JOURNAL OF FUNCTIONAL ANALYSIS 73, 1–37 (1987)

Wave Packets for the Relative Discrete Series I. The Holomorphic Case

REBECCA A. HERB*

Department of Mathematics, University of Maryland, College Park, Maryland 20742

AND

JOSEPH A. WOLF†

Department of Mathematics, University of California, Berkeley, California 94720

Communicated by the Editors

Received November 4, 1985

Contents. 1. Introduction, 2. Bounded symmetric domains, 3. Holomorphic relative discrete series, 4. Rapidly decreasing functions, 5. Matrix coefficients, 6. Universal cover of $SU(1, 1)$, 7. Wave packets.

1. INTRODUCTION

For a connected semisimple Lie group G with finite center, the discrete series representations play a central role in the decomposition of the regular representation on $L_2(G)$. Similarly, the space ${}^0\mathscr{C}(G)$ of cusp forms, which is spanned by the K-finite matrix coefficients of discrete series representations, is crucial in describing the Schwartz space $\mathscr{C}(G)$.

For G an arbitrary connected semisimple Lie group, the relative discrete series plays a role analogous to that of the discrete series in decomposing $L_2(G)$. See [3, 4, 9]. However, when G has infinite center the matrix coefficients of the relative discrete series are not Schwartz class. In this case the relative discrete series forms continuous families of representations, and it is necessary to form wave packets of matrix coefficients along the continuous parameter, analogous to those defined by Harish-Chandra for prin-

* Alfred P. Sloan Research Fellow and Member of the Mathematical Sciences Research Institute. Partially supported by the NSF under Grant DMS-8401374.

† Research partially supported by the Miller Institute for Basic Research in Science and by the NSF under Grant DMS-8200235.

0022-1236/87 $3.00

cipal series representations, to obtain Schwartz class functions. In this paper we construct these wave packets for holomorphic relative discrete series of the simply connected, simple Lie groups of hermitian type and show that they are Schwartz functions, in fact, cusp forms.

Suppose G is a connected, simply connected, simple Lie group of hermitian type, K a maximal compactly embedded subgroup. The holomorphic discrete series of G are parametrized by certain irreducible unitary representations $\chi \in \hat{K}$. The representation π_χ of G corresponding to (χ, V_χ) can be realized on the space of holomorphic sections of the vector bundle $\mathbf{V}_\chi = G \times_K V_\chi$. There is an extension of χ to a function $\chi: G \to GL(V_\chi)$ so that holomorphic sections f of $\mathbf{V}_\chi$ are in bijective correspondence to holomorphic functions $F: G/K \to V_\chi$ via

$$f(g) = \chi(g)^{-1} F(gK), \qquad g \in G. \tag{1.1}$$

Now $K = [K, K] \times Z_K^0$, where Z_K^0 is a one-dimensional real vector group, so that the representations of K form one-parameter families of the form $\chi_h = \chi_0 \otimes e^{h\nu}$, $h \in \mathbf{R}$, where e^ν is a nontrivial character of Z_K^0. We choose χ_0 and e^ν so that the χ_h correspond to holomorphic discrete series representations of G for $h > 0$. Now given a K-finite holomorphic function $F: G/K \to V_{\chi_0}$, we can define a one-parameter family of K-finite sections of the bundles $\mathbf{V}_{\chi_h}$ by

$$f_h(g) = \chi_h(g)^{-1} F(gK), \qquad g \in G, \quad h > 0. \tag{1.2}$$

Given two such one-parameter families of sections f_h and f'_h, we form a one-parameter family of K-finite matrix coefficients for the representations $\pi_h = \pi_{\chi_h}$ by

$$\phi_h(g) = \langle \pi_h(g) f_h, f'_h \rangle, \qquad g \in G, \quad h > 0. \tag{1.3}$$

Finally, given $\alpha \in \mathscr{C}(\mathbf{R}^+)$, a suitable space of smooth functions on $\mathbf{R}^+ = (0, \infty)$ which are rapidly decreasing at both 0 and ∞, we define

$$\phi_\alpha(g) = \int_0^\infty \alpha(h)\, \phi_h(g)\, dh, \qquad g \in G. \tag{1.4}$$

These are the wave packets of matrix coefficients. We show that they are in the Schwartz space $\mathscr{C}(G)$ of rapidly decreasing functions on G defined in [4] and that they are in fact cusp forms in the sense of Harish-Chandra.

Write $G = KAK$ in its Cartan decomposition. Standard growth estimates for discrete series matrix coefficients were extended to our situation in [4] to show that any K-finite matrix coefficient of a relative discrete series representation is rapidly decreasing along A. However, since K is not compact, and in fact contains a direct factor $Z_K^0 \cong \mathbf{R}$, this does not imply that they are rapidly decreasing on G. In fact, any such matrix coefficient transforms by a unitary character along the infinite cyclic group $Z = Z_G \cap Z_K^0$ of integral points in Z_K^0 so that it cannot vanish at infinity along Z_K^0.

In order to show that the wave packets ϕ_α defined in (1.4) are rapidly decreasing on G it is necessary to estimate $\phi_h(a)$ and all its derivatives with respect to h as functions of both $a \in A$ and $h > 0$. This cannot be done with the standard estimates obtained from the differential equations satisfied by $\phi_h(a)$ since they give no information on how the solutions grow as h goes to infinity. Instead we compute directly and obtain a formula (Theorem 5.1) for $\phi_h(a)$ which shows the dependence on h explicitly, modulo a term which is polynomial in h. Using this formula it is easy to obtain the necessary estimates. Important motivation for these results was provided by the explicit formulas for discrete series matrix coefficients for the universal covering group of $SL(2, \mathbf{R})$ obtained by Sally in [6].

In a second paper we define wave packets for the non-holomorphic relative discrete series. There we do not have explicit formulas for the one-parameter families of K-finite matrix coeficients $\phi_h(x)$. However, the parameter h lies in a bounded interval, and we are able to use the theory of asymptotics coming from the differential equations to obtain estimates that are uniform in the parameter h. It is interesting that, to handle the entire relative discrete series, we need both the theory of asymptotics for the non-holomorphic case, where explicit formulas are not available, and the explicit formulas for the holomorphic case, where we must control growth at infinity. In that second paper we also prove that finite sums of wave packets obtained in these two ways form a dense subspace of the space of cusp forms.

In Section 2 we describe the holomorphic trivialization of the bundles $\mathbf{V}_\chi$ corresponding to holomorphic relative discrete series. We also define a holomorphic local group containing G which plays the role that the complexification $G_\mathbf{C}$ of a real linear group $G_\mathbf{R}$ plays in the standard theory of holomorphic discrete series. The main results for this local group are summarized in Theorem 2.17.

In Section 3 we organize the holomorphic relative discrete series into one-parameter families and work out explicit formulas for the one-parameter families of K-finite sections.

In Section 4 we discuss the global and relative Schwartz spaces for G, and the corresponding spaces of cusp forms.

The main results of Section 5 are Theorem 5.1, which gives a formula for the matrix coefficients, and its corollary, which gives the estimates necessary to form wave packets.

In Section 6 we illustrate the results of Section 5 when G is the universal covering group of $SU(1, 1)$. The reader may prefer to read Section 6 before Section 5.

Finally, in Section 7 we define our wave packets and prove that they are Schwartz functions and cusp forms. These results are stated in Theorem 7.3.

2. Bounded Symmetric Domains

In this section we study an irreducible bounded symmetric domain D as homogeneous space G/K of the universal cover of its analytic automorphism group. We recall the Harish-Chandra embedding and factor of automorphy for D, based on certain structural facts

$$G_{\mathbf{R}} \subset P_+ K_{\mathbf{C}} P_- \subset G_{\mathbf{C}}$$

for a linear quotient group of G, and lift this picture up to G itself. The analogous structural basis

$$G \subset P_+ \tilde{K}_{\mathbf{C}} P_-$$

is given in Theorem 2.17. The technical point here is that G does not have a complexification per se. This problem is met by introducing a holomorphic local group structure on the universal cover $P_+ \tilde{K}_{\mathbf{C}} P_-$ of $P_+ K_{\mathbf{C}} P_-$.

Of course we depend very much on the fundamental results of Harish-Chandra [0, 1] for the holomorphic discrete series. But our approach and objective are rather different.

G is a connected, simply connected, simple Lie group of hermitian type. Fix a Cartan involution θ; so the fixed point set $K = G^\theta$ is a maximal compactly embedded subgroup of G, and its center Z_K is the direct product of a finite abelian group with a one-dimensional real vector group Z_K^0. We write $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{z}_K$ for the corresponding real Lie algebras.

Since G is of hermitian type, K is the centralizer of Z_K^0, and in particular G has a Cartan subgroup $T \subset K$. Note that T and K are connected, $K = [K, K] \times Z_K^0$ and $T = \{T \cap [K, K]\} \times Z_K^0$.

The root system $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ has a positive subsystem Φ^+ with the following properties. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, ± 1 eigenspaces of θ, as usual, and call a root $\gamma \in \Phi$ *compact* if the root space $\mathfrak{g}_\gamma \subset \mathfrak{k}_{\mathbf{C}}$, *noncompact* if $\mathfrak{g}_\gamma \subset \mathfrak{p}_{\mathbf{C}}$. Then there is a unique noncompact simple root, say α_0, and $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where $\mathfrak{p}_\pm$ is the sum of the root spaces for roots in which the coefficient (when the root is expressed as a linear combination of simple roots) of α_0 is ± 1. This corresponds to the choice of invariant complex structure on G/K such that $\mathfrak{p}_+$ represents the holomorphic tangent space.

We need to see this complex structure directly. Let $G_{\mathbf{C}}$ denote the connected simply connected complex Lie group for $\mathfrak{g}_{\mathbf{C}}$. Let $G_{\mathbf{R}}$, $K_{\mathbf{R}}$, $K_{\mathbf{C}}$, P_+ and P_- denote the analytic subgroups for $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_{\mathbf{C}}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$. Then $X = G_{\mathbf{C}}/K_{\mathbf{C}} P_-$ is a complex flag manifold. Let $q: G \to G_{\mathbf{R}}$ denote the (universal) covering induced by $\mathfrak{g} \subsetneq \mathfrak{g}_{\mathbf{C}}$. Then $K = q^{-1}(K_{\mathbf{R}})$, $q: K \to K_{\mathbf{R}}$ is a covering, $G_{\mathbf{R}} \subset P_+ K_{\mathbf{C}} P_-$ and $G_{\mathbf{R}} \cap (K_{\mathbf{C}} P_-) = K_{\mathbf{R}}$; so $G/K = G_{\mathbf{R}}/K_{\mathbf{R}}$ is an open $G_{\mathbf{R}}$-orbit in X. That is the "Borel embedding." Furthermore, define

$$\zeta_\pm : P_+ K_{\mathbf{C}} P_- \to \mathfrak{p}_\pm, \qquad p_\pm : P_+ K_{\mathbf{C}} P_- \to P_\pm, \qquad \bar{\kappa}: P_+ K_{\mathbf{C}} P_- \to K_{\mathbf{C}} \tag{2.1a}$$

by

$$x = p_+(x) \cdot \bar{\kappa}(x) \cdot p_-(x) \qquad \text{and} \qquad p_\pm(x) = \exp \zeta_\pm(x). \tag{2.1b}$$

We frequently write $\zeta(x)$ for $\zeta_+(x)$. If $g \in G$ we also write $\zeta_\pm(g)$ for $\zeta_\pm(q(g))$, $p_\pm(g)$ for $p_\pm(q(g))$, and $\bar{\kappa}(g)$ for $\bar{\kappa}(q(g))$.

$\zeta: G \to \mathfrak{p}_+$ induces a holomorphic diffeomorphism of G/K onto a bounded domain D in the complex vector space $\mathfrak{p}_+$. That is the "Harish-Chandra embedding."

EXAMPLE 2.2. Let $G_{\mathbf{R}} = SU(m, n)$. It consists of all block form matrices $\left(\begin{smallmatrix} U & X \\ Y & V \end{smallmatrix}\right)$ of determinant 1 such that $UY^* = XV^*$, $UU^* - XX^* = I_m$ and $VV^* - YY^* = I_n$. Here $G_{\mathbf{R}} \subset P_+ K_{\mathbf{C}} P_-$ is given by

$$\begin{pmatrix} U & X \\ Y & V \end{pmatrix} = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ W & I \end{pmatrix} = \begin{pmatrix} A + ZBW & ZB \\ BW & B \end{pmatrix}$$

so if $q(g) = \left(\begin{smallmatrix} U & X \\ Y & V \end{smallmatrix}\right)$ then $\zeta(g)$ can be identified with $Z = XV^{-1}$ in the space of all complex $m \times n$ matrices. This identifies D with

$$\{Z \in \mathbf{C}^{m \times n}: I_m - ZZ^* \gg 0\} = \{Z \in \mathbf{C}^{m \times n}: I_n - Z^*Z \gg 0\}$$

with the action of $SU(m, n)$ given by

$$\begin{pmatrix} U & X \\ Y & V \end{pmatrix}: Z \to (UZ + X)(YZ + V)^{-1}.$$

The factor of automorphy is given by

$$\bar{\kappa}(g) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} U - XV^{-1}Y & 0 \\ 0 & V \end{pmatrix}.$$

One can check that $\zeta(g)$ and $\bar{\kappa}(g)$ are related by $I_m - ZZ^* = AA^*$ and $I_n - Z^*Z = (BB^*)^{-1}$. (*End of example.*)

Let $q_K: \tilde{K}_{\mathbf{C}} \to K_{\mathbf{C}}$ denote the universal covering group. We may view $\tilde{K}_{\mathbf{C}}$ as the complexification of K, and then $q_K|_K = q|_K$. Since G is simply connected, $\bar{\kappa}: G \to K_{\mathbf{C}}$ has a unique lift

$$\kappa: G \to \tilde{K}_{\mathbf{C}} \qquad \text{such that} \qquad \kappa|_K: K \subsetneq \tilde{K}_{\mathbf{C}}. \tag{2.3}$$

κ is the *universal factor of automorphy*. Compare Tirao [7, p. 64].

Fix a (necessarily finite dimensional) irreducible unitary representation $\chi \in \hat{K}$. $V = V_\chi$ denotes the representation space, and

$$\mathbf{V} = \mathbf{V}_\chi = G \times_K V \to G/K$$

is the associated homogeneous holomorphic vector bundle. Here $G \times_K V$ is $G \times V$ modulo the relation $(gk, \chi(k)\, v) \sim (g, v)$, so sections correspond to functions $f: G \to V$ such that $f(gk) = \chi(k)^{-1} \cdot f(g)$. The section is holomorphic just when the right derivatives $f(g; \xi) = 0$ for all $\xi \in \mathfrak{p}_-$. This means that

$$\left.\frac{d}{dt}\right|_0 \{f(g \cdot \exp(t\xi_1)) + if(g \cdot \exp(t\xi_2))\} = 0$$

where $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in \mathfrak{g}$.

The universal factor of automorphy defines the *factor of automorphy* for $\mathbf{V}_\chi \to G/K$ as follows. First extend χ to a holomorphic representation $\chi: \tilde{K}_{\mathbf{C}} \to GL(V_\chi)$ of the complexification of K. Now define a function (not, of course, a representation)

$$\chi: G \to GL(V_\chi) \qquad \text{by} \quad \chi(g) = \chi(\kappa(g)). \tag{2.4}$$

That is the factor of automorphy. Note that $\chi(k)$, $k \in K$, retains its original meaning, because $\kappa(k) = k$. It gives a holomorphic trivialization of $\mathbf{V}_\chi \to G/K$ as follows. The functions $F: G/K \to V_\chi$ are in G-equivariant correspondence with the sections f of $\mathbf{V}_\chi \to G/K$ by

$$f(g) = \chi(g)^{-1} \cdot F(gK). \tag{2.5}$$

Since K normalizes $\mathfrak{p}_-$, f is a holomorphic section if and only if F is a holomorphic function.

We now lift the local group structure of the dense open subset $P_+ K_{\mathbf{C}} P_-$ in $G_{\mathbf{C}}$, to a local group structure on its universal cover, which we think of as $P_+ \tilde{K}_{\mathbf{C}} P_-$. This will be a key tool in our analysis of matrix coefficients. The result we need is Theorem 2.17 below.

The map $\mathfrak{p}_+ \times K_{\mathbf{C}} \times \mathfrak{p}_- \to P_+ K_{\mathbf{C}} P_-$, given by $(\xi, k, \eta) \to \exp(\xi) \cdot k \cdot \exp(\eta)$, is a holomorphic diffeomorphism. As $\tilde{K}_{\mathbf{C}}$ is simply connected,

$$\tilde{q} = (1 \times q_K \times 1): P_+ \times \tilde{K}_{\mathbf{C}} \times P_- \to P_+ K_{\mathbf{C}} P_- \tag{2.6}$$

is the universal covering space. It locally is a holomorphic diffeomorphism.

LEMMA 2.7. *Let* $x, y \in P_+ K_{\mathbf{C}} P_-$. *Then* $xy \in P_+ K_{\mathbf{C}} P_-$ *if and only if* $p_-(x) \cdot p_+(y) \in P_+ K_{\mathbf{C}} P_-$.

Proof. Write $xy = p_+(x) \cdot \bar{\kappa}(x) \cdot p_-(x) \cdot p_+(y) \cdot \bar{\kappa}(y) \cdot p_-(y)$. It is in $P_+ K_{\mathbf{C}} P_-$ if and only if $\bar{\kappa}(x) \cdot p_-(x) \cdot p_+(y) \cdot \bar{\kappa}(y) \in P_+ K_{\mathbf{C}} P_-$. Write the latter as $\mathrm{Ad}(\bar\kappa(x))\{p_-(x)\cdot p_+(y)\}\cdot\bar\kappa(x)\cdot\bar\kappa(y)$. As $\mathrm{Ad}(K_{\mathbf{C}})\{P_+K_{\mathbf{C}}P_-\}=P_+K_{\mathbf{C}}P_-$, and $P_+K_{\mathbf{C}}P_-K_{\mathbf{C}}=P_+K_{\mathbf{C}}P_-$, our assertion follows. Q.E.D.

Lemma 2.7 describes the local group structure. Note that $G_{\mathbf{R}}$, P_+, $K_{\mathbf{C}}$ and P_- are subgroups, and that $K_{\mathbf{C}}$ normalizes P_+ and P_- within $P_+K_{\mathbf{C}}P_-$. See Lemmas 2.11 and 2.14 below.

Write $^-$ for conjugation of $\mathfrak{g}_{\mathbf{C}}$ over $\mathfrak{g}$, $G_{\mathbf{C}}$ over $G_{\mathbf{R}}$. It interchanges $\mathfrak{p}_+$ and $\mathfrak{p}_-$, P_+ and P_-.

LEMMA 2.8. *The bounded symmetric domain*

$$D=\{\xi\in\mathfrak{p}_+ : g\in\exp(\xi)\cdot K_{\mathbf{C}}P_- \text{ for some } g\in G_{\mathbf{R}}\}$$

has complex conjugate

$$\bar D=\{\eta\in\mathfrak{p}_- : g\in P_+K_{\mathbf{C}}\cdot\exp(\eta) \text{ for some } g\in G_{\mathbf{R}}\}.$$

Proof. Let $\xi=\zeta_+(g)$, $g\in G_{\mathbf{R}}$. Then $g^{-1}=\overline{g^{-1}}\in G_{\mathbf{R}}$ and $\bar\xi=-\zeta_-(\overline{g^{-1}})$. Note $-D=D$ since $D=\mathrm{Ad}(Z_K^0)\,D=\{e^{i\theta}D\colon \theta \text{ real}\}$. Q.E.D.

LEMMA 2.9. *If $\xi,\xi'\in D$ then* $\exp(\bar\xi)\exp(\xi')\in P_+K_{\mathbf{C}}P_-$.

Proof. Let $g,g'\in G_{\mathbf{R}}$ such that $\bar\xi=\zeta_-(g)$ and $\xi'=\zeta_+(g')$. As $gg'\in G_{\mathbf{R}}\subset P_+K_{\mathbf{C}}P_-$, Lemma 2.7 says that $\exp(\bar\xi)\exp(\xi')\in P_+K_{\mathbf{C}}P_-$. Q.E.D.

LEMMA 2.10. *Let $(\xi,\xi')=-\langle\xi,\bar{\bar{\xi}}'\rangle$, the positive definite hermitian form on $\mathfrak{g}_{\mathbf{C}}$ invariant by the compact real form $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$, where $\eta\mapsto\bar{\bar\eta}=\theta(\bar\eta)$ is conjugation of $\mathfrak{g}_{\mathbf{C}}$ over $\mathfrak{g}_u$. Relative to $(\ ,\ )$, choose orthonormal bases $\{\xi_1,\dots,\xi_n\}$ of $\mathfrak{p}_+$, $\{\eta_1,\dots,\eta_m\}\subset\mathfrak{k}$ of $\mathfrak{k}_{\mathbf{C}}$, and $\{\bar{\bar\xi}_1,\dots,\bar{\bar\xi}_n\}$ of $\mathfrak{p}_-$. Write block form matrices for this basis of $\mathfrak{g}_{\mathbf{C}}$. If $\xi\in\mathfrak{p}_+$, then there is an $n\times m$ matrix Z such that*

$$\mathrm{ad}(\xi)=\begin{pmatrix}0&Z&0\\0&0&-{}^tZ\\0&0&0\end{pmatrix}\qquad\textit{and}\qquad \mathrm{ad}(\bar\xi)=\begin{pmatrix}0&0&0\\{}^t\bar Z&0&0\\0&-\bar Z&0\end{pmatrix}$$

where $\bar Z$ means the complex conjugate of Z.

Proof. Note that

$$\mathrm{ad}(\xi)=\begin{pmatrix}0&Z&0\\0&0&Z'\\0&0&0\end{pmatrix}\qquad\text{and}\qquad \mathrm{ad}(\bar\xi)=\begin{pmatrix}0&0&0\\W'&0&0\\0&W&0\end{pmatrix}$$

for some $n\times m$ matrices Z, W and $m\times n$ matrices Z', W'. Now $\mathrm{ad}(\xi)\,\eta_i=\sum z_{ij}\xi_j$ implies $\mathrm{ad}(\bar\xi)\,\eta_i=\overline{\mathrm{ad}(\xi)\,\eta_i}=\sum\overline{z_{ij}}\,\bar\xi_j=-\sum\overline{z_{ij}}\,\bar{\bar\xi}_j$, so

$W=-\bar{Z}$. Similarly $W'=-\bar{Z}'$. Check $\operatorname{ad}(\xi)^*=\operatorname{ad}(\bar{\xi})$ so that $(\operatorname{ad}(\xi)\,\bar{\bar{\xi}}_j, \eta_i)=(\bar{\bar{\xi}}_j, \operatorname{ad}(\bar{\xi})\,\eta_i)$, i.e., $W={}^t\bar{Z}'$. That does it. Q.E.D.

LEMMA 2.11. *Let $\xi, \xi' \in \mathfrak{p}_+$, say*

$$\operatorname{ad}(\xi)=\begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -{}^tZ \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \operatorname{ad}(\xi')=\begin{pmatrix} 0 & Z' & 0 \\ 0 & 0 & -{}^tZ' \\ 0 & 0 & 0 \end{pmatrix}$$

in the notation of Lemma 2.10. *Then* $\exp(\bar{\xi})\cdot\exp(\xi')\in P_+K_{\mathbf{C}}P_-$ *if and only if the $n\times n$ matrix* $\frac{1}{4}\bar{Z}\cdot{}^t\bar{Z}\cdot Z'\cdot{}^tZ'+\bar{Z}\cdot{}^tZ'+I$ *is invertible.*

Proof. It suffices to prove that $\operatorname{Ad}(\exp(\bar{\xi}))\cdot\operatorname{Ad}(\exp(\xi'))\in\operatorname{Ad}(P_+)\cdot\operatorname{Ad}(K_{\mathbf{C}})\cdot\operatorname{Ad}(P_-)$ if and only if $\frac{1}{4}\bar{Z}\cdot{}^t\bar{Z}\cdot Z'\cdot{}^tZ'+\bar{Z}\cdot{}^tZ'+I$ is invertible, where Ad means $\operatorname{Ad}_{G_{\mathbf{C}}}$ in every instance. To see this, we check that the conditions

(a) $\exp(\bar{\xi})\cdot\exp(\xi')\in P_+K_{\mathbf{C}}P_-$, and

(b) $\operatorname{Ad}(\exp(\bar{\xi}))\cdot\operatorname{Ad}(\exp(\xi'))\in\operatorname{Ad}(P_+)\cdot\operatorname{Ad}(K_{\mathbf{C}})\cdot\operatorname{Ad}(P_-)$

are equivalent. Clearly (a) implies (b). For the converse it suffices to show that $K_{\mathbf{C}}$ contains the kernel of Ad, which is the center of $G_{\mathbf{C}}$. As $K_{\mathbf{C}}P_-$ is a maximal parabolic subgroup of $G_{\mathbf{C}}$, $K_{\mathbf{C}}$ is the $G_{\mathbf{C}}$-centralizer of the circle group $\exp_{G_{\mathbf{C}}}(\mathfrak{z}_K)$, and thus contains the center of $G_{\mathbf{C}}$. Now (a) and (b) are equivalent.

Compute

$$\begin{aligned}
&\operatorname{Ad}(\exp\bar{\xi})\cdot\operatorname{Ad}(\exp\xi')\\
&=\begin{pmatrix} I & 0 & 0 \\ {}^t\bar{Z} & I & 0 \\ -\frac{1}{2}\bar{Z}\cdot{}^t\bar{Z} & -\bar{Z} & I \end{pmatrix}\begin{pmatrix} I & Z' & -\frac{1}{2}Z'\cdot{}^tZ' \\ 0 & I & -{}^tZ' \\ 0 & 0 & I \end{pmatrix}\\
&=\begin{pmatrix} I & Z' & -\frac{1}{2}Z'\cdot{}^tZ' \\ {}^t\bar{Z} & {}^t\bar{Z}\cdot Z'+I & -\frac{1}{2}{}^t\bar{Z}\cdot Z'\cdot{}^tZ'-{}^tZ' \\ -\frac{1}{2}\bar{Z}\cdot{}^t\bar{Z} & -\frac{1}{2}\bar{Z}\cdot{}^t\bar{Z}\cdot Z'-\bar{Z} & \frac{1}{4}\bar{Z}\cdot{}^t\bar{Z}\cdot Z'\cdot{}^tZ'+\bar{Z}\cdot{}^tZ'+I \end{pmatrix}.
\end{aligned}$$

This is in $\operatorname{Ad}(P_+K_{\mathbf{C}}P_-)$ if and only if it has expression

$$\begin{aligned}
&\begin{pmatrix} I & U & -\frac{1}{2}U\cdot{}^tU \\ 0 & I & -{}^tU \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ {}^t\bar{V} & I & 0 \\ -\frac{1}{2}\bar{V}\cdot{}^t\bar{V} & -\bar{V} & I \end{pmatrix}\\
&\quad=\begin{pmatrix} A+UB{}^t\bar{V}+\frac{1}{4}U\cdot{}^tU\cdot\bar{V}\cdot{}^t\bar{V} & UB+\frac{1}{2}U\cdot{}^tUC\bar{V} & -\frac{1}{2}U\cdot{}^tUC \\ B\cdot{}^t\bar{V}+\frac{1}{2}{}^tU\cdot C\cdot\bar{V}\cdot{}^t\bar{V} & B+{}^tUC\bar{V} & -{}^tUC \\ -\frac{1}{2}C\bar{V}\cdot{}^t\bar{V} & -C\bar{V} & C \end{pmatrix}.
\end{aligned}$$

If it has that expression, then

$$C = \mathrm{Ad}(\tilde{\kappa}((\exp \bar{\xi})(\exp \xi'))) |_{\mathfrak{p}_-} = \tfrac{1}{4}\bar{Z} \cdot {}^t\bar{Z} \cdot Z' \cdot {}^tZ' + \bar{Z} \cdot {}^tZ' + I$$

is invertible. Conversely, if the latter is invertible, we solve for U and V, then B, and finally A. Q.E.D.

LEMMA 2.12. *Let $\xi, \xi' \in \mathfrak{p}_+$. Suppose that there exists $r > 0$ such that $\xi \in rD$ and $\xi' \in (1/r) D$. Then* $\exp(\bar{\xi}) \cdot \exp(\xi') \in P_+ K_{\mathbf{C}} P_-$.

Proof. $(1/r)\,\xi, r\xi' \in D$, so Lemma 2.9 says that $\exp((1/r)\,\bar{\xi}) \cdot \exp(r\xi') \in P_+ K_{\mathbf{C}} P_-$, and now Lemma 2.11 says that

$$\tfrac{1}{4}\left(\frac{1}{r}\bar{Z}\right)\left(\frac{1}{r}{}^t\bar{Z}\right)(rZ')(r\,{}^tZ') + \left(\frac{1}{r}\bar{Z}\right)(r\,{}^tZ') + I$$

is invertible. All the r's cancel there, so Lemma 2.11 ensures that $\exp(\bar{\xi}) \cdot \exp(\xi') \in P_+ K_{\mathbf{C}} P_-$. Q.E.D.

We now define subsets of $P_+ K_{\mathbf{C}} P_-$ and $P_+ \times \tilde{K}_{\mathbf{C}} \times P_-$ by

$$\Omega = \{(x, y) \in (P_+ K_{\mathbf{C}} P_-) \times (P_+ K_{\mathbf{C}} P_-) \colon \exists r > 0 \text{ with } \zeta_-(x) \in r\bar{D} \text{ and } \zeta_+(y) \in (1/r)\, D\}; \tag{2.13a}$$

$$\tilde{\Omega} = (\tilde{q} \times \tilde{q})^{-1}(\Omega). \tag{2.13b}$$

LEMMA 2.14. *If $(x, y) \in \Omega$ then $xy \in P_+ K_{\mathbf{C}} P_-$. $\tilde{\Omega}$ is simply connected and $(\tilde{q} \times \tilde{q})\colon \tilde{\Omega} \to \Omega$ is the universal covering.*

Proof. For the first assertion, combine Lemmas 2.7 and 2.12.

Let $\sigma(t) = (\sigma_1(t), \sigma_2(t))$, $0 \leqslant t \leqslant 1$, be a closed curve in $\tilde{\Omega}$ starting at $(\tilde{1}, \tilde{1})$, where $\tilde{1} = (1, 1, 1) \in P_+ \times \tilde{K}_{\mathbf{C}} \times P_-$. If $0 \leqslant u \leqslant 1$ define $\sigma(u, t) = (\sigma_1(u, t), \sigma_2(u, t))$, where

$$\sigma_i(u, t) = (\exp((1-u)\,\zeta_+(\sigma_i(t))), \kappa(\sigma_i(t)), \exp((1-u)\,\zeta_-(\sigma_i(t)))).$$

For each u, $t \mapsto \sigma(u, t)$ again is a closed curve in $\tilde{\Omega}$ starting at $(\tilde{1}, \tilde{1})$. This gives a homotopy of the closed curve $\sigma(\cdot) = \sigma(0, \cdot)$ to the closed curve $\sigma(1, \cdot)$ in $\tilde{K}_{\mathbf{C}} \times \tilde{K}_{\mathbf{C}}$, which is simply connected. Thus σ is null-homotopic. Now $\tilde{\Omega}$ is simply connected. As

$$(\tilde{q} \times \tilde{q})\colon (P_+ \times \tilde{K}_{\mathbf{C}} \times P_-) \times (P_+ \times \tilde{K}_{\mathbf{C}} \times P_-) \to (P_+ K_{\mathbf{C}} P_-) \times (P_+ K_{\mathbf{C}} P_-)$$

is a covering and $\tilde{\Omega}$ is the inverse image of an open set, now $(\tilde{q} \times \tilde{q})\colon \tilde{\Omega} \to \Omega$ is the universal cover of Ω. Q.E.D.

LEMMA 2.15. *View G as a subset of $P_+ \times \tilde{K}_{\mathbf{C}} \times P_-$ under the injection $p_+ \times \kappa \times p_-$. Then $\tilde{q}\,|_G$ is the covering $q\colon G \to G_{\mathbf{R}}$ and $G \times G \subset \tilde{\Omega}$.*

Proof. The statement on $\tilde{q}\,|_G$ follows from (2.3) and (2.6). $D = \zeta_+(G)$ and $\bar{D} = \zeta_-(G)$ as a consequence of Lemma 2.8, so we have $G \times G \subset \tilde{\Omega}$. Q.E.D.

Multiplication $m\colon \Omega \to P_+ K_{\mathbf{C}} P_-$ is well defined by the first part of Lemma 2.14. In view of the rest of that lemma, there is a unique lift

$$\begin{array}{ccc} \tilde{\Omega} & \xrightarrow{\tilde{m}} & P_+ \times \tilde{K}_{\mathbf{C}} \times P_- \\ \downarrow{\scriptstyle \tilde{q} \times \tilde{q}} & & \downarrow{\scriptstyle \tilde{q}} \\ \Omega & \xrightarrow{m} & P_+ K_{\mathbf{C}} P_- \end{array} \tag{2.16}$$

such that $\tilde{m}(\tilde{1}, \tilde{1}) = \tilde{1}$.

THEOREM 2.17. *$(P_+ \times \tilde{K}_{\mathbf{C}} \times P_-, \tilde{m})$ is a complex analytic local group, which we denote $P_+ \tilde{K}_{\mathbf{C}} P_-$. G is a closed real analytic subgroup of $P_+ \tilde{K}_{\mathbf{C}} P_-$, and P_+, $\tilde{K}_{\mathbf{C}}$ and P_- are closed complex analytic subgroups. If $p \in P_\pm$ and $\tilde{k} \in \tilde{K}_{\mathbf{C}}$ then $\tilde{k}p = p'\tilde{k}$, where $p' = \mathrm{Ad}(\tilde{q}(\tilde{k})) \cdot p$. Finally,*

$$\tilde{q}\colon P_+ \tilde{K}_{\mathbf{C}} P_- \to P_+ K_{\mathbf{C}} P_-$$

is the universal local group covering.

Proof. Since $\tilde{m}$ is well defined, it must be given by lifting curves, as follows. If $(\tilde{x}, \tilde{y}) \in \tilde{\Omega}$ then there are curves $\{\tilde{x}_t\}$ from $\tilde{1}$ to $\tilde{x}$ and $\{\tilde{y}_t\}$ from $\tilde{1}$ to $\tilde{y}$ such that $\{(\tilde{x}_t, \tilde{y}_t)\}$ is a curve in $\tilde{\Omega}$ from $(\tilde{1}, \tilde{1})$ to $(\tilde{x}, \tilde{y})$. Now $\tilde{m}(\tilde{x}, \tilde{y})$ is the endpoint of the $\tilde{q}$-lift, starting at $\tilde{1}$, of the curve $\{\tilde{q}(\tilde{x}_t) \cdot \tilde{q}(\tilde{y}_t)\}$ in Ω. In particular, $\tilde{m}$ restricts to the original group laws on G, P_+, $\tilde{K}_{\mathbf{C}}$, P_-, and the semidirect products $\tilde{K}_{\mathbf{C}} \cdot P_\pm$. All are closed real analytic submanifolds, and all but G are complex submanifolds. The local group covering statement follows as well. Q.E.D.

We will need the local group result of Theorem 2.17 in order to avoid a monodromy problem in the proof of Lemma 5.4. That problem corresponds to the choice of a branch of the logarithm in the argument of Lemma 6.4.

3. HOLOMORPHIC RELATIVE DISCRETE SERIES

In this section we organize the holomorphic relative discrete series of G into one-parameter families along which we will later form our wave packets. We work out explicit formulas for the K-finite vectors in these one-parameter families and obtain some estimates that will be needed to analyze the wave packets.

Retain the set-up and notation of Section 2. In particular, $\chi \in \hat{K}$ and $\mathbf{V} = G \times_K V \to G/K$ is the associated holomorphic homogeneous vector bundle. Since χ is unitary, the fibres of $\mathbf{V} \to G/K$ are Hilbert spaces, so a section f has well defined pointwise norm $\| f(g) \| = (\langle f(g), f(g) \rangle_V)^{1/2}$. As usual, that gives a Hilbert space

$$L_2(G/K, \mathbf{V}) = \left\{ \text{measurable sections } f\colon \int_{G/K} \| f(g) \|^2 \, d(gK) < \infty \right\}$$

with global inner product $\langle f, f' \rangle = \langle f, f' \rangle_{G/K} = \int_{G/K} \langle f(g), f'(g) \rangle_V \, d(gK)$ and norm $\| f \| = \| f \|_{G/K} = \langle f, f \rangle^{1/2}$. G acts unitarily by $(g \cdot f)(x) = f(g^{-1}x)$. The subspace

$$H_2(G/K, \mathbf{V}) = \left\{ \text{holomorphic sections } f\colon \int_{G/K} \| f(g) \|^2 \, d(gK) < \infty \right\}$$

is a closed subspace. A famous result of Harish-Chandra [1] says that

$$H_2(G/K, \mathbf{V}) \neq 0 \Leftrightarrow \langle \lambda + \rho, \mu \rangle < 0 \tag{3.1}$$

where λ is the highest weight of χ, μ is the maximal root, and ρ is half the sum of the positive roots. When that condition holds, G acts on $H_2(G/K, \mathbf{V})$ by an irreducible unitary representation π_χ, which is the *holomorphic relative discrete series* representation of Harish-Chandra parameter $\lambda + \rho$.

We organize the holomorphic relative discrete series into *one-parameter families* as follows. Define $\nu \in i\mathfrak{z}_K^*$ by $2\langle \nu, \alpha_0 \rangle / \langle \alpha_0, \alpha_0 \rangle = -1$, so ν is the negative of the fundamental highest weight corresponding to the noncompact simple root α_0. Given $\chi \in \hat{K}$ with highest weight λ, we set $\lambda_0 = \lambda + a\nu$ and $\chi_0 = \chi \otimes e^{a\nu}$, where $a = 2\langle \lambda + \rho, \mu \rangle / \langle \mu, \mu \rangle$. The longest element w_0 of the Weyl group of K exchanges α_0 and μ. See the discussion around (3.13) below. As $w_0 \nu = \nu$, now $2\langle \nu, \mu \rangle / \langle \mu, \mu \rangle = -1$. Thus χ_0 has highest weight λ_0 such that $\langle \lambda_0 + \rho, \mu \rangle = 0$. Now

$$\begin{gathered} \chi_h = \chi_0 \otimes e^{h\nu} \in \hat{K} \text{ has highest weight } \lambda_h = \lambda_0 + h\nu, \\ \text{and } \chi_h \text{ satisfies the conditions of (3.1)} \Leftrightarrow h > 0. \end{gathered} \tag{3.2}$$

Any $\chi \otimes e^{b\nu}$ leads to the same *one-parameter family* $\{\chi_h\colon h > 0\}$, so we could have avoided duplication by choosing $\chi \in [K, K]\hat{}$. Thus we have the holomorphic relative discrete series partitioned into one-parameter families $\{\pi_h\colon h > 0\}$, where $\pi_h = \pi_{\chi_h}$, as χ ranges over $[K, K]\hat{}$. Here π_h has Harish-Chandra parameter $\lambda_h + \rho = \lambda_0 + h\nu + \rho$.

Harish-Chandra proved (3.1) in [1] by a method that amounts to use of the holomorphic trivialization of (2.4), carrying the pointwise norm from f to F. He noted that F is K-finite just when it is polynomial and showed that if some polynomial F is L_2 then the constant functions F are L_2. Going back, now, this shows that if (3.1) holds then every K-finite holomorphic section of $\mathbf{V} \to G/K$ is square integrable. So we take a close look at K-finite sections.

Let U be a finite-dimensional K-invariant space of sections of $\mathbf{V} \to G/K$, let $f \in U$, and let $\{f_i\}$ be any basis of U. Fix an abstract K-module W isomorphic to U. Let $w \in W$ correspond to f, let $\{w_i\}$ be the basis corresponding to $\{f_i\}$, and let $\{w_i^*\}$ be the dual basis of W^*. Then

$$f(kg) = \sum_i \langle k^{-1}w, w_i^* \rangle f_i(g) \tag{3.3}$$

for all $k \in K$ and $g \in G$. For $f(kg) = [k^{-1} \cdot f](g)$ and $k^{-1}w = \sum \langle k^{-1}w, w_i^* \rangle w_i$.

In (3.3) if we start with a K-finite section f, we may take $U = \mathcal{U}(\mathfrak{k}) f$. In particular, if f is holomorphic we may assume that the f_i are holomorphic.

Write $\Phi(\mathfrak{p}_+)$ for the set $\{\alpha \in \Phi \colon \mathfrak{g}_\alpha \subset \mathfrak{p}_+\}$ of noncompact positive roots. Choose root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi(\mathfrak{p}_+)$, such that $\langle E_\alpha, \bar{E}_\alpha \rangle = 1$ where $\bar{E}_\alpha = E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ results from conjugation of $\mathfrak{g}_{\mathbf{C}}$ over $\mathfrak{g}$. The monomial complex valued functions on $\mathfrak{p}_+$ are the

$$\sum z_\alpha E_\alpha \mapsto \left(\sum z_\alpha E_\alpha\right)^n = \prod z_\alpha^{n_\alpha}, \qquad n = (n_\alpha), \tag{3.4}$$

$\alpha \in \Phi(\mathfrak{p}_+)$ and n_α integers $\geqslant 0$. Fix a basis $\{v_b\}$ of V consisting of weight vectors. Then the K-finite holomorphic functions $G/K \to V$ are the finite linear combinations of the

$$F_{n,b} \colon gK \mapsto \zeta(g)^n \, v_b. \tag{3.5a}$$

Corresponding to this, the K-finite holomorphic sections of $\mathbf{V} \to G/K$ are the finite linear combinations of the

$$f_{n,b} \colon g \mapsto \zeta(g)^n \, \chi(g)^{-1} \, v_b. \tag{3.5b}$$

Note that $f_{n,b}$ is a vector of weight $\beta - \sum n_\alpha \cdot \alpha$, where β is the weight of v_b.

Now fix a K-finite holomorphic function $F = \sum c_{n,b} F_{n,b} \colon G/K \to V$. When we pass to the one-parameter family $\{\chi_h\} \subset \hat{K}$, F corresponds to a *one-parameter family of sections*

$$\begin{aligned} f_h(g) &= \sum c_{n,b} \zeta(g)^n \, \chi_h(g)^{-1} \, v_b \\ &= e^{-h\nu}(\kappa(g)) \sum c_{n,b} \zeta(g)^n \, \chi_0(g)^{-1} \, v_b \end{aligned} \tag{3.6}$$

of the respective bundles $\mathbf{V}_h \to G/K$. Here note that v_b has weight $\beta_0 + hv$ relative to χ_h, where β_0 is its weight with respect to χ_0.

In order to be more precise and to be able to estimate sections and coefficients, we need some information on the root structure and the Cartan decomposition $G = KAK$.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$ be a maximal strongly orthogonal subset of $\Phi(\mathfrak{p}_+)$ obtained by cascading down from the maximal root μ: $\gamma_1 = \mu$ and γ_{j+1} is any root in $\Phi(\mathfrak{p}_+)$ maximal with respect to the condition $\gamma_{j+1} \perp \{\gamma_1, \ldots, \gamma_j\}$. Let $E_j \in \mathfrak{g}_{\gamma_j}$ such that $\langle E_j, \bar{E}_j \rangle = 1$ and set $H_j = [E_j, \bar{E}_j]$. Define $\mathfrak{a}$ to be the real span of the $X_j = E_j + \bar{E}_j$ and $A = \exp_G(\mathfrak{a})$. Then $G = KAK$.

The "Cayley transform" element for G is $c = \exp_{G_{\mathbf{C}}}(\frac{1}{4}\pi i \sum Y_j)$, where $Y_j = i(E_j - \bar{E}_j) \in \mathfrak{p}$. If we split $\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^-$, where $\mathfrak{t}^+ = \Gamma^\perp$ and $i\mathfrak{t}^-$ is the span of the H_j, then $\mathrm{Ad}(c)$ fixes $\mathfrak{t}^+$ and carries $i\mathfrak{t}^-$ to $\mathfrak{a}$. Thus $\mathrm{Ad}(c)\, \mathfrak{t}_{\mathbf{C}} \cap \mathfrak{g} = \mathfrak{t}^+ + \mathfrak{a}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. We use the positive $\mathfrak{a}$-root system $\Phi_{\mathfrak{a}}^+$ consisting of all $\mathrm{Ad}(c)\, \alpha|_{\mathfrak{a}}$ such that $\alpha \in \Phi^+$ and $\alpha|_{\mathfrak{t}^-} \neq 0$. Write $\rho_{\mathfrak{a}}$ for half the sum (with multiplicities) of the elements of $\Phi_{\mathfrak{a}}^+$, so $\rho_{\mathfrak{a}} = \mathrm{Ad}(c)\, \rho|_{\mathfrak{a}}$.

The positive Weyl chamber is $\mathfrak{a}^+ = \{X \in \mathfrak{a}: \delta(X) > 0 \text{ for all } \delta \in \Phi_{\mathfrak{a}}^+\}$. Let $\mathfrak{g}[\gamma_j]$ be the span of E_j, $\bar{E}_j$ and H_j. It is the Lie algebra of a subgroup $G_{\mathbf{C}}[\gamma_j] \subset G_{\mathbf{C}}$ isomorphic to $SL(2; \mathbf{C})$ under $E_j \leftrightarrow (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$, $\bar{E}_j \leftrightarrow (\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ and $H_j \leftrightarrow (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$. Calculate in $SL(2; \mathbf{C})$

$$\mathrm{Ad}\left(\exp\left(\frac{\pi i}{4}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to see that $\mathrm{Ad}(c) H_j = X_j$. We will need this for (3.15) below, where we see that

$$\mathfrak{a}^+ = \left\{\sum s_j X_j : s_1 > \cdots > s_l > 0\right\};$$

then the form of the Cartan decomposition

$$G = K \cdot \mathrm{cl}(A^+) \cdot K, \qquad A^+ = \exp_G(\mathfrak{a}^+)$$

will be sufficiently precise so that we can make certain estimates.

We look in $G_{\mathbf{C}}[\Gamma] = \prod G_{\mathbf{C}}[\gamma_j] \subset G_{\mathbf{C}}$ to compute the decomposition $q(a) \in \exp(\zeta(a)) \cdot \tilde{\kappa}(a) \cdot P_-$ for an element $a \in A$. Note that

$$\exp\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} = \begin{pmatrix} 1 & \tanh(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\cosh(s) & 0 \\ 0 & \cosh(s) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh(s) & 1 \end{pmatrix}.$$

It follows that, for $a_s = \exp_G \sum s_j X_j$,

$$\zeta(a_s) = \sum \tanh(s_j)\, E_j$$

and (3.7a)

$$\tilde{\kappa}(a_s)^{-1} = \exp_{K_{\mathbf{C}}} \sum \log\cosh(s_j)\, H_j.$$

Lifting the curve $t \mapsto \exp_G\{t \sum s_j X_j\}$ from $P_+ K_{\mathbf{C}} P_-$ to $P_+ \tilde{K}_{\mathbf{C}} P_-$ as in Theorem 2.17, we conclude

$$\kappa(a_s)^{-1} = \exp_{\tilde{K}_{\mathbf{C}}} \sum \log \cosh(s_j)\, H_j. \tag{3.7b}$$

Combine (3.5) and (3.7) to see that $F_{n,b}(a_s K) = 0$ and $f_{n,b}(a_s) = 0$ if some $n_\alpha > 0$ for $\alpha \notin \Gamma$, and that if $n_\alpha = 0$ whenever $\alpha \notin \Gamma$ then

$$\begin{aligned} F_{n,b}(a_s K) &= \left\{ \prod \tanh(s_j)^{n_j} \right\} v_b, \qquad n_j = n_{\gamma_j}, \\ f_{n,b}(a_s) &= \left\{ \prod \tanh(s_j)^{n_j} \cosh(s_j)^{2\langle \beta, \gamma_j \rangle / \langle \gamma_j, \gamma_j \rangle} \right\} v_b \end{aligned} \tag{3.8}$$

where β is the weight of v_b.

Combine (3.3) and (3.8). Thus f is a K-finite holomorphic section of $\mathbf{V} \to G/K$, say $f = \sum c_{n,b} f_{n,b}$, U is a finite-dimensional K-invariant space of holomorphic sections which contains all the $f_{n,b}$ involved in this expression of f, and W is an abstract K-module isomorphic to U. We have $w \in W$ corresponding to f. We may assume that U has a basis of the form $\{f_{n,b}\}_{n \in N, b \in B}$. Let $\{w_{n,b}\}$ be the corresponding basis of W and $\{w^*_{n,b}\}$ the dual basis of W^*. Then for $k, k' \in K$,

$$f(k a_s k') = \sum_{\substack{n \in N' \\ b \in B}} \langle k^{-1} w, w^*_{n,b} \rangle\, \chi(k')^{-1} \cdot f_{n,b}(a_s), \tag{3.9}$$

where $N' = \{n \in N\colon n_\alpha = 0 \text{ whenever } \alpha \notin \Gamma\}$.

Now combine (3.6) and (3.9). Thus we start with a K-finite $F = \sum c_{n,b} F_{n,b}$ and it defines the one-parameter family of sections f_h of $\mathbf{V}_h \to G/K$ associated to $\chi_h = \chi_0 \otimes e^{h\nu} \in \hat{K}$. Here U_h has basis consisting of the

$$f_{h,n,b}\colon g \mapsto e^{-h\nu}(\kappa(g))\, f_{0,n,b}(g), \qquad n \in N,\ b \in B. \tag{3.10a}$$

Note, from (3.8), that

$$f_{h,n,b}(a_s) = \left\{ \prod \tanh(s_j)^{n_j} \cosh(s_j)^{2\langle \beta_h, \gamma_j \rangle / \langle \gamma_j, \gamma_j \rangle} \right\} v_b$$

where $\beta_h = \beta_0 + h\nu$ is the weight of v_b in $\chi_h = \chi_0 \otimes e^{h\nu}$. The same vector space W supports the abstract K-modules isomorphic to U_h. If τ_h denotes the action of K on W corresponding to U_h, then using (3.3) one sees that $\tau_h = \tau_0 \otimes e^{h\nu}$. Thus

$$f_h(kak') = \sum_{\substack{n \in N' \\ b \in B}} \langle \tau_h(k)^{-1} w, w^*_{n,b} \rangle\, \chi_h(k')^{-1} \cdot f_{h,n,b}(a) \tag{3.11}$$

where $N' = \{n \in N\colon n_\alpha = 0 \text{ for } \alpha \notin \Gamma\}$.

Finally, we put together some information on the growth properties of the $f_{h,n,b}$, in regard to the parameter h and the highest weight λ_0 of χ_0.

If λ is the highest weight for K on V_χ, then every weight of K on V_χ has the form

$$\beta = \lambda - (\text{sum of compact positive roots}).$$

Since the compact roots vanish on $\mathfrak{z}_K$, that gives

$$e^\beta(z) = e^\lambda(z) \qquad \text{for} \quad z \in Z_K^0 \text{ and } \beta \text{ any weight of } V_\chi. \tag{3.12}$$

Recall the definition of $v \in i\mathfrak{z}_K^*$: $2\langle v, \alpha_0\rangle/\langle \alpha_0, \alpha_0\rangle = -1$, where α_0 is the noncompact simple root. The longest element of the Weyl group W_K exchanges α_0 and the maximal root $\mu = \gamma_1$, and a subgroup of W_K permutes the γ_i transitively. See [1, 5, 8]. Thus

$$\frac{2\langle v, \gamma_i\rangle}{\langle \gamma_i, \gamma_i\rangle} = \frac{2\langle v, \mu\rangle}{\langle \mu, \mu\rangle} = \frac{2\langle v, \alpha_0\rangle}{\langle \alpha_0, \alpha_0\rangle}$$

so

$$\frac{2\langle v, \gamma_i\rangle}{\langle \gamma_i, \gamma_i\rangle} = -1 \qquad \text{for} \quad 1 \leqslant i \leqslant l. \tag{3.13a}$$

Combine this with (3.7) to see that

$$e^{hv}(\kappa(a_s)) = \prod \cosh(s_i)^h. \tag{3.13b}$$

The Restricted Root Theorem of Harish-Chandra [1] and Moore [5] is usually formulated relative to a set $\Gamma' = \{\gamma_1', ..., \gamma_l'\}$ of strongly orthogonal roots where γ_1' is the noncompact simple root and γ_{i+1}' is a lowest root in $\Phi(\mathfrak{p}_+)$ that is orthogonal to $\{\gamma_1', ..., \gamma_i'\}$. The longest element of the Weyl group of K, the one that interchanges $\Phi(\mathfrak{k})^+ \cup \Phi(\mathfrak{p}_+)$ with $\{-\Phi(\mathfrak{k})^+\} \cup \Phi(\mathfrak{p}_+)$, interchanges Γ' with our set Γ of strongly orthogonal roots. Thus, relative to Γ, the Restricted Root Theorem as stated in [8, p. 285] says that restriction of roots, from $\mathfrak{t}$ to the subspace $\mathfrak{t}^-$ such that $i\mathfrak{t}^-$ is the span of the H_j, has the following property.

Case 1. D is of tube type. Then $\text{rest}(\Phi) \cup \{0\} = \{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \colon 1 \leqslant i, j \leqslant l\}$, and

$$\begin{aligned} &\text{rest}(\Phi(\mathfrak{k})^+) \cup \{0\} = \{0\} \cup \{\tfrac{1}{2}(\gamma_i - \gamma_j) \colon 1 \leqslant i < j \leqslant l\}, \\ &\text{rest}(\Phi(\mathfrak{p}_+)) \cup \{0\} = \{0\} \cup \{\tfrac{1}{2}(\gamma_i + \gamma_j) \colon 1 \leqslant i \leqslant j \leqslant l\}. \end{aligned} \tag{3.14a}$$

Case 2. D is not of tube type. Then $\mathrm{rest}(\Phi) \cup \{0\} = \{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j, \frac{1}{2}\gamma_i\colon 1 \leqslant i \leqslant j \leqslant l\}$, and

$$\begin{aligned} \mathrm{rest}(\Phi(\mathfrak{k})^+) \cup \{0\} &= \{0\} \cup \{\tfrac{1}{2}(\gamma_i - \gamma_j)\colon i < j\} \cup \{\tfrac{1}{2}\gamma_i\}, \\ \mathrm{rest}(\Phi(\mathfrak{p}_+)) \cup \{0\} &= \{0\} \cup \{\tfrac{1}{2}(\gamma_i + \gamma_j)\colon i \leqslant j\} \cup \{\tfrac{1}{2}\gamma_i\}. \end{aligned} \tag{3.14b}$$

An immediate consequence is

$$\text{if } \alpha \in \Phi^+ \text{ and } s_1 \geqslant s_2 \geqslant \cdots \geqslant s_l \geqslant 0 \text{ then } \alpha\left(\sum s_i H_i\right) \geqslant 0. \tag{3.15a}$$

Since $\mathrm{Ad}(c) H_i = X_i$, we also get

$$\mathfrak{a}^+ = \left\{\sum s_i X_i\colon s_1 > s_2 > \cdots > s_l > 0\right\}. \tag{3.15b}$$

The highest weight λ for K on V_χ, and an arbitrary weight β, differ by a sum of compact positive roots. If $s_1' \geqslant \cdots s_l' \geqslant 0$ then (3.15a) gives $\lambda(\sum s_i' H_i) \geqslant \beta(\sum s_i' H_i)$. Use (3.15b) and take $s_i' = \log\cosh(s_i)$, where $a_s \in \mathrm{cl}(A^+)$; then, using (3.7), $-\lambda(\log \kappa(a_s)) \geqslant -\beta(\log \kappa(a_s))$. So,

$$e^{-\beta}(\kappa(a)) \leqslant e^{-\lambda}(\kappa(a)) \qquad \text{for all} \quad a \in \mathrm{cl}(A^+). \tag{3.16}$$

The condition (3.1) can be phrased $\langle \lambda + \rho, \gamma \rangle < 0$ for all $\gamma \in \Phi(\mathfrak{p}_+)$, in particular for $\gamma = \gamma_i$. Now from (3.2), $\langle \lambda_0 + \rho, \gamma_i \rangle \leqslant 0$. As above, if $a_s \in \mathrm{cl}(A^+)$, so $\log\cosh(s_1) \geqslant \cdots \geqslant \log\cosh(s_l) \geqslant 0$, we get $(\lambda_0 + \rho)(-\log \kappa(a_s)) \leqslant 0$, so

$$e^{-(\lambda_0 + \rho)}(\kappa(a)) \leqslant 1 \qquad \text{for all} \quad a \in \mathrm{cl}(A^+). \tag{3.17}$$

Again using (3.7) and (3.15), we have

$$e^{\alpha}(\kappa(a)) \leqslant 1 \qquad \text{for} \quad \alpha \in \Phi^+ \text{ and } a \in \mathrm{cl}(A^+). \tag{3.18a}$$

In particular,

$$e^{\rho}(\kappa(a)) \leqslant 1 \qquad \text{for} \quad a \in \mathrm{cl}(A^+). \tag{3.18b}$$

4. Rapidly Decreasing Functions

We recently defined and studied the space of rapidly decreasing functions on a general semisimple Lie group [4]. In this section we recall some definitions and facts needed to form our wave packets. These are somewhat simplified from [4] because here we are dealing with a connected group.

First, we normalize Haar measures. Let Z_G denote the center of G and define $Z = Z_G \cap Z_K^0$. Then $K/Z \cong [K, K] \times (Z_K^0/Z)$. Let $H \in \mathfrak{z}_K$ be the element such that $\mathrm{ad}(H)$ is $\pm i$ on $\mathfrak{p}_\pm$. Then the circle group $Z_K^0/Z = \{z_\theta Z\}$, $z_\theta = \exp_G(\theta H)$, as θ runs from 0 to 2π. Normalize Haar measures by

$$\int_{[K,K]} dk' = 1 \qquad \text{and} \qquad \int_{Z_K^0/Z} d(zZ) = 1$$

so that

$$\int_{K/Z} d(kZ) = 1 \qquad \text{and} \qquad \int_{Z_K^0/Z} \phi(zZ)\, d(zZ) = \frac{1}{2\pi} \int_0^{2\pi} \phi(z_\theta Z)\, d\theta.$$

Further normalize by

$$\int_K \phi(k)\, dk = \int_{K/Z} \sum_{z \in Z} \phi(kz)\, d(kZ).$$

Define D: $A \to \mathbf{R}$ by

$$D(a) = \prod_{\Phi_a^+} |e^\alpha(a) - e^{-\alpha}(a)|^{m_\alpha}, \qquad m_\alpha = \dim \mathfrak{g}_\alpha.$$

Let da denote the euclidean measure on A,

$$\int_A \phi(a)\, da = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(a_s)\, ds_1 \cdots ds_l.$$

Then we normalize the invariant measure on G/K by

$$\int_{G/K} \phi(gK)\, d(gK) = \int_{K/Z} \int_{A^+} \phi(ka)\, D(a)\, da\, d(kZ).$$

Now we normalize Haar measures on G and G/Z by

$$\int_{G/Z} \phi(gZ)\, d(gZ) = \int_{G/K} \int_{K/Z} \phi(gkZ)\, d(kZ)\, d(gK)$$

and

$$\int_G \phi(g)\, dg = \int_{G/K} \int_K \phi(gk)\, dk\, d(gK)$$

$$= \int_{G/Z} \sum_{z \in Z} \phi(gz)\, d(gZ).$$

This has no effect on the considerations in Section 3.

The first ingredient in the definition of rapidly decreasing function is a distance or norm, to keep track of polynomial growth. The Killing form of G defines the structure of riemannian symmetric space on G/K. Let $o = 1 \cdot K \in G/K$ and define σ: $G \to \mathbf{R}^+$ by

$$\sigma(x) = \text{distance}(o, x(o)). \tag{4.1a}$$

This is the norm used to define the relative Schwartz spaces $\mathscr{C}(G/Z, \zeta)$, $\zeta \in \hat{Z}$, which can be viewed as the direct integrands of the absolute Schwartz space $\mathscr{C}(G)$. As G is connected and $K = [K, K] \times Z_K^0$,

$$[K, K] \text{ is the group } K^\vee \text{ of [4, Lemma 6.3]},$$

$$Z_K^0 \text{ is the group } V \text{ of [4, Lemma 6.3]},$$

$$G \text{ is the group } G^\vee \text{ of [4, Lemma 6.3]}.$$

Of course $(z, k', \xi) \mapsto zk' \exp_G(\xi)$ is a diffeomorphism of $Z_K^0 \times [K, K] \times \mathfrak{p}$ onto G, and $\sigma(zk' \exp_G(\xi)) = \|\xi\|$. Define a norm $\|z_\theta\| = |\theta|$, $z_\theta = \exp_G(\theta H)$, on Z_K^0. Now define $\tilde{\sigma}$: $G \to \mathbf{R}^+$ by

$$\tilde{\sigma}(zk' \exp(\xi)) = \|z\| + \|\xi\| \qquad \text{for} \quad z \in Z_K^0,\ k' \in [K, K],\ \xi \in \mathfrak{p}. \tag{4.1b}$$

Note that

$$\tilde{\sigma}(x) \geqslant \sigma(x) \qquad \text{for all } x \in G, \qquad \text{and} \qquad \tilde{\sigma}(a) = \sigma(a) \qquad \text{for} \quad a \in A. \tag{4.2a}$$

Also, since $k(o) = o$ in G/K, and since

$$\mathrm{Ad}(zk'')(z_\theta k' \exp(\xi)) = z_\theta \cdot \mathrm{Ad}(k'')k' \cdot \exp(\mathrm{Ad}(zk'')\xi)$$

for $z \in Z_K^0$ and $k', k'' \in [K, K]$, we have

$$\sigma(kxk^{-1}) = \sigma(x) \qquad \text{and} \qquad \tilde{\sigma}(kxk^{-1}) = \tilde{\sigma}(x) \qquad \text{for} \quad k \in K. \tag{4.2b}$$

In addition, from [4, Sects. 2 and 6], if $k_i \in K$ and $k_i' \in [K, K]$, then

$$\sigma(k_1 x k_2) = \sigma(x) \qquad \text{and} \qquad \tilde{\sigma}(k_1' x k_2') = \tilde{\sigma}(x); \tag{4.2c}$$

$$\sigma(xy) \leqslant \sigma(x) + \sigma(y) \qquad \text{and} \qquad \tilde{\sigma}(xy) \leqslant 3(\tilde{\sigma}(x) + \tilde{\sigma}(y)). \tag{4.2d}$$

The second ingredient in the definition of a rapidly decreasing function is a term to compensate for negative curvature on G/K or $G/[K, K]$. That is the zonal spherical function on G for $0 \in \mathfrak{a}^*$,

$$\Xi(x) = \int_{K/Z} e^{-\rho_{\mathfrak{a}}(H(kx))}\, d(kZ) \tag{4.3}$$

where $x \in N \cdot \exp H(x) \cdot K$ with $H(x) \in \mathfrak{a}$, for the Iwasawa decomposition $G = NAK$. Among the properties of Ξ (see [2, Lemma 10.1] and [4, 2.5 and 6.12]) one finds

$$\Xi(a)^{-1} e^{-\rho_{\mathfrak{a}}}(a) \leqslant 1 \qquad \text{for} \quad a \in \mathrm{cl}(A^+). \tag{4.4}$$

Let $\zeta \in \hat{Z}$ and $C^\infty(G/Z, \zeta) = \{f \in C^\infty(G)\colon f(xz) = \zeta(z)^{-1} f(x) \text{ for } z \in Z, x \in G\}$. If $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$ and $r \in \mathbf{R}$, set

$$_{D_1}|f|_{r,D_2} = \sup_{x \in G} (1 + \sigma(x))^r \, \Xi(x)^{-1} \, |f(D_1; x; D_2)|. \tag{4.5a}$$

The *relative Schwartz spaces* on G are the

$$\mathscr{C}(G/Z, \zeta) = \{f \in C^\infty(G/Z, \zeta)\colon \text{if } r \in \mathbf{R} \text{ and } D_i \in \mathscr{U}(\mathfrak{g}) \text{ then } {}_{D_1}|f|_{r,D_2} < \infty\}. \tag{4.5b}$$

As shown in [4, Theorems 2.7 and 2.8],

$$C_c^\infty(G/Z, \zeta) \subset \mathscr{C}(G/Z, \zeta) \subset L_2(G/Z, \zeta) \tag{4.5c}$$

are continuous inclusions onto dense subspaces.

Similarly, for $f \in C^\infty(G)$, set

$$_{D_1}\|f\|_{r,D_2} = \sup_{x \in G} (1 + \tilde{\sigma}(x))^r \, \Xi(x)^{-1} \, |f(D_1; x; D_2)|. \tag{4.6a}$$

The *Schwartz space* on G is

$$\mathscr{C}(G) = \{f \in C^\infty(G)\colon \text{if } r \in \mathbf{R} \text{ and } D_i \in \mathscr{U}(\mathfrak{g}) \text{ then } {}_{D_1}\|f\|_{r,D_2} < \infty\}.$$

As shown in [4, Theorems 6.11 and 6.13],

$$C_c^\infty(G) \subset \mathscr{C}(G) \subset L_2(G) \tag{4.6c}$$

are continuous inclusions onto dense subspaces. The direct integral decomposition

$$L_2(G) = \int_{\hat{Z}} L_2(G/Z, \zeta) \, d\zeta$$

is implemented by $f_\zeta(x) = \int_Z f(xz) \zeta(z) \, dz$, say for $f \in C_c(G)$. In fact [4, Theorem 7.2], if $f \in \mathscr{C}(G)$, then each $f_\zeta \in \mathscr{C}(G/Z, \zeta)$, and $f \to f_\zeta$ is a continuous map from $\mathscr{C}(G)$ to $\mathscr{C}(G/Z, \zeta)$.

If $f \in \mathscr{C}(G)$ and if $P' = M'A'N'$ is a parabolic subgroup of G, then we set $f^{P'}(x) = \int_{N'} f(xn') \, dn'$. The integral converges absolutely, uniformly for x in compact subsets of G. f is a *cusp form* if $f^{P'} = 0$ for all proper parabolic subgroups P' of G. The space of cusp forms,

$$^0\mathscr{C}(G) = \{f \in \mathscr{C}(G) \colon f \text{ is a cusp form}\} \tag{4.7a}$$

is a closed G-invariant subspace of $\mathscr{C}(G)$.

Similarly, if $f \in \mathscr{C}(G/Z, \zeta)$, then $f^{P'}$ converges absolutely uniformly for xZ in compact subsets of G/Z. We say that f is a *relative cusp form* if $f^{P'} = 0$ for all proper parabolic subgroups P' of G. The space of relative cusp forms,

$$^0\mathscr{C}(G/Z, \zeta) = \{f \in \mathscr{C}(G/Z, \zeta) \colon f \text{ is a relative cusp form}\} \tag{4.7b}$$

is a closed G-invariant subspace of $\mathscr{C}(G/Z, \zeta)$.

LEMMA 4.8. *Let $f \in \mathscr{C}(G)$. Then $f \in {}^0\mathscr{C}(G)$ if and only if $f_\zeta \in {}^0\mathscr{C}(G/Z, \zeta)$ for all $\zeta \in \hat{Z}$.*

Proof. Let $f \in {}^0\mathscr{C}(G)$ and $\zeta \in \hat{Z}$. Then $f_\zeta \in \mathscr{C}(G/Z, \zeta)$, and

$$\begin{aligned}(f_\zeta)^{P'}(x) &= \int_{N'} f_\zeta(xn')\, dn' = \int_{N'} \int_Z f(xn'z)\, \zeta(z)\, dz\, dn' \\ &= \int_Z \zeta(z) \int_{N'} f(xzn')\, dn'\, dz = (f^{P'})_\zeta(x)\end{aligned}$$

because Z centralizes N' and all the integrals converge absolutely. If P' is a proper parabolic subgroup of G, now $f^{P'} = 0$ implies $(f_\zeta)^{P'} = 0$, so f_ζ is a relative cusp form.

Conversely let $f_\zeta \in {}^0\mathscr{C}(G/Z, \zeta)$ for all $\zeta \in \hat{Z}$. Then

$$\begin{aligned}f^{P'}(x) &= \int_{N'} f(xn')\, dn' = \int_{N'} \int_{\hat{Z}} f_\zeta(xn')\, d\zeta\, dn' \\ &= \int_{\hat{Z}} \int_{N'} f_\zeta(xn')\, dn'\, d\zeta = \int_{\hat{Z}} (f_\zeta)^{P'}(x)\, d\zeta,\end{aligned}$$

which vanishes, so f is a cusp form. Q.E.D.

We understand the spaces of relative cusp forms. See [4, Corollary 5.7 and Theorem 5.8]. ${}^0\mathscr{C}(G/Z, \zeta)$ contains every $\mathscr{Z}$-finite element of $\mathscr{C}(G/Z, \zeta)$. In particular, if $\pi \in \hat{G}$ is a relative discrete series representation, then every K-finite matrix coefficient of π is a relative cusp form. Moreover, if we denote

$\mathscr{C}_{\text{disc}}(G/Z, \zeta)$: closed subspace of $\mathscr{C}(G/Z, \zeta)$ spanned by relative discrete series matrix coefficients, (4.9) then ${}^0\mathscr{C}(G/Z,\zeta)=\mathscr{C}_{\mathrm{disc}}(G/Z,\zeta)$. In view of Lemma 4.8 one expects something similar for ${}^0\mathscr{C}(G)$. This paper is a step toward proving the corresponding results for ${}^0\mathscr{C}(G)$ and $\mathscr{C}_{\mathrm{disc}}(G)$.

5. Matrix Coefficients

Let ϕ_h be a one-parameter family of matrix coefficients corresponding to K-finite sections of the bundles $\mathbf{V}_h\to G/K$ as in (1.3). In this section we will obtain a formula for $\phi_h(a)$, $a\in \mathrm{cl}(A^+)$, which gives precise information about the asymptotic behavior of $\phi_h(a)$, and all its derivatives with respect to h, as a function of both a and h. The precise statement is as follows.

THEOREM 5.1. *For $a\in \mathrm{cl}(A^+)$, $\phi_h(a)$ is a finite linear combination of terms of the form*

$$e^{-\beta_h}(\kappa(a))\int_{A^+} e^{-2\beta'_h}(\kappa(a_1))\,D(a_1)\,p(a_1,a,h)\,da_1$$

where $\beta_h=\beta_0+h\nu$ and $\beta'_h=\beta'_0+h\nu$ are weights of χ_h, $p(a_1,a,h)$ is a polynomial in h of the form $\sum_{j=0}^m c_j(a_1,a)h^j$, where the coefficients $c_j(a_1,a)$ are bounded for a, $a_1\in \mathrm{cl}(A^+)$, and $D(a)=\prod_{\alpha\in\Phi_a^+}|e^{\alpha}(a)-e^{-\alpha}(a)|^{m_\alpha}$, for $a\in A$.

COROLLARY 5.2. *Let $r\geq 0$ be an integer. Then there are a constant $c_r>0$ and an integer $m\geq 0$ so that for all $a\in \mathrm{cl}(A^+)$, $h>0$,*

$$\left|\frac{\partial^r}{\partial h^r}\phi_h(a)\right|\leq c_r(1+h)^m\left(1+\frac{1}{h}\right)^{l(r+1)}(1+\sigma(a))^r\,e^{-(\lambda_0+h\nu)}(\kappa(a)).$$

The remainder of this section is devoted to the proofs of Theorem 5.1 and Corollary 5.2. Let f_h, f'_h be one-parameter families of K-finite sections of $\mathbf{V}_h\to G/K$. For $x\in G$,

$$\begin{aligned}\phi_h(x)=\langle L(x)f_h,f'_h\rangle_{G/K}&=\int_{G/K}\langle f_h(x^{-1}g),f'_h(g)\rangle_V\,d(gK)\\&=\int_{K/Z}\int_{A^+}\langle f_h(x^{-1}ka),f'_h(ka)\rangle_V\,D(a)\,dk\,da.\end{aligned}$$

Now writing f_h as a finite linear combination of monomials as in (3.6), and using (3.10b) and (3.11) to expand f'_h, we see that $\phi_h(x)$ is a finite linear combination of terms of the form

$$\begin{aligned}\psi_h(x)=\int_{K/Z}\int_{A^+}&\zeta(x^{-1}ka_1)^n\,\langle\overline{\tau_h(k^{-1})\,w,w^*}\rangle\\&\times\zeta(a_1)^m\,e^{-\beta'_h}(\kappa(a_1))\,\langle\chi_h(x^{-1}ka_1)^{-1}v,u\rangle_V\,D(a_1)\,dk\,da_1\qquad(5.3)\end{aligned}$$

where $u, v \in V$ are weight vectors of weights $\beta'_h = \beta'_0 + hv$ and $\beta_h = \beta_0 + hv$, respectively, for the action of χ_h on V, $n \in N$, $m \in N'$, $w \in W$ corresponds to f'_h under the action τ_h, and $w^* \in W^*$ is a weight vector of weight $-\beta'_h + \sum_{j=1}^{l} m_j \gamma_j$ for the action dual to τ_h. We want to show that for $a \in \mathrm{cl}(A^+)$, $\psi_h(a)$ is a term of the form given in Theorem 5.1.

LEMMA 5.4. *Define ψ_h as in (5.3). Then for $a \in \mathrm{cl}(A^+)$,*

$$\psi_h(a) = e^{-\beta_h}(\kappa(a)) \int_{[K,K]} \int_{A^+} \langle \overline{\tau_0(k^{-1})\, w, w^*} \rangle$$

$$\times \zeta(a_1)^m \, e^{-2\beta'_h}(\kappa(a_1))\, D(a_1)\, I_h(a, a_1, k)\, dk\, da_1$$

where

$$I_h(a, a_1, k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iM\theta} \zeta(a^{-1} \exp(e^{i\theta} \zeta(ka_1)))^n$$

$$\times \langle \chi_h(p_-(a^{-1}) \exp(e^{i\theta}\zeta(ka_1)))^{-1} v, \chi_h(k) u \rangle_V \, d\theta$$

and

$$M = \sum_{j=1}^{l} m_j.$$

Proof. Decompose K/Z as $[K, K] \times Z_K^0/Z$ and write $z_\theta = \exp(\theta H)$ as in Section 4. Now for $k \in [K, K]$, $\langle \overline{\tau_h(z_\theta^{-1} k^{-1}) w, w^*} \rangle$ $=$ $e^{\beta'_h - \sum m_j \gamma_j}(z_\theta) \langle \overline{\tau_0(k^{-1}) w, w^*} \rangle$ since w^* has weight $-\beta'_h + \sum m_j \gamma_j$. Now $e^{\beta'_h}(z_\theta) = e^{\lambda_h}(z_\theta)$ by (3.12) since λ_h is the highest weight of χ_h, and $e^{-\sum m_j \gamma_j}(z_\theta) = e^{-iM\theta}$ since $\gamma(H) = i$ for all $\gamma \in \Phi(\mathfrak{p}_+)$.

View G as a subgroup of $P_+ \tilde{K}_{\mathbf{C}} P_-$ as in (2.17). We can decompose $ka_1 = \exp(\zeta(ka_1))\, k\kappa(a_1)\, p_-(a_1)$. Then

$$z_\theta k a_1 = \exp(e^{i\theta} \zeta(ka_1))\, z_\theta k \kappa(a_1)\, p_-(a_1)$$

since $\mathrm{ad}\, z_\theta E = e^{i\theta} E$ for all $E \in \mathfrak{p}_+$. Thus

$$\zeta(a^{-1} z_\theta k a_1) = \zeta(a^{-1} \exp(e^{i\theta} \zeta(ka_1)))$$

and

$$\begin{aligned} \kappa(a^{-1} z_\theta k a_1) &= \kappa(\kappa(a)\, p_-(a^{-1}) \exp(e^{i\theta}\zeta(ka_1))\, z_\theta k \kappa(a_1)) \\ &= \kappa(a)\, \kappa(p_-(a^{-1}) \exp(e^{i\theta}\zeta(ka_1)))\, z_\theta k \kappa(a_1). \end{aligned}$$

So

$$\langle \chi_h(a^{-1}z_\theta k a_1)^{-1}v, u\rangle_V$$
$$= e^{-\lambda_h}(z_\theta)\,\langle \chi_h(p_-(a^{-1})\exp(e^{i\theta}\zeta(ka_1)))^{-1}\,\chi_h(a)^{-1}v, \chi_h(k)\,\chi_h(a_1)^{-1}u\rangle_V$$

since $\chi_h(a_1)^* = \chi_h(a_1)$ and $\chi_h(k)^* = \chi_h(k)^{-1}$. But $\chi_h(a)^{-1}v = e^{-\beta_h}(\kappa(a))v$ and $\chi_h(a_1)^{-1}u = e^{-\beta_h}(\kappa(a_1))u$. Q.E.D.

To complete the proof of Theorem 5.1 we need to show that $\int_{[K,K]} \langle \tau_0(k^{-1})w, w^*\rangle\, \zeta(a_1)^m\, I_h(a, a_1, k)\, dk$ is a polynomial in h with bounded coefficients. To do this we will evaluate the integral $I_h(a, a_1, k)$ over Z_K^0/Z by making a change of variables $s = e^{i\theta}$ and using residues at $s = 0$. Thus we rewrite

$$I_h(a, a_1, k) = \frac{1}{2\pi i}\int_{|s|=1} s^{-(M+1)}\zeta(a^{-1}\exp(s\zeta(ka_1)))^n$$
$$\times \langle \chi_h(p_-(a^{-1})\exp(s\zeta(ka_1)))^{-1}v, \chi_h(k)u\rangle_V\, ds. \quad (5.5)$$

LEMMA 5.6. *The function* $b(s) = b(s, a, a_1, k) = \zeta(a^{-1}\exp(s\zeta(ka_1)))^n$ *is holomorphic in a neighborhood of* $|s| \leqslant 1$. *For any integer* $r \geqslant 0$, $(\partial^r b/\partial s^r)\cdot(0, a, a_1, k)$ *is bounded for all* $a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$.

Proof. Write $E = \zeta(ka_1)$. Then E is in the bounded domain D of $\mathfrak{p}_+$ so there is $\varepsilon > 0$ so that $sE \in D$ for all $|s| < 1 + \varepsilon$. Thus for $|s| < 1 + \varepsilon$, $a^{-1}\exp(sE) \in P_+\tilde{K}_{\mathbf{C}}P_-$ so that $b(s)$ is defined.

Recall that $\zeta(g)^n = \prod_{\alpha \in \Phi(\mathfrak{p}_+)} \zeta_\alpha(g)^{n_\alpha}$, $n_\alpha \geqslant 0$, where $\zeta_\alpha(g)$ denotes the coefficient of E_α in $\zeta(g)$. It is enough to prove the lemma when $b(s) = \zeta_\alpha(a^{-1}\exp(sE))$, $\alpha \in \Phi(\mathfrak{p}_+)$. Then $b(s) = -\zeta_\alpha(a) + e^\alpha(\kappa(a))\,\zeta_\alpha(p_-(a^{-1})\cdot\exp(sE))$, where for $a \in \mathrm{cl}(A^+)$, using (3.7) and (3.18b), $\zeta_\alpha(a)$ and $e^\alpha(\kappa(a))$ are both bounded. Thus it is enough to look at $b_1(s) = \zeta_\alpha(p_-(a^{-1})\exp(sE))$. Now $s \mapsto p_-(a^{-1})\exp(sE)$ is a holomorphic map from $\{s \in \mathbf{C}: |s| < 1 + \varepsilon\}$ into $P_+\tilde{K}_{\mathbf{C}}P_-$. Further, the projection $x \to \zeta_\alpha(x)$, $\alpha \in \Phi(\mathfrak{p}_+)$, is a holomorphic map from $P_+\tilde{K}_{\mathbf{C}}P_-$ to $\mathbf{C}$. Thus b_1 is holomorphic for $|s| < 1 + \varepsilon$.

Let E, F denote arbitrary elements of $\mathrm{cl}(D)$. Then $\bar{F} \in 2\bar{D}$ and $sE \in \frac{1}{2}D$ for $|s| < \frac{1}{2}$, so that as in (2.12), $\exp(\bar{F})\exp(sE) \in P_+\tilde{K}_{\mathbf{C}}P_-$ for $|s| < \frac{1}{2}$.

Define $b_1'(s, E, F) = \zeta_\alpha(\exp(\bar{F})\exp(sE))$ for $|s| < \frac{1}{2}$, $E, F \in \mathrm{cl}(D)$. As above, b_1' is holomorphic for $|s| < \frac{1}{2}$ and continuous for $E, F \in \mathrm{cl}(D)$. Thus $b_1'(s, E, F) = \sum_{j=0}^\infty c_j(E, F)s^j$, where the coefficients are continuous functions of E, $F \in \mathrm{cl}(D)$, hence bounded. But now for $|s| < \frac{1}{2}$, $b_1(s, a, a_1, k) = b_1'(s, \zeta_+(ka_1), \overline{\zeta_-(a^{-1})})$ so that $(\partial^r b/\partial s^r)(0, a, a_1, k) = r!\, c_r(\zeta_+(ka_1), \overline{\zeta_-(a^{-1})})$ is bounded for all $a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$.

Q.E.D.

We next want to prove a result similar to that of Lemma 5.6 for the function of s in (5.5) involving χ_h. Thus we write

$$\begin{aligned} B_h(s) &= B_h(s, a, a_1, k) \\ &= \langle \chi_h(p_-(a^{-1}) \exp(s\zeta(ka_1)))^{-1} v, \chi_h(k) u \rangle_V, \end{aligned} \tag{5.7}$$

$a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$.

LEMMA 5.8. *The function $B_h(s)$ is holomorphic in a neighborhood of $|s| \leqslant 1$. For any integer $r \geqslant 0$, $(\partial^r B_h/\partial s^r)(0, a, a_1, k) = \sum_{j=0}^{r} c_j(a, a_1, k) h^j$ is a polynomial of degree $\leqslant r$ in h with coefficients $c_j(a, a_1, k)$ bounded for all $a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$.*

Proof. Pick $\varepsilon > 0$ so that $s\zeta(ka_1) \in D$ for $|s| < 1 + \varepsilon$. Then $\zeta_-(a^{-1}) \in \bar{D}$ and $s\zeta(ka_1) \in D$ so that the product $p_-(a^{-1}) \exp(s\zeta(ka_1))$ is defined in $P_+ \tilde{K}_{\mathbf{C}} P_-$ for $|s| < 1 + \varepsilon$, and as in (5.6), $s \mapsto p_-(a^{-1}) \exp(s\zeta(ka_1))$ is a holomorphic map from $\{s \in \mathbf{C}: |s| < 1 + \varepsilon\}$ into $P_+ \tilde{K}_{\mathbf{C}} P_-$. But $x \mapsto \kappa(x)$ is a holomorphic map from $P_+ \tilde{K}_{\mathbf{C}} P_-$ to $\tilde{K}_{\mathbf{C}}$ and χ_h is a holomorphic representation of $\tilde{K}_{\mathbf{C}}$ so that the matrix coefficient $B_h(s)$ is holomorphic for $|s| < 1 + \varepsilon$.

Let E, F be arbitrary elements of $\mathrm{cl}(D)$. For any $k \in [K, K]$, $\mathrm{ad}\, k(E) \in \mathrm{cl}(D)$ also, so, as in (5.6), we have the product $\exp(\bar{F}) \exp(s\, \mathrm{ad}\, kE)$ defined in $P_+ \tilde{K}_{\mathbf{C}} P_-$ for all $|s| < \frac{1}{2}$, $E, F \in \mathrm{cl}(D)$. Define $B_0'(s, E, F, k) = \langle \chi_0(\exp(\bar{F}) \exp(s\, \mathrm{ad}\, kE))^{-1} v, \chi_0(k) u \rangle_V$. B_0' is holomorphic for $|s| < \frac{1}{2}$ and continuous for $E, F \in \mathrm{cl}(D)$, $k \in [K, K]$. Thus as in (5.6), for any $r \geqslant 0$, $(\partial^r B_0/\partial s^r)(0, a, a_1, k) = (\partial^r B_0'/\partial s^r)(0, \zeta_+(a_1), \zeta_-(a^{-1}), k)$ is bounded for all $a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$.

Now $B_h(s, a, a_1, k) = e^{-h\nu}(p_-(a^{-1}) \exp(s\zeta(ka_1)))\, B_0(s, a, a_1, k)$. By the above we may as well assume that $\chi_0 = 1$ so that $B_h(s, a, a_1, k) = e^{-h\nu}(p_-(a^{-1}) \exp(s\zeta(ka_1)))$. Since $\tilde{K}_{\mathbf{C}} = [K, K]_{\mathbf{C}} \exp((\mathfrak{z}_K)_{\mathbf{C}})$, there is a holomorphic map from $P_+ \tilde{K}_{\mathbf{C}} P_-$ to $\mathbf{C}$ given by $x \to \kappa_Z(x)$, where $x \in P_+ [K, K]_{\mathbf{C}} \exp(\kappa_Z(x) H) P_-$. For $E, F \in \mathrm{cl}(D)$, $|s| < \frac{1}{2}$, define $\kappa_Z'(s, E, F) = \kappa_Z(\exp(\bar{F}) \exp(sE))$ and $B_h'(s, E, F) = \exp(-h\nu(H)\, \kappa_Z'(s, E, F))$. Clearly κ_Z' and B_h' are holomorphic for $|s| < \frac{1}{2}$ and continuous for $E, F \in \mathrm{cl}(D)$. Further, any derivative $(\partial^r B_h'/\partial s^r)(0, E, F)$ is a polynomial in h (of degree $\leqslant r$) and the derivatives $(\partial^j \kappa_Z'/\partial s^j)(0, E, F)$, $1 \leqslant j \leqslant r$, since $\kappa_Z'(0, E, F) = 0$. Thus $(\partial^r B_h/\partial s^r)(0, a, a_1, k) = (\partial^r B_h'/\partial s^r)(0, \zeta_+(ka_1), \zeta_-(a^{-1}))$ is a polynomial in h with coefficients bounded for all $a, a_1 \in \mathrm{cl}(A^+)$, $k \in [K, K]$, as required. Q.E.D.

Proof of Theorem 5.1. In the notation of (5.6) and (5.7),

$$I_h(a, a_1, k) = \frac{1}{2\pi i} \int_{|s|=1} s^{-(M+1)}\, b(s)\, B_h(s)\, ds.$$

Using Lemmas 5.6 and 5.8, b and B_h are holomorphic in a neighborhood of $|s| \leqslant 1$. Thus we can evaluate I_h by $I_h(a, a_1, k) = \mathrm{Residue}_{s=0}(s^{-(M+1)}\, b(s)\, B_h(s))$. Further, using the results of (5.6) and (5.8) regarding derivatives of b and B_h at $s=0$, we know that this residue is a polynomial in h with coefficients bounded as functions of a, $a_1 \in \mathrm{cl}(A^+)$ and $k \in [K, K]$. Now it is clear from Lemma 5.4 that Theorem 5.1 is proved with

$$p(a_1, a, h) = \int_{[K,K]} \overline{\langle \tau_0(k^{-1})w, w^* \rangle}\, \zeta(a_1)^m\, I_h(a, a_1, k)\, dk. \quad \text{Q.E.D.}$$

Proof of Corollary 5.2. By Theorem 5.1, $\phi_h(a)$ is a finite linear combination of terms of the form

$$\psi_h(a) = e^{-\beta_h}(\kappa(a)) \int_{A^+} e^{-2\beta'_h}(\kappa(a_1))\, D(a_1)\, p(a_1, a, h)\, da_1,$$

where $\beta_h = \beta_0 + h\nu$, $\beta'_h = \beta'_0 + h\nu$. For $a_s = \exp(\sum_{i=1}^{l} s_i X_i)$, $e^{-h\nu}(\kappa(a_s)) = \prod_{i=1}^{l} (\cosh s_i)^{-h}$ by (3.13b), so for $a_s \in \mathrm{cl}(A^+)$,

$$\psi_h(a_s) = e^{-\beta_0}(\kappa(a_s)) \int_{A^+} e^{-2\beta'_0}(\kappa(a_t))\, D(a_t) \times \left(\prod_{i=1}^{l} \cosh s_i \cosh^2 t_i \right)^{-h} p(a_t, a_s, h)\, da_t.$$

But clearly

$$\frac{\partial^r}{\partial h^r} \left(\prod_{i=1}^{l} \cosh s_i \cosh^2 t_i \right)^{-h} p(a_t, a_s, h) = \left(\prod_{i=1}^{l} \cosh s_i \cosh^2 t_i \right)^{-h} p_r(a_t, a_s, h)$$

where $p_r(a_t, a_s, h) = \sum_{k=0}^{m} \sum_{j=0}^{r} c_{j,k,r}(a_t, a_s)(\log \prod_i \cosh s_i \cosh^2 t_i)^j h^k$ is a polynomial in h and $\log \prod_i \cosh s_i \cosh^2 t_i$ with coefficients satisfying $\sup_{a_s, a_t \in \mathrm{cl}(A^+)} |c_{j,k,r}(a_t, a_s)| < \infty$. Thus for all a_s, $a_t \in \mathrm{cl}(A^+)$, $|p_r(a_t, a_s, h)| \leqslant C_r(1+h)^m(1+\sum_i \log \cosh s_i)^r(1+\sum_i \log \cosh t_i)^r$, where we use C_r generically to denote constants depending on r. Now

$$\left| \frac{\partial^r}{\partial h^r} \psi_h(a_s) \right| \leqslant C_r(1+h)^m \left(1 + \sum_i \log \cosh s_i \right)^r e^{-\beta_h}(\kappa(a_s)) \times \int_{A^+} e^{-2\beta_h}(\kappa(a_t))\, D(a_t) \left(1 + \sum_i \log \cosh t_i \right)^r da_t.$$

But $e^{-\beta_h}(\kappa(a_s)) \leqslant e^{-(\lambda_0 + h\nu)}(\kappa(a_s))$ using (3.16), since λ_h is the highest weight of χ_h, and $\sum_i \log\cosh s_i \leqslant C\sigma(a_s)$. Thus to prove the corollary it suffices to show that

$$J(r,h) = \int_{A^+} e^{-2\beta_h}(\kappa(a_t))\, D(a_t) \left(1 + \sum_i \log\cosh t_i\right)^r da_t$$

$$\leqslant C_r \left(1 + \frac{1}{h}\right)^{l(r+1)}.$$

But

$$D(a) = e^{2\rho_a}(a) \prod_{\alpha \in \Phi_a^+} (1 - e^{-2\alpha}(a))^{m_\alpha}$$

$$\leqslant e^{2\rho_a}(a) \qquad \text{for} \quad a \in \mathrm{cl}(A^+).$$

But for

$$a_t \in \mathrm{cl}(A^+),\ e^{2\rho_a}(a_t) = \exp\left(\sum_{i=1}^{l} 2t_i \rho_a(X_i)\right)$$

$$= \exp\left(\sum_{i=1}^{l} 2t_i \rho(H_i)\right)$$

$$\leqslant C \exp\left(\sum_{i=1}^{l} 2 \log\cosh t_i \rho(H_i)\right)$$

$$= C e^{-2\rho}(\kappa(a_t)).$$

Thus

$$D(a_t)\, e^{-2\beta_h}(\kappa(a_t)) \leqslant C e^{-2(\lambda_0 + \rho)}(\kappa(a_t)) \left(\prod_i \cosh t_i\right)^{-2h}$$

$$\leqslant C \left(\prod_i \cosh t_i\right)^{-2h},$$

since $e^{-2(\lambda_0+\rho)}(\kappa(a_t)) \leqslant 1$ for $a_t \in \mathrm{cl}(A^+)$ by (3.17). Now

$$J(r,h) \leqslant C \int_{A^+} \left(\prod_i \cosh t_i\right)^{-2h} \left(1 + \sum_i \log\cosh t_i\right)^r da_t$$

$$\leqslant C_r \prod_{i=1}^{l} \int_0^\infty (\cosh t_i)^{-2h} (1 + \log\cosh t_i)^r \, dt_i.$$

But

$$\begin{aligned}
&\int_0^\infty (\cosh t)^{-2h}(1+\log\cosh t)^r\,dt \\
&\qquad \leqslant 2^r + \coth(1)\int_1^\infty (\cosh t)^{-2h}(1+\log\cosh t)^r\,\frac{\sinh t}{\cosh t}\,dt \\
&\qquad = 2^r + \coth(1)\int_{\log\cosh 1}^\infty e^{-2hx}(1+x)^r\,dx \\
&\qquad \leqslant C_r\left(1+\frac{1}{h}\right)^{r+1}.
\end{aligned}$$

Thus $J(r, h) \leqslant C_r(1+1/h)^{(r+1)/l}$. Q.E.D.

6. Universal Cover of $SU(1, 1)$

In this section we will illustrate the method of Section 5 when G is the universal covering group of $SU(1, 1)$. Explicit formulas for the matrix coefficients in this case were obtained by Sally in [6].

In this case $K = Z_K^0$ so that there is a single one-parameter family $\chi_h = e^{(1+h)\nu} \in \hat{K}$ corresponding to holomorphic discrete series for $h > 0$. The K-finite holomorphic sections of $\mathbf{V}_h \to G/K$ are the finite linear combinations of the

$$f_{h,m}: g \to \zeta(g)^m\,\chi_h^{-1}(g), \qquad m = 0, 1, 2, \ldots \tag{6.1}$$

As in (5.4) we write $Z_K^0 = \{z_\theta : \theta \in \mathbf{R}\}$, where $q(z_\theta) = \left[\begin{smallmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}\end{smallmatrix}\right] \in SU(1, 1)$. Then for any $g \in G$, $\zeta(z_\theta^{-1}g) = e^{-i\theta}\zeta(g)$ and $\kappa(z_\theta^{-1}g) = z_\theta^{-1}\kappa(g)$, so

$$f_{h,m}(z_\theta^{-1}g) = e^{im\theta}\chi_h(z_\theta)\,f_{h,m}(g). \tag{6.2}$$

Thus $f_{h,m}$ is a weight vector of weight $(1+h)\nu - m\alpha$, for the left action of $K = Z_K^0$, where $\{\alpha\} = \Phi(\mathfrak{p}_+)$.

For fixed n, $m \geqslant 0$ we will compute $\phi_h(a) = \langle L(a) f_{h,n}, f_{h,m}\rangle$, $a \in \mathrm{cl}(A^+)$. Using (6.1), (6.2) and the integration formulas of Section 4,

$$\begin{aligned}
\phi_h(a) = \int_{Z_K^0/Z_G}\int_{A^+} &\zeta(a^{-1}z_\theta a_1)\,\chi_h(a^{-1}z_\theta a_1)^{-1}\,e^{-im\theta}\chi_h(z_\theta) \\
&\times \zeta(a_1)^m\,\chi_h(a_1)^{-1}\,D(a_1)\,da_1\,dz_\theta.
\end{aligned} \tag{6.3}$$

Write $A^+ = \{a_t\colon t>0\}$ where

$$q(a_t)=\begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}\in SU(1,1).$$

LEMMA 6.4. *For* $y\geqslant 0$

$$\phi_h(a_y)=(\cosh y)^{-(h+1)}\int_0^\infty (\tanh t)^m(\cosh t)^{-2(h+1)}(e^{2t}-e^{-2t})\, I_h(a_y,a_t)\,dt$$

where

$$I_h(a_y,a_t)$$
$$=\frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\left[\frac{e^{i\theta}\tanh t-\tanh y}{1-e^{i\theta}\tanh y\tanh t}\right]^n(1-e^{i\theta}\tanh y\tanh t)^{-(h+1)}\,d\theta.$$

Proof. As in example (2.2), if $q(g)=\left(\begin{smallmatrix}\bar v & x\\ \bar x & v\end{smallmatrix}\right)\in SU(1,1)$, $\zeta(g)=\left[\begin{smallmatrix}0 & xv^{-1}\\ 0 & 0\end{smallmatrix}\right]$ and $\bar\kappa(g)=\left[\begin{smallmatrix}v^{-1} & 0\\ 0 & v\end{smallmatrix}\right]$. We will identify $\mathfrak{p}_+$ with $\mathbf{C}$ and write $\zeta(g)=xv^{-1}$. Since $\nu=-\alpha/2$ is the negative of the fundamental highest weight we have $e^\nu(\bar\kappa(g))=v$. It follows that $\zeta(a_t)^m=(\tanh t)^m$, $\chi_h(a_t)^{-1}=(\cosh t)^{-(h+1)}$, and $\chi_h(z_\theta)=e^{-i(h+1)\theta/2}$. Recall that $D(a)=\prod_{\alpha\in\Phi_\mathfrak{a}^+}|e^\alpha(a)-e^{-\alpha}(a)|^{m_\alpha}$, so that in the present case, $D(a_t)=|e^{2t}-e^{-2t}|$.

Now $q(a_y^{-1}z_\theta a_t)=\left(\begin{smallmatrix}\bar v & x\\ \bar x & v\end{smallmatrix}\right)$, where $x=e^{i\theta/2}\cosh y\sinh t-e^{-i\theta/2}\sinh y\cosh t$ and $v=e^{-i\theta/2}\cosh y\cosh t-e^{i\theta/2}\sinh y\sinh t$. Thus $\zeta(a_y^{-1}z_\theta a_t)=xv^{-1}=(e^{i\theta}\tanh t-\tanh y)(1-e^{i\theta}\tanh y\tanh t)^{-1}$ and $e^\nu(\bar\kappa(a_y^{-1}z_\theta a_t))=v=e^{-i\theta/2}\cosh y\cosh t(1-e^{i\theta}\tanh y\tanh t)$. Now the lifting $\kappa(a_y^{-1}z_\theta a_t)$ must satisfy $e^{-(h+1)\nu}(\kappa(a_y^{-1}z_\theta a_t))=e^{i(h+1)\theta/2}(\cosh y)^{-(h+1)}(\cosh t)^{-(h+1)}(1+e^{i\theta}\tanh y\tanh t)^{-(h+1)}$, where the last term is defined by taking the principal branch of the logarithm since $\mathrm{Re}\,(1-e^{i\theta}\tanh y\tanh t)>0$. Q.E.D.

Using the change of variables $s=e^{i\theta}$ we can rewrite

$$I_h(a_y,a_t)=\frac{1}{2\pi i}\int_{|s|=1} s^{-(m+1)}b(s,a_y,a_t)\,B_h(s,a_y,a_t)\,ds \qquad (6.5)$$

where

$$b(s,a_y,a_t)=\left(\frac{s\tanh t-\tanh y}{1-s\tanh y\tanh t}\right)^n$$

and

$$B_h(s,a_y,a_t)=(1-s\tanh y\tanh t)^{-(h+1)}.$$

For fixed y, $t \geqslant 0$, $0 \leqslant \tanh y \tanh t < 1$ so that b and B_h are holomorphic for s in the neighborhood of $|s| \leqslant 1$ given by $|s| < 1/\tanh y \tanh t$. For any $r \geqslant 0$, $(\partial^r b/\partial s^r)$ $(0, a_y, a_t)$ is a polynomial in $\tanh y$ and $\tanh t$ and $(\partial^r B_h/\partial s^r)(0, a_y, a_t) = (h+1)\cdots(h+r)(\tanh y \tanh t)^r$. Thus $I_h(a_y, a_t) = \text{Residue}_{s=0}\{s^{-(m+1)}(bB_h)(s)\} = (1/m!)(\partial^m/\partial s^m)(b \cdot B_h)(0)$ is a polynomial in h, $\tanh y$ and $\tanh t$. We have proved the following version of Theorem 5.1.

THEOREM 6.6. *For* $y \geqslant 0$,

$$\phi_h(a_y) = (\cosh y)^{-(h+1)} \int_0^\infty (\cosh t)^{-2(h+1)}(e^{2t} - e^{-2t})\, p(h, \tanh y, \tanh t)\, dt$$

where $p(h, \tanh y, \tanh t) = (\tanh t)^m I_h(a_y, a_t)$ *is a polynomial in* h, $\tanh y$, *and* $\tanh t$.

7. WAVE PACKETS

Let f_h, f'_h, $h > 0$ be one-parameter families of K-finite sections of the bundles $\mathbf{V}_h \to G/K$ as in (3.6), ϕ_h the corresponding one-parameter family of matrix coefficients given by $\phi_h(x) = \langle L(x) f_h, f'_h \rangle$, $x \in G$. Then each $\phi_h \in {}^0\mathscr{C}(G/Z, \zeta_h)$ for an appropriate $\zeta_h \in \hat{Z}$. In this section we will show how to form "wave packets" of the ϕ_h which are in the space of global cusp forms, ${}^0\mathscr{C}(G)$.

Write $\mathbf{R}^+$ for the open interval $(0, \infty)$. Given $\alpha \in C^\infty(\mathbf{R}^+)$ and $r, s \geqslant 0$, $k = 0, 1, 2,\ldots$, we define

$$\|\alpha\|_{r,s,k} = \sup_{h>0} (1+h)^r \left(1 + \frac{1}{h}\right)^s \left|\frac{d^k}{dh^k} \alpha(h)\right|. \tag{7.1a}$$

Then we set

$$\mathscr{C}(\mathbf{R}^+) = \{\alpha \in C^\infty(\mathbf{R}^+) \colon \|\alpha\|_{r,s,k} < \infty \text{ for all } r, s \geqslant 0,\ k = 0, 1, 2,\ldots\} \tag{7.1b}$$

Corresponding to a one-parameter family ϕ_h of matrix coefficients we define wave packets ϕ_α by

$$\phi_\alpha(x) = \int_0^\infty \alpha(h)\, \phi_h(x)\, dh, \qquad x \in G, \alpha \in \mathscr{C}(\mathbf{R}^+). \tag{7.2}$$

THEOREM 7.3. (a) *The integral defining* $\phi_\alpha(x)$ *is absolutely convergent for any* $\alpha \in \mathscr{C}(\mathbf{R}^+)$, *uniformly for* $x \in G$.

(b) $\phi_\alpha \in \mathscr{C}(G)$ *and* $\alpha \to \phi_\alpha$ *is a continuous mapping of* $\mathscr{C}(\mathbf{R}^+)$ *into* $\mathscr{C}(G)$; *i.e. given* $r \geqslant 0$, D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$, *there is a continuous seminorm* μ *on* $\mathscr{C}(\mathbf{R}^+)$ *so that for all* $\alpha \in \mathscr{C}(\mathbf{R}^+)$,

$$\sup_{x \in G} \Xi^{-1}(x)(1+\tilde{\sigma}(x))^r \,|\phi_\alpha(D_1;x;D_2)| \leqslant \mu(\alpha).$$

(c) $\phi_\alpha \in {}^0\mathscr{C}(G)$.

The proof of the theorem will be given as a series of lemmas.

LEMMA 7.4. *For all* $\alpha \in \mathscr{C}(\mathbf{R}^+)$, $\int_0^\infty \alpha(h)\,\phi_h(x)\,dh$ *converges absolutely, uniformly for* $x \in G$.

Proof. Let U_h be a one-parameter family of K-invariant spaces containing both f_h and f'_h, (τ_h, W) the corresponding abstract K-modules. Let w, $w' \in W$ correspond to f_h and f'_h, respectively. As in (3.11), we write, for $k \in K$ and $x \in G$, $f_h(kx) = \sum_i \langle \tau_h(k^{-1})\,w, w_i^*\rangle\, f_{h,i}(x)$ and $f'_h(kx) = \sum_i \langle \tau_h(k^{-1})w', w_i^*\rangle\, f_{h,i}(x)$. Then if we write $x \in G$ as $x = k_1 a k_2$, k_1, $k_2 \in K$, $a \in \mathrm{cl}(A^+)$,

$$\begin{aligned}\phi_h(k_1 a k_2) &= \int_{G/K} \langle f_h(k_2^{-1}a^{-1}y), f'_h(k_1 y)\rangle\, dy\\ &= \sum_{i,j} \langle \tau_h(k_2)w, w_i^*\rangle \overline{\langle \tau_h(k_1^{-1})w', w_j^*\rangle}\\ &\quad\times \int_{G/K} \langle f_{h,i}(a^{-1}y), f_{h,j}(y)\rangle\, dy.\end{aligned}$$

Thus $|\phi_h(k_1 a k_2)| \leqslant C \sum_{i,j} |\phi_{h,i,j}(a)|$ where

$$C = \max_{i,j} \sup_{k_1,k_2 \in K} |\langle \tau_h(k_2)w, w_i^*\rangle|\, |\langle \tau_h(k_1^{-1})w', w_j^*\rangle|$$

is finite and independent of h since $|e^{h\nu}(k)| = 1$ for all $k \in K$, and $\phi_{h,i,j}$ denotes the one-parameter family of matrix coefficients corresponding to $f_{h,i}$ and $f_{h,j}$. Now using Corollary 5.2 for each of the $\phi_{h,i,j}$ we find a constant $C \geqslant 0$ and an integer $m \geqslant 0$ so that $|\phi_h(k_1 a k_2)| \leqslant C(1+h)^m(1+1/h)^l\, e^{-(\lambda_0 + h\nu)}(\kappa(a))$. But using (3.13b), (3.17), and (3.18b), $e^{-(\lambda_0+h\nu)}(\kappa(a)) \leqslant e^{-\lambda_0}(\kappa(a)) \leqslant e^{\rho}(\kappa(a)) \leqslant 1$ for all $a \in \mathrm{cl}(A^+)$. Thus for all $x \in G$,

$$\begin{aligned}\int_0^\infty |\alpha(h)|\,|\phi_h(x)|\,dh &\leqslant C\int_0^\infty |\alpha(h)|\,(1+h)^m\left(1+\frac{1}{h}\right)^l dh\\ &\leqslant C'\,\|\alpha\|_{m+2,l,0}\end{aligned}$$

since $\int_0^\infty (1+h)^{-2}\,dh < \infty$. Q.E.D.

LEMMA 7.5. *There is a constant* $C>0$ *so that for all* $x\in G$, $\alpha\in\mathscr{C}(\mathbf{R}^+)$, $\sup_{k_1,k_2\in[K,K]}|\phi_\alpha(k_1xk_2)|\leqslant C\sum_{i,j}|\phi_{\alpha,i,j}(x)|$ *where the* $\phi_{\alpha,i,j}$ *are a finite collection of wave packets corresponding to one-parameter families of matrix coefficients* $\phi_{h,i,j}$.

Proof. As in the proof of Lemma 7.4 we write for k_1, $k_2\in[K,K]$, $x\in G$, $\phi_h(k_1xk_2)=\sum_{i,j}\langle\tau_h(k_2)w, w_i^*\rangle\,\overline{\langle\tau_h(k_1^{-1})w', w_j^*\rangle}\,\phi_{h,i,j}(x)$. But for $k\in[K,K]$, $e^{h\nu}(k)=1$ so that $\tau_h(k)=\tau(k)$ is independent of h. Thus

$$\begin{aligned}\sup_{k_1,k_2\in[K,K]}&|\phi_\alpha(k_1xk_2)|\\&=\sup_{k_1,k_2\in[K,K]}\left|\sum_{i,j}\langle\tau(k_2)w, w_i^*\rangle\overline{\langle\tau(k_1^{-1})w', w_j^*\rangle}\,\phi_{\alpha,i,j}(x)\right|\\&\leqslant C\sum_{i,j}|\phi_{\alpha,i,j}(x)|,\end{aligned}$$

where $C=\max_{i,j}\sup_{k_1,k_2\in[K,K]}|\langle\tau(k_2)w, w_i^*\rangle\overline{\langle\tau(k_1^{-1})w', w_j^*\rangle}|$. Q.E.D.

LEMMA 7.6. *For each nonnegative integer* r *there is a constant* $C_r>0$ *so that for all* $x\in G$ *and* $\alpha\in\mathscr{C}(\mathbf{R}^+)$,

$$\begin{aligned}\sup_{z_1,z_2\in Z_K^0}&(1+\tilde\sigma(z_1z_2))^r\,|\phi_\alpha(z_1xz_2)|\\&\leqslant C_r\sum_{i,j}\max\left\{\int_0^\infty\left|\frac{\partial^r}{\partial h^r}(\alpha(h)\,\phi_{h,i,j}(x))\right|dh,\ \int_0^\infty|\alpha(h)\,\phi_{h,i,j}(x)|\,dh\right\}\end{aligned}$$

where the $\phi_{h,i,j}$ *are a finite collection of one-parameter families of matrix coefficients.*

Proof. We use the notation from the proof of Lemma 7.4. The one-parameter families $f_{h,j}$ can be chosen to be weight vectors so that for each j there is an integer n_j so that if $z_t=\exp(tH)\in Z_K^0$, $f_{h,j}(z_t^{-1}x)=e^{i(h+n_j)t}f_{h,j}(x)$ for all $x\in G$. Write $z_i=\exp(t_iH)$, $i=1,2$. Then $\tilde\sigma(z_1z_2)=|t_1+t_2|$ and

$$\begin{aligned}\phi_h(z_1xz_2)&=\sum_{i,j}\langle w, w_i^*\rangle\overline{\langle w', w_j^*\rangle}\int_{G/K}\langle f_{h,i}(z_2^{-1}x^{-1}y), f_{h,j}(z_1y)\rangle\,dy\\&=\sum_{i,j}\langle w, w_i^*\rangle\overline{\langle w', w_j^*\rangle}\,e^{i(h+n_i)t_2}\overline{e^{-i(h+n_j)t_1}}\phi_{h,i,j}(x).\end{aligned}$$

Thus $|\phi_\alpha(z_1 x z_2)| \leqslant C \sum_{i,j} |\int_0^\infty \alpha(h)\, e^{ih(t_1+t_2)} \phi_{h,i,j}(x)\, dh|$. Now for $t = t_1 + t_2 \neq 0$, and any integer $r \geqslant 0$,

$$\left| \int_0^\infty \alpha(h)\, e^{iht} \phi_{h,i,j}(x)\, dh \right|$$

$$= \left| \int_0^\infty \frac{e^{iht}}{(it)^r} \frac{\partial^r}{\partial h^r} (\alpha(h)\, \phi_{h,i,j}(x))\, dh \right|$$

$$\leqslant |t|^{-r} \int_0^\infty \left| \frac{\partial^r}{\partial h^r} (\alpha(h)\, \phi_{h,i,j}(x)) \right| dh.$$

Thus for $|t| \geqslant 1$,

$$(1 + |t|)^r\, |\phi_\alpha(z_1 x z_2)|$$

$$\leqslant 2^r C \sum_{i,j} \int_0^\infty \left| \frac{\partial^r}{\partial h^r} (\alpha(h)\, \phi_{h,i,j}(x)) \right| dh.$$

But for $|t| \leqslant 1$, $(1+|t|)^r\, |\phi_\alpha(z_1 x z_2)| \leqslant 2^r C \sum_{i,j} \int_0^\infty |\alpha(h)\, \phi_{h,i,j}(x)|\, dh$. Thus the conditions of the lemma are satisfied by $C_r = 2^r C$. Q.E.D.

LEMMA 7.7. *For any $r \geqslant 0$ there is a continuous seminorm μ on $\mathscr{C}(\mathbf{R}^+)$ so that*

$$\sup_{x \in G} \Xi^{-1}(x)(1 + \tilde{\sigma}(x))^r\, |\phi_\alpha(x)| \leqslant \mu(\alpha) \qquad \text{for all} \quad \alpha \in \mathscr{C}(\mathbf{R}^+).$$

Proof. Write $x \in G$ as $k_1 z_1 a z_2 k_2$, where $k_1, k_2 \in [K, K]$, $z_1, z_2 \in Z_K^0$, and $a \in \mathrm{cl}(A^+)$. Then by (4.2c),

$$\Xi^{-1}(x)(1 + \tilde{\sigma}(x))^r\, |\phi_\alpha(x)| = \Xi^{-1}(a)(1 + \tilde{\sigma}(z_1 a z_2))^r\, |\phi_\alpha(k_1 z_1 a z_2 k_2)|.$$

By Lemma 7.5, $\sup_{k_1, k_2 \in [K,K]} |\phi_\alpha(k_1 z_1 a z_2 k_2)| \leqslant C \sum_{i,j} |\phi_{\alpha,i,j}(z_1 a z_2)|$, where we assume the $f_{h,i}$ are weight vectors.

Now by (4.2b) and (4.2d), $\tilde{\sigma}(z_1 a z_2) = \tilde{\sigma}(z_2 z_1 a) \leqslant 3(\tilde{\sigma}(z_2 z_1) + \sigma(a))$. Thus $(1 + \tilde{\sigma}(z_1 a z_2))^r \leqslant C_r (1 + \tilde{\sigma}(z_1 z_2))^r (1 + \sigma(a))^r$, where we write C_r generically for any constant depending on r. We may as well assume that r is an integer. Using Lemma 7.6,

$$\sup_{z_1, z_2 \in Z_K^0} (1 + \tilde{\sigma}(z_1 z_2))^r\, |\phi_{\alpha,i,j}(z_1 a z_2)|$$

$$\leqslant C_r \max \left\{ \int_0^\infty \left| \frac{\partial^r}{\partial h^r} (\alpha(h)\, \phi_{h,i,j}(x)) \right| dh, \int_0^\infty |\alpha(h) \phi_{h,i,j}(x)|\, dh \right\}.$$

Thus it suffices to show that for any $r \geqslant 0$, any nonnegative integer s, and any one-parameter family ϕ_h of K-finite matrix coefficients, there is a continuous seminorm μ so that

$$\sup_{a \in \mathrm{cl}(A^+)} \Xi^{-1}(a)(1+\sigma(a))^r \int_0^\infty \left| \frac{\partial^s}{\partial h^s} (\alpha(h)\,\phi_h(a)) \right| dh \leqslant \mu(\alpha)$$

for all $\alpha \in \mathscr{C}(\mathbf{R}^+)$, or equivalently, that for any two integers $p, q \geqslant 0$, there is μ so that

$$\sup_{a \in \mathrm{cl}(A^+)} \Xi^{-1}(a)(1+\sigma))^r \int_0^\infty |\alpha^{(p)}(h)\,\phi_h^{(q)}(a)|\, dh \leqslant \mu(\alpha).$$

But, using Corollary 5.2,

$$\begin{aligned} &\sup_{a \in \mathrm{cl}(A^+)} \Xi^{-1}(a)(1+\sigma(a))^r \int_0^\infty |\alpha^{(p)}(h)|\,|\phi_h^{(q)}(a)|\, dh \\ &\quad \leqslant \sup_{a \in \mathrm{cl}(A^+)} C_q \Xi^{-1}(a)\, e^{-\lambda_0}(\kappa(a))(1+\sigma(a))^{r+q} \\ &\quad\quad \times \int_0^\infty (1+h)^m \left(1+\frac{1}{h}\right)^{l(q+1)} e^{-h\nu}(\kappa(a))\,|\alpha^{(p)}(h)|\, dh. \end{aligned}$$

By (4.4), $\Xi^{-1}(a) \leqslant e^{\rho_a}(a)$ for all $a \in \mathrm{cl}(A^+)$ so that as in Corollary 5.2, $e^{-\lambda_0}(\kappa(a))\,\Xi^{-1}(a) \leqslant Ce^{-(\lambda_0+\rho)}(\kappa(a)) \leqslant C$. Further, for all $h > 0$,

$$\begin{aligned} &\sup_{a \in \mathrm{cl}(A^+)} (1+\sigma(a))^{r+q} e^{-h\nu}(\kappa(a)) \\ &\quad \leqslant C_{r+q} \sup_{s_i \geqslant 0} \left(\prod_{i=1}^{l} \cosh s_i\right)^{-h} \left(1+\sum_{i=1}^{l} s_i\right)^{r+q} \\ &\quad \leqslant C_{r+q} \textstyle\prod_{i=1}^{l} \{\sup_{s_i \geqslant 0} (\cosh s_i)^{-h}(1+s_i)^{r+q}\}. \end{aligned}$$

But

$$\begin{aligned} &\sup_{s \geqslant 0} (\cosh s)^{-h}(1+s)^{r+q} \\ &\quad \leqslant C_{r+q} \sup_{s \geqslant 0} (\cosh s)^{-h}(1+\log\cosh s)^{r+q} \\ &\quad \leqslant C_{r+q} \sup_{x \geqslant 0} e^{-hx}(1+x)^{r+q} \leqslant C_{r+q}\left(1+\frac{1}{h}\right)^{r+q}. \end{aligned}$$

Thus

$$\sup_{a \in \mathrm{cl}(A^+)} \Xi^{-1}(a)(1+\sigma(a))^r \int_0^\infty |\alpha^{(p)}(h)|\, |\phi_h^{(q)}(a)|\, dh$$

$$\leqslant C_{r,q} \int_0^\infty (1+h)^m \left(1+\tfrac{1}{h}\right)^{l(2q+r+1)} |\alpha^{(p)}(h)|\, dh$$

$$\leqslant C_{r,q} \|\alpha\|_{m+2,l(2q+r+1),p}$$

which gives a continuous seminorm on $\mathscr{C}(\mathbf{R}^+)$. Q.E.D.

LEMMA 7.8. *Let $D \in \mathscr{U}(\mathfrak{g})$ and let f_h be a one-parameter family of K-finite sections. Then there are a finite collection of one-parameter families of K-finite sections $f_{h,i}$ and polynomials p_i of degree bounded by* $\deg D$, *so that $\pi_h(D) f_h = \sum_i p_i(h) f_{h,i}$.*

Proof. It suffices to prove the lemma when $D = X \in \mathfrak{g}$. Write $f_h(g) = \chi_h(g)^{-1} F(gK)$, where $F: G/K \to V$ is a V-valued polynomial. Now for each fixed $h > 0$, $\pi_h(X) f_h$ is a K-finite holomorphic section of $\mathbf{V}_h \to G/K$, so it is of the form $\pi_h(X) f_h(g) = \chi_h(g)^{-1} F_h^X(gK)$, where F_h^X: $G/K \to V$ is a V-valued polynomial with coefficients depending on h, i.e., $F_h^X(gK) = \sum_{n,b} c_{n,b}(h) F_{n,b}(gK)$, where the $F_{n,b}$: $G/K \to V$ are monomials defined in (3.5a).

But differentiating directly,

$$\pi_h(X) f_h(g) = \frac{d}{dt}\Big|_{t=0} f_h(\exp(-tX)\, g)$$

$$= \frac{d}{dt}\Big|_{t=0} \chi_h(\exp(-tX)\, g)^{-1} F(gK)$$

$$+ \chi_h(g)^{-1} \frac{d}{dt}\Big|_{t=0} F(\exp(-tX)\, gK).$$

Now

$$\kappa(\exp(-tX)\, g) = \kappa(\exp(-tX) \exp(\zeta(g)))\, \kappa(g)$$

and $\chi_h = \chi_0 \otimes e^{hv}$ so that

$$\frac{d}{dt}\Big|_{t=0} \chi_h(\exp(-tX)\, g)^{-1} = \chi_h(g)^{-1} \left\{ \frac{d}{dt}\Big|_{t=0} \chi_0(\exp(-tX) \exp(\zeta(g)))^{-1} \right.$$

$$\left. + h \frac{d}{dt}\Big|_{t=0} e^{-v}(\exp(-tX) \exp(\zeta(g))) \right\}.$$

Thus

$$F_h^X(gK) = \frac{d}{dt}\Big|_{t=0} F(\exp(-tX)\,gK)$$
$$+\frac{d}{dt}\Big|_{t=0} \chi_0^{-1}(\exp(-tX)\exp(\zeta(g)))\,F(gK)$$
$$+h\frac{d}{dt}\Big|_{t=0} e^{-\nu}(\exp(-tX)\exp(\zeta(g)))\,F(gK)$$

is a linear function of h. Thus the coefficients $c_{n,b}(h)$ of F_h^X are linear functions of h, and since for each h, $c_{n,b}(h)=0$ for all but finitely many pairs (n, b), there are only finitely many that are not identically zero. Thus the collection $f_{h,n,b}(g)=\chi_h(g)^{-1}F_{n,b}(gK)$ satisfies $\pi_h(X)f_h = \sum_{n,b} c_{n,b}(h)\,f_{h,n,b}$. Q.E.D.

LEMMA 7.9. *Given $r \geqslant 0$, D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$, there is a continuous seminorm μ on $\mathscr{C}(\mathbf{R}^+)$ so that for all $\alpha \in \mathscr{C}(\mathbf{R}^+)$,*

$$\sup_{x\in G} \Xi^{-1}(x)(1+\tilde{\sigma}(x))^r\,|\phi_\alpha(D_1;x;D_2)| \leqslant \mu(\alpha).$$

Proof. Let D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$. Then since the integral converges absolutely, uniformly for $x\in G$, $\phi_\alpha(D_1;x;D_2) = \int_0^\infty \alpha(h)\,\phi_h(D_1;x;D_2)\,dh$ where $\phi_h(D_1;x;D_2) = \langle \pi_h(D_1)\,\pi_h(x)\,\pi_h(D_2)f_h, f'_h\rangle = \langle \pi_h(x)\,\pi_h(D_2)f_h, \pi_h(D_1)^* f'_h\rangle$. Using Lemma 7.8 we write $\pi_h(D_2)f_h = \sum_i p_i(h)f_{h,i}$ and $\pi_h(D_1)^* f'_h = \sum_j g_j(h)f'_{h,j}$, where the $f_{h,i}$, $f'_{h,j}$ are one-parameter families of K-finite sections and the p_i, q_j are polynomials. Then $\phi_h(D_1;x,D_2) = \sum_{i,j} p_i(h)\,\overline{q_j(h)}\,\phi_{h,i,j}(x)$, where $\phi_{h,i,j}(x) = \langle L(x)f_{h,i}, f'_{h,j}\rangle$. Thus

$$|\phi_\alpha(D_1;x;D_2)| \leqslant \sum_{i,j}\left|\int_0^\infty p_i(h)\,\overline{q_j(h)}\,\alpha(h)\,\phi_{h,i,j}(x)\,dh\right|.$$

But since p_i, q_j are polynomials, $\alpha_{ij} = p_i\bar{q}_j\alpha \in \mathscr{C}(\mathbf{R}^+)$. Using Lemma 7.7 there are continuous seminorms μ_{ij} on $\mathscr{C}(\mathbf{R}^+)$ so that $\sup_{x\in G}\Xi^{-1}(x)(1+\tilde{\sigma}(x))^r\,|\phi_\alpha(D_1;x;D_2)| \leqslant \sum_{i,j}\mu_{ij}(\alpha_{ij})$ for all $\alpha\in\mathscr{C}(\mathbf{R}^+)$. But $\alpha\mapsto\alpha_{ij}$ is a continuous map of $\mathscr{C}(\mathbf{R}^+)$ into itself so there are continuous seminorms ν_{ij} so that $\mu_{ij}(\alpha_{ij}) \leqslant \nu_{ij}(\alpha)$ for all $\alpha\in\mathscr{C}(\mathbf{R}^+)$. Thus we can take $\mu = \sum \nu_{ij}$. Q.E.D.

We now show that $\phi_\alpha \in {}^0\mathscr{C}(G)$. Because of Lemma 4.8 and Theorem 7.3(b), it suffices to show that $(\phi_\alpha)_\zeta \in {}^0\mathscr{C}(G/Z,\zeta)$ for all $\zeta\in\hat{Z}$. Recall that $Z = Z_G \cap Z_K^0 = \{z_\theta : \theta\in 2\pi\mathbf{Z}\}$. Write $z(n) = z_{2\pi n}$, $n\in\mathbf{Z}$. Let $\zeta_h =$

$e^{-(\lambda_0+h\nu)}|_Z$. Then $\zeta_h=\zeta_{h+dn}$ for all $n\in\mathbf{Z}$ and $\hat{Z}=\{\zeta_n\colon 0<h\leqslant d\}$, where $\nu(H)=i/d$. Note

$$\zeta_h(z(n))=\zeta_0(z(n))\,e^{-\pi inh2/d}, \qquad h\in\mathbf{R},\, n\in\mathbf{Z}; \tag{7.10}$$

$$\phi_h(xz(n))=\overline{\zeta_h(z(n))}\,\phi_h(x), \qquad x\in G,\, n\in\mathbf{Z},\, h>0. \tag{7.11}$$

LEMMA 7.12. *For $0<h\leqslant d$ and any $x\in G$, $(\phi_\alpha)_{\zeta_n}(x)=d\sum_{n=0}^{\infty}\alpha(h+dn)\,\phi_{h+dn}(x)$. The sum converges absolutely, uniformly for $x\in G$ and $0<h\leqslant d$.*

Proof. The statement about convergence of the sum is proved using the estimate $|\phi_h(x)|\leqslant C(1+h)^m(1+1/h)^l$ of (7.4) and the definition of $\mathscr{C}(\mathbf{R}^+)$.

The equality is the Poisson summation formula for

$$F(h)=\begin{Bmatrix}\alpha(h)\,\phi_h(x), & h>0\\ 0, & h\leqslant 0\end{Bmatrix}.$$

$F\in L_1(\mathbf{R})\cap C^\infty(\mathbf{R})$ and for any $n\in\mathbf{Z}$, using (7.10) and (7.11),

$$\begin{aligned}\hat{F}(-\pi n2/d)&=\int_0^\infty \alpha(h)\,\phi_h(x)\,e^{\pi inh2/d}\,dh\\ &=\zeta_0(z(n))\int_0^\infty \alpha(h)\,\phi_h(xz(n))\,dh\\ &=\zeta_0(z(n))\,\phi_\alpha(xz(n)).\end{aligned}$$

Define $H(h)=d\sum_{j=-\infty}^{\infty}F(h+dj)$. Then $H\in C^\infty(\mathbf{R}/d\mathbf{Z})$ and $\hat{H}(n)=\hat{F}(\pi n2/d)$ for all $n\in\mathbf{Z}$ so that for $0<h\leqslant d$,

$$\begin{aligned}H(h)&=\sum_{n=-\infty}^{\infty}\hat{F}(-\pi n2/d)\,e^{-\pi inh2/d}\\ &=\sum_{n=-\infty}^{\infty}\phi_\alpha(xz(n))\,\zeta_h(z(n))=(\phi_\alpha)_{\zeta_h}(x).\end{aligned}$$ Q.E.D.

LEMMA 7.13. $\phi_\alpha\in{}^0\mathscr{C}(G)$.

Proof. Since $\phi_\alpha\in\mathscr{C}(G)$ we know that $(\phi_\alpha)_\zeta\in\mathscr{C}(G/Z,\zeta)$ for all $\zeta\in\hat{Z}$. It remains to show tht $(\phi_\alpha)_\zeta^P(x)=0$ for every $x\in G$ and proper parabolic subgroup P of G. Write $\zeta=\zeta_h$, $0<h\leqslant d$, and $P=MAN$. Then using Lemma 7.12,

$$\begin{aligned}(\phi_\alpha)_{\zeta_h}^P(x)&=\int_N(\phi_\alpha)_{\zeta_h}(xn)\,dn\\ &=d\int_N\sum_{m=0}^{\infty}\alpha(h+dm)\,\phi_{h+dm}(xn)\,dn.\end{aligned}$$

But the sum converges absolutely, so that we can integrate term by term, and for all $m \geqslant 0$, $\int_N \phi_{h+dm}(xn)\,dn = 0$ since $\phi_{h+dm} \in {}^0\mathscr{C}(G/Z, \zeta_h)$ by Corollary 5.7 of [4]. Q.E.D.

REFERENCES

0. HARISH-CHANDRA, Representations of semisimple Lie groups, V, *Amer. J. Math.* **78** (1956), 1–41.
1. HARISH-CHANDRA, Representations of semisimple Lie groups, VI, *Amer. J. Math.* **78** (1956), 564–628.
2. HARISH-CHANDRA, Harmonic analysis on real reductive groups, I, *J. Funct. Anal.* **19** (1975), 104–204.
3. R. A. HERB AND J. A. WOLF, The Plancherel theorem for general semisimple groups, *Compositio Math.* **57** (1986), 271–355.
4. R. A. HERB AND J. A. WOLF, Rapidly decreasing functions on general semisimple groups, *Compositio Math.* **58** (1986), 73–110.
5. C. C. MOORE, Compactifications of symmetric spaces, II. The Cartan domains, *Amer. J. Math.* **86** (1964), 358–378.
6. P. J. SALLY, JR., "Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group of $SL(2; \mathbf{R})$," Memoirs Amer. Math. Soc., No. 69, Amer. Math. Soc., Providence, RI, 1967.
7. J. A. TIRAO, Square integrable representations of semisimple Lie groups, *Trans. Amer. Math. Soc.* **190** (1974), 57–75.
8. J. A. WOLF, Fine structure of hermitian symmetric spaces, *in* "Symmetric Spaces" (Boothby and Weiss, Eds.), pp. 271–357, Dekker, New York, 1972.
9. J. A. WOLF, "Unitary Representations on Partially Holomorphic Cohomology Spaces," Memoirs Amer. Math. Soc., No. 138, Amer. Math. Soc., Providence, RI, 1974.