L}_P$ so that given any $r' \geqslant 0$ there are $C > 0$ and $t \geqslant 0$ so that

$$\sup_{\substack{k \in K \\ a \in \mathrm{cl}(A_0^+(P))}} (1+\sigma(a))^r \, \Xi_P^{-1}(a)(1+\tilde{\sigma}(k))^{r'}$$

$$\times \left\| \int_{\mathcal{D} \times \bar{F}} (e^h)(k)\, \psi_\varphi(h:v:l_1;a;l_2)\, \pi_R^{-1}(v)\, dh\, dv \right\|$$

$$\leqslant C\, {}^0S_{F,r',t}(\varphi) \qquad \text{for all} \quad \varphi \in I'(\mathcal{D}, L_P).$$

Here $A_0^+(P)$ is a positive chamber of A_0 with respect to $\Delta^+(L_P, A_0)$, a set of positive roots for (L_P, A_0), and Ξ_P is the spherical function Ξ for L_P.

Before we start the proof of Theorem 8.5, we will need some lemmas.

Let $E = \mathbb{R}^n$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ put $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and denote by M the set of all multi-indices. For W a finite dimensional vector space with norm $\| \ \|$, put $\mathscr{C}(E:W) = \{ f \in C^\infty(E:W) : s_{\alpha,r}(f) = \sup_E (1+|x|)^r \|D^\alpha f(x)\| < \infty$ for all $r \geqslant 0$, $\alpha \in M\}$. Let $p \neq 0$ be the product of N real linear forms on E.

LEMMA 8.6. *Fix $\alpha \in M$ and let $F = \{\beta \in M : |\beta| \leqslant |\alpha| + N\}$. Then for every $r \geqslant 0$, we can choose $C_r \geqslant 1$ with the following property. Suppose $f \in \mathscr{C}(E:W)$ and $p^{-1}f$ is locally bounded on E. Then $f = pg$, where $g \in \mathscr{C}(E:W)$ and $s_{\alpha,r}(g) \leqslant C_r \sum_{\beta \in F} s_{\beta,r}(f)$.* Q.E.D.

Proof. See [4, Lemma 22.2].

LEMMA 8.7. *Let $\varepsilon(h)$ be a continuous, piecewise affine function on $\mathcal{D}$ so that $\varepsilon(h) > 0$ for all $h \in \mathcal{D}$. Then for all $r \geqslant 0$ there are a $C > 0$ and an $r_1 > 0$ so that*

$$\sup_{a \in \mathrm{cl}(A_0^+(P)) \cap M_P} (1+\sigma(a))^r \, \Xi_P^{\varepsilon(h)}(a)$$

$$\leqslant C_r(1 + d(h)^{-1})^{r_1} \qquad \text{for all} \quad h \in \mathcal{D}.$$

Proof. There are constants $q \geqslant 0$ and $C \geqslant 0$ so that $\Xi_P(a) \leqslant C(1+\sigma(a))^q e^{-\rho_P(\log a)}$ for all $a \in \mathrm{cl}(A_0^+(P)) \cap M_P$, $\rho_P = \frac{1}{2}\sum m(\alpha)\alpha$, $\alpha \in \Delta^+(L_P, A_0)$. Let $\alpha_1, \ldots, \alpha_d$ be the set of simple roots of (L_P, A_0) determining $A_0^+(P)$. Pick $H_1, \ldots, H_d \in (\mathfrak{a}_0 \cap \mathfrak{m}_P)$ so that $\alpha_i(H_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant d$. Then $\mathfrak{a}_0^+(P) \cap \mathfrak{m}_P = \{\sum_{i=1}^d t_i H_i : t_i > 0$ for $1 \leqslant i \leqslant d\}$ and $\rho_P = \sum_{i=1}^d n_i \alpha_i$ for some $n_i > 0$, $1 \leqslant i \leqslant d$. Thus $\sup_{a \in \mathrm{cl}(A_0^+(P)) \cap M_P} (1+\sigma(a))^r \, \Xi_P^{\varepsilon(h)}(a) \leqslant C \prod_{i=1}^d [\sup_{t_i \geqslant 0} (1+t_i)^{r+q} e^{-n_i t_i \varepsilon(h)}] \leqslant C[\sup_{t \geqslant 0} (1+t)^{r+q} e^{-nt\varepsilon(h)}]^d$, where $n = \min_{1 \leqslant i \leqslant d} n_i > 0$. Write $\mathcal{D} = \bigcup_{i=1}^k \mathcal{D}_i$, where $\varepsilon_i(h) = n\varepsilon(h)|_{\mathcal{D}_i}$ is affine for $1 \leqslant i \leqslant k$, $\varepsilon_i(h) > 0$ on $\mathcal{D}_i$. Then these are constants C_i so that

$\sup_{t\geqslant 0}(1+t)^{r+q}e^{-t\varepsilon_i(h)}\leqslant C_i(1+1/\varepsilon_i(h))^{r+q}$ for all $h\in\mathscr{D}_i$. But since $\{h\in\mathfrak{v}^*:\varepsilon_i(h)=0\}$ is outside of $\mathscr{D}$, there is a constant C_i' so that for all $h\in\mathscr{D}_i$, $d(h)\leqslant C_i'\varepsilon_i(h)$. Q.E.D.

Proof of Theorem 8.5. Let $\{\alpha_1, ..., \alpha_d\}$ be the simple roots for the set of positive roots of (L_P, A_0) determining $A_0^+(P)$. The proof will be by induction on d, the number of simple roots.

Case I. Suppose $d=0$. Then A_0 is central in L_P so that $P=M_0A_0N_0$ is a minimal parabolic. Since we assume that $A_P\subseteq A_H\subseteq A_0$, this occurs only when $A_0=A_H$. Note then $\Xi_P\equiv 1$, $\mathrm{cl}(A_0^+(P))=A_0$, and $\psi_\varphi=\varphi$.

Every element of $\mathscr{U}(\mathfrak{l}_0)$ is a finite sum of terms of the form $l_i=k_iu_i$, where $k_i\in\mathscr{U}(\mathfrak{m}_0\cap\mathfrak{k})$ and $u_i\in S(\mathfrak{a}_0)$. Since φ is $(K\cap M_0)$-spherical, $\varphi(h:v:k_1u_1; a; u_2k_2)=d\tau_{1,h}(k_1)\,\varphi(h:v:u_1; a; u_2)\,d\tau_{2,h}(k_2)$, where $d\tau_{i,h}(k_i)$, $i=1,2$, depends polynomially on h. Thus there are finitely many polynomials $P_j(h)$ such that $\|\int(e^h)(k)\,\varphi(h:v:l_1; a; l_2)\,\pi_R^{-1}(v)\,dh\,dv\|\leqslant\sum_j\|\int(e^h)(k)\,\varphi(h:v:a; u_1u_2)\,P_j(h)\,\pi_R^{-1}(v)\,dh\,dv\|$.

Further, since $S(\mathfrak{a}_0)\subseteq\mathscr{Z}_P$ and φ is an eigenfunction for $\mathscr{Z}_P$, $\varphi(h:v:a)=e^{\Lambda_{h,v}(\log a)}\,\varphi(h:v:1)$ for all $a\in A_0$. But $\mathfrak{a}_0=\mathfrak{a}_H$ so that $\Lambda_{h,v}(\log a)=iv(\log a)$. Thus $\varphi(h:v:a; u_1u_2)=u_1u_2(iv)\,e^{iv(\log a)}\,\varphi(h:v:1)$ for all $a\in A_0$. Note that $u_1u_2(iv)=Q(v)$ is a polynomial in v. Now since $\varphi\in I'(\mathscr{D}, L_P)$, $f_j(h:v)=P_j(h)\,Q(v)\,\varphi(h:v:1)\in\mathscr{C}(i\mathfrak{v}^*\times\mathscr{F}, W)$ and π_R is a product of real linear forms on $\mathscr{F}$ for which $\pi_R^{-1}f_j$ is locally bounded. Thus by (8.6), $g_j=\pi_R^{-1}f_j\in\mathscr{C}(i\mathfrak{v}^*\times\mathscr{F}, W)$. Now $\mathscr{F}$ is the dual of A_0 and $\mathscr{D}$ is a subset of the dual of V so that $g_j(h:v)$ is supported in $\mathrm{cl}(\mathscr{D})\times\mathscr{F}$. Pick a polynomial $R(h, v)$ such that $C=\int_{\mathscr{D}\times\mathscr{F}}|R(h, v)|^{-1}\,dh\,dv<\infty$. Then by abelian Fourier analysis, there is $D_j\in\mathscr{P}$ so that

$$\sup_{\substack{k\in V\\ a\in A_0}}(1+\sigma(a))^r\,(1+\tilde\sigma(k))^r\left\|\int_{\mathscr{D}\times\mathscr{F}}(e^h)(k)\,e^{iv(\log a)}\,g_j(h:v)\,dh\,dv\right\|$$

$$\leqslant\int_{\mathscr{D}\times\mathscr{F}}\|g_j(h:v; D_j)\|\,|R(h:v)|^{-1}\,dh\,dv.$$

But, by (8.6) there is a finite subset F_j of $\mathscr{P}$ so that $\|g_j(h:v; D_j)\|\leqslant\sum_{D'\in F_j}\|f_j(h:v; D')\|$. Thus

$$\sup_{\substack{K\in V\\ a\in A_0}}(1+\sigma(a))^r\,(1+\tilde\sigma(k))^r\left\|\int_{\mathscr{D}\times\mathscr{F}}(e^h)(k)\,\varphi(h:v:l_1; a; l_2)\,\pi_R^{-1}(v)\,dh\,dv\right\|$$

$$\leqslant C\sum_j\sum_{D'\in F_j}\sup_{\mathscr{D}\times\mathscr{F}}\|f_j(h:v; D')\|$$

$$\leqslant C\sum_{D\in F}\sup\|\varphi(h:v; D:1)\|$$

for some finite subset F of $\mathscr{P}$. But for any $r' \geqslant 0$, $\sup_{\mathscr{S} \times \mathscr{F}} \|\varphi(h:v; D:1)\| \leqslant {}^0S_{D,r',0}(\varphi)$. This finishes the case $d=0$.

Case II. Pick $d \geqslant 1$ and assume inductively that the theorem is true when $d' < d$. For $1 \leqslant i \leqslant d$, let $\mathfrak{a}_i = \{H \in \mathfrak{a}_0 : \alpha_j(H) = 0 \text{ for } j \neq i\}$ and let L_i be the centralizer in L_P of $\mathfrak{a}_i$. Let $\Delta_i^+ = \Delta(L_P, A_0) \backslash \Delta(L_i, A_0) \cap \Delta^+(L_P, A_0)$. Let $Q_i = L_i N_i$ be the maximal parabolic subgroup of L_P for which N_i corresponds to Δ_i^+. For $H \in \mathfrak{a}_0$, let $\rho^i(H) = \frac{1}{2} \sum m(\alpha)\, \alpha(H)$, $\alpha \in \Delta_i^+$, $\rho_P(H) = \frac{1}{2} \sum m(\alpha)\, \alpha(H)$, $\alpha \in \Delta^+(L_P, A_0)$, $\rho_i(H) = \rho_P(H) - \rho^i(H)$. For $b > 0$, $1 \leqslant i \leqslant d$, let $A_i^+(b) = \{a \in A_0^+(P) : \alpha_i(\log a) > b\rho_P(\log a)\}$. Fix b small enough that $A_i^+(P) \subseteq \bigcup_{1 \leqslant i \leqslant d} A_i^+(b)$.

Let $l_1, l_2 \in \mathscr{U}(\mathfrak{l}_P)$, $r \geqslant 0$. Since $\mathrm{cl}(A_0^+(P)) \subseteq \bigcup_{i=1}^d \mathrm{cl}(A_i^+(b))$, we must show for each $1 \leqslant i \leqslant d$ that there is $F_i \subseteq \mathscr{S}_P$ so that given any $r' \geqslant 0$ there are $C > 0$ and $t \geqslant 0$ so that

$$\sup_{\substack{k \in V \\ a \in \mathrm{cl}(A_i^+(b))}} (1+\sigma(a))^r\, \Xi_P^{-1}(a)(1+\tilde{\sigma}(k))^r$$

$$\times \left\| \int_{\mathscr{S} \times \mathscr{F}} (e^h)(k)\, \psi_\varphi(h:v:l_1; a; l_2)\, \pi_R^{-1}(v)\, dh\, dv \right\|$$

$$\leqslant C\, {}^0S_{F_i, r', t}(\varphi).$$

Fix an i, and drop it from the notation so that $Q = LN = Q_i$.

Write $\mathscr{U}(\mathfrak{l}_P) = \mathscr{U}(\mathfrak{m}_P)\, S(\mathfrak{a}_P)$, where $S(\mathfrak{a}_P) \subseteq \mathscr{Z}_P$. Then if $l_i = m_i u_i$, $m_i \in \mathscr{U}(\mathfrak{m}_P)$, $u_i \in S(\mathfrak{a}_P)$, $i = 1, 2$, and if $a = a_1 a_2$, where $a_1 \in M_P \cap A_0$, $a_2 \in A_P$, then, as in the case $d = 0$, recalling that $A_P \subseteq A_H$, $\psi_\varphi(h:v:m_1u_1; a_1a_2; m_2u_2) = Q(v)\, e^{iv(\log a_2)}\, \psi_\varphi(h:v:m_1; a_1; m_2)$, where $u_1u_2(iv) = Q(v)$ is a polynomial in v.

Write $\mathscr{U}(\mathfrak{m}_P) = \mathscr{U}(\mathfrak{k}_P)\, \mathscr{U}(\mathfrak{l} \cap \mathfrak{m}_P)\, \mathscr{U}(\mathfrak{n}) = \mathscr{U}(\theta(\mathfrak{n}))\, \mathscr{U}(\mathfrak{l} \cap \mathfrak{m}_P)\, \mathscr{U}(\mathfrak{k}_P)$. There exist $b_1 \in \mathscr{U}(\mathfrak{k}_P)\, \mathscr{U}(\mathfrak{l} \cap \mathfrak{m}_P)$, $b_2 \in \mathscr{U}(\mathfrak{l} \cap \mathfrak{m}_P)\, \mathscr{U}(\mathfrak{k}_P)$ and $m_1' \in \mathscr{U}(\mathfrak{m}_P)\mathfrak{n}$, $m_2' \in \theta(\mathfrak{n})\, \mathscr{U}(\mathfrak{m}_P)$ such that $m_i = b_i + m_i'$, $i = 1, 2$. Thus $\psi_\varphi(h:v:m_1u_1; a_1a_2; m_2u_2) = \psi_\varphi(h:v:m_1'u_1; a_1a_2; m_2u_2) + \psi_\varphi(h:v:b_1u_1; a_1a_2; m_2'u_2) + \psi_\varphi(h:v:b_1u_1; a_1a_2; b_2u_2)$. We will estimate each of these terms separately. First

$$\sup_{\substack{k \in V \\ a \in \mathrm{cl}(A_i^+(b))}} (1+\sigma(a))^r\, \Xi_P^{-1}(a)(1+\tilde{\sigma}(k))^r$$

$$\times \left\| \int_{\mathscr{S} \times \mathscr{F}} (e^h)(k)\, \psi_\varphi(h:v:m_1'u_1; a; m_2u_2)\, \pi_R^{-1}(v)\, dh\, dv \right\|$$

$$\leqslant \sup_{k \in V} \sup_{a_2 \in A_P} \sup_{a_1 \in \mathrm{cl}(A_i^+(b)) \cap M_P} (1+\sigma(a_2))^r$$
$$\times (1+\tilde{\sigma}(k))^r (1+\sigma(a_1))^r \Xi_P^{-1}(a_1)$$
$$\times \left\| \int_{\mathcal{S} \times \mathcal{F}} (e^h)(k)\, e^{i\nu(\log a_2)}\, Q(\nu) \right.$$
$$\left. \times\, \psi_\varphi(h:\nu:m_1';\, a_1;\, m_2)\, \pi_R^{-1}(\nu)\, dh\, d\nu \right\|.$$

As in the case $d=0$, we can use (8.6) and abelian Fourier analysis to find a finite subset F_1 of $\mathscr{P}$ and $c_1>0$ so that this last expression is bounded by

$$C_1 \sup_{a \in \mathrm{cl}(A_i^+(b)) \cap M_P} (1+\sigma(a_1))^r\, \Xi_P^{-1}(a_1)$$
$$\times \sum_{D \in F_1} \sup_{\mathcal{S} \times \mathcal{F}} \|\psi_\varphi(h:\nu;D:m_1';\, a_1;\, m_2)\|.$$

But now using (7.11), there are a finite subset $F \subseteq \mathscr{L}_P$ and $r_0 \geqslant 0$ so that for all $r' \geqslant 0$ this is bounded by

$$^0S_{F,r',0}(f) \sup_{a \in \mathrm{cl}(A_i^+(b)) \cap M_P} (1+\sigma(a_1))^r$$
$$\times\, \Xi_P^{-1}(a_1)\, \Xi_Q(a_1)\, e^{\beta_Q(\log a_1)} (1+\sigma(a_1))^{r'+r_0}\, d_Q^{-1}(a_1).$$

But there are constants $D \geqslant 0$ and $q \geqslant 0$ so that $\Xi_Q(a_1) \leqslant De^{-\rho_i(\log a_1)}(1+\sigma(a_1))^q$. Further, $d_Q^{-1}(a_1) = e^{-\rho^i(\log a_1)}$ and $e^{\beta_Q(\log a_1)} = e^{-\alpha_i(\log a_1)} \leqslant e^{-b\rho_P(\log a_1)}$ since $a_1 \in A_i^+(b)$. Thus $\Xi_Q(a_1)\, d_Q^{-1}(a_1)\, e^{-\beta_Q(\log a_1)} \leqslant De^{-\rho_P(\log a_1)(1+b)}(1+\sigma(a_1))^q \leqslant D\Xi_P(a_1)^{1+b}(1+\sigma(a_1))^q$. Thus

$$\sup_{a \in \mathrm{cl}(A_i^+(b)) \cap M_P} (1+\sigma(a_1))^{r+r'+r_0}\, \Xi_P^{-1}(a_1)$$
$$\times\, \Xi_Q(a_1)\, e^{-\beta_Q(\log a_1)}\, d_Q^{-1}(a_1)$$
$$\leqslant D \sup_{a \in \mathrm{cl}(A_i^+(b)) \cap M_P} \Xi_P(a_1)^b (1+\sigma(a_1))^{r+r'+r_0+q} = C_{r'} < \infty,$$

since $b>0$.

Thus the term involving $\psi_\varphi(h:\nu:m_1'u_1;\, a;\, m_2u_2)$ can be bounded by $C_{r'}\, {}^0S_{F,r',0}(f)$ for any $r' \geqslant 0$. The same argument also works for $\psi_\varphi(h:\nu:b_1u_1;\, a;\, m_2'u_2)$. It remains to estimate the terms with $\psi_\varphi(h:\nu:b_1u_1;\, a;\, b_2u_2)$, where $b_1 \in \mathscr{U}(\mathfrak{k}_P)\,\mathscr{U}(\mathfrak{m}_P \cap \mathfrak{l})$, $b_2 \in \mathscr{U}(\mathfrak{m}_P \cap \mathfrak{l})\,\mathscr{U}(\mathfrak{k}_P)$. Write $b_1 = \kappa_1\beta_1'$, $b_2 = \beta_2'\kappa_2$, where $\kappa_i \in \mathscr{U}(\mathfrak{k}_P)$, $\beta_i \in \mathscr{U}(\mathfrak{m}_P \cap \mathfrak{l})$, $i=1,2$, $\beta_i' = d_Q^{-1} \circ \beta_i \circ d_Q$, as in Section 7. As in the $d=0$ case, since φ is K_P-spherical, there are polynomials $P_j(h)$ so that $\|\int_{\mathscr{V}\times\mathscr{F}}(e^h)(k)\,\pi_R^{-1}(v)\,\psi_\varphi(h:v:\kappa_1\beta_1' u_1;a;u_2\beta_2'\kappa_2)\,dh\,dv\| \leqslant \sum_j \|\int_{\mathscr{V}\times\mathscr{F}}(e^h)(k)\,\pi_R^{-1}(v)\,P_j(h)\,\psi_\varphi(h:v:\beta_1' u_1;a;u_2\beta_2')\,dh\,dv\|$. Thus it suffices to estimate terms of the form

$$\sup_{\substack{k\in V\\ a\in \mathrm{cl}(A_i^+(b))}} (1+\sigma(a))^r\,\Xi_P^{-1}(a)(1+\tilde\sigma(k))^r$$

$$\times\left\|\int_{\mathscr{V}\times\mathscr{F}}(e^h)(k)\,\pi_R^{-1}(v)\,P(h)\,\psi_\varphi(h:v:\beta_1' u_1;a;u_2\beta_2')\,dh\,dv\right\|,$$

where $P(h)$ is a polynomial. We will split this up further. Write $d_\varphi(h:v:\beta_1u_1;a;\beta_2u_2)=\psi_\varphi(h:v:\beta_1'u_1;a;\beta_2'u_2)-\sum_{s\in W(\mathfrak{a}_H,\mathfrak{a}_Q)}\det sd_Q^{-1}(a)\times \psi_{\varphi,s}(h:v:\beta_1u_1;a;\beta_2u_2)$. Then, as before,

$$\sup_{\substack{k\in V\\ a\in \mathrm{cl}(A_i^+(b))}} (1+\sigma(a))^r\,\Xi_P^{-1}(a)(1+\tilde\sigma(k))^r$$

$$\times\left\|\int_{\mathscr{V}\times\mathscr{F}}(e^h)(k)\,d_\varphi(h:v:\beta_1u_1;a;\beta_2u_2)\,\pi_R^{-1}(v)\,P(h)\,dk\,dv\right\|$$

$$\leqslant \sup_{k\in V}\ \sup_{a_2\in A_P}\ \sup_{a_1\in \mathrm{cl}(A_i^+(b))\cap M_P}(1+\sigma(a_1))^r$$

$$\times(1+\sigma(a_2))^r\,(1+\tilde\sigma(k))^r\,\Xi_P^{-1}(a_1)$$

$$\times\left\|\int_{\mathscr{V}\times\mathscr{F}}(e^h)(k)\,e^{iv(\log a_2)}\right.$$

$$\left.\times\, d_\varphi(h:v:\beta_1;a;\beta_2)\,\pi_R^{-1}(v)\,P(h)\,Q(v)\,dh\,dv\right\|,$$

where $Q(v)=(u_1u_2)(iv)$. Again, since $\varphi\in I'(\mathscr{L},L_P)$, we can use (8.6) and abelian Fourier analysis to find a subset F_2 of $\mathscr{P}$ and $C_2>0$ so that this expression is bounded by

$$C_2 \sup_{a_1\in \mathrm{cl}(A_i^+(b))\cap M_P}(1+\sigma(a_1))^r\,\Xi_P^{-1}(a_1)$$

$$\times\sum_{D\in F_2}\ \sup_{\mathscr{V}\times\mathscr{F}}\|d_\varphi(h:v;D:\beta_1;a_1;\beta_2)\|.$$

Write $a_1=a_1'\exp(TH)$, where $H\in\mathfrak{a}_0\cap\mathfrak{m}_P$ satisfies $\alpha_j(H)=0$, $j\neq i$, and $\alpha_i(H)=1$, $T=\alpha_i(\log a_1)$, and $a_1'\in A_0\cap M_P$ satisfies $e^{\alpha_i}(a_1')=1$. Then $a_1'\in L_Q^+$. Take $\Omega=\{H\}$, let $\delta(h)$ be the function given by (7.21), and write

$\varepsilon(h) = \frac{1}{2}\delta(h)$. Then using Theorem 7.31 there exists a finite subset F of $\mathscr{L}_P$ and an $r_1 > 0$ so that for all $r' \geqslant 0$, $t \geqslant 0$, there is a $C' > 0$ so that this is bounded by

$$C_2 C' \sup_{a_1 \in \mathrm{cl}(A_i^+(h)) \cap M_P,\, h \in \mathscr{S}} (1+\sigma(a_1))^r$$

$$\times \Xi_P^{-1}(a_1)\, d_Q^{-1}(a_1)\, {}^0S_{F,r',t}(f)\, e^{-\varepsilon(h)T}$$

$$\times \Xi_Q(a_1)(1+\sigma(a_1))^{r'+r_1}(1+d(h)^{-1})^{-t}.$$

But as before, for $a_1 \in \mathrm{cl}(A_i^+(b)) \cap M_P$, $\Xi_P^{-1}(a_1)\, d_Q^{-1}(a_1)\, \Xi_Q(a_1)\, e^{-\varepsilon(h)\alpha_i(\log a_1)} \leqslant D\Xi_P(a_1)^{\varepsilon(h)b}(1+\sigma(a_1))^q$ for some constants $D \geqslant 0$, $q \geqslant 0$. But by (8.7), there exists $C_{r'}$ and t such that $\sup_{a_1 \in \mathrm{cl}(A_i^+(b)) \cap M_P} (1+\sigma(a_1))^{r+r'+r_1+q}\, \Xi_P(a_1)^{b\varepsilon(h)} \leqslant C_{r'}(1+d(h)^{-1})^t$. Thus for any $r' \geqslant 0$ we can find C and t so that we can bound our expression involving d_φ by $C\, {}^0S_{F,r',t}(f)$.

Finally, for each $s \in W(\mathfrak{a}_H, \mathfrak{a}_Q)$, we look at

$$\sup_{\substack{k \in V \\ a \in \mathrm{cl}(A_i^+(b))}} (1+\sigma(a))^r\, \Xi_P^{-1}(a)(1+\tilde{\sigma}(k))^r\, d_Q^{-1}(a)$$

$$\times \left\| \int_{\mathscr{S} \times \mathscr{F}} (e^h)(k)\, \psi_{\varphi,s}(h:v:\beta_1 u_1;\, a;\, \beta_2 u_2)\, P(h)\, \pi_R^{-1}(v)\, dh\, dv \right\|.$$

Now $\Xi_P^{-1}(a)\, d_Q^{-1}(a) \leqslant D\Xi_Q^{-1}(a)(1+\sigma(a))^q$, $\mathrm{cl}(A_i^+(b)) \subseteq \mathrm{cl}(A_0^+(Q))$, and $\psi_{\varphi,s}(h:v:\beta_1 u_1;\, a;\, \beta_2 u_2) = \psi^1_{\varphi,s}(h:v:\beta_1 u_1;\, a;\, \beta_2 u_2)\, \pi_Q(s_{i(s)} A_{h,v})$, so we can bound this by

$$D \sup_{\substack{k \in V \\ a \in \mathrm{cl}(A_0^+(Q^s))}} (1+\sigma(a))^{r+q}\, \Xi_{Q^s}^{-1}(a)(1+\tilde{\sigma}(k))^r$$

$$\times \left\| \int_{\mathscr{S} \times \mathscr{F}} (e^h)(k)\, \pi_R^{-1}(v)\, P(h)(\psi^1_{\varphi,s})^s \right.$$

$$\left. \times (h:v:d_1;\, a;\, d_2)\, \pi_{Q^s}(A_{h,v})\, dh\, dv \right\|,$$

where Q^s is a parabolic subgroup of G with $A_{Q^s} \subseteq A_H$, $d_1, d_2 \in \mathscr{U}(\mathfrak{l}_{Q^s})$, and $P \cdot (\psi^1_{Q,s})^s \in I'(\mathscr{D}, L_{Q^s})$. Thus by the induction hypothesis, there is $F' \subset \mathscr{L}_{Q^s}$ such that for any $r', r_1 > 0$ there are C' and t so that the above is bounded by $C'\, {}^0S_{F',r_1,t}(P(\psi^1_{\varphi,s})^s) = C'\, {}^0S_{F'',r'+r_1,t}((\psi^1_{\varphi,s})^s)$, where $F'' = \{DP : D \in F'\}$. But now using Theorem 7.33, there is a finite subset F of $\mathscr{L}_P$ and an $r_1 \geqslant 0$ and $C > 0$ so that ${}^0S_{F'',r'+r_1,t}((\psi^1_{\varphi,s})^s) \leqslant C\, {}^0S_{F,r',t}(f)$. Q.E.D.

9. Wave Packets of Eisenstein Integrals

In this section we relate the abstract families defined in Section 7 and used to form Schwartz wave packets in Section 8 to the holomorphic families of Eisenstein integrals defined in Section 6. The first main result is Theorem 9.10 which gives the a priori estimates which are needed to show that Eisenstein integrals are functions of type $H(\mathscr{S}, G)$. The second main result is Theorem 9.14 which characterizes those wave packets of Eisenstein integrals which are Schwartz functions on G.

Finally, we show in Theorem 9.18 that the Schwartz wave packets of Eisenstein integrals corresponding to the H-series of representations are in $\mathscr{C}_H(G)$, the closed subspace of $\mathscr{C}(G)$ consisting of functions whose Plancherel formula expansions involve only the H-series of tempered representations.

Let $F:\mathfrak{v}_{\mathbb{C}}^* \times G \to W = W(\tau_1:\tau_2)$ be a holomorphic family of $K_M^\dagger$-spherical functions on G coming from a holomorphic family of matrix coefficients on $M^\dagger$ and a $K_{M,1}^\prime$-endomorphism of $W(\tau_1:\tau_2)$. Then, as in (6.6), for $(h, \nu, x) \in \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G$, we define the Eisenstein integral

$$E(P:F:h:\nu:x) = \int_{K/Z} F(h:xk)\, \tau_{2,h}(k^{-1})\, e^{(i\nu-\rho_P)H_P(xk)}\, d(kZ). \tag{9.1}$$

Lemma 9.2. *Let $\Phi:\mathfrak{v}_{\mathbb{C}}^* \times G \to W$ be any smooth family of τ-spherical functions. Then given $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$, there are a finite subset S of $\mathscr{U}(\mathfrak{g})$ and an $r \geqslant 0$ so that $\|\Phi(h:D_1;x;D_2)\| \leqslant (1+|h|)^r \sum_{D\in S} \|\Phi(h:D;x)\|$ for all $(h, x) \in \mathfrak{v}_{\mathbb{C}}^* \times G$.*

Proof. For fixed $h \in \mathfrak{v}_{\mathbb{C}}^*$, a similar estimate is proved in [2, Lemma 17]. The constants involved are independent of h except for terms of the form $\|d\tau_{1,h}(\kappa)\|$ for some $\kappa \in \mathscr{U}(\mathfrak{k})$, depending on D_1, D_2. These grow polynomially in h. Q.E.D.

Corollary 9.3. *For any $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$, there are a finite subset S of $\mathscr{U}(\mathfrak{g})$ and an $r \geqslant 0$ so that*

$$\|E(P:F:h:\nu:D_1;x;D_2)\| \leqslant (1+|h|)^r \sum_{D\in S} \|E(P:F:h:\nu:D;x)\|.$$

Write $F_\nu(h:x) = F(h:x)\, e^{(i\nu-\rho_P)H_P(x)}$. Then, since $\|w\tau_{2,h}(k)\| = |e^h(k)|\, \|w\|$ for all $k \in K$, $w \in W$, we have

$$\|E(P:F:h:\nu:D;x)\| \leqslant \int_{K/Z} \|F_\nu(h:D;xk)\|\, |e^h(k^{-1})|\, d(kZ) \tag{9.4}$$

for all $(h, \nu, x) \in \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G$. For $x \in G$, write $x = \mathbf{k}(x)\mathbf{m}(x)\exp H_P(x)\mathbf{n}(x)$ as in (6.5a). For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, write $\nu = \nu_R + i\nu_I$, where $\nu_R, \nu_I \in \mathfrak{a}^*$.

LEMMA 9.5. *For any $D \in \mathscr{U}(\mathfrak{g})$, there are a finite subset $S \subset \mathscr{U}(\mathfrak{m})$, and constants $C > 0$, $r \geqslant 0$ so that $\|F_\nu(h:D;x)\| \leqslant C(1+|h|)^r\, |e^h(\mathbf{k}(x))|\, (1+|\nu|)^r\, e^{|\nu_I H_P(x)|}\, e^{-\rho_P H_P(x)} \sum_{v \in S} \|F_M(h:v;\mathbf{m}(x))\|$ for all $(h, \nu, x) \in \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G$.*

Proof. Let $k \in K$, $m \in M^\dagger$, $a \in A$, $n \in N$. Then using (6.5c), $\|F_\nu(h:D;kman)\| = \|\tau_{1,h}(k)\, F_\nu(h:{}^{k^{-1}}D;ma)\| \leqslant C|(e^h)(k)| \sum_i \|F_\nu(h:D_i;ma)\|$, where we express ${}^{k^{-1}}D = \sum_i a_i(k)\, D_i$ and let $C = \sup_{k \in K,i} |a_i(k)|$. Now fix i and write $D_i = \kappa\mu b$, where $\kappa \in \mathscr{U}(\mathfrak{k})$, $\mu \in \mathscr{U}(\mathfrak{m}+\mathfrak{a})$, and $b \in \mathscr{U}(\mathfrak{n})$. Then $\|F_\nu(h:\kappa\mu b;ma)\| = \|d\tau_{1,h}(\kappa)\, F_\nu(h:\mu;ma;{}^{m^{-1}a^{-1}}b)\| \leqslant C(1+|h|)^r\, \|F_\nu(h:\mu;ma;{}^{m^{-1}a^{-1}}n)\|$, since $d\tau_{1,h}(\kappa)$ is a polynomial in h. But F_ν is right N-invariant and ${}^{m^{-1}a^{-1}}b \in \mathscr{U}(\mathfrak{n})$, so $F_\nu(h:\mu;ma;{}^{m^{-1}a^{-1}}b) = \varepsilon(b)\, F_\nu(h:\mu;ma)$, where $\varepsilon(b)$ is the constant term of b. Finally, we write $\mu \in \mathscr{U}(\mathfrak{m}+\mathfrak{a})$ as $\mu = vv'$, where $v \in \mathfrak{U}(\mathfrak{m})$ and $v' \in S(\mathfrak{a})$. Then differentiating with respect to v' gives a polynomial in ν. Thus $\|F_\nu(h:vv';ma)\| \leqslant C(1+|\nu|)^r\, \|F_\nu(h:v;ma)\| = C(1+|\nu|)^r\, |e^{(i\nu-\rho_P)(\log a)}|\, \|F_\nu(h:v;m)\|$. Q.E.D.

LEMMA 9.6. *There is a constant C_0 so that*

$$\|E(P:F:h:\nu:D;x)\| \leqslant C(1+|h|)^r\,(1+|\nu|)^r\, e^{C_0|\nu_I|\sigma(x)}\, \Xi(x) \sup_{k \in K} |e^h(\mathbf{k}(kxk^{-1}))| \times \sup_k \Big\{ \Xi_M(\mathbf{m}(xk))^{-1} \sum_{v \in S} \|F(h:v;\mathbf{m}(xk))\| \Big\}$$

for all $(h, \nu, x) \in \mathfrak{v}_{\mathbb{C}}^* \times \mathfrak{a}_{\mathbb{C}}^* \times G$.

Proof. Combine (9.4) and (9.5) to obtain

$$\|E(P:F:h:\nu:D;x)\| \leqslant C(1+|h|)^r\,(1+|\nu|)^r \times \sum_{v \in S} \int_{K/Z} |e^h(k^{-1})|\, |e^h(\mathbf{k}(xk))|\, e^{|\nu_I H_P(xk)|}\, e^{-\rho_P H_P(xk)} \times \|F(h:v;\mathbf{m}(xk))\|\, d(kZ).$$

But by [10, p. 275], there is $C_0 \geqslant 0$ so that $|\nu_I H_P(xk)| \leqslant C_0\, |\nu_I|\, \sigma(x)$ for all $k \in K$. Also, by [10, p. 275], $\int_{K/Z} e^{-\rho_P H_P(xk)}\, \Xi_M(\mathbf{m}(xk))\, d(kZ) = \Xi(x)$. Q.E.D.

LEMMA 9.7. *There are constants C and c so that $\sup_{k \in K} |e^h(\mathbf{k}(kxk^{-1}))| \leqslant Ce^{|h_I|(\sigma_1(x)+c)}$.*

Proof. Write $x = k_1 a k_2$, where $k_1, k_2 \in K$, $a \in A_0$. Then $\sup_{k \in K} |e^h(\mathbf{k}(kxk^{-1}))| = |e^h(k_1 k_2)| \sup_{k \in K} |e^h(\mathbf{k}(kak^{-1}))|$. Now $|e^h(k_1 k_2)| \leqslant e^{|h_I| \sigma_I(x)}$, so it suffices to show $\sup_{k \in K, a \in A_0} |e^h(\mathbf{k}(kak^{-1}))| < \infty$. But $|e^h(\mathbf{k}(kak^{-1}))| \leqslant e^{|h_I| \sigma_{V}(\mathbf{k}(kak^{-1}))}$ and by (2.11), there is a constant c so that $\sup_{k,a} \sigma_V(\mathbf{k}(kak^{-1})) \leqslant c$. Q.E.D.

Let $\mathscr{C}$ be the chamber in $i\mathfrak{v}^*$ for which $F_M(h)$ is a spherical function of matrix coefficients and $\mathscr{C}_{\mathbb{C}} = \mathscr{C} + i\omega$, where ω is a relatively compact neighborhood of 0 in $i\mathfrak{v}^*$. Let $d(h)$ denote the distance from h_R to the boundary of $\mathscr{C}$, where $h = h_R + ih_I$, $h_R, h_I \in i\mathfrak{v}^*$.

LEMMA 9.8. *For any $v \in \mathscr{U}(\mathfrak{m})$, there are constants $C > 0$, $r, t \geqslant 0$ so that $\sup_k \Xi_M(\mathbf{m}(xk))^{-1} \|F(h:v; \mathbf{m}(xk))\| \leqslant C(1+|h|)^r (1+d(h)^{-1})^t$ for all $(h, x) \in \mathscr{C}_{\mathbb{C}} \times G$.*

Proof. Write $M^+ = K_M^+ \operatorname{cl}(A_{0,M}^+) K_M^+$. When we decompose $xk = \mathbf{k}(xk)\, \mathbf{m}(xk) \exp H_P(xk)\, \mathbf{n}(xk)$, we can assume $\mathbf{m}(xk) \in \operatorname{cl}(A_{0,M}^+) K_M^+$ as $K_M^+ \subset K$. Also, since Z is central in M, elements of Z can be commuted past $\operatorname{cl}(A_{0,M}^+) K_M^+$ into K, so we can assume that $\sigma_V(\mathbf{m}(xk))$ is bounded.

Thus we can assume that $\mathbf{m}(xk) = ak_1$, where $a \in \operatorname{cl}(A_{0,M}^+)$ and $\sigma(k_1)$ is bounded. Now $\Xi_M(\mathbf{m}(xk)) = \Xi_M(a)$ and $\|F(h:v; \mathbf{m}(xk))\| = \|F(h:v; a)\|$ $|e^h(k_1)| \leqslant C \|F(h:v; a)\|$. Now by Theorem 5.12, there are constants $C > 0$, $r \geqslant 0$, $m \geqslant 0$ so that $\|F(h:v; a)\| \leqslant C(1+|h|)^r (1+\sigma(a))^m e^{\operatorname{Re} \omega(h)}(a)$ for all $a \in \operatorname{cl}(A_{0,M}^+)$, $h \in \mathscr{C}_{\mathbb{C}}$. Thus $\Xi_M^{-1}(a) \|F(h:v; a)\| \leqslant C(1+|h|)^r (1+\sigma(a))^m$ $e^{(\operatorname{Re} \omega(h) - \rho_M)}(a)$. But as in Lemma 8.7 there are constants $C > 0$ and $t \geqslant 0$ so that

$$\sup_{a \in \operatorname{cl}(A_{0,M}^+)} (1+\sigma(a))^m e^{(\operatorname{Re} \omega(h) - \rho_M)}(a) \leqslant C(1+d(h)^{-1})^t. \qquad \text{Q.E.D.}$$

LEMMA 9.9. *Let $D \in \mathscr{U}(\mathfrak{g})$. Then there are constants $C, c_0, r, t \geqslant 0$ so that $\|E(P:F:h:\nu:D:x)\| \leqslant C(1+|h|)^r (1+|\nu|)^r (1+d(h)^{-1})^t \Xi(x) e^{c_0 |\nu_I| \sigma(x)}$ $e^{|h_I| \sigma_V(x)}$ for all $(h, \nu, x) \in \mathscr{C}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}^* \times G$.*

Proof. This follows from combining (9.6), (9.7), and (9.8) since $|h_I|$ is bounded in $\mathscr{C}_{\mathbb{C}}$. Q.E.D.

THEOREM 9.10. *Let $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$ and $D \in \mathscr{P}$. Then there are constants C, r, c_0 so that $\|E(P:F:h:\nu; D:g_1: x; g_2)\| \leqslant C(1+|h|)^r (1+|\nu|)^r (1+d(h)^{-1})^r (1+\tilde{\sigma}(x))^r \Xi(x) e^{c_0 |\nu_I| \sigma(x)} e^{|h_I| \sigma_V(x)}$ for all $(h, \nu, x) \in \mathscr{C}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}^* \times G$.*

Proof. We know from Theorem 6.7 that $E(P:F:h:\nu:x)$ is holomorphic as a function of $(h, \nu) \in \mathscr{C}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}^*$. We use the same method to estimate derivatives in (h, ν) that Harish-Chandra used to estimate derivatives in ν of the ordinary Eisenstein integral. Namely, if f is a holomorphic function in a neighborhood of $|z - w| \leqslant C$, then $|(d^n/dz^n) f(z)| \leqslant$

$(n!/C^n) \sup_{|z-w|=C} |f(w)|$. For derivatives in v, we use radius $(1+\sigma(x))^{-1}$ and for derivatives in h we use radius $C \leqslant \min\{(1+\sigma_{V}(x))^{-1}, \frac{1}{2}d(h)\}$. Combined, this gives us the polynomial growth in $\tilde{\sigma}(x) = \sigma(x) + \sigma_V(x)$ in addition to the terms needed to estimate $E(P:F:h:v:g_1;x;g_2)$ coming from (9.3) and (9.9). Q.E.D.

COROLLARY 9.11. $E(P, F) \in II(\mathscr{D}, G)$.

Proof. Combine (6.7), (6.8), and (9.10). Q.E.D.

We are now ready to discuss wave packets of Eisenstein integrals. Since $E(P:F) \in II(\mathscr{D}, G)$, $E(P:F) \cdot \alpha \in I(\mathscr{D}, G)$ for any $\alpha \in \mathscr{C}(\mathscr{D} \times \mathscr{F})$. (See the remark following Definition 7.7.) However, there is no reason to expect that $E(P:F) \cdot \alpha \in I'(\mathscr{D}, G)$. This, as in the finite center case, is because of "poles of the c-function." In order to eliminate these poles, we bring in the Plancherel measure. Recall that the Eisenstein integral is a spherical function of matrix coefficients for a series of induced representations. Thus for each $(h, v) \in (\mathscr{D} \times \mathscr{F})$, $E(P:F:h:v)$ is associated to a representation $\pi_{h,v}$ defined as in (3.8). By the results of [6], the Plancherel measure for this representation is, up to a constant factor independent of (h, v), given by

$$m(h:v) = \pi_G(\Lambda_{h,v})\, \pi_R^{-1}(v) \prod_{\alpha \in \Phi_R^{+}(\mathfrak{g},\mathfrak{h})} m_\alpha(h:v), \tag{9.12}$$

where $m_\alpha(h:v) = v_\alpha \sinh \pi v_\alpha / (\cosh \pi v_\alpha - \cos \pi h_\alpha)$, $v_\alpha = 2\langle v, \alpha\rangle / \langle \alpha, \alpha\rangle$, and $h \to h_\alpha$ is an affine linear functional on $\mathscr{D}$ for which we do not need the exact formula (see [6]). Write $m_R(h:v) = \prod_{\alpha \in \Phi_R^+} m_\alpha(h:v)$.

Multiplying $E(P:F) \cdot \alpha$ by m_R will eliminate the problem of poles of the c-function. However, in our situation, it introduces new difficulties because $m_\alpha(h:v)$ is not jointly continuous at points (h, v), where $h_\alpha \in \mathbb{Z}$ and $v_\alpha = 0$ for some $\alpha \in \Phi_R^+$ with h_α not constant. (These are points corresponding to principal series which are reducible, or which fail to be reducible because certain limits of discrete series are zero.) Thus we will need to assume that α is chosen so that $E(P:F) \cdot \alpha \cdot m_R$ is jointly smooth. This will certainly be true if $\alpha \cdot m_R$ is jointly smooth. We will need the following lemma.

Let $E = \mathbb{R}^{n+2}$, $n \geqslant 0$, and denote the coordinates by (x, y, z), where $x, y \in \mathbb{R}$, $z \in \mathbb{R}^n$. Define $\mathscr{C}(E, W)$ as in (8.6).

LEMMA 9.13. *Suppose $f \in \mathscr{C}(E, W)$ satisfies $g(x, y, z) = x \sinh \pi x / (\cosh \pi x - \cos \pi y)\, f(x, y, z)$ is jointly smooth on E. Then $g \in \mathscr{C}(E, W)$, and given $\alpha \in M$, $r \geqslant 0$, there exist constants $C > 0$, $t \geqslant 0$ and a finite subset F of M so that $s_{\alpha,r}(g) \leqslant C \sum_{\beta \in F} s_{\beta,t}(f)$.*

Proof. Fix $r \geqslant 0$ and $\alpha \in M$. Then $s_{\alpha,r}(g) = \sup_{m \in \mathbb{Z}} \sup_{-1 \leqslant y \leqslant 1, x, z} (1 + |(x, y+2m, x)|)^r\, |D^\alpha g(x, y+2m, z)|$. For $m \in \mathbb{Z}$, write $g_m(x, y, z) =$

$h(x, y) f(x, y+2m, z)$, where $h(x, y) = x \sinh \pi x(x^2+y^2)/(\cosh \pi x - \cos \pi y)$ is jointly smooth on $\mathbb{R} \times [-1, 1]$ and satisfies the condition that for all $\beta \in M$ there exist constants $C_\beta \geqslant 0$, $t_\beta \geqslant 0$ so that $|D^\beta h(x, y)| \leqslant C_\beta(1+|x|)^{t_\beta}$ for all $x \in \mathbb{R}$, $-1 \leqslant y \leqslant 1$. Then $g(x, y+2m, z) = (1/(x^2+y^2))\, g_m(x, y, z)$. Write $k = |\alpha_1| + |\alpha_2|$ for the total degree of D^α in x and y. Then there are finite subsets F_1, F_2 of M, for each $\beta \in F_1$ a polynomial $P_\beta(x, y)$, and for each $\beta \in F_2$ a constant C'_β so that

$$D^\alpha g(x, y+2m, z) = \begin{cases} \displaystyle\sum_{\beta \in F_1} P_\beta(x, y)\, D^\beta g_m(x, y, z)(x^2+y^2)^{-k} & \text{if } (x, y) \neq (0, 0), \\ \displaystyle\sum_{\beta \in F_2} C'_\beta D^\beta g_m(0, 0, z) & \text{if } (x, y) = (0, 0). \end{cases}$$

Further, by Taylor's theorem there is a finite subset F_3 of M and for each $\beta \in F_3$ a polynomial $P'_\beta(x, y)$ so that for all $m \in \mathbb{Z}$, $|D^\alpha g(x, y+2m, z) - D^\alpha g(0, 2m, z)| \leqslant \sqrt{x^2+y^2}\, \sup_{(x_1, y_1)} \sum_{\beta \in F_3} |P'_\beta(x_1, y_1)\, D^\beta g_m(x_1, y_1, z)|$, where the sup is taken over $(x_1, y_1) \in \mathbb{R}^2$ such that $|x_1| \leqslant |x|$, $|y_1| \leqslant |y|$.

Now for each $m \in \mathbb{Z}$,

$$\begin{aligned} &\sup_{-1 \leqslant y \leqslant 1, x, z} (1+|(x, y+2m, z)|)^r\, |D^\alpha g(x, y+2m, z)| \\ &\quad \leqslant \sup_{x^2+y^2 \leqslant 1, z} (1+|(x, y+2m, z)|)^r\, |D^\alpha g(x, y+2m, z)| \\ &\quad\quad + \sup_{x^2+y^2 \geqslant 1,\, |y| \leqslant 1, z} (1+|(x, y+2m, z)|)^r\, |D^\alpha g(x, y+2m, z)|. \end{aligned}$$

But

$$\begin{aligned} &\sup_{x^2+y^2 \geqslant 1,\, |y| \leqslant 1, z} (1+|(x, y+2m, z)|)^r\, |D^\alpha g(x, y+2m, z)| \\ &\quad \leqslant \sup_{x^2+y^2 \geqslant 1,\, |y| \leqslant 1, z} (1+|(x, y+2m, z)|)^r \\ &\quad\quad \times \sum_{\beta \in F_1} |P_\beta(x, y)|\, |D^\beta g_m(x, y, z)| \\ &\quad \leqslant \sup_{|y| \leqslant 1, x, z} (1+|(x, y+2m, z)|)^r \\ &\quad\quad \times \sum_{\beta \in F_1} |P_\beta(x, y)| \sum_{(\gamma, \gamma') \in F_4(\beta)} |D^\gamma h(x, y)| \\ &\quad\quad \times |D^{\gamma'} f(x, y+2m, z)|, \end{aligned}$$

where for each $\beta \in F_1$, $F_4(\beta)$ is a finite subset of $M \times M$. Pick $C_1, t_1 \geqslant 0$ so that $|P_\beta(x, y)| \leqslant C_1(1+|x|)^{t_1}$ for all $\beta \in F_1$, $x \in \mathbb{R}$, $|y| \leqslant 1$. Then this last expression is bounded by

$$C_1 \sum_{\beta \in F_1} \sum_{(\gamma,\gamma') \in F_4(\beta)} C_\gamma$$
$$\times \sup_{|y| \leqslant 1, x, z} (1 + |(x, y+2m, z)|)^{r+t_1+t_\gamma} |D^{\gamma'} f(x, y+2m, z)|$$
$$\leqslant C_1 C_2 \sum_{\beta \in F_1} \sum_{(\gamma,\gamma') \in F_4(\beta)} s_{\gamma', r+t_1+t_\gamma}(f).$$

Now

$$\sup_{x^2+y^2 \leqslant 1, z} |(1 + |(x, y+2m, z)|)^r \, |D^\alpha g(x, y+2m, z)|$$
$$\leqslant \sup_{x^2+y^2 \leqslant 1, z} |(1 + |(x, y+2m, z)|)^r \, |D^\alpha g(0, 2m, z)|$$
$$+ \sup_{x^2+y^2 \leqslant 1, z} |(1 + |(x, y+2m, z)|)^r$$
$$\times |D^\alpha g(x, y+2m, z) - D^\alpha g(0, 2m, z)|.$$

But there is a constant C_3 so that the first of these is bounded by

$$C_3 \sup_z |(1 + |(0, 2m, z)|)^r \sum_{\beta \in F_2} C'_\beta \, |D^\beta g_m(0, 0, z)|$$
$$\leqslant C_3 \sup_z \sum_{\beta \in F_2} C'_\beta \sum_{(\gamma,\gamma') \in F_4(\beta)} |(1 + |(0, 2m, z)|)^r$$
$$\times |D^\gamma h(0, 0)| \, |D^{\gamma'} f(0, 2m, z)|,$$

where for each $\beta \in F_2$, $F_4(\beta)$ is a finite subset of $M \times M$. Let $C_4 = \max_{\beta \in F_2, (\gamma,\gamma') \in F_4(\beta)} C_3 C'_\beta \, |D^\gamma h(0, 0)|$. Then this expression is bounded by $C_4 \sum_{\beta \in F_2} C'_\beta \sum_{(\gamma,\gamma') \in F_4(\beta)} s_{\gamma', r}(f)$.

Finally

$$\sup_{x^2+y^2 \leqslant 1, z} (1 + |(x, y+2m, z)|)^r$$
$$\times |D^\alpha g(x, y+2m, z) - D^\alpha g(0, 2m, z)|$$
$$\leqslant \sup_{x^2+y^2 \leqslant 1, z} (1 + |(x, y+2m, z)|)^r$$
$$\times \sum_{\beta \in F_3} \sup_{x_1^2+y_1^2 \leqslant 1} |P'_\beta(x_1, y_1)| \, |D^\beta g_m(x_1, y_1, z)|$$

$$\leqslant \sup_{x^2+y^2\leqslant 1, z} (1+|(x, y+2m, z)|)^r$$

$$\times \sum_{\beta\in F_3} C''_\beta \sum_{(\gamma,\gamma')\in F_4(\beta)} \sup_{x_1^2+y_1^2\leqslant 1} |D^\gamma h(x_1, y_1)|$$

$$\times |D^{\gamma'} f(x_1, y_1+2m, z)|$$

where for each $\beta\in F_3$, $C''_\beta = \sup_{x_1^2+y_1^2\leqslant 1} |P'_\beta(x_1, y_1)| < \infty$ and $F_4(\beta)$ is a finite subset of $M\times M$. Write

$$C_5 = \max_{\beta\in F_3, (\gamma,\gamma')\in F_4(\beta)} C''_\beta \sup_{x_1^2+y_1^2\leqslant 1} |D^\gamma h(x_1, y_1)|.$$

Then the above is bounded by

$$C_5 \sup_{x^2+y^2\leqslant 1, z} (1+|(x, y+2m, z)|)^r \sum_{\beta\in F_3, (\gamma,\gamma')\in F_4(\beta)} s_{\gamma',r}(f)$$
$$\times (1+((2|m|-1)^2+|z|^2)^{1/2})^{-r}.$$

But there is a constant C_r so that for all $m\in\mathbb{Z}$, $\sup_{x^2+y^2\leqslant 1, z}(1+|(x, y+2m, z)|)^r (1+((2|m|-1)^2 + |z|^2)^{1/2})^{-r}\leqslant C_r$. Thus we have a bound of the desired form. Q.E.D.

THEOREM 9.14. *Suppose $E(P:F)$ is a holomorphic family of Eisenstein integrals defined as in (6.6) and $\alpha\in\mathscr{C}(\mathscr{D}\times\mathscr{F})$ such that $\alpha\cdot m_R$ is jointly smooth as a function on $\mathscr{D}\times\mathscr{F}$. Then $E(P:F)\cdot\alpha\cdot m_R\in I'(\mathscr{D}, G)$. Given any $D\in\mathscr{L}_G$ there is $r\geqslant 0$ so that given any $t\geqslant 0$ there is a continuous seminorm μ on $\mathscr{C}(\mathscr{D}\times\mathscr{F})$ so that ${}^0S_{D,r,t}(E(P:F)\,\alpha m_R)\leqslant\mu(\alpha)$ for all α as above.*

COROLLARY 9.15. *Suppose $E(P:F)$ and α are as above. Then $F_\alpha(x) = \int_{\mathscr{D}\times\mathscr{F}} E(P:F:h:v:x)\,\alpha(h:v)\,m(h:v)\,dh\,dv$ is in $\mathscr{C}(G, W)$. Given any $g_1, g_2\in\mathscr{U}(\mathfrak{g})$, $r\geqslant 0$, there is a continuous seminorm μ on $\mathscr{C}(\mathscr{D}\times\mathscr{F})$ so that ${}_{g_1}\|F_\alpha\|_{r, g_2}\leqslant\mu(\alpha)$ for all α as above.*

Proof. By Theorem 9.14, $\varphi = E(P:F)\,\alpha m_R\in I'(\mathscr{D}, G)$. Then by Theorem 8.2, I_φ is a Schwartz function on G and there is $F\subseteq\mathscr{L}_G$, so that given any $r'\geqslant 0$, there are $C, t\geqslant 0$ so that ${}_{g_1}\|I_\varphi\|_{r, g_2}\leqslant C\,{}^0S_{F,r',t}(\varphi)$. But $I_\varphi(x) = \int_{\mathscr{D}\times\mathscr{F}} E(P:F:h:v:x)\,\alpha(h:v)\,m_R(h:v)\,\pi_G(A_{h,v})\,\pi_R(v)^{-1}\,dh\,dv$. But by (9.12), $m_R(h:v)\,\pi_G(A_{h,v})\,\pi_R(v)^{-1} = m(h:v)$ so that $I_\varphi = F_\alpha$. Further, there is $r'\geqslant 0$ so that given any $t\geqslant 0$ there is μ so that ${}^0S_{F,r',t}(\varphi)\leqslant\mu(\alpha)$. Q.E.D.

Rather than prove Theorem 9.14 as stated, we will prove a slightly generalized version.

THEOREM 9.16. *Suppose $E(P:F_1), \ldots, E(P:F_k)$ are holomorphic families of Eisenstein integrals, $\alpha_1, \ldots, \alpha_k\in\mathscr{C}(\mathscr{D}\times\mathscr{F})$, and $\varphi = m_R\sum_{i=1}^k E(P:F_i)\alpha_i$ is* *jointly smooth on* $\mathscr{D} \times \mathscr{F} \times G$. *Then* $\varphi \in I'(\mathscr{D}, G)$, *and given* $D \in \mathscr{L}_G$, *there is* $r \geqslant 0$ *so that given any* $t \geqslant 0$, *there are continuous seminorms* $\mu_1, ..., \mu_k$ *on* $\mathscr{C}(\mathscr{D} \times \mathscr{F})$ *so that* ${}^0S_{D,r,t}(\varphi) \leqslant \sum_{i=1}^k \mu_i(\alpha_i)$.

Proof. We know that $\varphi_1 = \sum_{i=1}^k E(P:F_i)\,\alpha_i \in I(\mathscr{D}, G)$. But using Lemma 9.13, we see that $\varphi \in I(\mathscr{D}, G)$ also, and that for any $D \in \mathscr{L}_G$, $r, t \geqslant 0$, there are a finite subset F of $\mathscr{L}_G$ and $r_1 \geqslant 0$ so that ${}^0S_{D,r,t}(\varphi) \leqslant {}^0S_{F,r_1,t}(\varphi_1) \leqslant \sum_{i=1}^k {}^0S_{F,r_1,t}(E(P:F_i)\alpha_i)$. But since each $E(P:F_i) \in II(\mathscr{D}, G)$, by making r_1 sufficiently large, given $t \geqslant 0$ there are continuous seminorms μ_i on $\mathscr{C}(\mathscr{D} \times \mathscr{F})$ so that ${}^0S_{F,r_1,t}(E(P:F_i)\alpha_i) \leqslant \mu_i(\alpha_i)$.

Now let Q be any parabolic subgroup of G, $s \in W(\mathfrak{a}, \mathfrak{a}_Q)$. We must show that $v \mapsto \pi_R^{\,1}(v)\,\psi_{\varphi,s}(h:x)$ has a smooth extension from $\mathscr{F}'$ to $\mathscr{F}$ for all $(h, x) \in \mathscr{D} \times L_Q$. By [4, Lemma 22.1], it is enough to show that $\psi_{\varphi,s}(h:v_0:x) = 0$ for all $(h, x) \in \mathscr{D} \times L_Q$ if $\pi_R(v_0) = 0$. But $\varphi \in I(\mathscr{D}, G)$ so that $\psi_{\varphi,s}(h:v:x)$ is jointly smooth on $\mathscr{D} \times \mathscr{F} \times L_Q$. Thus it is enough to show that for all $h_0 \in \mathscr{D}$ such that φ factors through a group of Harish-Chandra class, and all v_0 such that $\pi_R(v_0) = 0$, $\lim_{v \to v_0} \psi_{\varphi,s}(h_0:v:x) = 0$ for all $x \in L_Q$, $v \to v_0$ through regular elements of $\mathscr{F}$. Now assume $Q = P' \in \mathscr{P}(A)$. Then $\psi_{\varphi,s}(h_0:v:x) = \sum_{i=1}^k \alpha_i(h_0:v)\, m_R(h_0:v)\, \psi_{E(P:F_i),s}(h_0:v:x)$. But $m_R(h_0:v)\, \psi_{E(P:F_i),s}(h_0:v) = \pi_R(v)\, m(h_0:v)\, E(P:F_i:h_0:v)_{P',s}$ where $E(P:F_i:h_0:v)_{P',s}$ is Harish-Chandra's constant term. Now by [5, Lemma 14.4], there is a constant c so that $\|\pi_R(v)\, m(h_0:v)\, E(P:F_i:h_0:v)_{P',s}\|_{M/Z}^2 = c^2 m(h_0:v)\, \|F_i(h_0)\|_{M/Z}^2\, \pi_R(v)^2$. But $\lim_{v \to v_0} m(h_0:v)$ exists and is finite even if $m(h:v)$ is not jointly continuous at (h_0, v_0), so that $\lim_{v \to v_0} \|m_R(h_0:v)\, \psi_{E(P:F_i),s}(h_0:v)\|_{M/Z}^2 = 0$. Thus $\psi_{\varphi,s}(h_0:v_0:x) = 0$ for all $x \in L_{P'} = MA$. Now by [4, Cor. of 11.1], $\psi_\varphi(h_0:v_0) = \pi(A_{h_0,v_0})\, \varphi(h_0:v_0) = 0$ on G.

Now suppose Q is any parabolic subgroup of G with $W(\mathfrak{a}, \mathfrak{a}_Q) \neq \varnothing$. Then for $s \in W(\mathfrak{a}, \mathfrak{a}_Q)$, let $g = (\psi^1_{\varphi,s})^s$. Let $P' \in \mathscr{P}(A)$, ${}^*P' = P' \cap L_Q$. As in (8.2), for $t \in W(\mathfrak{a}, \mathfrak{a}^s)$ there is $t' \in W(\mathfrak{a}, \mathfrak{a})$ so that $(\psi^{*P'}_{g,t})^t = (\psi^{P'}_{\varphi,t'})^{t'}$. But $\psi^{P'}_{\varphi,t'}(h_0:v_0) = 0$ as above. Thus $(\psi^{*P'}_{g,t})(h_0:v_0) = 0$ for all $P' \in \mathscr{P}(A)$, $t \in W(\mathfrak{a}, \mathfrak{a})$. Thus, again using [4, Lemma 11.1], $\pi_R(s_{i(s)} A_{h_0,v_0})\, \psi^1_{\varphi,s}(h_0:v_0)$ $= \psi_{\varphi,s}(h_0:v_0) = 0$ on L_Q. Q.E.D.

For $s \in W(\mathfrak{a}, \mathfrak{a})$, $h, h' \in \mathscr{D}$, write $h' = sh$ if $\lambda(h') = s\lambda(h)$. Let $W_0(h) = \{s \in W(\mathfrak{a}, \mathfrak{a}) : sh = h\}$ and $W_1(h) = \{s \in W(\mathfrak{a}, \mathfrak{a}) : sh = h'$ for some $h' \in \mathscr{D}\}$. Note that $W_1 = W_1(h)$ is independent of $h \in \mathscr{D}$. Let $\Theta(H:h:v)$ denote the distribution character of the representation $\pi_{h,v}$.

LEMMA 9.17. *There is a constant* c *so that for* F_α *as above, and any* $h, v \in \mathscr{D} \times \mathscr{F}$, $\Theta(H:h:v:R(x)F_\alpha) = c \sum_{s \in W_1} \alpha(sh:sv)\, E(P:F_M:sh:sv:x)$.

Proof. For fixed $h \in \mathscr{D}$, define $F_\alpha(h:x) = \int_{\mathscr{F}} E(P:F_M:h:v:x)\ \alpha(h:v)\, m(h:v)\, dv$, and define $\zeta(h) \in \hat{Z}$ by $\zeta(h) = e^h|_Z$. Let $\mathscr{D}_h = \{h' \in \mathscr{D} : e^{h}|_Z =$

$e^{h'}|_Z\}$. Then it follows from a Poisson summation argument similar to [8, 7.12] that for all $h \in \mathscr{D}$, $\hat{F}_\alpha(x:\zeta(h)) = \int_Z F_\alpha(xz)\,\zeta(h)(z)\,dz = c\sum_{h' \in \mathscr{D}_h} F_\alpha(h':x)$. Now $\Theta(H:h:v:R(x)F_\alpha) = \int_{G/Z} \Theta(H:h:v:y)\,\hat{F}_\alpha(yx:\zeta(h))\,d(yZ) = c\sum_{h' \in \mathscr{D}_h} \int_{G/Z} \Theta(H:h:v:y)\,F_\alpha(h':xy)\,d(yZ)$. Fix h rational, that is, for which $(\tau_{1,h}, \tau_{2,h})$ factors through a group of Harish-Chandra class. Then h' is rational for all $h' \in \mathscr{D}_h$, and using [5, Theorems 20.1 and 27.1]

$$\int_{G/Z} \Theta(H:h:v:y)\,F_\alpha(h':xy)\,d(yZ)$$

$$= \begin{cases} 0 & \text{unless} \quad h' = sh \text{ for some } s \in W(\mathfrak{a}, \mathfrak{a}) \\ c' \sum\limits_{s \in W_0(h)} \alpha(h':s^{-1}v)\,E(P:F:h':s^{-1}v:x) & \\ \qquad \text{if} \quad h' = sh \text{ for some } s \in W(\mathfrak{a}, \mathfrak{a}). & \end{cases}$$

Thus in this case $\Theta(H:h:v:R(x)F_\alpha) = cc'\sum_{s \in W_1} \alpha(sh:sv)\,E(P:F_M:sh:sv:x)$. But both sides are smooth functions of h, so equality persists for all $h \in \mathscr{D}$.

Q.E.D.

THEOREM 9.18. *$F_\alpha \in \mathscr{C}_H(G, W)$. In fact there is a constant c so that $F_\alpha(x) = c\int_{\mathscr{D} \times \mathscr{F}} \Theta(H:h:v)(R(x)F_\alpha)\,m(h:v)\,dh\,dv$.*

Proof. Using (9.17) $\int_{\mathscr{D} \times \mathscr{F}} \Theta(H:h:v)(R(x)F_\alpha)\,m(h:v)\,dh\,dv = c\sum_{s \in W_1} \int_{\mathscr{D} \times \mathscr{F}} \alpha(sh:sv)\ E(P:F_M:sh:sv:x)\,m(h:v)\,dh\,dv = c[W_1]\int_{\mathscr{D} \times \mathscr{F}} \alpha(h:v)\,E(P:F_M:h:v:x)\,m(h:v)\,dh\,dv$ by changing variables $(h, v) \mapsto (s^{-1}h, s^{-1}v)$ in the integration. Q.E.D.

We have constructed wave packets

$$F_\alpha(x) = \int_{\mathscr{D} \times \mathscr{F}} E(P:F:h:v:x)\,\alpha(h:v)\,m(h:v)\,dh\,dv \tag{9.19a}$$

and shown they are elements of $\mathscr{C}_H(G:W)$. If we want scalar-valued wave packets, we need only take

$$\begin{aligned} f_\alpha(x) &= F_\alpha(x)(1:1) \\ &= \int_{\mathscr{D} \times \mathscr{F}} E(P:F:h:v:x)(1:1)\,\alpha(h:v)\,m(h:v)\,dh\,dv. \end{aligned} \tag{9.19b}$$

Since $\varphi \to \varphi(1:1)$ is a linear functional on the finite dimensional vector space W, we will have ${}_{g_1}\|f_\alpha\|_{r,g_2} \leqslant C\,{}_{g_1}\|F_\alpha\|_{r,g_2}$ for all $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$, $r \geqslant 0$. Thus $f_\alpha \in \mathscr{C}(G)$ whenever $F_\alpha \in \mathscr{C}(G:W)$. We can also evaluate both sides of (9.18) at $(1:1) \in K_1 \times K_1$ to obtain:

THEOREM 9.20. *Suppose $E(P:F)$ is a holomorphic family of Eisenstein integrals defined as in* (6.6) *and $\alpha \in \mathscr{C}(\mathscr{D} \times \mathscr{F})$ such that $\alpha \cdot m_R$ is jointly smooth on $\mathscr{D} \times \mathscr{F}$. Define the wave packet f_α as in* (9.19b). *Then $f_\alpha \in \mathscr{C}_H(G)$. More precisely*:

(i) *Given any $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$, $r \geqslant 0$, there is a continuous seminorm μ on $\mathscr{C}(\mathscr{D} \times \mathscr{F})$ so that ${}_{g_1}\|f_\alpha\|_{r, g_2} \leqslant \mu(\alpha)$ for all α as above.*

(ii) *There is a constant c so that for all $x \in G$,*

$$f_\alpha(x) = c \int_{\mathscr{D} \times \mathscr{F}} \Theta(H:h:v)(R(x) f_\alpha)\, m(h:v)\, dh\, dv.$$

10. EXTENSION TO DISCONNECTED GROUPS

Finally we extend our results for connected reductive Lie groups to the class [11, 6, 7, 8, 9] of real reductive Lie groups G such that

$$G \text{ has a closed normal abelian subgroup } Z \text{ such that } Z \subset Z_G(G^0) \text{ and } |G/ZG^0| < \infty, \tag{10.1a}$$

$$\text{if } x \in G \text{ then } \operatorname{Ad}(x) \in \operatorname{Int}(\mathfrak{g}_{\mathbb{C}}), \tag{10.1b}$$

and

$$G/G^0 \text{ is finitely generated.} \tag{10.1c}$$

The Harish-Chandra class consists of the groups (10.1) such that $[G^0, G^0]$ has finite center and G/G^0 is finite.

The first step is to show that there is a particularly good choice of Z.

Fix a Cartan involution θ of G as in [11]. The fixed point set $K = G^\theta$ is the inverse image of a maximal compact subgroup of the linear semisimple group $G/Z_G(G^0)$. K meets, and has connected intersection with, every component of G. As in the connected case every Cartan subgroup of G is G^0-conjugate to a θ-stable one. So every cuspidal parabolic subgroup of G is G^0-conjugate to one of the form MAN, where M and A are θ-stable, $MA = M \times A = Z_G(A)$, and M and MA satisfy (10.1).

Proposition 2.1 says, here, that K^0 has a unique maximal compact subgroup K_1^0 and a closed normal vector subgroup V such that $K^0 = K_1^0 \times V$ and $Z_{G^0} \cap V$ is co-compact in both V and Z_{G^0}. Since $\operatorname{Ad}_G(K)$ is compact we may assume that it stabilizes the Lie algebra of V, and thus that V is normal in K.

PROPOSITION 10.2. *The group* Z *of* (10.1a) *can be chosen so that* $Z = (Z \cap G^0) \times E$, *where*

(a) E *is a finitely generated free abelian group,*

(b) E *is a closed normal subgroup of* G, *and*

(c) $Z \cap G^0 = Z_{G^0} \cap V$.

Then $ZG^0 = G^0 \times E$ *and* $ZK^0 = K^0 \times E = K_1^0 \times V \times E$.

Proof. Start with Z_1 that satisfies (10.1a). Then $Z_2 = Z_1 Z_{G^0}$ does also, and Z_2/Z_2^0 is finitely generated by (10.1c). Now [7, Lemma 6.3] $Z_2 = \{Z_{G^0} \cdot F\} \times E'$, F finite abelian, E' finitely generated free abelian.

$Z_3 = \{Z_{G^0} \cap V\} \times E'$ has finite index in Z_2, $Z_{G^0} \cap V$ is normal in $G = KG^0$ because it is normal in K and centralized by G^0, and the finite index subgroup $Z_2 G^0 \subset G$ centralizes Z_3. So Z_3 has a finite index subgroup $Z_4 = \{Z_{G^0} \cap V\} \times E''$ that is normal in G and thus satisfies (10.1a).

Split $V = V' \times V''$, where $V' = V \cap [G^0, G^0]$ and $V'' \subset Z_{G^0}$ is normal in K. Then $Z_{G^0} \cap V = L' \times V''$, where L' is a lattice in V', normal in $G = KG^0$ because it is normal in K and central in G^0. By (10.1b), V'' is central in G. So is any lattice $L'' \subset V''$. Now $L = L' \times L''$ is a lattice in V, normal in G. So $Z'' = L \times E''$ is a finitely generated free abelian group that satisfies (10.1a).

The action of G on Z'' by conjugation defines a linear representation φ of G on the rational vector space $\mathbb{Z}''_{\mathbb{Q}} = Z'' \otimes_{\mathbb{Z}} \mathbb{Q}$. As $\varphi(G)$ is finite the invariant subspace $L_{\mathbb{Q}}$ has an invariant complement B. Let $E = Z'' \cap B$ and $Z' = L \times E$. Then L and E are normal in G so Z' satisfies (10.1a). Proposition 10.2 follows with $Z = (Z_{G^0} \cap V) \times E$. Q.E.D.

From now on, we choose Z as in Proposition 10.2. For convenience we write G'' for ZG^0 and use ^{0} and ${}''$ to indicate items pertaining to G^0 and G''.

Recall that the Schwartz spaces for G'' and G were defined in [7, Sect. 6] as follows. For $x \in G^0$, define Ξ and $\tilde{\sigma}$ as in (2.4) and (2.7). Since V is normal in K we can assume that σ_V is K-invariant. Let σ_E be a norm on E coming from an $\mathrm{Ad}_G(K)$-invariant positive definite inner product on $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. Now we extend $\tilde{\sigma}$ to $G'' = G^0 \times E$ by

$$\tilde{\sigma}(xe) = \tilde{\sigma}(x) + \sigma_E(e), \qquad x \in G^0,\ e \in E. \tag{10.3a}$$

This is equivalent to the definition of $\tilde{\sigma}$ in [7, Sect. 6]. Using (2.9c), we see that

$$\tilde{\sigma}(x''y'') \leqslant 3(\tilde{\sigma}(x'') + \tilde{\sigma}(y'')) \qquad \text{for all} \quad x'', y'' \in G'' \tag{10.3b}$$

and since σ_V and σ_E are chosen to be K-invariant we have

$$\tilde{\sigma}(kx''k^{-1}) = \tilde{\sigma}(x'') \qquad \text{for all} \quad k \in K. \tag{10.3c}$$

Extend Ξ to G'' by

$$\Xi(xe) = \Xi(x), \qquad x \in G^0,\ e \in E. \tag{10.3d}$$

Then clearly

$$\Xi(k''x'') = \Xi(x''k'') = \Xi(x'') \qquad \text{for all} \quad x'' \in G'',\ k'' \in K''. \tag{10.3e}$$

Finally, because of (10.1b), every element of $N_G(A_0)/Z_G(A_0)$ can be represented by an element of K^0. Now since coset representatives of K/K^0 can be chosen to normalize A_0, we have $\Xi(kak^{-1}) = \Xi(a)$ for all $k \in K$, $a \in A_0$. Thus

$$\Xi(kx''k^{-1}) = \Xi(x'') \qquad \text{for all} \quad k \in K,\ x'' \in G''. \tag{10.3f}$$

For $f \in C^\infty(G'')$, $g_1, g_2 \in \mathcal{U}(\mathfrak{g})$, $r \geqslant 0$, we define

$$_{g_1}\|f\|_{r, g_2} = \sup_{x \in G''} |f(g_1; x; g_2)|\, \Xi(x)^{-1}(1 + \tilde{\sigma}(x))^r. \tag{10.3g}$$

Then

$$\mathscr{C}(G'') = \{f \in C^\infty(G''): {}_{g_1}\|f\|_{r, g_2} < \infty \text{ for all } g_1, g_2 \in \mathcal{U}(\mathfrak{g}), r \geqslant 0\}, \tag{10.3h}$$

and

$$\mathscr{C}(G) = \{f \in C^\infty(G): (L(x)f)|_{G''} \in \mathscr{C}(G'') \qquad \text{for all} \quad x \in G\}. \tag{10.3i}$$

Let $\{b_1, ..., b_n\}$ be coset representatives for G/G''. For $f \in C^\infty(G)$ and $1 \leqslant t \leqslant n$, define $f_t = (L(b_t^{-1})f)|_{G''}$. Then $\mathscr{C}(G) = \{f \in C^\infty(G): f_t \in \mathscr{C}(G''), 1 \leqslant t \leqslant n\}$, and it can be topologized by the seminorms

$$_{g_1, t}\|f\|_{r, g_2} = {}_{g_1}\|f_t\|_{r, g_2}, \qquad g_1, g_2 \in \mathcal{U}(\mathfrak{g}), r \geqslant 0, 1 \leqslant t \leqslant n. \tag{10.3j}$$

The next step is to extend Theorem 9.20 from G^0 to $G'' = G^0 \times E$.

Fix a θ-stable Cartan subgroup $H'' \subset G''$ and let $P'' = M''AN$ be an associated cuspidal parabolic. Let

$$\boldsymbol{\pi}_G^0 = \{\pi_{h, \nu}^0 : h \in \mathscr{D} \text{ and } \nu \in \mathfrak{a}^*\} \tag{10.4a}$$

be a continuous family of $H'' \cap G^0$-series representations of G^0 as in (3.8). The corresponding continuous family of H''-series representations of G'' is

$$\boldsymbol{\pi}_G'' = \{\pi''_{h, \nu, \eta} = \pi_{h, \nu}^0 \otimes \eta : h \in \mathscr{D}, \nu \in \mathfrak{a}^*, \eta \in \hat{E}\}. \tag{10.4b}$$

Let $E^0(P'' \cap G^0 : F : h : v : x)$ be a family of Eisenstein integrals on G^0 as in (6.6) corresponding to the family π_G^0 and a family F of τ-spherical functions on G^0 coming from a holomorphic family of matrix coefficients on $M'' \cap G^0$ as in (6.5). Now

$$\varepsilon^0(P'' \cap G^0 : F : h : v : x) = E^0(P'' \cap G^0 : F : h : v : x)(1:1) \tag{10.5a}$$

is a smooth family of matrix coefficients of π_G^0. Since every matrix coefficient of $\pi''_{h,v,\eta}$ is of the form $f(x:e) = f^0(x)\,\eta(e)$, where f^0 is a matrix coefficient of $\pi^0_{h,v}$, we obtain a smooth family of coefficients of π''_G by defining

$$\varepsilon''(P'' : F : h : v : \eta : x : e) = \eta(e)\,\varepsilon^0(P'' \cap G^0 : F : h : v : x). \tag{10.5b}$$

Let $\mathscr{P}^0$ and $\mathscr{P}''$ denote the respective algebras of differential operators on $\mathscr{S} \times \mathscr{F}$ and $\mathscr{S} \times \mathscr{F} \times \hat{E}$ whose coefficients are polynomials on $\mathscr{S} \times \mathscr{F}$ and constant on $\hat{E}$. If $h \in \mathscr{S}$ then, as before, $d(h)$ is the distance from h to $bd(\mathscr{S})$. The seminorms on $C^\infty(\mathscr{S} \times \mathscr{F})$

$$\|\alpha\|^0_{D,t} = \sup_{\mathscr{S} \times \mathscr{F}} |D\alpha(h:v)|\,(1 + d(h)^{-1})^t \tag{10.6a}$$

and on $C^\infty(\mathscr{S} \times \mathscr{F} \times \hat{E})$

$$\|\beta\|''_{D,t} = \sup_{\mathscr{S} \times \mathscr{F} \times \hat{E}} |D\beta(h:v:\eta)|\,(1 + d(h)^{-1})^t \tag{10.6b}$$

define Schwartz spaces

$$\mathscr{C}(\mathscr{S} \times \mathscr{F}) = \{\alpha \in C^\infty(\mathscr{S} \times \mathscr{F}) : \|\alpha\|^0_{D,t} < \infty \text{ for } D \in \mathscr{P}^0,\ t \geqslant 0\} \tag{10.7a}$$

and

$$\mathscr{C}(\mathscr{S} \times \mathscr{F} \times \hat{E}) = \{\beta \in C^\infty(\mathscr{S} \times \mathscr{F} \times \hat{E}) : \|\beta\|''_{D,t} < \infty \quad \text{for} \quad D \in \mathscr{P}'',\ t > 0\}. \tag{10.7b}$$

The space (10.7a) was used to form the wave packets for G^0 in Theorem 9.20. We will use (10.7b) to form the analogous wave packets for G''.

Two remarks:

LEMMA 10.8. *Let d be a constant coefficient differential operator on $\hat{E}$ and μ a continuous seminorm on $\mathscr{C}(\mathscr{S} \times \mathscr{F})$. If $\beta \in \mathscr{C}(\mathscr{S} \times \mathscr{F} \times \hat{E})$ define $\alpha_{d,\eta}(h:v) = \beta(h:v:\eta; d)$. Then $\alpha_{d,\eta} \in \mathscr{C}(\mathscr{S} \times \mathscr{F})$ and $\beta \mapsto \sup_\eta \mu(\alpha_{d,\eta})$ is a continuous seminorm on $\mathscr{C}(\mathscr{S} \times \mathscr{F} \times \hat{E})$.*

LEMMA 10.9. *Let $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$ and let $\alpha_{d,\eta}$ be as in Lemma 10.8. If $m_R(h:v)\,\beta(h:v:\eta)$ extends to be C^∞ on $\mathscr{D} \times \mathscr{F} \times \hat{E}$ then each $m_R(h:v)\,\alpha_{d,\eta}(h:v)$ extends to be C^∞ on $\mathscr{D} \times \mathscr{F}$.*

Now we extend Theorem 9.20 to G''. Let $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$ such that $m_R(h:v)\,\beta(h:v:\eta)$ extends C^∞ on $\mathscr{D} \times \mathscr{F} \times \hat{E}$. Form the wave packets

$$\varphi''_\beta(x:e) = \int_{\mathscr{D} \times \mathscr{F} \times \hat{E}} \varepsilon''(P'':F:h:v:\eta:x:e)\,\beta(h:v:\eta)\,dh\,dv\,d\eta. \quad (10.10)$$

THEOREM 10.11. *Let $\varepsilon''(P'':F)$ be a smooth family of matrix coefficients on $G'' = G^0 \times E$ defined as in (10.5b). Let $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$ such that $m_R(h:v)\,\beta(h:v:\eta)$ extends to be C^∞ on $\mathscr{D} \times \mathscr{F} \times \hat{E}$. Then $\varphi''_\beta \in \mathscr{C}_H(G'')$. More precisely:*

(i) *Let $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$, $r \geqslant 0$. Then there is a continuous seminorm μ on $\mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$, independent of $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$, such that ${}_{g_1}\|\varphi''_\beta\|_{r,g_2} \leqslant \mu(\beta)$.*

(ii) *There is a constant c so that for all $x \in G''$,*

$$\varphi''_\beta(x) = c \int_{\mathscr{D} \times \mathscr{F} \times \hat{E}} \Theta(H'':h:v:\eta)(R(x)\varphi''_\beta)\,m(h:v)\,dh\,dv\,d\eta.$$

Proof. For (i) split $\tilde{\sigma}(xe) = \tilde{\sigma}(x) + \sigma_E(e)$, where $x \in G^0$ and $e \in E$. Then the integral defining ${}_{g_1}\|\varphi''_\beta\|_{r,g_2}$ is bounded by

$$\begin{aligned} &\sup_{x \in G^0} (1 + \tilde{\sigma}(x))^r\,\Xi(x)^{-1} \sup_{e \in E} (1 + \sigma_E(e))^r \\ &\qquad \times \left| \int_{\mathscr{D} \times \mathscr{F} \times \hat{E}} \varepsilon''(P'':F:h:v:\eta:g_1;x;g_2:e)\,\beta(h:v:\eta)\,m(h:v)\,dh\,dv\,d\eta \right| \\ &\quad \leqslant \sup_{x \in G^0} (1 + \tilde{\sigma}(x))^r\,\Xi(x)^{-1} \sup_{e \in E} (1 + \sigma_E(e))^r \left| \int_{\hat{E}} \eta(e)\,\psi_\beta(x:\eta)\,d\eta \right|, \end{aligned}$$

where

$$\psi_\beta(x:\eta) = \int_{\mathscr{D} \times \mathscr{F}} \varepsilon^0(P'' \cap G^0:F:h:v:g_1;x;g_2)\,\beta(h:v:\eta)\,m(h:v)\,dh\,dv.$$

Now $\psi_\beta(x) \in C^\infty(\hat{E})$. For given $r \geqslant 0$ we have a constant coefficient operator d on $\hat{E}$ such that

$$\sup_{e \in E} (1 + \sigma_E(e))^r \left| \int_{\hat{E}} \eta(e)\,\psi_\beta(x:\eta)\,d\eta \right| \leqslant \sup_{\eta \in \hat{E}} |d\psi_\beta(x:\eta)|$$

independent of x. So the integral defining ${}_{g_1}\|\varphi''_\beta\|_{r,g_2}$ is bounded by

$$\sup_{\substack{x\in G^0\\ \eta\in\hat{E}}} (1+\tilde{\sigma}(x))^r\,\Xi(x)^{-1}\left|\int_{\mathscr{S}\times\mathscr{F}} \varepsilon^0(P''\cap G^0:F:h:v:g_1;x;g_2)\right.$$

$$\left.\cdot\,\beta(h:v:\eta;d)\,m(h:v)\,dh\,dv\right|.$$

Since $\alpha_{d,\eta}(h:v)=\beta(h:v:\eta;d)\in\mathscr{C}(\mathscr{S}\times\mathscr{F})$, Theorem 9.20(i) gives us a continuous seminorm μ^0 on $\mathscr{C}(\mathscr{S}\times\mathscr{F})$, depending only on g_1, g_2, and r, such that this is bounded by $\sup_{\eta\in\hat{E}}\mu^0(\alpha_{d,\eta})$. Thus we have a continuous seminorm $\mu(\beta)=\sup_{\eta\in\hat{E}}\mu^0(\alpha_{d,\eta})$ on $\mathscr{C}(\mathscr{S}\times\mathscr{F}\times\hat{E})$ such that the integral defining ${}_{g_1}\|\varphi''_\beta\|_{r,g_2}$ is bounded by $\mu(\beta)$.

For (ii) note $\Theta(H'':h:v:\eta:x:e)=\eta(e)\,\Theta(H''\cap G^0:h:v:x)$ and note that the Plancherel density functions for G'' and G^0 are related by $m''(h:v:\eta)=m(h:v)$. Now, using Theorem 9.20(ii),

$$\varphi''_\beta(xe)=\int_{\hat{E}}\eta(e)\left\{\int_{\mathscr{S}\times\mathscr{F}}\varepsilon^0(P''\cap G^0:F:h:v:x)\,\beta(h:v:\eta)\,m(h:v)\,dh\,dv\right\}d\eta$$

$$=\int_{\hat{E}}\eta(e)\,\varphi^0_{\alpha_\eta}(x)\,d\eta,$$

where $\alpha_\eta(h:v)=\beta(h:v:\eta)\in\mathscr{C}(\mathscr{S}\times\mathscr{F})$ and

$$\varphi^0_{\alpha_\eta}(x)=\int_{\mathscr{S}\times\mathscr{F}}\varepsilon^0(P''\cap G^0:F:h:v:x)\,\alpha_\eta(h:v)\,m(h:v)\,dh\,dv.$$

So

$$\varphi^0_{\alpha_\eta}(x)=c\int_{\mathscr{S}\times\mathscr{F}}\Theta(H''\cap G^0:h:v)(R(x)\,\varphi^0_{\alpha_\eta})\,m(h:v)\,dh\,dv$$

and thus

$$\varphi''_\beta(xe)=c\int_{\mathscr{S}\times\mathscr{F}\times\hat{E}}\Theta(H''\cap G^0:h:v)(R(x)\,\varphi^0_{\alpha_\eta})\,\eta(e)\,m(h:v)\,dh\,dv\,d\eta$$

$$=c\int_{\mathscr{S}\times\mathscr{F}\times\hat{E}}\Theta(H'':h:v:\eta)(R(xe)\,\varphi''_\beta)\,m(h:v)\,dh\,dv\,d\eta$$

using the Fourier inversion formula on E. Q.E.D.

Our final step is to extend Theorem 10.11 from $G''=ZG^0$ to G. First note that G'' is normal in G and of finite index.

Our continuous family of H-series representations, as in (10.4), will be

$$\pi_G = \{\pi_{h,\nu,\eta} = \mathrm{Ind}_{G''}^{G}(\pi''_{h,\nu,\eta}) : h \in \mathscr{D},\ \nu \in \mathfrak{a}^*,\ \eta \in \hat{E}\}. \tag{10.12}$$

Note that in general $G'' \subsetneq Z_G(G^0)G^0$. Thus the representations $\pi_{h,\eta,\nu}$ are in general finite sums of irreducible H-series representations.

Choose coset representatives $\{b_1, ..., b_n\}$ for G/G''. Since K meets every component of G, we can assume that $b_i \in K$, $1 \leqslant i \leqslant n$.

Now $\pi_{h,\nu,\eta}|_{G''} = \sum_{i=1}^{n} \pi''_{h,\nu,\eta} \cdot \mathrm{Ad}(b_i)^{-1}$. In fact, given $v \in \mathscr{H}(\pi''_{h,\nu,\eta})$ we construct $\{v_1, ..., v_n\} \subset \mathscr{H}(\pi_{h,\nu,\eta})$ as follows:

$$v_k : G \to \mathscr{H}(\pi''_{h,\nu,\eta}) \text{ is supported in } b_k G'', \tag{10.13a}$$

$$v_k(b_k x) = \pi''_{h,\nu,\eta}(x)^{-1} \cdot v \qquad \text{for} \quad x \in G''. \tag{10.13b}$$

Similarly, if $w \in \mathscr{H}(\pi''_{h,\nu,\eta})$ we have $\{w_1, ..., w_n\}$ in $\mathscr{H}(\pi_{h,\nu,\eta})$.

LEMMA 10.14. *The coefficients* $x \mapsto \langle \pi_{h,\nu,\eta}(x) w_i, v_k \rangle$ *of* $\pi_{h,\nu,\eta}$ *is supported in the coset* $b_s G''$ *for which* $b_k \equiv b_s b_i \bmod G''$. *On* $b_s G''$ *it is given by* $b_s x'' \mapsto \langle \pi''_{h,\nu,\eta}(a_{si}) \cdot \pi''_{h,\nu,\eta}(b_i^{-1} x'' b_i) \cdot w, v \rangle_{\mathscr{H}(\pi''_{h,\nu,\eta})}$, *where* $a_{si} = b_k^{-1} b_s b_i \in K''$.

Proof. Drop the subscripts on $\pi_{h,\nu,\eta}$ and $\pi''_{h,\nu,\eta}$ and compute

$$\langle \pi(x) w_i, v_k \rangle_{\mathscr{H}(\pi)} = \sum_{1 \leqslant t \leqslant n} \langle w_i(x^{-1} b_t), v_k(b_t) \rangle_{\mathscr{H}(\pi'')}.$$

Let $x = b_s x''$, $x'' \in G''$, so $w_i(x^{-1} b_t) = 0$ unless $b_s^{-1} b_t \in b_i G''$. Note $v_k(b_t) = 0$ unless $k = t$. So $x \mapsto \langle \pi(x) w_i, v_k \rangle$ is supported in the coset $b_s G''$ with $b_k \equiv b_s b_i$. Given that, $b_s b_i = b_k a_{si}$ and we compute

$$\begin{aligned}\langle \pi(x) w_i, v_k \rangle_{\mathscr{H}(\pi)} &= \langle w_i(x^{-1} b_k), v_k(b_k) \rangle_{\mathscr{H}(\pi'')} \\ &= \langle w_i(b_i \cdot (b_i^{-1} x'' b_i)^{-1} \cdot a_{si}^{-1}), v \rangle_{\mathscr{H}(\pi'')} \\ &= \langle \pi''(a_{si}) \cdot \pi''(b_i^{-1} x'' b_i) \cdot w, v \rangle_{\mathscr{H}(\pi'')}\end{aligned}$$

as asserted. Q.E.D.

Lemma 10.14 tells us how the family (10.5b) of matrix coefficients of G'' defines families for G. Let $\varepsilon''(P'' : F : h : \eta : \nu : x'')$ be as in (10.5b) with x'' in place of $(x : e)$. For $1 \leqslant i, k \leqslant n$ define

$\varepsilon_{ik}(P : F : h : \eta : \nu) : G \to \mathbb{C}$ supported in the coset $b_s G''$ for which $b_k \equiv b_s b_i \bmod G''$ (10.15a)

by the formula

$\varepsilon_{ik}(P : F : h : \eta : \nu : b_s x'') = \varepsilon''(P'' : F : h : \eta : \nu : a_{si} b_i^{-1} x'' b_i)$ for $x'' \in G''$, b_s as specified in (10.15a) and $a_{si} = b_k^{-1} b_s b_i$. (10.15b)

The $\varepsilon_{ik}(P:F:h:\eta:\nu)$ are matrix coefficients of $\pi_{h,\eta,\nu}$. Now, as in (10.10), let $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$ such that $m_R(h:\nu)\,\beta(h:\nu:\eta)$ extends to C^∞ on $\mathscr{D} \times \mathscr{F} \times \hat{E}$, and form the wave packets

$$\varphi_{ik\beta}(x) = \int_{\mathscr{D} \times \mathscr{F} \times \hat{E}} \varepsilon_{ik}(P:F:h:\nu:\eta:x)\,\beta(h:\nu:\eta)\,m(h:\nu)\,dh\,d\nu\,d\eta. \qquad (10.16)$$

THEOREM 10.17. *Let $E^0(P \cap G^0:F)$ be a family of Eisenstein integrals on G^0 as in (6.6) and let $\beta \in \mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$ such that $m_R(h:\nu)\,\beta(h:\nu:\eta)$ extends to be C^∞ on $\mathscr{D} \times \mathscr{F} \times \hat{E}$. Then the $\varphi_{ik\beta}$ of (10.16) belong to $\mathscr{C}_H(G)$. More precisely:*

(i) *Let $g_1, g_2 \in \mathscr{U}(\mathfrak{g})$, $r \geq 0$, $1 \leq t \leq n$. Define s as in (10.15). Then if $t \neq s$, ${}_{g_1,t}\|\varphi_{ik\beta}\|_{r,g_2} = 0$. For $t = s$, there is a continuous seminorm μ on $\mathscr{C}(\mathscr{D} \times \mathscr{F} \times \hat{E})$, independent of β, so that ${}_{g_1,s}\|\varphi_{ik\beta}\|_{r,g_2} \leq \mu(\beta)$.*

(ii) *There is a constant c so that for all $x \in G$,*

$$\varphi_{ik\beta}(x) = c \int_{\mathscr{D} \times \mathscr{F} \times \hat{E}} \Theta(H:h:\nu:\eta:R(x)\,\varphi_{ik\beta})\,m(h:\nu)\,dh\,d\nu\,d\eta.$$

Proof. Using (10.15) we see that $\varphi_{ik\beta}$ is supported in the coset $b_s G''$ and $\varphi_{ik\beta}(b_s x'') = \varphi''_\beta(a_{si} b_i^{-1} x'' b_i)$. Now using (10.3b, c, e, f) it is easy to see that there are a constant C and $g'_1, g'_2 \in \mathscr{U}(\mathfrak{g})$, depending on r, g_1, g_2, a_{si}, and b_i, but not on β, so that ${}_{g_1,s}\|\varphi_{ik\beta}\|_{r,g_2} \leq C\,{}_{g'_1}\|\varphi''_\beta\|_{r,g'_2}$. Part (i) now follows from Theorem 10.11.

For (ii), we first fix h, η, ν and write Θ and Θ'' for the characters $\Theta(H:h:\nu:\eta)$ and $\Theta(H'':h:\nu:\eta)$, respectively. Since Θ is supported on G'', we have $\Theta(R(x)\,\varphi_{ik\beta}) = 0$ unless $x \in b_s G''$. Thus it suffices to prove (ii) for $x \in b_s G''$. On G'' we have $\Theta = \sum_{1 \leq t \leq n} (\Theta'')^{b_t}$, where $(\Theta'')^{b_t}(y'') = \Theta''(b_t y'' b_t^{-1})$, $y'' \in G''$. Thus we have $\Theta(R(b_s x'')\,\varphi_{ik\beta}) = \sum_{1 \leq t \leq n} (\Theta'')^{b_t}(R(a_{si} b_i^{-1} x'' b_i)\,\varphi''_\beta)$. We can assume as in [6, 6.5] that $b_1 = 1, b_2, \ldots, b_n$ normalize H. Let $w_t \in W(G, H)$ be the Weyl group element represented by b_t. Then $(\Theta(H'':h:\nu:\eta))^{b_t}$ is the character of the H''-series representation with parameters $w_t(\lambda_0 + h_M(h)) \in it^*$, $w_t\chi(h) \in Z_M(M^0)^\wedge$, $w_t\nu \in i\mathfrak{a}^*$, and $w_t\eta \in \hat{E}$. By part (ii) of Theorem 10.11 and the uniqueness of the Plancherel formula representation of φ''_β, $\Theta(H'':h:\nu:\eta)^{b_t}(R(y'')\,\varphi''_\beta) \equiv 0$ unless

$$\begin{aligned}&\text{there are } s \in W(G'', H'') \text{ and } h' \in \mathscr{D} \text{ so that}\\ &w_t(\chi(h)) = s\chi(h') \text{ and } w_t(\lambda_0 + h_M(h)) = s(\lambda_0 + h_M(h')).\end{aligned} \qquad (10.18)$$

Let $S = \{1 \leq t \leq n : w_t \text{ satisfies } 10.18\}$. Note $[S] \geq 1$ since $1 \in S$. Fix $t \in S$ and write $w = w_t$. Since w and $s^{-1}w$ represent the same coset of G/G'', we may as well assume that $w\lambda_0 = \lambda_0$, $w\chi_0 = \chi_0$, and $w\mathscr{D} = \mathscr{D}$, so that $\Theta(H'':h:\nu:\eta)^{b_t} = \Theta(H'':wh:w\nu:w\eta)$. As in [6, 6.8], $m(wh:w\nu) = m(h:\nu)$ so we can change variables in the integration and apply part (ii) of Theorem 10.11 to write

$$c \int_{\mathcal{S} \times \mathcal{F} \times \hat{E}} \Theta(H'':h:v:\eta)^{b_i}(R(a_{si}b_i^{-1}x''b_i)\,\varphi''_\beta)\,m(h:v)\,dh\,dv\,d\eta$$

$$= c \int_{\mathcal{S} \times \mathcal{F} \times \hat{E}} \Theta(H'':h:v:\eta)(R(a_{si}b_i^{-1}x''b_i)\,\varphi''_\beta)\,m(h:v)\,dh\,dv\,d\eta$$
$$= \varphi''_\beta(a_{si}b_i^{-1}x''b_i) = \varphi_{ik\beta}(b_s x'').$$

Thus

$$\varphi_{ik\beta}(b_s x'') = c' \int_{\mathcal{S} \times \mathcal{F} \times \hat{E}} \Theta(H:h:v:\eta)(R(b_s x'')\,\varphi_{ik\beta})\,m(h:v)\,dh\,dv\,d\eta,$$

where $c' = c/[S]$. Q.E.D.

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