Progress in Systems & Control Theory 1
Berkhauser 1989

Computation and Control
Proceedings Bozeman 1988

OBSERVABILITY AND GROUP REPRESENTATION THEORY

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1 Introduction

In this note we interpret some questions of discrete observability of linear systems $dx/dt = Ax$ in terms of finite dimensional group representation theory. Some results of Martin and Smith [3] and of Martin and Wallace [4] are reformulated as results concerning cyclic representations, and then completed in some sense when we give a complete characterization of cyclic representations and their cyclic vectors. We then examine the case where the set of observation points generates a subgroup with the Selberg density property ([2], [5]).

Thanks to George Bergman for discussions and examples of cyclic representations.

2 Group Theoretic Interpretation of Constant Coefficient Systems

We start by casting some of the basic definitions of [3] and [4] into group theoretic language.

2.1. Definition. Let π be a representation of a group G on a vector space V of dimension $n < \infty$. Fix a vector $x_0 \in V$, a (co)vector c' in the linear dual space V' of V, and a subset $S = \{g_1, ..., g_n\} \subset G$. The triple (π, c', S) is **discretely observable** if we can always solve for x_0 in the system of equations

$$c' \cdot \pi(g_i)x_0 = e_i, \qquad 1 \leq i \leq n \tag{2.2}$$

*Research partially supported by NSF Grant DMS-85-13467

Discrete observability of (π, c', S) is equivalent to nonsingularity of the matrix

$$M = M(\pi, c', S) = \begin{pmatrix} c' \cdot \pi(g_1) \\ \vdots \\ c' \cdot \pi(g_n) \end{pmatrix}. \qquad (2.3)$$

The notion of discrete observability for a linear system $dx/dt = Ax$ with constant coefficients, corresponds to the case of a 1-parameter linear group, where G is the additive group of real numbers, A is an $n \times n$ matrix, $\pi(t) = exp(tA)$, and $g_i = t_i$ for some real numbers $t_1, \ldots, t_n$, so that $\pi(g_i) = exp(t_i A)$. See [4].

The point of the group-theoretic interpretation is that it has a useful formulation, as follows.

2.4. Theorem. Let π' denote the dual of π , representation of G on the linear dual V' of V. Let H denote the subgroup of G generated by S. If (π, c', S) is discretely observable then c' is a cyclic vector for $\pi'|_H$.

Proof. If (π, c', S) is discretely observable then the matrix M of (2.3) is nonsingular, so $\pi'(S)c'$ is a basis of V'. Then $\pi'(H)c'$ spans V', i.e. c' is cyclic for $\pi'|_H$. QED

In particular, in Theorem 2.4, c' is a cyclic vector for π', so π' is a cyclic representation.

2.5. Definition. A cocyclic representation is a representation whose dual is cyclic.

2.6. Corollary. There exist $c' \in V'$ and $S \subset G$ such that (π, c', S) is discretely observable, if and only if the representation π is cocyclic.

Proof. The vector c' is cyclic for π' if and only if one can choose $S = \{g_1, \ldots, g_n\}$ such that the $\pi'(g_i)c'$ form a basis of V' . QED

Combining Corollary 2.6 with Corollary 3.10 below, in the case of a linear system $dx/dt = Ax$ with constant coefficients as described above, we see that π is cocyclic if and only if A is admissible in the sense of [4].

2.7. Counterexamples. First, the converse to Theorem 2.4 fails, for example in the case where

$$G = \langle g_0 \rangle \text{ infinite cyclic}, \quad S = \{1, g_0, g_0{}^3\} \text{ and } \quad \pi'(g_0) = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Second, a cyclic semisimple (completely reducible) representation is cocyclic, but this fails without semisimplicity, as in the case where

$$G \text{ is the linear group consisting of all real } \begin{pmatrix} a & 0 & d \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \quad \text{with } a, b \neq 0 .$$

There G is cyclic on $V = \mathbf{R}^3$ (column vectors) but is not cyclic on V' (row vectors). Dually, cocyclic does not imply cyclic in general.

In order to apply Theorem 2.4 and Corollary 2.6 one needs to be able to recognize cyclic and cocyclic representations. For that, see Corollaries 3.9 and 3.10 below.

3 Characterization of Cyclic Representations

We first reduce to the case of semisimple representations.

3.1. Theorem. Let π be a finite dimensional representation of a group G. Then π has a unique maximal semisimple quotient[1] representation, say ψ , and π is cyclic if and only if ψ is cyclic.

Proof. Since the representation space V of π is finite dimensional, it has a minimal invariant subspace U such that the representation of G on V/U is semisimple. The representation ψ of G on V/U is a maximal semisimple quotient representation of π . If T is any invariant subspace such that G acts semisimply on V/T then G is semisimple on $V/(T \cap U)$, because $V/(T \cap U)$ embeds equivariantly in $V/T \oplus V/U$ by the direct sum of the projections. So $T \subset U$. This proves uniqueness of the maximal semisimple quotient of π .

If π is cyclic, so is every quotient representation, including ψ .

Now suppose that the maximal semsimple quotient representation ψ is

[1] This quotient, when it exists (as in our case) is the *cosocle* of π

cyclic. Let V/U be its representation space. If $U = 0$ then $\pi = \psi$, so suppose $U \neq 0$ and choose an irreducible invariant subspace $T \subset U$. Then V/U is a quotient of V/T , so ψ is the maximal semisimple quotient of the representation ϕ of G on V/T. By induction on the length of the composition series now ϕ is cyclic. Let $v \in V$ represent a cyclic vector for ϕ and let W denote the span of $\pi(G)v$.

If $W \cap T = 0$, then $V = W \oplus T$. As W maps onto V/T it maps onto V/U, so $V/U = W/(W \cap U)$, and the representation of G on $(W/(W \cap U)) \oplus T = V/(W \cap U)$ is a semisimple quotient of π that is larger than ψ . That is a contradiction.

Now $W \cap T \neq 0$, so $T \subset W$ by irreducibility of T . Thus $W = V$ and v is a cyclic vector for π . This completes the proof. QED

Theorem 3.1 dualizes to

3.2. Theorem. Let π be a finite dimensional representation of a group G. Then π is cocyclic if and only if its maximal semisimple subrepresentation[2] is cyclic.

The rest of this section, characterization of semisimple cyclic representations, is essentially standard. It is included here because there does not seem to be a good reference.

If $\mathbf{A}$ is a simple finite dimensional associative algebra over a field $\mathbf{F}$ then $\mathbf{A}$ is isomorphic to the algebra of $n \times n$ matrices with coefficients in a finite dimensional division algebra $\mathbf{D}$ over $\mathbf{F}$. See ([1], p. 39). The number $n = deg_F \mathbf{A}$, the degree of $\mathbf{A}$ over $\mathbf{F}$, is the maximal size of a set of commuting idempotents in $\mathbf{A}$. The left regular representation of $\mathbf{A}$ (on itself, by left multiplications) is n times the canonical representation of $\mathbf{A}$ on $\mathbf{D}^n$, and the latter is irreducible over $\mathbf{F}$.

3.3. Definition. Let π be an $\mathbf{F}$–irreducible representation of a group G on a finite dimensional vector space V over a field $\mathbf{F}$, so the set $\pi(G)$ of linear transformations of V generates a simple finite dimensional associative algebra $\mathbf{A}$ over $\mathbf{F}$. Then $deg_F \pi$ denotes $deg_F \mathbf{A}$, the integer n such that $\mathbf{A}$ is isomorphic to the algebra of $n \times n$ matrices with entries in a division algebra $\mathbf{D}$ over $\mathbf{F}$. *Note that $deg_F \pi = dim_D V$. This is slightly nonstandard.*

3.4. Proposition. Let β be a semisimple representation of a group G on a finite dimensional vector space V over a field $\mathbf{F}$. Decompose $\beta = k_1\beta_1 \oplus \ldots \oplus k_r\beta_r$ where the β_i are mutually inequivalent $\mathbf{F}$–irreducible representations. Then *beta* is cyclic if and only if $k_i \leq deg_F \beta_i$ for $1 \leq i \leq r$.

Proof. Th algebra $\mathbf{A}$ of linear transformations of V generated by $\pi(G)$ is of the form $\mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_r$ where $\mathbf{A}_i$ is the $\mathbf{F}$–simple algebra generated by π_i . So the condition

$$k_i \leq deg_F \beta_i \quad for \quad 1 \leq i \leq r \tag{3.5}$$

is equivalent to

$$\beta \textit{ is a quotient of } \lambda|_G \tag{3.6}$$

where λ is the left regular representation of $\mathbf{A}$.

Let J be any finite group of order prime to the characteristic of $\mathbf{F}$, such that $\mathbf{A}$ is isomorphic to an ideal in the group algebra of J over $\mathbf{F}$. Then we have a representation α of J on V such that $\mathbf{A}$ is the algebra of linear transformations of V generated by $\alpha(J)$. Each of the conditions, cyclicity and (3.6), depends only on $\mathbf{A}$. So we may replace G by J.

Now G is a finite group whose order is prime to the characteristic of $\mathbf{F}$. So (3.5) is equivalent to

$$\beta \textit{ is a quotient of the left regular representation of } G. \tag{3.7}$$

The left regular representation of a finite group is cyclic because any nonzero function supported at a single group element is a cyclic vector. Any cyclic representation of a finite group is a quotient of the left regular representation; just apply the group algebra to the cyclic vector. Now (3.7) is equivalent to the condition that β be cyclic. That completes the proof of Proposition 3.4. QED

3.8. Remark. In Proposition 3.4, decompose $V = V_1 \oplus \ldots \oplus V_r$ and $V_i = \mathbf{F}^{k_i} \otimes U_i$ with the U_i inequivalent and irreducible under β . In other words, U_i is the representation space of β_i where $\beta = k_1\beta_1 \oplus \ldots \oplus k_r\beta_r$ such that the β_i are mutually inequivalent and $\mathbf{F}$–irreducible. Express the algebra of linear transformations of U_i generated by $\beta_i(G)$, as the algebra of $n_i \times n_i$ matrices over a division algebra $\mathbf{D}_i$. When β is cyclic, that is each $k_i \leq n_i$, one can check that its cyclic vectors are exactly the

[2]The maximal semisimple subrepresentation, which always exists, is the *socle* of π

$$v = v_1 + \dots + v_r, \qquad v_i = (f_{i,1} \otimes u_{i,1}) + \dots + (f_{i,k_i} \otimes u_{i,k_i}),$$

such that $\{u_{i,j}\} \subset U_i$ is linearly independent over $\mathbf{D}_i$ and $\{f_{i,j}\}$ is a basis of $\mathbf{F}^{k_i}$.

Combine Theorems 3.1 and 3.2 with Proposition 3.4:

3.9. Corollary. Let π represent a group G on a finite dimensional vector space over a field $\mathbf{F}$. Then π is cyclic if and only if every $\mathbf{F}$–irreducible summand of the maximal semisimple quotient of π has multiplicity bounded by its $\mathbf{F}$–degree (Definition 3.3). In that case the cyclic vectors for π are just the vectors whose images in the maximal semisimple quotient o π are cyclic, and the latter are specified in Remark 3.8.

3.10. Corollary. Let π represent a group G on a finite dimensional vector space over a field $\mathbf{F}$. Then π is cocyclic if and only if every $\mathbf{F}$–irreducible summand of the maximal semisimple subrepresentation of π has multiplicity bounded by its $\mathbf{F}$–degree. In that case the cyclic vectors for π' are just the vectors whose images in the maximal semisimple quotient of π' are cyclic, and the latter are specified in Remark 3.8.

4 Dense Observation Subgroups

When one knows the structure of the group G, Corollary 3.10 leads to a quick decision as to whether the representation π is cocyclic, and thus (Corollary 2.6) whether there exist $c' \in V'$ and $S \subset G$ such that (π, c', S) is discretely observable. If one tries to apply Corollary 3.10 to the subgroup H generated by S, one runs into problems because the structure and representation theory of H may be far from clear. However, if H has the Selberg density property inside G then a result of Borel shows that enough of the representation theory of H is clear so that there is no problem in applying Corollary 3.10.

In ([5], Lemma 1) Selberg shows that if G is locally compact, and if the quotient G/H has finite invariant measure, then H has the density property: if $g \in G$ and U is an open neighborhood of the identity element of G then there exist u_1, $u_2 \in U$ and an integer $n > 0$ such that $u_1 g^n u_2{}^{-1} \in H$. Borel named the property and proved ([2], Corollary 4.5) that if G is a connected semisimple Lie group without compact factor, H is a Selberg dense subgroup of G, and π is a finite dimensional real (resp. complex) linear representation of G, then every element of $\pi(G)$ is a real (resp. complex) linear combination of elements of $\pi(H)$. Since π is automatically semisimple, it is cyclic if and only if it is cocyclic, so Borel's result gives us

4.1. Lemma. Let H be a Selberg dense subgroup in a connected semisimple Lie group G without compact factor, and let π be a finite dimensional real or complex linear representation of G . Then the following are equivalent: (i) π is cyclic, (ii) $\pi|_H$ is cyclic,(iii) π is cocyclic, and (iv) $\pi|_H$ is cocyclic.

Now combine Corollary 3.9 and Lemma 4.1 with Corollary 2.6 to see

4.2. Theorem Let G be a connected semisimple Lie group without compact factor, let π represent G on a vector space V of finite dimension n over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, and let $S_0 = \{g_1, \dots, g_n\} \subset G$ generate a Selberg dense subgroup H of G. Then the following are equivalent: (i) there exist $c' \in V'$ and $S = \{h_1, \dots, h_n\} \subset H$ such that (π, c', S) is discretely observable, (ii) there exists $S = \{h_1, \dots, h_n\} \subset H$ such that (π, c', S) is discretely observable for almost all $c' \in V'$, and (iii) every $\mathbf{F}$–irreducible summand of π has multiplicity bounded by its $\mathbf{F}$–degree.

4.3. Remark. In (i) and (ii), c' can be any cyclic vector for π' . Recall that those are described in Remark 3.8.

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