NEW CONSTRUCTIONS FOR REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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0. INTRODUCTION

This is a survey of some new methods in the representation theory of semisimple Lie groups. At first acquaintance these methods are not completely straightforward, but they already have proved to be quite powerful. Much of their power is the fact that these methods have rather different perspectives but turn out to be equivalent.

Roughly speaking, the three methods discussed here are

i) methods of differential geometry which on the surface are variations of classical geometric quantization,

ii) methods of homological algebra such as the Zuckerman derived functor construction, and

iii) methods of algebraic geometry, specifically $\mathcal{D}$-modules and the Beilinson-Bernstein localization theory.

There are a number of methods that I won't discuss in any serious way, but I'll try to point out which of them fit into the same framework as the ones that will be described in some detail. In this regard, there are some interesting open questions, and some of those will be described at the end of this note.

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1. THE BASIC DATUM

The groups we consider will be those of "Harish-Chandra class." In other words we consider real Lie groups G, say with real Lie algebra $\mathfrak{g}_0$, complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and topological identity component G^0, such that

$$\text{(1.1)}\qquad \begin{array}{l} G \text{ is reductive, i.e., } \mathfrak{g}_0 = (\text{semisimple}) \oplus (\text{commutative}), \\ G/G^0 \text{ is finite and the commutator } [G^0,G^0] \text{ has finite center,} \\ \text{and if } x \in G \text{ then } \mathrm{Ad}(x) \text{ is an } \textbf{inner} \text{ automorphism of } \mathfrak{g}. \end{array}$$

This last condition ensures that the Casimir operators act by scalars in any reasonable category of irreducible representations of G.

Let H be a Cartan subgroup of G. Thus, its real Lie algebra $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, that is, $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a maximal ad diagonalizable subalgebra of $\mathfrak{g}$, and H is the centralizer of $\mathfrak{h}_0$ in G.

Let χ be a finite dimensional representation of H. Denote the representation space by $E = E_\chi$ and the associated homogeneous vector bundle by $\mathbf{E} = \mathbf{E}_\chi \to G/H$. So sections of $\mathbf{E}$ can be identified with functions $f\colon G \to E$ such that $f(gh) = \chi(h)^{-1}\cdot f(g)$ for $g \in G$ and $h \in H$.

Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ that contains $\mathfrak{h}$. In other words, choose a positive root system $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{h})$ and define

$$\text{(1.2)}\qquad \mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha} \qquad \text{and} \qquad \mathfrak{b} = \mathfrak{h} + \mathfrak{n} \quad .$$

Then $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$ and our choice amounts to a choice of polarization on G/H.

We refer to the triple $(H,\mathfrak{b},\chi)$ as a **basic datum**. The various constructions of representations will be described in terms of basic data. This will make it convenient to compare constructions.

2. CLASSICAL CONSTRUCTIONS

Fix a basic datum $(H,\mathfrak{b},\chi)$ as above. Note that $d\chi$ extends to a representation of $\mathfrak{b}$ on E by $d\chi(\mathfrak{n}) = 0$. So we have a system of differential equations on the sections of $\mathbf{E} \to G/H$,

$$(2.1) \qquad f(g;\xi) = 0 \quad \text{for} \quad g \in G \quad \text{and} \quad \xi \in \mathfrak{n} \ .$$

Here, if $\xi = \xi' + i\xi''$ with $\xi', \xi'' \in \mathfrak{g}_0$ then $f(g;\xi)$ means $\frac{d}{dt}\Big|_0 f(g\cdot\exp(t\xi')) + i\,\frac{d}{dt}\Big|_0 f(g\cdot\exp(t\xi''))$. So the system (2.1) defines a sheaf

$$(2.2) \qquad \mathcal{O}_{\mathfrak{n}}(\mathbf{E}) \to G/H : \text{ germs of } C^\infty \text{ sections annihilated by } \mathfrak{n} \ .$$

G acts naturally on sections of $\mathbf{E}$ by $(x\cdot f)(g) = f(x^{-1}g)$. This action commutes with differentiation from $\mathfrak{n}$, so G acts naturally on $\mathcal{O}_{\mathfrak{n}}(\mathbf{E})$ and thus on the cohomologies of that sheaf.

Classical geometric quantization leads to the natural representations of G on the cohomologies $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$, especially on the space $H^0(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ of sections of $\mathbf{E}$ that are annihilated by $\mathfrak{n}$ in the sense of (2.1).

Example: totally complex polarization. This is the case $\mathfrak{n} \cap \bar{\mathfrak{n}} = 0$. Then [14] G/H has an invariant complex structure for which (2.1) is the Cauchy-Riemann equation, and [28] $\mathbf{E} \to G/H$ has a G-invariant holomorphic vector bundle structure, again defined by (2.1), such that $\mathcal{O}_{\mathfrak{n}}(\mathbf{E})$ is the sheaf of germs of holomorphic sections. Thus $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ can be calculated as Dolbeault cohomology. If G has a compact Cartan subgroup, i.e., if H is compact, then the resulting representations of G are those of the "discrete series" and its limits ([1],[23],[24],[26],[27]). In general in this case, H is maximally compact among Cartan subgroups of G and the representations in question are those of the "fundamental series" and its limits ([26],[27]).

Example: totally real polarization. This is the case $\mathfrak{n} = \bar{\mathfrak{n}}$ where G is quasi-split and H is maximally noncompact among Cartan subgroups of G. Note that $\mathfrak{n} = \mathfrak{n}_0 \otimes_R \mathbb{C}$ where $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$, so G has a (minimal) parabolic subgroup P with Lie algebra $\mathfrak{p}_0 = \mathfrak{h}_0 + \mathfrak{n}_0$, and $P = HN$ where $N = \exp(\mathfrak{n}_0)$. As $\mathfrak{n}$ (and thus N) acts trivially on E, the bundle $\mathbf{E} \to G/H$ pushes down to $\mathbf{E} \to G/P$. The Poincaré Lemma along the fibres of $G/H \to G/P$ shows that $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ vanishes for $q > 0$. By definition of induced representation, $H^0(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E})) = \mathrm{Ind}_P^G(E)$ in the C^∞

category, i.e. the action of G on $H^0(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ is just the C^∞ induced representation $\mathrm{Ind}_P^G(\chi)$. The resulting representations of G are those of the "principal series".

Example: H general, $\mathfrak{h}$ chosen to maximize $\mathfrak{n}\cap\bar{\mathfrak{n}}$. Then the resulting representations of G include the ones that occur in Harish-Chandra's Plancherel formula for G. Specifically, let θ be a Cartan involution of G, i.e. an automorphism with square 1 whose fixed point set is a maximal compact subgroup K of G. Without loss of generality one may assume $\theta(H) = H$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ where $d\theta$ is $+1$ on $\mathfrak{t}$ and is -1 on $\mathfrak{a}$. That splits $H = T \times A$ where $T = H \cap K$ and $A = \exp(\mathfrak{a}_0)$. The respective centralizers of $\mathfrak{a}$ in $\mathfrak{g}$ and A in G are of the form $\mathfrak{m} + \mathfrak{a}$ and $M \times A$ with $d\theta(\mathfrak{m}) = \mathfrak{m}$ and $\theta(M) = M$. Choose a system of positive $\mathfrak{a}_0$-roots on $\mathfrak{g}_0$, let $\mathfrak{n}_H$ be the sum of the negative root spaces and $N_H = \exp(\mathfrak{n}_H)$, and define $P = MAN_H$. The P is a **cuspidal parabolic subgroup** of G associated to H. If η is a discrete series representation of M, say with Harish-Chandra parameter $\nu + \rho_M$ where ρ_M is half the sum of the positive roots of M, and if $\sigma \in \mathfrak{a}_0^*$ so that $\exp(i\sigma)$ is a unitary character on A, then we define $\chi = \exp(\nu + i\sigma)$. For a certain index $q = q(\eta)$, G acts on the cohomology $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ by the standard tempered representation $\mathrm{Ind}_P^G(\eta \otimes \exp(i\sigma))$ (see [24] and [32]).

The third case described above combines the first two by using Dolbeault cohomology to the greatest extent possible and ordinary induction for the other "variables". This strongly suggests that one should try to compute $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(\mathbf{E}))$ as follows. First, the map

$$(2.3) \qquad (\mathfrak{g}/\mathfrak{h})^* \otimes E \otimes \Lambda^q\mathfrak{n}^* \to E \otimes \Lambda^{q+1}\mathfrak{n}^* \quad \text{by} \quad \phi \otimes e \otimes \omega \to e \otimes (\phi|_{\mathfrak{n}} \wedge \omega)$$

induces an operator

$$(2.4) \qquad d_{\mathfrak{n}} : C^\infty(G/H, \mathbf{E} \otimes \Lambda^q \mathbf{N}^*) \to C^\infty(G/H, \mathbf{E} \otimes \Lambda^{q+1}\mathbf{N}^*) \quad .$$

The pullback to G is a complex

$$(2.5) \qquad (C^\infty(G) \otimes E \otimes \Lambda^{\cdot}\mathfrak{n}^*)^H, \ d$$

where d is the coboundary for Lie algebra cohomology of $\mathfrak{n}$.

The complex (2.5) does compute the cohomologies $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(E))$ in the cases

$\mathfrak{n}$ is real or $\mathfrak{n}$ is maximally complex for the choice of $\mathfrak{h}$

because in those cases the complex $\mathcal{C}^{\infty}(G/H, \mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*)$, $d_{\mathfrak{n}}$ of sheaves of local C^{∞} sections of $\mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*$ is acyclic. But in general that complex is not acyclic, and so in general (2.5) does not compute the cohomology of $\mathcal{O}_{\mathfrak{n}}(\mathbf{E})$ but rather computes the hypercohomology of a complex of sheaves.

There is also a topological problem with (2.5). Experience shows that, in the cases where one can use (2.5), one must work very hard to show that d has closed range. And it is very likely that d does not always have closed range. This range problem is entangled with the question of whether a decomposition of E (as $(H,\mathfrak{h})$-module) will always be reflected in decompositions of the $H^q(G/H, \mathcal{O}_{\mathfrak{n}}(E))$.

These problems are addressed [24],[26],[27] by combining the methods mentioned in the introduction with the notion of maximal globalization [25] for Harish-Chandra modules.

3. METHODS OF HOMOLOGICAL ALGEBRA

The Zuckerman derived functor construction (see [29]) is defined in a rather abstract way but in fact fits into the picture sketched above.

H is a θ-stable Cartan subgroup of G as in the third example above. We write $\mathcal{M}(\mathfrak{g}, H\cap K)$ for the category $(\mathfrak{g}, H\cap K)$-modules, i.e. of $\mathfrak{g}$-modules which are, in a consistent way on $\mathfrak{h}\cap\mathfrak{k}$, modules for $H\cap K$. Similarly, write $\mathcal{M}(\mathfrak{g}, H\cap K)_{(H\cap K)}$ for the subcategory that consists of the $(H\cap K)$-finite modules in $\mathcal{M}(\mathfrak{g}, H\cap K)$. $\mathcal{M}(\mathfrak{g},K)_{(K)}$ has an analogous meaning. Consider the "functor"

$$\Gamma\colon \mathcal{M}(\mathfrak{g}, H\cap K)_{(H\cap K)} \longrightarrow \mathcal{M}(\mathfrak{g},K)_{(K)} \tag{3.1}$$

that maps $V\in\mathcal{M}(\mathfrak{g}, H\cap K)_{(H\cap K)}$ to its maximal $\mathfrak{k}$-semisimple $\mathfrak{k}$-finite submodule. It is left-exact, but generally has nontrivial right derived functors $R^q\Gamma$. The basic datum $(H,\mathfrak{h},X)$ specifies the **derived functor modules**

$$(3.2) \qquad A^q(H,\mathfrak{h},X) = (R^q\Gamma)\{\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}),E)_{(H\cap K)}\} \quad .$$

To interpret these in bundle language, write C^{for} for formal power series sections at the base point $1.H$ in G/H. Evaluation at $1.H$ defines an isomorphism $C^{for}(G/H, \mathbf{E}\otimes\Lambda^q\mathbf{N}^*) \cong \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), E \otimes \Lambda^q\mathfrak{n}^*)$. The operator $d_{\mathfrak{n}}$ of (2.4) acts on the first term, defining a complex of $(\mathfrak{g},H)$-modules. That gives us

$$\mathrm{Ker}(d_{\mathfrak{n}}\colon C^{for}(G/H,\mathbf{E})_{(H\cap K)} \to C^{for}(G/H, \mathbf{E}\otimes\mathbf{N}^*)_{(H\cap K)}) \cong \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}),E)_{(H\cap K)} \, .$$

Resolve the right-hand side by the complex $C^{for}(G/H, \mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*)_{(H\cap K)}, d_{\mathfrak{n}}$. The result is

$$(3.3) \qquad A^q(H,\mathfrak{h},X) \cong H^q(C^{for}(G/H, \mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*)_{(K)}, d_{\mathfrak{n}}) \quad .$$

Note that the coefficient homomorphisms, defined by Taylor series at $1.H$, gives a $(\mathfrak{g},K)$-module homomorphism

$$(3.4) \qquad H^q(C^{for}(G/H, \mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*)_{(K)}, d_{\mathfrak{n}}) \to H^q(C^{for}(G/H, \mathbf{E}\otimes\Lambda^{\cdot}\mathbf{N}^*)_{(K)}, d_{\mathfrak{n}}) \; .$$

4. COMPLETIONS OF HARISH-CHANDRA MODULES

At this point we have to be a little bit careful about what we mean by a representation. By **representation** of G we will mean a strongly continuous representation of finite length (finite composition series) on a complete locally convex topological vector space. By **Harish-Chandra module** for G we will mean a $(\mathfrak{g},K)$-module that is $\mathcal{U}(\mathfrak{g})$-finite and K-semisimple, and in which every vector is K-finite.

If $(\pi,\tilde{V})$ is a representation of G, then $\tilde{V}_{(K)} = V$ is a Harish-Chandra module for G. If V is a Harish-Chandra module for G, then any representation $(\pi,\tilde{V})$ of G such that $\tilde{V}_{(K)} = V$ is called a **globalization** of V. Globalizations always exist.

We are interested in a particular functorial globalization, the **maximal globalization** V_{max} [25]. It has the properties

(4.1) If $\tilde{V}$ is any globalization of V then $\tilde{V}_{(K)} \to V$ induces a continuous inclusion of $\tilde{V}$ into V_{max}.

(4.2) $V \to V_{max}$ is an exact functor .

If $\tilde{V}$ is a globalization on a reflexive Banach space, $\tilde{V}'$ is the dual Banach space, and $(\tilde{V}')^{\omega}$ is the space of analytic vectors in $\tilde{V}'$, then we define the space $\tilde{V}^{-\omega}$ of **hyperfunction vectors** to be the strong topological dual of $(\tilde{V}')^{\omega}$.

(4.3) If $\tilde{V}$ is a globalization on a reflexive Banach space then $\tilde{V}^{-\omega} \to V_{max}$ is an isomorphism of topological vector spaces.

The construction itself is straightforward. Let $\{v_i'\}_{1\le i\le n}$ generate the dual Harish-Chandra module $V' = V^*_{(K)}$. Map $v \in V$ to the n tuple $(f_{v_i',v}) \in C^{\infty}(G)^n$ of matrix coefficients, $f_{v_i',v}(x) = \langle v_i', \pi(x)v\rangle$. This injects V into $C^{\infty}(G)^n$. V_{max} is defined as the completion of V in the induced topology. Since any two choices of finite generating sets for V' differ by a $\mathcal{U}(\mathfrak{g})$-valued matrix, V_{max} is well defined.

5. DIFFERENTIAL-GEOMETRIC METHODS

Using the maximal globalization we can describe some variations ([26],[27]) on the classical construction (2.5) that will solve both the acyclicity problem and the closed range problem.

First note that the operator of (2.4) works perfectly well with hyperfunction sections, giving us a complex

(5.1) $\{C^{-\omega}(G/H, \mathbf{E} \otimes \Lambda^{\bullet}\mathbf{N}^*), d_{\mathfrak{n}}\}$: hyperfunction version of (2.4) .

The pullback to G is the hyperfunction version $\{C^{-\omega}(G) \otimes E \otimes \Lambda^{\bullet}\mathfrak{n}^*)\ )^H, d\}$ of (2.5). The point of hyperfunctions here, at least at first glance, is to get around the closed range problem. There is, however, a more serious point as well, which we will see in the next section, that the cohomology resulting from (5.1) will be the maximal globalization of its underlying Harish-Chandra module.

Let X denote the **flag variety** of Borel subalgebras of $\mathfrak{g}$. X is a compact complex manifold, in fact is a complex projective variety. The real group G acts on X by conjugation and there are only finitely many orbits [33]. Let S denote the orbit $G\cdot\mathfrak{b} \subset X$. As in the principal series example in Section 2, the bundle $\mathbf{E} \to G/H$ pushes down to a bundle $\mathbf{E} \to S$. There it has a Cauchy-Riemann structure from the Cauchy-Riemann structure on S that is induced by the complex structure on X. Let $\mathbf{N}_S \to S$ denote the antiholomorphic tangent bundle of S, that is, the part of the antiholomorphic tangent bundle of X that is tangent to S. It has typical fibre represented by $\mathfrak{n}/\mathfrak{n}\cap\bar{\mathfrak{n}}$. Let $\bar{\partial}_S$ denote the Cauchy-Riemann operator for S and Cauchy-Riemann bundles over S; it is the part of the $\bar{\partial}$ operator of X that involves differentiations only in the $\mathbf{N}_S$ directions. Now we have the Cauchy-Riemann complex for $\mathbf{E} \to S$ with hyperfunction coefficients:

(5.2) $\{C^{-\omega}(S, \mathbf{E} \otimes \Lambda^{\bullet}\mathbf{N}_S^*), \bar{\partial}_S\}$: hyperfunction Cauchy-Riemann complex .

In effect, (5.2) will make it possible to describe a topology on hyperfunction forms despite the fact that G/H is generally noncompact.

One can spread (5.2) out a little bit inside X as follows. $\mathbf{E} \to S$ extends uniquely to a $\mathfrak{g}$-equivariant holomorphic vector bundle $\tilde{\mathbf{E}} \to \tilde{S}$ where $\tilde{S}$ is a germ of a neighborhood of S in X. Then we have the usual Dolbeault cohomology of $\tilde{\mathbf{E}} \to \tilde{S}$. That, however, loses track of some of the structure of S inside X. So, instead, we look at Dolbeault cohomology with coefficients that are hyperfunctions on $\tilde{S}$ (or any open neighborhood of S, even all of X) with support in S:

(5.3) $\{C_S^{-\omega}(\tilde{S}, \tilde{\mathbf{E}} \otimes \Lambda^{\bullet}\mathbf{N}_{\tilde{S}}^*), \bar{\partial}\}$: hyperfunction local Dolbeault complex .

It turns out that one needs (5.3) for technical reasons: one can calculate the infinitesimal character of G in its action on the cohomology of (5.3). Note that $H^q(C_S^{-\omega}(\tilde{S}, \tilde{\mathbf{E}} \otimes \Lambda^{\bullet}\mathbf{N}_{\tilde{S}}^*), \bar{\partial})$ is just the local cohomology $H_S^{q+c}(\tilde{S}, \mathcal{O}(\tilde{\mathbf{E}}))$ where c is the real codimension of S in X.

6. EQUIVALENCE OF DIFFERENTIAL-GEOMETRIC AND HOMOLOGICAL METHODS

The result here is

6.1. Theorem (Schmid-Wolf [27]). *There are canonical* G-*equivariant isomorphisms between the cohomologies of the complexes* (5.1), (5.2) *and* (5.3) *for the same basic datum* $(H,\mathfrak{b},X)$,

$$H^q(C^{-\omega}(G/H,\ \tilde{\mathbf{E}}\otimes\Lambda^\bullet\mathbf{N}^*),\ d_{\mathfrak{n}}),\quad H^q(C^{-\omega}(S,\ \mathbf{E}\otimes\Lambda^\bullet\mathbf{N}_S^*),\ \bar\partial_S)\ ,\quad \textit{and}$$

$$H^q(C_S^{-\omega}(\tilde S,\ \tilde{\mathbf{E}}\otimes\Lambda^\bullet\mathbf{N}_{\tilde S}^*),\ \bar\partial) = H_S^{q+c}(\tilde S,\ \mathcal{O}(\tilde{\mathbf{E}})),\quad c = \mathrm{codim}_R(S \text{ in } X)\ .$$

These cohomologies carry natural Fréchet topologies such that the action of G *is continuous. The resulting representations of* G *are canonically and topologically isomorphic to the representation of* G *on* $A^q(H,\mathfrak{b},X)_{max}$, *the maximal globalization of the Zuckerman derived functor module* (3.2) *defined by the same basic datum.*

As mentioned above, the complex (5.1) is closest to what one expects for geometric quantization, while (5.2) carries the topology of and (5.3) gives access to the infinitesimal character.

The topological part of the theorem must be understood in one of two equivalent ways. First, the cohomology of the Cauchy-Riemann complex (5.2) can be calculated from a certain subcomplex that has a natural Fréchet topology in which $\bar\partial_S$ has closed range. Thus $H^q(C^{-\omega}(S,\ \mathbf{E}\otimes\Lambda^\bullet\mathbf{N}^*),\ \bar\partial_S)$ inherits a Fréchet topology, and natural isomorphisms carry the topology over to the cohomologies of (5.1) and (5.3). This makes the statement precise. Second, the topology is determined by the underlying Harish-Chandra module $A^q(H,\mathfrak{b},X)$ because [25] the topology on its globalization $A^q(H,\mathfrak{b},X)_{max}$ can be defined in purely algebraic terms.

The subcomplex of (5.2) to which I alluded to just above, is given as follows. Let P be the cuspidal parabolic subgroup of G mentioned in Section 2. Then S fibres over G/P, and in (5.2) one restricts the hyperfunction coefficients to hyperfunctions that are C^∞ along the fibres of $S \to G/P$. That defines a subcomplex $\{C_{G/P}^{-\omega}(S,\ \mathbf{E}\otimes\Lambda^\bullet\mathbf{N}_S^*),\ \bar\partial_S\}$ of (5.2). The inclusion is isomorphism on cohomology. G/P is compact so hyperfunctions on G/P do not have a natural topology. The C^∞ objects along the fibre are Fréchet. That leads to the topology on $H^q(C^{-\omega}(S,\ \mathbf{E}\otimes\Lambda^\bullet\mathbf{N}_S^*),\ \bar\partial_S)$.

7. EQUIVALENCE OF HOMOLOGICAL AND ALGEBRAIC-GEOMETRIC METHODS

The complexified group† $G_{\mathbb{C}}$ acts on the flag variety X of Borel subalgebras of $\mathfrak{g}$. $K_{\mathbb{C}}$ acts with only finitely many orbits [33]. Bernstein showed that every orbit is affinely embedded in X (see [15]). Comparing the description [31] of the G-orbits on X with the description ([19] or [33]) of the $K_{\mathbb{C}}$-orbits on X, one sees‡ that the relation

$$\text{a } G\text{-orbit } S \subset X \text{ and a } K_{\mathbb{C}}\text{-orbit } Q \subset X \text{ are } \textbf{dual} \text{ if } S \cap Q \text{ is a } K\text{-orbit} \tag{7.1}$$

defines a one-to-one order-reversing correspondence between G-orbits and $K_{\mathbb{C}}$-orbits on X.

Recall the bundle $\mathbf{E} \to S$ specified by the basic datum $(H,\mathfrak{b},X)$. Let Q be the $K_{\mathbb{C}}$-orbit dual to S. We may, and do, assume that $\mathfrak{b} \in S \cap Q$. Then $\mathbf{E} \to S$ restricts to $S \cap Q$ and extends to an algebraic $K_{\mathbb{C}}$ homogeneous vector bundle over Q. That extension is unique provided that one includes the $\mathcal{D}$-module structure defined by the basic datum $(H,\mathfrak{b},X)$.

See [2] for the original work on $\mathcal{D}$-modules, Bernstein's University of Maryland lectures for the first exposition, [7] for a more recent exposition of the general theory of $\mathcal{D}$-modules, and [20] for the basic applications of $\mathcal{D}$-module theory to representation theory.

For simplicity of exposition suppose that $G \subset G_{\mathbb{C}}$, so H is commutative, and assume that X is irreducible. Then $X = \exp(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Write $\mathbf{E}_\lambda$ for $\mathbf{E}$, ρ for half the sum of the positive roots. Now $\mathcal{U}(\mathfrak{g})$ specifies a sheaf $\mathcal{D}_{\lambda+\rho} \to X$ of twisted differential operators that act on $\mathbf{E}_\lambda \to Q$. The $\mathcal{D}_{\lambda+\rho}$-module direct image sheaf $j_+\mathcal{O}_Q(\mathbf{E}_\lambda \to Q)$

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†The centralizer of G^0 in G acts trivially on X, so G acts on X as if it were a linear group. We can view $G_{\mathbb{C}}$ as the complexification of that linear group $Ad(G)$, still acting on X of course, and $K_{\mathbb{C}}$ as the complex analytic subgroup of $G_{\mathbb{C}}$ such that $Ad_G(K)$ is contained in $K_{\mathbb{C}}$ and meets every topological component. See [15] for details. Here, in order to avoid technicalities that are essentially irrelevant, we will speak as if G were a linear group, $G \subset G_{\mathbb{C}}$, with $K_{\mathbb{C}} = K \cdot K_{\mathbb{C}}^0$, $K_{\mathbb{C}}^0$ connected with Lie algebra $\mathfrak{k}$.

‡These descriptions and the duality were given for connected groups G, but the passage to our case presents no difficulty. See [15] for details.

is a $K_{\mathbf{C}}$-invariant sheaf of $\mathcal{D}_{\lambda+\rho}$-modules. Its cohomologies give us a family of Harish-Chandra modules $H^q(X, j_+\mathcal{O}_Q(E_\lambda \to Q))$.

The result here is

7.2. Theorem (Hecht-Miličić-Schmid-Wolf [15]). *Let* $s = \dim_R(S \cap Q) - \dim_{\mathbf{C}} S$ *and* $X = \exp(\lambda)$. *Then* $A^q(H, \mathfrak{h}, X)$ *and* $H^{s-q}(X, j_+\mathcal{O}_Q(\mathbf{E}_{-\lambda-2\rho} \to Q))$ *are canonically dual in the category of Harish-Chandra modules.*

The pairing of (7.2) can be described intuitively as follows. Using (3.3) we can view the Zuckerman module $A^q(H, \mathfrak{h}, X)$ as cohomology of $\mathbf{E} \to S \cap Q$ with function coefficients in the S-directions transversal to $S \cap Q$. Using the definition of the $\mathcal{D}_{-\lambda-\rho}$-module $\mathcal{O}_Q(\mathbf{E}_{-\lambda-2\rho} \to Q)$ we can view the Beilinson-Bernstein modúle $H^{s-q}(X, j_+\mathcal{O}_Q(\mathbf{E}_{-\lambda-2\rho} \to Q)$ as cohomology of $\mathbf{E}_{-\lambda-2\rho} \to S \cap Q$ with differential operator coefficients in the Q-directions transversal to $S \cap Q$. The idea is to pair the two by pairing the differential operators against the functions obtaining a real differential form in degree $\dim_R(S \cap Q)$, which we then integrate over the compact manifold $S \cap Q$. The proof [15] has to be somewhat more technical, but given the result (7.2) this description is valid [27].

The result (7.2) has a number of interesting consequences. The basic point is that many things are easy from one of the Zuckerman or Beilinson-Berstein viewpoints and difficult or previously unknown from the other. For example, the classification of irreducible Harish-Chandra modules and the irreducibility question for standard modules are relatively easy in the Beilinson-Bernstein picture [16] but quite difficult in the Zuckerman picture, the the Beilinson-Bernstein picture has made it possible [10] to extend the Knapp-Zuckerman classification of tempered representations from linear semisimple groups to general semisimple groups.

Between (6.1) and (7.2) we see that the methods of geometric quantization, Zuckerman derived functor modules, and Beilinson-Bernstein localization, are essentially equivalent.

8. ENVELOPING ALGEBRA METHODS

Enveloping algebra methods are another general sort of method for constructing and analyzing representations of semisimple Lie groups.

Enveloping algebra methods start with a finite dimensional (usually irreducible) K-module F and construct Harish-Chandra modules V for G as quotients of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})^K \cdot \mathcal{U}(\mathfrak{k})} F$ where $\mathcal{U}(\mathfrak{g})^K$ is the centralizer of K in $\mathcal{U}(\mathfrak{g})$. If we decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under the Cartan involution then typically one may try to build up V by successively applying elements of $\mathfrak{p}$, i.e. as a quotient of the tensor product $\mathcal{S}(\mathfrak{p}) \otimes_{\mathbf{C}} F$ of the symmetric algebra with F. This method works best if other conditions are required. Important examples of such additional conditions are (i) unitarity of V with concrete choice of G (as in the work of Angelopoulos, Sijacki and others); (ii) a requirement that V be a highest weight $(\mathfrak{g},K)$-module (as in [8],[11],[12],[18] and [30]); (iii) unitarity and the stipulation that V come from a highest weight module by the derived functor construction as in [13]; and (iv) the condition that F be a minimal K-type of V in the sense of Vogan [29].

It is unfortunate that mathematicians and physicists using enveloping algebra methods do not communicate well. There is some psychological reason for this in that physicists tend to work with specific groups and often see no reason to have a lot of general machinery, while mathematicians tend to feel that the theory is defective unless it can treat all semisimple Lie groups. Nevertheless, both do roughly the same thing with enveloping algebra methods, and so communication certainly is possible.

In [9], Vogan's minimal K-type classification of irreducible Harish-Chandra modules is put into correspondence with the Beilinson-Bernstein classification. In view of the discussion leading up to (7.2) it thus can be formulated in terms of basic data $(H,\mathfrak{h},X)$. In view of the results described in (6.1) and (7.2) it also is in correspondence with the Langlands classification and with the Vogan-Zuckerman classification. Now it should be feasible, and it certainly would be interesting, to translate the parameterizations in some important classifications (perhaps Sijacki's classifications of the unitary dual of the double cover of SL(n;R) and Angelopoulos' classification for SO(p,q)) into the parameterization of the basic datum $(H,\mathfrak{h},X)$. And that certainly would be a positive step in improving communication between group theoretical mathematicians and physicists.

9. SPECIAL METHODS

By "special method" I mean a method for constructing representations that that has somewhat restricted validity. Howe's theory of dual reductive pairs applies to semisimple groups of classical type. The Kostant-Sternberg-Blattner method of moving polarizations ([4],[5],[6],[17]) applies to the Segal-Shale-Weil oscillator representation of the double cover of the real symplectic group [17] and to a ladder representation of the double cover of SL(3;R) [22]. The Rawnsley-Schmid-Wolf method of indefinite harmonic forms applies to certain semisimple groups G for which G/K is an hermitian symmetric space. It should be possible to reformulate these methods, at least the second and the third, in terms of general methods and special circumstances.

REFERENCES

[1] R. Aguilar-Rodriguez, Connections between representations of Lie groups and sheaf cohomology, Ph.D. thesis, Harvard (1987).

[2] A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris 292 (1981), 15-18.

[3] A. Beilinson and J. Bernstein, A generalization of Casselman's submodule theorem. In *Representation Theory of Reductive Groups*, Progress in Math 40 (Birkhauser, 1983), 35-52.

[4] R. Blattner, Pairing of half form spaces. In *Geometrie Symplectique et Physique Mathematique*, Editions C.N.R.S. (1975), 175-186.

[5] R. Blattner, Intertwining operators and the half-density pairing. In *Noncommutative Harmonic Analysis*, Lecture Notes in Mathematics 466 (Springer, 1975), 1-12.

[6] R. Blattner, The metalinear geometry of non-real polarizations. In *Differential Geometrical Methods in Mathematical Physics*, Lecture Notes in Mathematics 570 (Springer, 1977), 11-45.

[7] A. Borel et al., *Algebraic $\mathcal{D}$-Modules*, (Academic Press, 1986).

[8] A. L. Carey and K. C. Hannabus, Twistors and geometric quantisation theory, Rep. Math. Phys. 13 (1978), 199-231.

[9] J.-T. Chang, Special K-types and the Beilinson-Bernstein realization, Ph.D. thesis, Harvard (1985).

[10] J.-T. Chang, Special K-types, tempered characters and the Beilinson-Bernstein realization, Duke Math. J., to appear.

[11] T. J. Enright, R. Howe and N. R. Wallach, A classification of unitary highest weight modules. In *Representation Theory of Reductive Groups* (Birkhauser, 1983), 97-143.

[12] T. J. Enright and J. A. Wolf, Continuation of unitary derived functor modules out of the canonical chamber. In *Analyse Harmonique sur les Groupes de Lie et les Espaces Symmetriques*, Mem. Soc. Math. de France 15 (1984), 139-156.

[13] T. J. Enright, R. Parthasarathy, N. R. Wallach and J. A. Wolf, Unitary derived functors with small spectrum, Acta Math. 154 (1985), 105-136.

[14] A. Frölicher, Zur Differentialgeometrie der komplexen Structuren, Math. Ann. 129 (1955), 50-95.

[15] H. Hecht, D. Miličić, W. Schmid and J. A. Wolf, Localization and standard modules for real semisimple Lie groups, I: The duality theorem, Inventiones Math. 90 (1987), 297-332.

[16] H. Hecht, D. Miličić, W. Schmid and J. A. Wolf, Localization and standard modules for real semisimple Lie groups, II: Vanishing and irreducibility theorems, to appear.

[17] B. Kostant, Symplectic spinors, Symposia Math. 14 (1974), 139-152.

[18] G. Mack and I. Todorov, Irreducibility of the ladder representations of U(2,2) when restricted to the Poincaré subgroup, J. Math. Phys. 10 (1969), 2078-2085.

[19] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan 31 (1979), 332-357.

[20] D. Miličić, *Localization and Representation Theory of Reductive Lie Groups*, book to appear.

[21] J. H. Rawnsley, A non-unitary pairing of polarizations for the Kepler problem, Trans. Amer. Math. Soc. 250 (1979), 167-180.

[22] J. H. Rawnsley and S. Sternberg, On representations associated to the minimal nilpotent coadjoint orbit of SL(3,R), Amer. J. Math. 104 (1982), 1153-1180.

[23] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Ph.D. thesis, Berkeley (1967).

[24] W. Schmid, L^2 cohomology and the discrete series, Ann. of Math. (2) 103 (1976), 375-394.

[25] W. Schmid, Boundary value problems for group invariant differential equations, Proc. Cartan Symposium, Lyon 1984 (Asterix, 1985), 311-322.

[26] W. Schmid and J. A. Wolf, Globalizations of Harish-Chandra modules, Bull. Amer. Math. Soc. (NS) 17 (1987), 117-120.

[27] W. Schmid and J. A. Wolf, Geometric quantization and derived functor modules for semisimple Lie groups, to appear.

[28] J. A. Tirao and J. A. Wolf, Homogeneous holomorphic vector bundles, Indiana Univ. Math. J. 20 (1970), 15-31.

[29] D. Vogan, *Representations of Real Reductive Lie Groups*, Progress in Mathematics 15 (Birkhauser, 1981).

[30] I. Todorov, Discrete series of hermitian representations of the Lie algebra of U(p,q), Lecture Notes I.C.T.P., Trieste, 1975.

[31] J. A. Wolf, The action of a real semisimple group on a complex flag manifold, I: Orbit structure and holomorphic arc components, Bull. Amer. Math. Soc. 75 (1969), 1121-1237.

[32] J. A. Wolf, The action of a real semisimple group on a complex flag manifold, II: Unitary representation on partially holomorphic cohomology spaces, Memoirs Amer. Math. Soc. 138 (1974).

[33] J. A. Wolf, Finiteness of orbit structure for real flag manifolds, Geometriae Dedicata 3 (1974), 619-628.