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BOUNDED ISOMETRIES AND HOMOGENEOUS RIEMANNIAN QUOTIENT MANIFOLDS

ABSTRACT. Let M be a Riemannian manifold that admits a transitive semisimple group G' of isometries, G' of noncompact type. Then every bounded isometry of M centralizes G' and so is a Clifford translation (constant displacement). Thus a Riemannian quotient $\Gamma\backslash M$ is homogeneous if and only if Γ consists of Clifford translations of M. The technique of proof also leads to a determination of the group of all isometries of M.

1. INTRODUCTION

Let M be a connected homogeneous Riemannian manifold, Γ a properly discontinuous group of isometries acting freely on M, and $\bar{M} = \Gamma\backslash M$ the Riemannian quotient manifold. In a variety of cases (see the discussion below) it is known that the conditions

(1.1) Γ is a group of Clifford translations (isometries of constant displacement) of M

(1.2) $\bar{M}$ is a homogeneous Riemannian manifold

are equivalent. In this paper we prove that (1.1) and (1.2) are equivalent when M admits a transitive semisimple group of isometries that has no compact local factor. Examples of such manifolds M include the flag domains used in representation theory [16] and automorphic cohomology [9].

Equivalence of (1.1) and (1.2) was proved first for manifolds of constant curvature ([10]. [11]), later for locally symmetric spaces ([12]; also see [5] in the case where the Clifford translations form a group, and [5] and [7] for cyclic groups of Clifford translations), then for manifolds of nonpositive curvature [14]. The equivalence is also known for Riemannian nilmanifolds [13] and for some classes of nonsymmetric compact manifolds [1].

It is straightforward [10] to see that (1.2) implies (1.1). The converse depends on the structure of M and its isometry group.

When M has sufficient negative curvature, all bounded (i.e. bounded displacement) isometries of M are trivial, so Γ is reduced to $\{1\}$ in (1.1), and (1.2)

* IMAF, Córdoba, Argentina. Partially supported by Conicet, Argentina, and by IMPA, Rio de Janeiro, Brazil.
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Geometriae Dedicata **21** (1986), 21–27.

is immediate. In case M is a symmetric space of noncompact type this is equivalent to Jacques Tits' result [8] that bounded automorphisms are trivial for semisimple groups of noncompact type.

When M has sufficient nonpositive curvature, bounded isometries of M are ordinary translations along the Euclidean factor of the de Rham decomposition, so Γ is under control in (1.1), and (1.2) is immediate.

In the presence of positive curvature the matter is much more delicate, especially in the cases where the Clifford translations of M do not form a group. In our case, positive curvature may be present, but we are able to push it aside by using some Lie group structure theory. The new ingredient is a structural result [6, Th. 4.4] of Carolyn Gordon.

2. Statement of main result

G' is a connected semisimple Lie group without compact local factors. Let G' act transitively and effectively by isometries on a Riemannian manifold M.

(2.1) THEOREM. *View G' as a subgroup of the isometry group $I(M)$. Let B denote the centralizer of G' in $I(M)$. Then B is the set of all bounded isometries of M. In particular, every bounded isometry of M is a Clifford translation.*

To be more precise, let G denote the closure of G' in $I(M)$. Then G is a connected reductive Lie subgroup of $I(M)$ and G' is its derived group. Fix a base point $x_0 \in M$ and view

$$(2.2) \qquad M = G/H \quad \text{where} \quad H = \{g \in G : gx_0 = x_0\}.$$

H is compact because G is transitive on M and is closed in $I(M)$. The normalizer of H in G,

$$(2.3) \qquad N_G(H) = \{g \in G : gHg^{-1} = H\},$$

acts differentiably on M by

$$(2.4) \qquad R(u): gH \mapsto gu^{-1}H, \quad \text{i.e.} \quad gx_0 \mapsto gu^{-1}x_0$$

so we have a closed subgroup of G given by

$$(2.5) \qquad U = \{u \in N_G(H) : R(u) \in I(M)\}$$

Theorem 2.1 will be proved as a consequence of

(2.6) THEOREM. *$R(U)$ is the set of all bounded isometries of M.*

Evidently, $R(U)$ centralizes G, thus G', in $I(M)$. On the other hand, if $b \in I(M)$ centralizes G', then it is a Clifford translation since $\operatorname{dist}(bgx_0, gx_0) = \operatorname{dist}(gbx_0, gx_0) = \operatorname{dist}(bx_0, x_0)$ for every $g \in G'$. So Theorem 2.6 gives

(2.7) COROLLARY. *$R(U)$ is the centralizer of G' in $I(M)$, and it consists of Clifford translations of M.*

Now Theorem 2.1 follows from Theorem 2.6, and we have the result which is the main point of this paper:

(2.8) COROLLARY. *Let Γ be a discrete subgroup of $I(M)$. Then the following are equivalent:*

(1) *Γ consists of Clifford translations of M,*
(2) *Γ consists of bounded isometries of M,*
(3) *Γ is a subgroup of $R(U)$, and*
(4) *$\bar{M} = \Gamma\backslash M$ is a homogeneous Riemannian manifold.*

3. PROOF OF MAIN RESULT

Retain the notation of Section 2. View $G \times U$ as an abstract Lie group and $G' \times U$ as a dense subgroup. Define.

$$(3.1) \qquad \varphi: G \times U \to I(M) \quad \text{by} \quad \varphi(g,u): yx_0 \to gyu^{-1}x_0$$

for all $y \in G$. Write U_0 and $I_0(M)$ for the identity components of U and $I(M)$. Our first main tool is a result of Gordon [6, Th. 4.4]:

(3.2) LEMMA. *φ is a continuous homomorphism and $\varphi(G \times U_0) = \varphi(G' \times U_0) = I_0(M)$.*

In fact Gordon proves $\varphi(G' \times U_0) = I_0(M)$, and the other equality follows since φ is continuous and G is connected. Note, since H is a compact subgroup of G, that

$$(3.3) \qquad \varphi(H \times \{1\}) \text{ is a compact subgroup of } I_0(M).$$

The point of the passage from G' to G was so that we would have (3.3) available for use with our second main tool, which is a minor variation on a result of Tits [8, Th. 3, Cor. 2]:

(3.4) LEMMA. *Let α be a bounded automorphism of G, i.e. suppose that G has a compact subset $\mathscr{C}$ such that $\alpha(g^{-1})\cdot g \in \mathscr{C}$ for all $g \in G$. Then $\alpha = 1$.*

In effect, if $Z(\cdot)$ denotes the center, then α induces a bounded automorphism $\bar{\alpha}$ on $G/Z(G) = G'/Z(G')$, and $\bar{\alpha} = 1$ by [8]; then $d\alpha = 1$ on the derived algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$, so $\alpha = 1$ on the corresponding analytic subgroup which is G', and finally $\alpha = 1$ because G' is dense.

Now we can start to study the map φ.

(3.5) LEMMA. *φ has kernel $\{(z, zh): z \in Z(G) \text{ and } h \in H\}$.*

Proof. Let $\varphi(g,u)=1$. If $y\in G$ then $gyu^{-1}H=yH$, i.e. $y^{-1}gyu^{-1}\in H$. The case $y=1$ gives $gu^{-1}\in H$, so $ug^{-1}\cdot y^{-1}gyu^{-1}\in H$. As $u\in N_G(H)$ now $\alpha(y^{-1})\cdot y\in H$ where α is conjugation by g^{-1}. By (3.3), α is a bounded automorphism of G, thus is trivial by Lemma 3.4. Now $g\in Z(G)$ and $u=gh$ where $h=g^{-1}u=u^{-1}\cdot(gu^{-1})^{-1}\cdot u\in H$.

Conversely, if $z\in Z(G)$ and $h\in H$ then

$$\varphi(z,zh)\colon yx_0\mapsto zyh^{-1}z^{-1}x_0=yh^{-1}x_0=yx_0,$$

so (z,zh) is in the kernel of φ. Q.E.D.

(3.6) LEMMA. *$\varphi(G\times\{1\})$ is a closed normal subgroup of $I(M)$.*

Proof. It is closed by construction of G as the closure of G' in $I(M)$.

Gordon [6, Th. 4.1] showed that $(G'\times\{1\})$ is the subgroup of $I_0(M)$ generated by the noncompact normal simple analytic subgroups. Now $\varphi(G'\times\{1\})$ is invariant under every automorphism of $I_0(M)$, hence is normal in $I(M)$. Thus its closure $\varphi(G\times\{1\})$ is normal in $I(M)$. Q.E.D.

Proof of Theorem 2.6. Let $b\colon M\to M$ be a bounded isometry. Then $g\mapsto bgb^{-1}$ is a bounded automorphism of $I(M)$, hence also of G by Lemma 3.6, hence trivial on G by Lemma 3.4. Now b centralizes G. Let $g_0\in G$ with $bx_0=g_0x_0$. Then, for all $g\in G$,

$$b(gx_0)=gb(x_0)=gg_0(x_0). \tag{3.7}$$

If $g\in H$, replace g by gh in (3.7) to see $ghg_0x_0=gg_0x_0$, i.e. $g_0\in N_G(H)$. Thus $b=R(u)$ where $u=g_0^{-1}\in U$.

Conversely, if $u\in U$ then, as noted just before Corollary 2.7, $R(u)$ is a Clifford translation and thus is a bounded isometry. Q.E.D.

(3.8) *Remark.* $U/Z(G)$ is a compact subgroup of $G/Z(G)$. In effect, $\mathrm{Ad}_G(U)$ is closed in $\mathrm{Ad}(G)$ because it is defined by equations, and it preserves a positive definite bilinear form on the Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

(3.9) *Remark.* If G' has finite center, then $G'=G$ and U is compact.

(3.10) *Example.* Here is a family of examples for which $H\cap G'$ is noncompact, and so $G'\neq G$, i.e. G' is not closed in $I(M)$. Let G' be a noncompact simply connected semisimple Lie group corresponding to a bounded symmetric domain G'/K'. Then $K'=[K',K']\times Z(K')_0$, $Z(K')_0$ is a real vector group, and $Z(G')\cap Z(K')_0$ is a lattice in $Z(K')_0$. Let H' be any nontrivial discrete subgroup of $Z(K')_0$ such that $Z(G')\cap H'=\{1\}$. Then H' is infinite, any $\mathrm{Ad}_{G'}(H')$-invariant positive definite bilinear form on $\mathfrak{g}'$ defines a Riemannian metric on $M=G'/H'$, and G' acts transitively, effectively, and by isometries, on the

Riemannian manifold M, with noncompact isotropy group H'. Here $H' = H \cap G'$ where G is the closure of G' in $I(M)$ and H is its isotropy subgroup.

More generally, let V' be any closed subgroup of $[K', K']$, let Z' denote the finite group $Z(G') \cap V'$, and set

$$G'' = G'/Z' \quad \text{and} \quad H'' = (V'H')/Z'.$$

Then $\mathfrak{g}'$ has $\mathrm{Ad}_{G''}(H'')$-invariant positive definite bilinear forms, any such form defines a Riemannian metric on $M'' = G''/H''$, and G'' acts on M'' as a transitive group of isometries with noncompact isotropy subgroup H''.

4. Structure of the isometry group

Decompose the Lie algebra of G as

(4.1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m} = \mathfrak{h}^{\perp}$ relative to the Killing form.

Then $\mathrm{Ad}_G(H)\mathfrak{m} = \mathfrak{m}$, the projection $\pi\colon G \to M$ by $\pi(g) = gx_0$ maps $\mathfrak{m}$ isomorphically onto the tangent space $T_{x_0}(M)$, and we lift the inner product there to an inner product $\langle\, , \rangle$ on $\mathfrak{m}$. Define

(4.2) $F = \{\gamma \in \mathrm{Aut}(G) : \gamma(H) = H \text{ and } d\gamma|_{\mathfrak{m}} \text{ preserves } \langle , \rangle\}$,

(4.3) $F^0 = \{\gamma \in F : \gamma \text{ is an inner automorphism of } G\}$.

Then F^0 is a normal subgroup and F/F^0 is finite.

Let $b \in I(M)$. Then we have $g \in G$ such that $bgx_0 = x_0$. In view of Lemma 3.6, bg normalizes G and H inside $I(M)$, and thus define an automorphism

(4.4) $\gamma\colon G \to G$ by $\gamma(y) = (bg)y(bg)^{-1}$

that belongs to F. If g' is another element of G and $bg'x_0 = x_0$, and if $\gamma' \in F$ is conjugation of G by bg', then $\gamma^{-1}\gamma' \in F^0$. So (4.4) defines a continuous homomorphism

(4.5) $p\colon I(M) \to F/F^0$.

If $p(b) = 1$, then $y \mapsto byb^{-1}$ is an inner automorphism of G, so we have $g \in G$ such that bg centralizes G. Now bg is a Clifford translation of M, so by Theorem 2.6 we have $bg \in R(U)$. Thus $b \in \varphi(G \times U)$. Conversely, if $b \in \varphi(G \times U)$, then the automorphism γ of (4.4) is inner, $\gamma \in F^0$, so $p(b) = 1$. We have just proved that

(4.6) $\varphi(G \times U)$ is the kernel of p.

If $\gamma \in F$, then $g \to \gamma(g)$ defines an isometry b of $M = G/H$, and $p(b) = \gamma F^0$. In summary, we have proved

(4.7) THEOREM. *There is an exact sequence*

$$\{1\} \to \varphi(G \times U) \to I(M) \to F/F^0 \to \{1\}.$$

The case of Theorem 4.7, where M is a Riemannian symmetric space, is due to Cartan ([2] and [3]; or see [15, Th. 8.8.1]).

Consider the case where M is the group manifold G with a left invariant Riemannian metric, i.e. where G is simply transitive on M. Lemma 3.6 says that G is normal in $I(M)$. Let $x_0 = 1_G$. Simple transitivity says $I(M) = G \cdot I(M)_{x_0}$ and $G \cap I(M)_{x_0} = \{1\}$. Thus $I(M)$ is a semidirect product,

$$I(M) = G \cdot I(M)_{x_0} \text{ semidirect,} \tag{4.8}$$

and Theorem 4.7 says that

$$I(M)_{x_0} = \{\gamma \in \mathrm{Aut}(G) \colon \gamma \text{ preserves } \langle , \rangle\}. \tag{4.9}$$

This is especially interesting for the Riemannian metrics studied in [4].

One can prove (4.8) and (4.9) for arbitrary semisimple group manifolds, provided that the metric is not bi-invariant on any simple factor, so that $g \mapsto g^{-1}$ is not isometric on any simple factor. See [2] and [15, Th. 8.8.1] for bi-invariant metrics.

REFERENCES

1. Cámpoli, O. A., 'Clifford Isometries of the Real Stiefel Manifolds' (IMAF preprint, 1984).
2. Cartan, É., 'La géométrie des groupes simples', *Annali Mat.* **4** (1927), 209–256.
3. Cartan, É., 'Sur certaines forms riemanniènnes rémarquables des géométries à groupe fondamental simple', *Ann. Sci. École. Norm. Sup* **44** (1927), 345–367.
4. Dotti Miatello, I., Leite, M. L. and Miatello, R. J., 'Negative Ricci Curvature on Complex Simple Lie Groups', *Geom. Dedicata* **17** (1984), 207–218.
5. Freudenthal, H., 'Clifford–Wolf–Isometrien symmetrischer Raüme', *Math. Ann.* **150** (1963), 136–149.
6. Gordon, C. 'Riemannian Isometry Groups Containing Transitive Reductive Subgroups', *Math. Ann.* **248** (1980), 185–192.
7. Ozols, V. 'Critical Points of the Displacement Function of an Isometry', *J. Diff. Geom.* **3** (1969), 411–432.
8. J. Tits, 'Automorphismes à déplacement borné des groupes de Lie', *Topology* **3** (1964), Suppl. 1, 97–107.
9. Wells, Jr. R. O. and Wolf, J. A., 'Poincaré Series and Automorphic Cohomology on Flag Domains', *Ann. Math.* **105** (1977), 397–448.
10. Wolf, J. A., 'Sur la classification des variétés riemanniènnes homogènes à courbure constante', *C. R. Acad. Sci. Paris* **250** (1960), 3443–3445.
11. Wolf, J. A., 'Vincent's Conjecture on Clifford Translations of the Sphere', *Comment. Math. Helv.* **36** (1961), 33–41.
12. Wolf, J. A., 'Locally Symmetric Homogeneous Spaces', *Comment. Math. Helv.* **37** (1962), 65–101.

13. Wolf, J. A., 'On Locally Symmetric Spaces of Non-negative Curvature and Certain Other Locally Homogeneous Spaces', *Comment. Math. Helv.* **37** (1963), 266–295.
14. Wolf, J. A. 'Homogeneity and Bounded Isometries in Manifolds of Negative Curvature', *Illinois J. Math.* **8** (1964), 14–18.
15. Wolf, J. A., *Spaces of Constant Curvature*, (McGraw-Hill, New York, 1967); fifth edition: Publish or Perish, Wilmington, DE (1984).
16. Wolf, J. A., *Unitary Representations on Partially Holomorphic Cohomology Spaces, Memoirs AMS*, No 138 (1974).

(Received, May 24, 1985)