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THE PLANCHEREL THEOREM FOR GENERAL SEMISIMPLE GROUPS

Rebecca A. Herb * and Joseph A. Wolf **

§0. Introduction

Harish-Chandra's celebrated Plancherel formula (see [6b,c,d]) applies to real reductive Lie groups G whose component group G/G^0 is finite and such that $[G^0, G^0]$ has finite center. That class includes all finite coverings of real reductive linear algebraic groups, but it does not include all real connected semisimple Lie groups. In this paper we obtain the Plancherel formula for a somewhat larger class of groups that does contain all real semisimple groups. The method is necessarily quite different from that of Harish-Chandra.

We deal with the class of Lie groups introduced in [17a,b,c] by one of us in 1972. See §1 below for the precise definition. This is a class of reductive groups that contains all the connected real semisimple Lie groups and that contains all Levi components of cuspidal parabolic subgroups. For example, it contains the simply connected covering groups of the analytic automorphism groups of the bounded symmetric domains; those are not in Harish-Chandra's class. The stability under passage to Levi components of cuspidal parabolic subgroups is, as in Harish-Chandra, necessary for consideration of the various series of representations used in harmonic analysis on the group.

The point of [17c] was a geometric realization of certain unitary representations. This required a "quick and dirty" extension of Harish-Chandra's Plancherel formula. While that extension was sufficient for the geometric applications, the Plancherel densities $m_{j,\xi,\nu}$ were only analysed to the extent of their measure-theoretic properties, and it is stated there that "In fact our proof should be considered provisional: if one carefully follows the details of Harish-Chandra's argument, he should be able to extend that argument to our case, proving the $m_{j,\xi,\nu}$ meromorphic as well." When Harish-Chandra's proof was finally made available ([6,b,c,d]) this prognosis turned out to be too optimistic.

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However an alternate proof of the Plancherel theorem by one of us for the class of linear semisimple groups suggested a method for computing the Plancherel densities explicitly. Our approach has two components. First, we use a technique from [17c] for replacing the center of a reductive group by a circle group, and reduce the explicit Plancherel formula for the general class (1.1) to a special class (1.2). Second, we extend the character theory and orbital integral methods of [7a,b,c,d,e] to prove the Plancherel theorem for this special class. This method can be summarized as follows.

Let G be a group in the special class (1.2) and T a fundamental Cartan subgroup of G. For $t \in T'$, the set of regular elements of T, $f \in C_c^\infty(G)$, the integral of f over the orbit of t is given by

$$F_f(t) = \Delta(t) \int_{G/T} f(xtx^{-1})\, \mathrm{d}\dot{x} \tag{0.1}$$

where Δ is the usual Weyl denominator and $\mathrm{d}\dot{x}$ is a G-invariant measure on G/T. Harish-Chandra proved that there is a differential operator D on T and a constant c so that $\mathrm{D}F_f$ extends continuously to all of T and

$$f(1) = c\mathrm{D}F_f(1). \tag{0.2}$$

This formula can be used to obtain the Plancherel formula as follows. We first obtain a Fourier inversion formula for the orbital integral. That is, we find a measure $\mathrm{d}\pi$ on $\hat{G}$ and functions $c(\pi, t)$, $t \in T'$, so that

$$F_f(t) = \int_{\hat{G}} \Theta_\pi(f) c(\pi, t)\, \mathrm{d}\pi, \quad f \in C_c^\infty(G). \tag{0.3}$$

We then differentiate and take the limit as $t \to 1$ on both sides of equation (0.3) to obtain the Plancherel formula.

This method was first used by P. Sally and G. Warner to prove the Plancherel theorem for linear groups of real rank one [11]. Obtaining the Fourier inversion formula for orbital integrals in groups of higher rank was a difficult problem since it required formulas for discrete series characters on noncompact Cartan subgroups. In [7b] the approach of Sally and Warner was modified to obtain the Plancherel formula for linear groups of arbitrary real rank via the stable orbital integrals given by

$$F_f^{(s)}(t) = \sum_{w \in W} \det w\, F_f(wt) \tag{0.4}$$

where W is the full complex Weyl group. The Fourier inversion formula for stable orbital integrals required only formulas for stable discrete series characters, that is the sums of discrete series characters with the same infinitesimal character. The determination of these stable characters in terms of two-structures allowed the Fourier inversion problem for arbitrary linear groups to be reduced to the cases of $SL(2, \mathbb{R})$ and $Sp(2, \mathbb{R})$.

For non-linear groups the action of the complex Weyl group on the Lie algebra of T does not always lift to the group, so that stable orbital integrals and stable characters do not exist. Thus it is necessary to work with the orbital integrals directly using formulas in [7d] which recover discrete series characters in terms of stable discrete series of Shelstad's endoscopic groups. The Fourier inversion formula for orbital integrals on linear groups was obtained in [7d] using these discrete series character formulas. The methods used there can be extended to the groups satisfying (1.2). As in the linear case the final Fourier inversion formula is expressed in terms of two-structures. Once the Fourier inversion formula (0.3) has been obtained, the differentiation to obtain a formula for $f(1)$ is routine.

In the case that G is of Harish-Chandra class, our result agrees with the product formula of Harish-Chandra [6d]. Indeed the product formula persists for the general class of groups considered here. However our methods are completely different from those of Harish-Chandra, his method being to use the formidable analytic machinery of intertwining operators, c-functions, and Eisenstein integrals to first prove the product formula. Then he needs to make direct computations only in the rank one situation. Our method is to reduce to the class of groups (1.2) where we derive a Plancherel formula in terms of the two-structures used for the Fourier inversion of the orbital integrals. Thus our formula appears initially as a product of factors corresponding to groups G such that G/S is isomorphic to one of the groups $SL(2, \mathbb{R})$ or $Sp(2, \mathbb{R})$, S the central circle subgroup of G. It is a messy computation to check that in the $Sp(2, \mathbb{R})$ case our formula agrees with the Harish-Chandra product formula. It should be emphasized that although our formula is more general than that of Harish-Chandra and our proof different, his product formula served as an inspiration as to how simple and elegant the final formula should be.

In the case that G is the simply connected covering group of $SL(2, \mathbb{R})$, our result agrees with that of L. Pukânszky [9]. His second proof has some elements in common with our proof since it uses the tools of orbital integrals and character theory.

Finally, P. Dourmashkin has recently proved the Poisson-Plancherel formula for simply connected equal rank groups of type B_n [1]. This formula expresses the Plancherel formula for G in terms of Fourier analysis on its Lie algebra and the orbit method. (See [13].)

Our class of real reductive groups arises naturally in several areas. Its prototype members are the universal covering groups of the analytic automorphism groups of bounded symmetric domains. These, of course, come up in Riemannian and Kaehlerian geometry and in automorphic function theory. They also play an important role in unitary representation theory.

If $D = G/K$ is a bounded symmetric domain, then passage from G to its universal covering group $\tilde{G}$ replaces one of the discrete series parameters by a continuous parameter. This allows continuation arguments, both for analysis on G and $\tilde{G}$, and for construction of the holomorphic discrete series. See Sally [10], Wallach [15] and Enright-Howe-Wallach [3]. More generally, if G is a real semisimple group and θ is a Cartan involution, one considers θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{m} + \mathfrak{u}_+$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, applies continuation methods to Harish-Chandra modules of $\mathfrak{m} = \mathfrak{q} \cap \mathfrak{g}$, and constructs and analyses representations of G obtained from those of $\mathfrak{m}$ by the Zuckerman functor technique. See Enright-Parthasarathy-Wallach-Wolf [4a,b], Enright-Wolf [5], and Vogan [14]. This sort of analytic continuation technique is, at the moment, one of the principal methods for constructing singular unitary representations. The analytic consequences of that construction have not yet been explored.

Continuation techniques have been standard for some time in the study of nilpotent groups. Now that the theories of solvable groups and of semisimple groups are being joined, we expect that our general class of reductive groups will be a natural setting for "the reductive part" in a more extended context.

For example, Duflo's general Plancherel formula [2] for finite coverings of real linear algebraic groups depends on reduction to Harish-Chandra's formula for the semisimple case. Our results can be used to study groups locally isomorphic to real linear algebraic groups.

In §1 we discuss our general class (1.1) of reductive groups and a subclass (1.2) to which the orbital integral method will be extended. We collect some structural information needed for the study of the special class (1.2) and some structural information used in carrying the Plancherel theorem from the class (1.2) to the general class (1.1).

In §2 we establish notation and set up character formulas for the orbital integral approach, in §§3 through 5, to the Plancherel theorem for the special class (1.2) of reductive groups.

In §3 we indicate our general method by deriving the explicit Plancherel formula for the universal covering group of SL(2; $\mathbb{R}$). In any case, much of this material is needed in §§4 and 6. The main results are Theorem 3.14 (for associated groups in class (1.2)), Theorem 3.25 (the relative Plancherel formula) and Theorem 3.26 (the global Plancherel formula). Here "relative" refers to a unitary character on the center of the group.

In §4 we derive the Fourier inversion formula (Theorem 4.11) and the Plancherel formula (Theorem 4.18) for the special class (1.2) of reductive groups. This is the main analytic part of the paper. In order to carry it out we develop appropriate extensions of a number of results on orbital integrals and unitary characters.

In §5 we carry out some specific calculations on Sp(2; $\mathbb{R}$) and its universal covering group, to which some results were reduced in §4 using the theory of two-structures.

In §6 we carry the Plancherel formula from the special class (1.2) to our general class of reductive groups. This goes as in §3 but is somewhat more delicate. The main results are Theorem 6.2, the relative Plancherel formula, and Theorem 6.17, the global Plancherel formula. These results are the goal of the paper.

§1. Group structure preliminaries

The Lie groups for which we obtain Plancherel formulas are the reductive Lie groups G such that

$$\text{if } g \in G \text{ then } Ad(g) \text{ is an inner automorphism of } \mathfrak{g}_{\mathbb{C}} \tag{1.1a}$$

and G has a closed normal abelian subgroup Z with the properties

$$\begin{aligned}&Z \text{ centralizes the identity component } G^0,\\ &ZG^0 \text{ has finite index in } G, \text{ and}\\ &Z \cap G^0 \text{ is co-compact in the center } Z_{G^0} \text{ of } G^0.\end{aligned} \tag{1.1b}$$

As discussed in [17c,§0], this is a convenient class of reductive groups that contains every connected semisimple group and is stable under passage to Levi components of cuspidal parabolic subgroups. The "Harish-Chandra class" of groups is the case where G/G^0 and the center of $[G^0, G^0]$ are finite.

In §4 we will obtain our formulas for the special case where

$$\begin{aligned}&G \text{ is a connected reductive Lie group,}\\ &G \text{ has a central circle subgroup } S\\ &\quad \text{with } G_1 = G/S \text{ semisimple,}\\ &G_1 \text{ is an analytic subgroup of the simply connected}\\ &\quad \text{complex Lie group } (G_1)_{\mathbb{C}}.\end{aligned} \tag{1.2}$$

Then in §6 we "lift" the Plancherel formula from the special case (1.2) to the general case (1.1). In this section we work out some structural information, not contained in [17c], which is needed to treat the case (1.2) and to carry the result to the general case (1.1).

PROPOSITION 1.3: *Let G_1 be a connected noncompact simple Lie group that is not of hermitian type, i.e. such that G_1/K_1 is not an hermitian symmetric space. Suppose $G_1 \subset G_{1\mathbb{C}}$ where the complex group $G_{1\mathbb{C}}$ is simply connected. Let q_1: $\tilde{G}_1 \to G_1$ be the universal covering. Then either q_1 is one to one or q_1 is two to one.*

TABLE 1

G_1	$[Z(G_1)]$	$\tilde{K}_1$	$[Z(\tilde{G}_1)]$	$[\tilde{Z}_1]$
Spin(m, n) m, n odd	2	Spin$(m)\times$Spin(n)	4	2
SL$(n; \mathbb{R})$ $n > 2$	1 for n odd 2 for n even	Spin(n)	2 for n odd 4 for n even	2
SL$(n; \mathbb{H})$	2	Sp(n)	2	1
E_{6,F_4}	1	F_4	1	1
E_{6,C_4}	1	Sp(4)	2	2

PROOF: Let $\overline{G}_1$ be the adjoint group of G_1. Let $\overline{K}_1$, K_1 and $\tilde{K}_1$ be the respective maximal compact subgroups of $\overline{G}_1$, G_1 and $\tilde{G}_1$. Let $Z(\cdot)$ denote the center, and $\tilde{Z}_1 = \ker q_1$, so $Z(G_1) \cong Z(\tilde{G}_1)/\tilde{Z}_1$.

Case 1: rank $K_1 <$ rank G_1. Then the Cartan involution of G_1 is an outer automorphism and $Z(\overline{K}_1) = \{1\}$. So $Z(\overline{K}_1) = Z(\overline{G}_1)$, $Z(K_1) = Z(G_1)$ and $Z(\tilde{K}_1) = Z(\tilde{G}_1)$. If G_1 is a complex group then $\tilde{G}_1 = G_1$ and $\tilde{Z}_1 = \{1\}$, i.e. q_1 is one to one. The other cases are as shown in table 1. The only non-obvious fact here, that $[Z(G_1)] = 1$ for $G_1 = E_{6,C_4}$, is because in that case

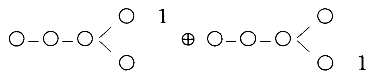
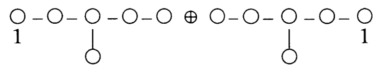
$$\left[Z(\tilde{G}_1)\right] = 2 \quad \text{while } \left[Z(E_6)\right] = 3.$$

Case 2: rank $K_1 =$ rank G_1. Then the Cartan involution of G_1 is an inner automorphism so, since G_1 is not of hermitian type, $[Z(\overline{K}_1)] = 2$. So $Z(\overline{K}_1)/Z(\overline{G}_1)$, $Z(K_1)/Z(G_1)$ and $Z(\tilde{K}_1)/Z(\tilde{G}_1)$ all are of order 2, and $Z(G_1) = Z(G_{1\mathbb{C}})$. The cases are shown in table 2.

TABLE 2

G_1	$[Z(G_1)]$	$\tilde{K}_1$	$[Z(\tilde{G}_1)]$	$[\tilde{Z}_1]$
Spin(m, n) m odd, n even, $n > 2$	2	Spin$(m)\times$Spin(n)	4	2
Spin(m, n) m, n even, $m \geqslant n > 2$	4	Spin$(m)\times$Spin(n)	8	2
Sp(m, n)	2	Sp$(m)\times$Sp(n)	2	1
E_{6,A_1A_5}	3	SU(2)$\times$SU(6)	6	2
E_{7,A_7}	2	SU(8)	4	2
E_{7,A_1D_6}	2	SU(2)$\times$Spin(12)	4	2
E_{8,D_8}	1	Spin(16)	2	2
E_{8,A_1E_7}	1	SU(2)$\times E_7$	2	2
F_{4,A_1C_3}	1	SU(2)$\times$Sp(3)	2	2
F_{4,B_4}	1	Spin(9)	1	1
G_{2,A_1A_1}	1	SU(2)$\times$SU(2)	2	2

Table 3

G_1	$[K_1, K_1]$	isotropy representation
$SU(p, q)$	$SU(p)\times SU(q)$	
$Spin(2, l)$	$Spin(l)$	
$Spin^*(2n)$	$SU(n)$	
$Sp(n; \mathbb{R})$	$SU(n)$	
$E_{6, D_5 T_1}$	$Spin(10)$	
$E_{7, E_6 T_1}$	E_6	

Since $[\tilde{Z}_1]$ is 1 or 2 in all cases, q_1 is one to one or two to one in all cases.

QED

Proposition 1.4: *Let G_1 be a connected noncompact simple Lie group of hermitian type. Suppose $G_1 \subset G_{1\mathbb{C}}$ where the complex group $G_{1\mathbb{C}}$ is simply connected. Let q_1: $\tilde{G}_1 \to G_1$ be the universal covering and let $\tilde{Z}_1 = \ker q_1$. Let K_1 be a maximal compact subgroup of G_1 and $\tilde{K}_1 = q_1^{-1}(K_1)$ the corresponding subgroup of $\tilde{G}_1$. Then $[K_1, K_1]$ is simply connected, $\tilde{Z}_1 \cap [\tilde{K}_1, \tilde{K}_1] = \{1\}$, and $\tilde{Z}_1$ is infinite cyclic.*

Proof: The fastest way to see that $[K_1, K_1]$ is simply connected is to run through the cases shown in table 3. Now q_1: $[\tilde{K}_1, \tilde{K}_1] \to [K_1, K_1]$ is one to one so $\tilde{Z}_1 \cap [\tilde{K}_1, \tilde{K}_1] = \{1\}$. As $\tilde{K}_1 \cong [\tilde{K}_1, \tilde{K}_1] \times \mathbb{R}$, $\tilde{Z}_1$ projects isomorphically to a discrete subgroup of the additive reals, and thus $\tilde{Z}_1$ is infinite cyclic.

QED

As an example of Proposition 1.4, we derive a specific result needed in §5. Let G satisfy (1.2) with

$$G_1 = G/S = Sp(2; \mathbb{R}), \tag{1.5a}$$

the group of all real 4×4 matrices m such that $m \cdot \xi \cdot {}^t m = \xi$ where $\xi = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Its maximal compact subgroup $K_1 = U(2)$ has $[K_1, K_1] = SU(2)$ simply connected, so the universal cover $\tilde{G}_1 \to G_1 = \tilde{G}_1/Z$ gives an isomorphism of $[\tilde{K}_1, \tilde{K}_1]$ onto $[K_1, K_1]$, and ξ spans the center of $\mathfrak{k}_1$. Write E_{ij} for the 4×4 matrix with 1 in the (i, j) place, 0 elsewhere, so the split Cartan is

$$\mathfrak{a}_1 = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_1 & \\ & & & -a_2 \end{pmatrix} : a_i \text{ real} \right\},$$

$$\text{with } \epsilon_i \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_1 & \\ & & & -a_2 \end{pmatrix} = a_i$$

and the roots and root vectors are given by

$$\begin{array}{lll} \text{root } \delta & X_\delta & Y_\delta \\ \alpha_1 = 2\epsilon_1 & E_{13} & E_{31} \\ \alpha_2 = 2\epsilon_2 & E_{24} & E_{42} \\ \beta_1 = \epsilon_1 + \epsilon_2 & E_{14} + E_{23} & E_{41} + E_{32} \\ \beta_2 = \epsilon_1 - \epsilon_2 & E_{12} - E_{43} & E_{21} - E_{34} \end{array} \tag{1.5b}$$

For any root δ, define $Z_\delta = X_\delta - Y_\delta$ and $\gamma_\delta = \exp(\pi Z_\delta)$. Notice Z_{β_1}, Z_{β_2}, $Z_{\alpha_1} - Z_{\alpha_2} \in [\mathfrak{k}_1, \mathfrak{k}_1]$ and $Z_{\alpha_1} + Z_{\alpha_2} = \xi$. In G_1 we compute $\gamma_{\alpha_1} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} = -\gamma_{\alpha_2}$ and $\gamma_{\beta_1} = \gamma_{\beta_2} = -I$. Since $[\tilde{K}_1, \tilde{K}_1] \cong [K_1, K_1]$ now we still have $\gamma_{\beta_1} = \gamma_{\beta_2} = \gamma_{\alpha_1}\gamma_{\alpha_2}^{-1} = -I \in \mathrm{SU}(2) = [\tilde{K}_1, \tilde{K}_1] \subset \tilde{G}_1$, and $\gamma_{\alpha_1}\gamma_{\alpha_2}$ generates the (infinite cyclic) center of $\tilde{G}_1$. This persists into G, where we thus have

in the groups G that satisfy (1.5a), $\gamma_{\beta_1} = \gamma_{\beta_2} = \gamma_{\alpha_1}\gamma_{\alpha_2}^{-1}$ is a central element of order 2, and $\gamma_{\alpha_1}\gamma_{\alpha_2}$ is a central element whose square lies in S. (1.5c)

Here is an indication of the route along which we carry the Plancherel formula from the special class (1.2) to the general class (1.1).

Let G be a reductive Lie group in our general class (1.1). Without loss of generality we replace Z by ZZ_{G^0}, so $Z \cap G^0 = Z_{G^0}$. In other words,

$$\bar{G}_1 = ZG^0/Z \text{ is a centerless connected semisimple group.} \tag{1.6}$$

For every unitary character $\zeta \in \hat{Z}$ we have

$$G[\zeta] = \{S \times ZG^0\}/\{(\zeta(z)^{-1}, z) : z \in Z\} \tag{1.7}$$

where $S = \{e^{i\theta}\}$ is the circle group. Note that S is the center of $G[\zeta]$ because $ZG^0 \to G[\zeta]$ induces $G[\zeta]/S \cong ZG^0/Z = \bar{G}_1$.

LEMMA 1.8: *There is a finite covering q_ζ: $\tilde{G}[\zeta] \to G[\zeta]$ with the following property. The identity component of the center of $\tilde{G}[\zeta]$ is a circle group $\tilde{S}$, and the semisimple group*

$$G_1 = \tilde{G}[\zeta]/\tilde{S} \tag{1.9}$$

is an analytic subgroup of the simply connected group $G_{1\mathbb{C}}$.

PROOF: $G[\zeta]$ has universal cover φ: $\tilde{G}_1 \times \mathbb{R} \to G[\zeta]$ where $\tilde{G}_1$ is the universal cover of $\bar{G}_1$. Note that $(\ker \varphi) \subset \tilde{Z}_1 \times \mathbb{R}$ where $\tilde{Z}_1$ is the center of $\tilde{G}_1$. Let Z_1' be the finite index subgroup of $\tilde{Z}_1$ such that $\tilde{G}_1/Z_1'$ is an analytic subgroup of the complex simply connected group with Lie algebra $\mathfrak{g}_{1\mathbb{C}}$. Define $B = (\ker \varphi) \cap (Z_1' \times \mathbb{R})$ and $\tilde{G}[\zeta] = (\tilde{G}_1 \times \mathbb{R})/B$. Then φ factors as

$$\tilde{G}_1 \times \mathbb{R} \xrightarrow{\beta} \tilde{G}[\zeta] \xrightarrow{\alpha} G[\zeta].$$

The multiplicity of α is the index of B in $\ker \varphi$, which is bounded by the index of Z_1' in $\tilde{Z}_1$ and thus is finite. As $\varphi(\mathbb{R}) = S$ now $\beta(\mathbb{R})$ is a central circle subgroup $\tilde{S} \subset \tilde{G}[\zeta]$ with $\alpha(\tilde{S}) = S$. Since $G[\zeta]$ has center S, $\tilde{G}[\zeta]$ has center $\alpha^{-1}(S)$, in which $\tilde{S}$ has finite index; so $\tilde{S}$ is the identity component of the center of $\tilde{G}[\zeta]$. Also $(\tilde{G}_1 \times \mathbb{R})/((\ker \varphi)\mathbb{R}) = G[\zeta]/\varphi(\mathbb{R}) = \bar{G}_1$ is centerless, so $\tilde{Z}_1 \subset \ker \varphi$ and thus $\tilde{G}[\zeta]/\tilde{S} = \tilde{G}[\zeta]/\beta(\mathbb{R}) = (\tilde{G}_1 \times \mathbb{R})/B\mathbb{R} = \tilde{G}_1/Z_1'$; so $\tilde{G}[\zeta]/\tilde{S}$ is in the simply connected complex group $G_{1\mathbb{C}}$.

QED

We now proceed as follows. For every $\zeta \in \hat{Z}$, $\tilde{G}[\zeta]$ belongs to the special class (1.2) of reductive groups, so its Plancherel formula comes out of the results of §4. The Plancherel formulas for the finite quotients $G[\zeta]$ follow. We transfer those to the relative Plancherel formulas for ZG^0, sum to get the global Plancherel formula for ZG^0, and extend that to G.

The representations and distributions involved in these Plancherel formulas all are constructed in terms of conjugacy classes of Cartan subgroups. In order that these constructions be coherent, we need

PROPOSITION 1.10: *There are natural one to one correspondences between the sets of all Cartan involutions of G, of ZG^0, of G^0, of the $G[\zeta]$, of the $\tilde{G}[\zeta]$, of the $G[\zeta]/S$, and of the $\tilde{G}[\zeta]/\tilde{S}$, specified by: two Cartan involutions correspond if they agree on $[\mathfrak{g}, \mathfrak{g}]$. There are natural one to one* *correspondences between the sets of all Cartan subgroups of those groups, specified by: two Cartan subgroups correspond if their Lie algebras have the same intersection with* $[\mathfrak{g}, \mathfrak{g}]$. *Given a corresponding collection of Cartan involutions and a corresponding collection of Cartan subgroups, if one of the Cartan subgroups is invariant then all are invariant. If two Cartan subgroups in one of* G, ZG^0, G^0, $G[\zeta]$, $\tilde{G}[\zeta]$, $G[\zeta]/S$ *or* $\tilde{G}[\zeta]/\tilde{S}$, *are conjugate, then the two corresponding Cartan subgroups in any other are conjugate.*

PROOF: This is clear except possibly that G-conjugacy of Cartan subgroups H_1 and H_2 of G implies G^0-conjugacy of $H_1 \cap G^0$ and $H_2 \cap G^0$. By (1.1a) [17c, Remark 4.2.4], that is known.

QED

We will normalize Haar measures on the above groups as follows, assuming as in (1.6) that Z has been enlarged if necessary so that $Z \cap G^0 = Z_{G^0}$. $\tilde{G}[\zeta]$ is a group of type (1.2) and its Haar measure is normalized as in [16,8.1.2]. Haar measure on $G[\zeta]$ is then normalized so that

$$\int_{\tilde{G}[\zeta]} \Phi(x)\, \mathrm{d}x = \int_{G[\zeta]} \sum_{b \in F} \Phi(xb)\, \mathrm{d}x \tag{1.11}$$

where F is the kernel of the finite covering from (1.8). Normalize Haar measure on S so that it has volume one and define Haar measure on $G[\zeta]/S$ so that

$$\int_{G[\zeta]} \Phi(x)\, \mathrm{d}x = \int_{G[\zeta]/S} \int_S \Phi(xs)\, \mathrm{d}s\, \mathrm{d}(xS). \tag{1.12}$$

We then transfer the measure $\mathrm{d}(xS)$ on $G[\zeta]/S$ to a measure $\mathrm{d}(xZ)$ on the isomorphic group ZG^0/Z. Fix a Haar measure on $Z_G(G^0)$; if $Z_G(G^0)$ is compact, use the Haar measure of volume 1; and use counting measure if it is an infinite discrete group. Specify measures on Z and ZG^0 by

$$\int_{Z_G(G^0)} \Phi(z)\, \mathrm{d}z = \sum_{z_0 \in Z_G(G^0)/Z} \int_Z \Phi(z_0 z)\, \mathrm{d}z \text{ and} \tag{1.13}$$

$$\int_{ZG^0} \Phi(x)\, \mathrm{d}x = \int_{ZG^0/Z} \int_Z \Phi(xz)\, \mathrm{d}z\, \mathrm{d}(xZ). \tag{1.14}$$

If we now normalize Haar measure on G so that

$$\int_G \Phi(x)\, \mathrm{d}x = \sum_{y \in G/ZG^0} \int_{ZG^0} \Phi(yx)\, \mathrm{d}x, \tag{1.15}$$

this will be independent of the choice of Z satisfying (1.1b). Finally, we normalize a measure on $\hat{Z}$ so that if

$$f_\zeta(x) = \int_Z f(xz)\zeta(z)\mathrm{d}z, \quad \zeta \in \hat{Z}, \quad f \in C_c^\infty(G), \tag{1.16}$$

we have

$$f(x) = \left[Z_G(G^0)/Z\right] \int_{\hat{Z}} f_\zeta(x)\mathrm{d}\zeta. \tag{1.17}$$

If $Z_G(G^0)$ is compact then $\mathrm{d}\zeta$ is counting measure on $\hat{Z}$.

§2. Preliminaries on characters and orbital integrals

In this section we set up notation and state formulas which will be needed in sections 3, 4, and 5. Throughout this and the next three sections G will be a group satisfying (1.2).

Let K be a maximal compact subgroup of G with Cartan involution θ. For any θ-stable Cartan subgroup J of G we will write $J = J_K J_p$ where $J_K = J \cap K$ and the Lie algebra of J_p is in the (-1)-eigenspace for θ. The set of roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{j}_\mathbb{C}$ will be denoted by $\Phi = \Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. The subsets of Φ taking real and pure imaginary values on $\mathfrak{j}$ will be denoted by $\Phi_R(\mathfrak{g}, \mathfrak{j})$ and $\Phi_I(\mathfrak{g}, \mathfrak{j})$ respectively. Let $\Phi_{CPX}(\mathfrak{g}, \mathfrak{j}) = \{\alpha \in \Phi \mid \alpha \notin \Phi_R(\mathfrak{g}, \mathfrak{j}) \cup \Phi_I(\mathfrak{g}, \mathfrak{j})\}$. For $\alpha \in \Phi_{CPX}(\mathfrak{g}, \mathfrak{j})$, there is $\bar{\alpha} \in \Phi_{CPX}(\mathfrak{g}, \mathfrak{j})$ satisfying $\bar{\alpha}(X) = \overline{\alpha(X)}$ for all $X \in \mathfrak{j}$. Φ^+ denotes a choice of positive roots in Φ. We always assume that Φ^+ is chosen so that for $\alpha \in \Phi^+_{CPX}(\mathfrak{g}, j)$, $\bar{\alpha} \in \Phi^+_{CPX}(\mathfrak{g}, \mathfrak{j})$.

For any root system Φ, $L(\Phi)$ denotes the weight lattice, $W(\Phi)$ the Weyl group of Φ, and $\rho(\Phi^+) = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi^+$.

Let $W(G, J) = N_G(J)/J$ where $N_G(J)$ is the normalizer in G of J. Then $W(G, J)$ acts on $\mathfrak{j}$, but not necessarily on J since J need not be abelian. Write J_0 for the center of J and define $W(G, J_0) = N_G(J)/J_0$. As before, we write superscript ^{0} to denote the identity component. Write $Z(\mathfrak{j}_p) = \pi^{-1}(K_1 \cap \exp(i\mathfrak{j}_p))$ where $K_1 = K/S \subseteq G_1$, $\exp(i\mathfrak{j}_p) \subseteq G_{1\mathbb{C}}$, and π: $G \to G_1$ denotes the projection of G onto $G_1 = G/S$.

For $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{j})$ let H^*_α be the element of $\mathfrak{j}_p$ dual to $\check{\alpha} = 2\alpha/\langle\alpha, \alpha\rangle$ under the Killing form. Let X_α, Y_α be elements of the roots spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ respectively so that $\theta(X_\alpha) = Y_\alpha$ and $[X_\alpha, Y_\alpha] = H^*_\alpha$. Write $Z_\alpha = X_\alpha - Y_\alpha$ and set $\gamma_\alpha = \exp(\pi Z_\alpha)$. Then $\gamma_\alpha \in Z(\mathfrak{j}_p)$. Further, $Z(\mathfrak{j}_p) \subseteq J_K$ and $J_K = Z(\mathfrak{j}_p)J_K^0$. This decomposition is not direct since, in particular, $S \subseteq Z(\mathfrak{j}_p) \cap J_K^0$. Let $L_J = Z_G(J_p)$, the centralizer in G of J_p, and write $L_J = M_J J_p$ in its Langlands decomposition.

Fix a θ-stable Cartan subgroup H of G and write $T = H_K$, $A = H_p$, $L = L_H$, $M = M_H$. There is a series of unitary representations of G

associated to H as follows (For details see [17c]). T^0 is a compact Cartan subgroup of the reductive group M^0. Let $L = \{\tau \in i\mathfrak{t}^* \mid \xi_\tau(\exp H) = \exp(\tau(H))$ gives a well-defined character of $T^0\}$, $L' = \{\tau \in L \mid \langle \tau, \alpha \rangle \neq 0$ for all $\alpha \in \Phi\}$ where $\Phi = \Phi(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Choose a set Φ^+ of positive roots for Φ. For $\tau \in L'$, set $\epsilon(\tau) = \operatorname{sign} \prod_{\alpha \in \Phi^+} \langle \alpha, \tau \rangle$. If $\tau \in L^s = L \setminus L'$, we set $\epsilon(\tau) = 1$. Let $q_H = \frac{1}{2} \dim(M/M \cap K)$. Corresponding to each $\tau \in L$ there is an invariant eigendistribution on M^0 which is given on $(T^0)'$, the regular set of T^0, by

$$\Theta_\tau(t) = \epsilon(\tau)(-1)^{q_H} \Delta^M(t)^{-1} \sum_{w \in W(M^0, T^0)} \det w \, \xi_{w\tau}(t) \tag{2.1}$$

where $\Delta^M(t) = \xi_\rho(t) \prod_{\alpha \in \Phi^+}(1 - \xi_{-\alpha}(t))$, $\rho = \rho(\Phi^+)$.

For $\tau \in L'$, Θ_τ is the character of a discrete series representation π_τ of M^0. For $\tau \in L^s$, Θ_τ is a singular invariant eigendistribution which is an alternating sum of characters of limits of discrete series representations [8]. We will need formulas for Θ_τ on noncompact Cartan subgroups of M^0. As in [17c] the formula of Harish-Chandra for this situation which we state as (2.2) can be extended to the compact center case without difficulties.

Let J be a θ-stable Cartan subgroup of M^0. We assume that $J_K^0 \subseteq T^0$ and let y denote an element of $\operatorname{Int}(\mathfrak{m}_{\mathbb{C}})$ which gives the Cayley transform $\operatorname{Ad} y\colon \mathfrak{t}_{\mathbb{C}} \to \mathfrak{j}_{\mathbb{C}}$. For $j \in J$, let $\Delta^M(j) = \xi_\rho(j) \prod_{\alpha \in {}^y\Phi^+}(1 - \xi_{-\alpha}(j))$, $\rho = \rho({}^y\Phi^+)$. Set $\Phi_R = \Phi_R(\mathfrak{m}, \mathfrak{j})$ and for $\gamma \in Z(\mathfrak{j}_p)$, $a \in J_p$, let $\Phi_R(\gamma) = \{\alpha \in \Phi_R \mid \xi_\alpha(\gamma) = 1\}$ and $\Phi_R^+(\gamma a) = \{\alpha \in \Phi_R(\gamma) \mid \alpha(\log a) > 0\}$. Write $W_R(\gamma) = W(\Phi_R(\gamma))$, $W_K(\gamma) = W_R(\gamma) \cap {}^yW(M^0, T^0)$. Then for $\gamma \in Z(\mathfrak{j}_p) \cap T^0$, $j_K \in J_K^0$, and $a \in J_p$ such that $\gamma j_K a \in J'$ we have

$$\begin{aligned}\Theta_\tau(\gamma j_K a) = {} & \epsilon(\tau)(-1)^{q_H} \Delta^M(\gamma j_K a)^{-1} \sum_{w \in W(M^0, T^0)} \det w \, \xi_{w\tau}(\gamma j_K) \\ & \times \sum_{s \in W_R(\gamma)/W_K(\gamma)} \det s \, c(s\colon w\tau\colon \Phi_R^+(\gamma a)) \\ & \times \exp(s^y(w\tau)(\log a)),\end{aligned} \tag{2.2}$$

(see [6a]), where

$$c(s\colon \tau\colon \Phi_R^+(\gamma s a)) = c(1\colon \tau\colon \Phi_R^+(\gamma a))$$

$$\text{for all } s \in W_R(\gamma),\ \tau \in L. \tag{2.3}$$

The constants $c(s\colon \tau\colon \Phi_R^+(\gamma a))$ which occur in (2.2) can be described as follows. Via the Cayley transform we can think of $\tau|_{\mathfrak{j}_p}$ as an element of $\mathfrak{j}_p{}^*$. Thus abstractly we have a root system Φ defined on a vector space

E, a Weyl group $W = W(\Phi)$, an element $\tau \in E$ and a choice Φ^+ of positive roots for Φ. For each $w \in W$, $\tau \in E$, we have a constant $c(w\colon \tau\colon \Phi^+)$. We also have a subroot system, Φ_{CPT}, of "compact" roots of Φ which would be given for $\Phi = \Phi_R(\gamma)$ by $\Phi_{\mathrm{CPT}} = \{\alpha \in \Phi_R(\gamma) \mid {}^{y^{-1}}\alpha$ is a compact root of $(\mathfrak{m}, \mathfrak{t})\}$. Let $W_K = W(\Phi_{\mathrm{CPT}})$.

The constant $c(w\colon w^{-1}\tau\colon \Phi^+)$ depends only on the coset of w in W/W_K and we define

$$\bar{c}(\tau\colon \Phi^+) = \sum_{w \in W/W_K} c(w\colon w^{-1}\tau\colon \Phi^+). \tag{2.4}$$

The constants $\bar{c}(\tau\colon \Phi^+)$ correspond to "stable" discrete series and can be described in terms of two-structures. For details see [7c,d]. We say a root system $\varphi \subseteq \Phi$ is a two-structure for Φ if (i) all simple factors of φ are of type A_1 or $B_2 \cong C_2$; (ii) $\{w \in W(\Phi) \mid w\varphi^+ = \varphi^+\}$ contains no elements of determinant -1 where $\varphi^+ = \Phi^+ \cap \varphi$. Let $T(\Phi)$ denote the set of all two-structures for Φ. For $\varphi \in T(\Phi)$ there are signs $\epsilon(\varphi\colon \Phi^+) = \pm 1$ uniquely determined by: (i) if $\varphi \in T(\Phi)$ and $w \in W$ satisfy $w\varphi^+ \subseteq \Phi^+$, then $\epsilon(w\varphi\colon \Phi^+) = \det w\, \epsilon(\varphi\colon \Phi^+)$; (ii) $\sum_{\varphi \in T(\Phi)} \epsilon(\varphi\colon \Phi^+) = 1$.

It is proved in [7c] that for all $\tau \in E$,

$$\bar{c}(\tau\colon \Phi^+) = \sum_{\varphi \in T(\Phi)} \epsilon(\varphi\colon \Phi^+)\bar{c}(\tau\colon \varphi^+) \quad \text{or equivalently}$$

$$\bar{c}(\tau\colon \Phi^+) = \epsilon(\varphi\colon \Phi^+)[W_1]^{-1} \sum_{w \in W_0} \det w\, \bar{c}(w^{-1}\tau\colon \varphi^+) \tag{2.5}$$

where

$$\varphi \in T(\Phi),\ W_1 = W_1(\varphi\colon \Phi^+) = \{w \in W \mid w\varphi^+ = \varphi^+\},$$

and

$$W_0 = W_0(\varphi\colon \Phi^+) = \{w \in W \mid w\varphi^+ \subseteq \Phi^+\}.$$

If $\varphi = \varphi_1 \cup \dots \cup \varphi_s$ is the decomposition of φ into simple root systems of type A_1 or B_2, then $\bar{c}(\tau\colon \varphi^+) = \prod_{i=1}^{s} \bar{c}(\tau\colon \varphi_i^+)$ where if $\varphi_i^+ = \{\alpha\}$ is of type A_1,

$$\bar{c}(\tau\colon \varphi_i^+) = \begin{cases} 2 & \text{if } \langle \tau, \alpha \rangle < 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

If φ_i is of type B_2 the formula for $\bar{c}(\tau\colon \varphi_i^+)$ is given in the proof of (5.1). Thus the constants $\bar{c}(\tau\colon \Phi^+)$ for any root system Φ are given in terms of constants for root system of type A_1 or B_2.

We now describe briefly how to recover the original constants $c(w\colon \tau\colon \Phi^+)$ in terms of "averaged" constants. For details see [7d]. The motivation behind this is work of Shelstad on endoscopic groups [12]. Let Λ denote the root lattice of Φ, $\Lambda_0 = \{\lambda \in \Lambda \mid \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in Z$ for all $\alpha \in \Phi\}$. For $\lambda \in \Lambda$, let $\Phi(\lambda) = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle \in Z\}$. Define a homomorphism χ: $\Lambda \to Z/2Z$ by setting

$$\chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Phi_{\mathrm{CPT}} \\ 1 & \text{if } \alpha \notin \Phi_{\mathrm{CPT}} \end{cases}$$

and extending linearly. For $\lambda \in \Lambda$ and $w \in W$ set

$$\kappa_\lambda(w) = \begin{cases} 1 & \text{if } \chi(w^{-1}\lambda) = \chi(\lambda); \\ -1 & \text{if } \chi(w^{-1}\lambda) \neq \chi(\lambda). \end{cases}$$

Then $\Phi(\lambda)$ and $\kappa_\lambda(w)$ depend only on the cosets of $\lambda \in \Lambda/\Lambda_0$ and $w \in W(\Phi(\lambda)) \backslash W / W_K$. There is a way of assigning to each $\lambda \in \Lambda/\Lambda_0$ a sign $\epsilon(\lambda: \Phi^+)$ so that

$$c(w: \tau: \Phi^+) = 2^{-n} \sum_{\lambda \in \Lambda/\Lambda_0} \epsilon(\lambda: \Phi^+) \kappa_\lambda(w) \bar{c}(w\tau: \Phi^+(\lambda)) \qquad (2.7)$$

where $n = \operatorname{rank} \Phi$.
Let $\Lambda_1 \subseteq \Lambda$ be a complete set of representatives for the orbits of W in Λ/Λ_0. For $\lambda \in \Lambda_1$ let $W_1(\lambda: \Phi^+) = \{w \in W \mid w\Phi^+(\lambda) = \Phi^+(\lambda)\}$, $W_0(\lambda: \Phi^+) = \{w \in W \mid w\Phi^+(\lambda) \subseteq \Phi^+\}$. Then (2.7) can be rewritten as

$$c(w: \tau: \Phi^+) = 2^{-n} \sum_{\lambda \in \Lambda_1} [W_1(\lambda: \Phi^+)]^{-1} \epsilon(\lambda: \Phi^+)$$
$$\times \sum_{s \in W_0(\lambda: \Phi^+)} \det s \; \kappa_\lambda(s^{-1}w) \bar{c}(s^{-1}w\tau: \Phi^+(\lambda)). \qquad (2.8)$$

This completes the description of the discrete series characters on M^0. We now finish the description of the characters associated to $H = TA$. Let $M^\dagger = Z(\mathfrak{a})M^0 = Z_M(M^0)M^0$, $\Gamma_0 = Z(\mathfrak{a}) \cap T^0$. Note that Γ_0 is central in $Z(\mathfrak{a})$. For $\chi \in Z(\mathfrak{a})\hat{}$, the set of irreducible unitary representations of $Z(\mathfrak{a})$, let

$$L_\chi = \{\tau \in L \mid \operatorname{tr} \chi|_{\Gamma_0} = \deg \chi \xi_{\tau-\rho}|_{\Gamma_0}\}, \quad \rho = \rho(\Phi^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})).$$

Then for $\chi \in Z(\mathfrak{a})\hat{}$, $\tau \in L_\chi$, there is a character of M, supported on $M^\dagger$, given by

$$\Theta_{\chi,\tau}(zm) = \sum_{x \in M/M^\dagger} \operatorname{tr} \chi(xzx^{-1}) \Theta_\tau(xmx^{-1}), \quad z \in Z(\mathfrak{a}), \; m \in M^0.$$
$$(2.9)$$

It is the character of the discrete series representation $\pi_{\chi,\tau} = \mathrm{Ind}_{M^\dagger}^{M}(\chi \otimes \pi_\tau)$ of M.

Now for any $\nu \in \mathfrak{a}^*$, $\Theta_{\chi,\tau} \otimes e^{i\nu}$ is a character of $L = MA$. If $P = MAN$ is a parabolic subgroup with levi factor L, we denote by $\Theta(H\colon \chi\colon \tau\colon \nu)$ the character of G induced from $\Theta_{\chi,\tau} \otimes e^{i\nu} \otimes 1$ on P, i.e. the character of $\pi(H\colon \chi\colon \tau\colon \nu) = \mathrm{Ind}_P^G(\pi_{\chi,\tau} \otimes e^{i\nu} \otimes 1)$. It is supported on the Cartan subgroups of G conjugate to those in L. Let $\mathrm{Car}(G, H)$ denote a complete set of such Cartan subgroups up to G-conjugacy. We can assume all Cartan subgroups in $\mathrm{Car}(G, H)$ are θ-stable.

For $J \subseteq L$ a Cartan subgroup of G, let $J_1, \ldots, J_k$ denote a complete set of representatives for the $L-$ conjugacy classes of Cartan subgroups of L which are conjugate to J in G. For $j \in J$, write $j_i = x_i j x_i^{-1}$ where $x_i \in G$ satisfies $x_i J x_i^{-1} = J_i$. Then for $j \in J'$

$$\Theta(H\colon \chi\colon \tau\colon \nu)(j) = \sum_{i=1}^{k} \left[W(L, J_{i,0})\right]^{-1} |\Delta^G(j_i)|^{-1} \times \sum_{w \in W(G, J_{i,0})} |\Delta^L(wj_i)| \left(\Theta_{\chi,\tau} \otimes e^{i\nu}\right)(wj_i). \quad (2.10)$$

Here for any Cartan subgroup $J \subseteq L$,

$$\Delta^G(j) = \xi_{\rho_G}(j) \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} (1 - \xi_{-\alpha}(j))$$

and

$$\Delta^L(j) = \xi_{\rho_L}(j) \prod_{\alpha \in \Phi^+(\mathfrak{l}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} (1 - \xi_{-\alpha}(j)),$$

$$\rho_G = \rho(\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})), \quad \rho_L = \rho(\Phi^+(\mathfrak{l}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})).$$

It will be convenient to write $Z(\mathfrak{a})^* = \{\mathrm{tr}\chi \mid \chi \in Z(\mathfrak{a})^\wedge\}$. If $\eta = \mathrm{tr}\,\chi$, we write $\Theta(H\colon \eta\colon \tau\colon \nu) = \Theta(H\colon \chi\colon \tau\colon \nu)$, $L_\eta = L_\chi$.

We note that in [7a,b,c,d,e] the characters above are indexed by elements $b^* \in \hat{T}$ rather than pairs (χ, τ), $\chi \in Z(\mathfrak{a})^\wedge$, $\tau \in L_\chi$, and the signs $\epsilon(\tau)$ and $(-1)^{q_H}$ are not included in the definition. In this notation

$$\Theta(H\colon b^*\colon \nu) = \epsilon(\tau)(-1)^{q_H}\Theta\left(H\colon \chi \cdot \xi_{\rho_G}^{-1}\colon \tau\colon \nu\right)$$

where $\chi = b^*|_{Z(\mathfrak{a})}$, $\tau = \log b^*$, and $\rho_G = \rho(\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}))$.

Note that Harish-Chandra in [6d] also indexes characters by elements of $\hat{T}$. But in this case the character given by $b^* \in \hat{T}$ would correspond to

$\Theta(H\colon \chi\colon \tau+\rho_M\colon \nu)$ where again $\chi = b^*|_{Z(\mathfrak{a})}$ and $\tau = \log b^*$. The shift is given by $\rho_M = \rho(\Phi^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}))$.

We now turn to a discussion of orbital integrals. A convenient reference for the results given here is section 8.5.1 of [16]. Although only groups of Harish-Chandra class are considered, it is easy to see that the results extend to the compact center case since all the hard work is done at the Lie algebra level and only at the end lifted to the group.

We assume that all Haar measures on G are normalized as in [16,8.1.2]. For H any Cartan subgroup of G define

$$F_f^H(h) = \epsilon_R(h)\Delta(h)\int_{G/H_0} f(xhx^{-1})\,\mathrm{d}\dot{x}, \quad h\in H',\ f\in C_c^\infty(G). \tag{2.11}$$

Here $\Delta = \Delta^G$ is defined as in (2.9) and $\epsilon_R(h) = \text{sign} \prod_{\alpha\in\Phi_R^+(\mathfrak{g},\mathfrak{h})}(1-\xi_{-\alpha}(h))$. Then F_f^H is a C^∞ function on H' with at worst jump discontinuities across singular hyperplanes and has compact support on H. Further, F_f^H is clearly a class function on H. For $\eta\in T^*$, the set of characters of irreducible unitary representations of T, and $\mu\in\mathfrak{a}^*$ define

$$\hat{F}_f^H(\eta\colon \mu) = \int_T\int_A F_f^H(ta)\eta(t)a^{\mathrm{i}\mu}\,\mathrm{d}t\,\mathrm{d}a. \tag{2.12}$$

Then for all $ta\in H'$ we have the Fourier inversion formula for $F_f^H(ta)$ as a function on H given by

$$F_f^H(ta) = \frac{1}{\text{vol } T(2\pi)^{d(A)}}\sum_{\eta\in T^*}\overline{\eta(t)}\int_{\mathfrak{a}^*} a^{-\mathrm{i}\mu}\hat{F}_f^H(\eta\colon \mu)\,\mathrm{d}\mu \tag{2.13}$$

where vol T denotes the total mass of T and $\mathrm{d}(A)$ denotes the dimension of A. The sum over $\eta\in T^*$ is not absolutely convergent since F_f^H has jump discontinuities.

Let $\sigma\in\hat{S}$ be a unitary character of the central circle subgroup S of G and let $C_c^\infty(G/S;\ \sigma) = \{f\in C_c^\infty(G)\,|\, f(xs) = \overline{\sigma(s)}f(x) \text{ for all } s\in S,\ x\in G\}$. Let $T_\sigma^* = \{\eta\in T^*\,|\,\eta|_S = \deg\eta\cdot\sigma\}$. Then clearly $F_f^H(hs) = \overline{\sigma(s)}F_f^H(h)$ for all $s\in S$, $h\in H$, $f\in C_c^\infty(G/S,\ \sigma)$, so that $\hat{F}_f^H(\eta:\mu) = 0$ unless $\eta\in T_\sigma^*$. Thus for $f\in C_c^\infty(G/S,\ \sigma)$, the sum in (2.13) need be taken only over T_σ^*.

Let $\mathrm{Car}(G)$ denote a complete set of representatives for conjugacy classes of Cartan subgroups of G, chosen to be θ-stable. Then we have the Weyl integral formula

$$\int_G f(x)\,\mathrm{d}x = \sum_{H\in\mathrm{Car}(G)}[W(G,\ H_0)]^{-1}\int_H \epsilon_R(h)\overline{\Delta(h)}F_f^H(h)\,\mathrm{d}h,$$

$$f\in C_c^\infty(G). \tag{2.14}$$

If H is a fundamental Cartan subgroup of G, let D be the differential operator on H given by $D = \prod_{\alpha \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})} H_\alpha$ where $H_\alpha \in \mathfrak{h}_\mathbb{C}$ is dual to α under the Killing form. Then for $f \in C_c^\infty(G)$,

$$f(1) = \frac{(-1)^q}{(2\pi)^r} \lim_{h \to 1} F_f^H(h; D) \tag{2.15}$$

where $r = [\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}]$ and $q = \frac{1}{2}(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$.

We will need the following identities which are valid for any $H \in \operatorname{Car}(G)$.

$$\Delta(wh) = \det w \, \Delta(h) \quad \text{for } w \in W(G, H_0),\ h \in H; \tag{2.16}$$

$$\epsilon_R(wh) F_f^H(wh) = \det w \, \epsilon_R(h) F_f^H(h)$$

$$\text{for } w \in W(G, H_0),\ h \in H; \tag{2.17}$$

$$\overline{\Delta(h)} = (-1)^{r_I(H)} \Delta(h), \quad h \in H;\ r_I(H) = \tfrac{1}{2}[\Phi_I(\mathfrak{g}, \mathfrak{h})]. \tag{2.18}$$

Suppose $M = M_H$, $H = TA$. Then for $z \in Z(\mathfrak{a})$, $t \in T^0$, $a \in A$

$$\operatorname{sign}\left\{ \frac{\Delta^G(zta)}{\Delta^M(t)} \right\} = \epsilon_R(zta) \xi_\rho(z)$$

$$\text{where } \rho = \rho(\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})). \tag{2.19}$$

Suppose $J \subseteq L$ is a Cartan subgroup. Then for $z \in Z(\mathfrak{a})$, $j_M \in J \cap M^0$, $a \in A$,

$$\operatorname{sign}\left\{ \frac{\Delta^G(zj_M a)}{\Delta^M(j_M)} \right\} = \epsilon_R(zj_M a) \epsilon_R^M(j_M) \xi_\rho(z) \tag{2.20}$$

where $\rho = \rho(\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}))$ and $\epsilon_R^M(j_M) = \operatorname{sign} \prod_{\alpha \in \Phi_R^+(\mathfrak{m}, \mathfrak{j}_M)} (1 - \xi_{-\alpha}(j_M))$.

§3. The universal cover of SL(2; ℝ)

In this section we indicate our general method by deriving the explicit Plancherel formula for the universal covering group of SL(2; ℝ). We also need this special result in order to do the general case. Of course our formula agrees with that of L. Pukánszky [9,p.117].

The proof proceeds in several stages. The heart of the matter is the use of orbital integrals to prove the explicit Plancherel formula for connected Lie groups G such that

$$G \text{ has a central circle subgroup } S \text{ with } G/S = \mathrm{SL}(2; \mathbb{R}). \tag{3.1}$$

The next step is a formal "transfer" of the Plancherel formula for groups (3.1) to a Plancherel formula for the universal cover

$$\tilde{G} \to \mathrm{SL}(2;\mathbb{R}), \tag{3.2}$$

relative to each of the unitary characters $\zeta \in \hat{Z}_{\tilde{G}}$. Finally, these relative Plancherel formulas are combined to give the explicit Plancherel formula for $\tilde{G}$.

A. The relative Plancherel formula for groups satisfying (3.1)

Suppose G is a connected Lie group satisfying (3.1). Let $\sigma \in \hat{S}$ be a unitary character of the central circle subgroup S of G and let $C_c^\infty(G/S, \sigma) = \{f \in C_c^\infty(G) \mid f(xs) = \overline{\sigma(s)} f(x) \text{ for all } s \in S,\ x \in G\}$. We will derive the Plancherel formula for $f \in C_c^\infty(G/S, \sigma)$ using the technique of Fourier inversion of orbital integrals. The presentation is not designed to be the most efficient possible for SL(2, $\mathbb{R}$), but rather to parallel the general case of §4 and to provide certain formulas which we will need for the general case.

Let H be the noncompact Cartan subgroup of G. Then $H = Z(\mathfrak{a})A$ where $Z(\mathfrak{a})$ in this case is the center of G and A is the split component of H. For $\chi \in Z(\mathfrak{a})^*$, $\mu \in \mathfrak{a}^*$, the corresponding principal series representation $\pi(H: \chi: \mu)$ has character $\Theta(H: \chi: \mu)$ given on regular elements of H by:

$$\begin{aligned}&\Theta(H: \chi: \mu)(za) = |\Delta(za)|^{-1}\chi(z)(a^{i\mu} + a^{-i\mu}),\\ &z \in Z(\mathfrak{a}),\ a \in A.\end{aligned} \tag{3.3}$$

Let T be the compact Cartan subgroup of G. Then

$$\Theta(H: \chi: \mu)(t) = 0 \quad \text{for all } t \in T'. \tag{3.4}$$

T is connected so that $\hat{T}$ is parameterized just by its weight lattice L. The discrete series representation $\pi(T: \lambda)$ of G corresponding to $\lambda \in L$ has character given by

$$\Theta(T: \lambda)(t) = -\epsilon(\lambda)\Delta(t)^{-1}\xi_\lambda(t), \quad t \in T'; \tag{3.5}$$

$$\begin{aligned}\Theta(T: \lambda)(za) = -\epsilon(\lambda)\Delta(za)^{-1}\xi_\lambda(z) \sum_{w \in W} \det w\, c(w: \lambda: \Phi^+(a))\\ \times \exp(w^y\lambda(\log a))\end{aligned} \tag{3.6}$$

where $\Phi = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm\alpha\}$ and $W = W(\Phi)$. The constants $c(w: \lambda: \Phi^+(a))$ are given in this case by

$$c(w: \lambda: \Phi^+(a)) = \begin{cases} 1 & \text{if } w^y\lambda(\log a) < 0; \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

LEMMA 3.8: *Let $za \in H'$. Then for $f \in C_c^\infty(G/S, \sigma)$,*

$$F_f^H(za) = \frac{1}{2\pi \operatorname{vol} Z(\mathfrak{a})} \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{\chi \cdot \xi_\rho(z)} \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f)$$
$$\times a^{-i\mu} d\mu.$$

PROOF: Note that $\epsilon_R(za)\Delta(za) = \xi_\rho(z)\,|\,\Delta(za)\,|$. Thus using the Weyl integral formula (2.14) and (3.3), (3.4), and (2.18)

$$\Theta(H\colon \chi\colon \mu)(f) = \tfrac{1}{2}\int_{Z(\mathfrak{a})A} \xi_\rho(z)\chi(z)(a^{i\mu} + a^{-i\mu}) F_f^H(za) dz da.$$

But $F_f^H(za) = F_f^H(za^{-1})$ so that the contributions of $a^{i\mu}$ and $a^{-i\mu}$ to the integral are equal and we see that $\Theta(H\colon \chi\colon \mu)(f) = \hat{F}_f^H(\chi \cdot \xi_\rho\colon \mu)$. The lemma now follows from the Fourier inversion formula (2.13) for $F_f^H(za)$.

QED

LEMMA 3.9: *Let $t \in T'$, $f \in C_c^\infty(G/S, \sigma)$. Then*

$$F_f^T(t) = \sum_{\lambda \in L_\sigma} \epsilon(\lambda)\overline{\xi_\lambda(t)}\Theta(T\colon \lambda)(f) + I_f^T(t) \text{ where}$$

$$I_f^T(t) = -\tfrac{1}{2}\sum_{\lambda \in L_\sigma} \epsilon(\lambda)\overline{\xi_\lambda(t)}\int_H \epsilon_R(h)\Delta(h)F_f^H(h)\Theta(T\colon \lambda)(h)\, dh$$

and where $L_\sigma = \{\lambda \in L\colon \xi_\lambda\,|_S = \sigma\}$.

PROOF: Using the Weyl integral formula and (3.5),

$$\Theta(T\colon \lambda)(f) = \epsilon(\lambda)\int_T \xi_\lambda(t)F_f^T(t)\, dt$$
$$+ \tfrac{1}{2}\int_H \epsilon_R(h)\Delta(h)F_f^H(h)\Theta(T\colon \lambda)(h)\, dh.$$

But the integral over T is exactly $\epsilon(\lambda)\hat{F}_f^T(\lambda)$. The result now follows using the Fourier inversion formula (2.13) for $F_f^T(t)$ and the fact that $\Sigma_{\lambda \in L_\sigma}\epsilon(\lambda)\overline{\xi_\lambda(t)}\Theta(T\colon \lambda)(f)$ converges absolutely [6a].

QED

Let $\Gamma(\mathfrak{a}) = \{I, \gamma_\alpha\}$ where $\gamma_\alpha = \exp(-i\pi^{y^{-1}}H_\alpha^*) = \exp(\pi Z_\alpha) \in Z(\mathfrak{a}) \cap T$. Then $Z(\mathfrak{a}) = \Gamma(\mathfrak{a}) \cdot S$.

LEMMA 3.10:

$$\tfrac{1}{2}\int_H \epsilon_R(h)\Delta(h)F_f^H(h)\Theta(T\colon \lambda)(h)\,\mathrm{d}h$$

$$= -\epsilon(\lambda)\sum_{\gamma\in\Gamma(\mathfrak{a})}\int_{SA^+} F_f^H(\gamma sa)\xi_\lambda(\gamma)\sigma(s)c(\lambda\colon a)\,\mathrm{d}a\,\mathrm{d}s$$

where $c(\lambda\colon a)=\sum_{w\in W}\det w\, c(w\colon \lambda\colon \Phi^+)\exp(w^y\lambda(\log a))$ *and* A^+ *is the positive chamber of* A *with respect to* Φ^+.

PROOF: Denote the left hand side of the equation by LHS. Then using (3.6)

$$\mathrm{LHS}=\sum_{w\in W}\det w\frac{-\epsilon(\lambda)}{2}\int_{Z(\mathfrak{a})A}\epsilon_R(za)F_f^H(za)\xi_\lambda(z)$$

$$\times c(w\colon \lambda\colon \Phi^+(a))\exp(w^y\lambda(\log a))\,\mathrm{d}z\,\mathrm{d}a.$$

For $w\in W$, make the change of variables $za\to w(za)=z(wa)$. Then using (2.17) and (2.3), $\epsilon_R(wza)F_f^H(wza)=\det w\,\epsilon_R(za)F_f^H(za)$ and $c(w\colon \lambda\colon \Phi^+(wa))=c(1\colon \lambda\colon \Phi^+(a))$. Thus

$$\mathrm{LHS}=-\epsilon(\lambda)\int_{Z(\mathfrak{a})A}\epsilon_R(za)F_f^H(za)\xi_\lambda(z)c(1\colon \lambda\colon \Phi^+(a))$$

$$\times\exp({}^y\lambda(\log a))\,\mathrm{d}z\,\mathrm{d}a.$$

Now write $A=\bigcup_{w\in W}w^{-1}A^+$. For each $a\in A^+$, $w\in W$, $\epsilon_R(zw^{-1}a)F_f^H(zw^{-1}a)=\det w\;F_f^H(za)$ and $c(1\colon \lambda\colon \Phi^+(\mathrm{w}^{-1}\mathrm{a}))=\mathrm{c}(\mathrm{w}\colon \lambda\colon \Phi^+)$. Thus $\mathrm{LHS}=-\epsilon(\lambda)\int_{Z(\mathfrak{a})A^+}F_f^H(za)\xi_\lambda(z)c(\lambda\colon a)\mathrm{d}a\mathrm{d}z$. The lemma now follows since $Z(\mathfrak{a})=\Gamma(\mathfrak{a})S$ and $\xi_\lambda(\gamma s)=\xi_\lambda(\gamma)\sigma(s)$ for $\lambda\in L_\sigma$, $\gamma\in\Gamma(\mathfrak{a})$, $s\in S$.

QED

Write $T=ST_1$ where $T_1=\exp({}^{y^{-1}}i\mathfrak{a})$ and decompose $t\in T'$ as $t=s_0t_1$, $s_0\in S$, $t_1\in T_1$. Write $L_R=\{\lambda\in L\mid \lambda|_{\mathfrak{s}}=0\}$. Then $L_R\cong(T_1/T_1\cap S)\hat{}$. For each $\chi\in Z(\mathfrak{a})^*_\sigma$, pick $\delta=\delta(\chi)\in L$ such that $\xi_\delta|_{Z(\mathfrak{a})}=\chi\cdot\xi_\rho$. Then $\delta\in L_\sigma$ and $L_\sigma=\delta+L_R$.

LEMMA 3.11:

$$I_f^T(s_0t_1)=\frac{1}{4\pi}\sum_{\gamma\in\Gamma(\mathfrak{a})}\sum_{\chi\in Z(\mathfrak{a})^*_\sigma}\overline{\chi(s_0)}$$

$$\times\int_{-\infty}^{\infty}\Theta(H\colon \chi\colon \mu)(f)I(\gamma\colon t_1\colon \chi\colon \mu)\,\mathrm{d}\mu$$

where for any $t_1 \in T_1$,

$$I(\gamma\colon t_1\colon \chi\colon \mu) = \overline{\xi_\delta(t_1)} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma^{-1}t_1)} \int_{A^+} a^{-i\mu} c(\lambda+\delta\colon a)\, da.$$

PROOF: Using Lemma 3.10 we can write

$$I_f^T(s_0 t_1) = \overline{\sigma(s_0)} \sum_{\lambda \in L_\sigma} \overline{\xi_\lambda(t_1)} \sum_{\gamma \in \Gamma(\mathfrak{a})} \xi_\lambda(\gamma) \int_{SA^+} F_f^H(\gamma s a)\sigma(s)$$

$$\times c(\lambda\colon a)\, da\, ds.$$

Using Lemma 3.8 we can write

$$F_f^H(\gamma s a) = \frac{1}{2\pi \operatorname{vol} Z(\mathfrak{a})} \overline{\sigma(s)} \sum_{\chi \in Z(\mathfrak{a})^*_\sigma} \overline{\chi \cdot \xi_\rho(\gamma)}$$

$$\times \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f) a^{-i\mu}\, d\mu$$

where the sum is finite and the integral converges absolutely. Thus

$$\int_{SA^+} F_f^H(\gamma s a)\sigma(s) c(\lambda\colon a)\, da\, ds$$

$$= \frac{\operatorname{vol} S}{2\pi \cdot \operatorname{vol} Z(\mathfrak{a})} \sum_{\chi \in Z(\mathfrak{a})^*_\sigma} \overline{(\chi \cdot \xi_\rho)(\gamma)} \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f)$$

$$\times \int_{A^+} a^{-i\mu} c(\lambda\colon a)\, da\, d\mu.$$

The lemma now follows from noting that $\operatorname{vol} S/\operatorname{vol} Z(\mathfrak{a}) = [Z(\mathfrak{a})\colon S]^{-1} = \frac{1}{2}$ and decomposing L_σ as $\delta + L_R$ as above.

QED

We must now simplify $I(\gamma\colon t_1\colon \chi\colon \mu)$. Note that $I(\gamma\colon t_1\colon \chi\colon \mu)$ occurs in the Fourier inversion formula as

$$\int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f) I(\gamma\colon t_1\colon \chi\colon \mu)\, d\mu$$

$$= \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f) \cdot \tfrac{1}{2} \sum_{w \in W} I(\gamma\colon t_1\colon \chi\colon w\mu)\, d\mu$$

since $\Theta(H\colon \chi\colon -\mu)(f) = \Theta(H\colon \chi\colon \mu)(f)$. If g and h are functions of μ

we will write $g \equiv h$ if $g(\mu)+g(-\mu)=h(\mu)+h(-\mu)$. Then it will be harmless to replace one by the other in the Fourier inversion formula.

Write $I(\gamma\colon t_1\colon \chi\colon \mu)=\overline{\xi_\delta(t_1)}S(\mu\colon \delta\colon \gamma^{-1}t_1)$ where for any $t_1 \in T_1$

$$S(\mu\colon \delta\colon t_1)=\sum_{w\in W} \det w \sum_{\lambda \in L_R} \overline{\xi_\lambda(t_1)}c(w\colon \lambda+\delta\colon \Phi^+)$$

$$\times \int_{\mathfrak{a}^+} \exp(w^{\nu}(\lambda+\delta)-\mathrm{i}\mu)(H)\,\mathrm{d}H.$$

Since $L_R \cong (T_1/T_1 \cap S)\hat{}$, $S(\mu\colon \delta\colon t_1)$ depends only on the coset of t_1 in $T_1/T_1 \cap S$ which is isomorphic to the compact Cartan subgroup of $\mathrm{SL}(2, \mathbb{R})$. There is a unique θ, $0<|\theta|<\pi$ so that $t_1(T_1 \cap S)= \exp(-\mathrm{i}\theta^{\nu^{-1}}H_\alpha^*)(T_1 \cap S)$.

Lemma 3.12: $S(\mu\colon \delta\colon t_1) \equiv (-2/\|\alpha\|)\pi \mathrm{i}(\mathrm{e}^{-(\mu+\mathrm{i}p)(\theta-\epsilon\pi)}/\sinh \pi(\mu+\mathrm{i}p))$ *where θ is defined as above, $\epsilon=\operatorname{sign}(\theta)=\pm 1$, $p=2\langle {}^{\nu}\delta, \alpha\rangle/\langle \alpha, \alpha\rangle$, and by abuse of notation the real number μ is defined by $\mu=2\langle \mu, \alpha\rangle/\langle \alpha, \alpha\rangle$.*

Proof: Write $\beta={}^{\nu^{-1}}\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then $L_R=\{n\beta/2 \mid n \in Z\}$ and for $\lambda=n\beta/2$, using (3.7)

$$c(1\colon \delta+\lambda\colon \Phi^+)=\begin{cases} 1 & \text{if } n+p<0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$c(s_\alpha\colon \delta+\lambda\colon \Phi^+)=\begin{cases} 1 & \text{if } n+p>0 \\ 0 & \text{otherwise.} \end{cases}$$

Write $\mathfrak{a}^+=\{rH_\alpha^* \mid r>0\}$. Then $\mathrm{d}H=c\mathrm{d}r$ where $c=2/\|\alpha\|$ and $S(\mu\colon \delta\colon t_1)=$

$$c\sum_{n+p<0} \mathrm{e}^{\mathrm{i}n\theta}\int_0^\infty \mathrm{e}^{(n+p-\mathrm{i}\mu)r}\,\mathrm{d}r - c\sum_{n+p>0} \mathrm{e}^{\mathrm{i}n\theta}\int_0^\infty \mathrm{e}^{-(n+p+\mathrm{i}\mu)r}\,\mathrm{d}r$$

$$=-c\sum_{n+p<0} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+p-\mathrm{i}\mu} - c\sum_{n+p>0} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+p+\mathrm{i}\mu}$$

$$\equiv -c\sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+p-\mathrm{i}\mu}$$

since we can replace μ by $-\mu$ in the second sum and the term for $n=0$ is $(1/-\mathrm{i}\mu)\equiv 0$. But this last expression is the Fourier series expansion of the function given in the lemma.

QED

LEMMA 3.13: *Let* $f \in C_c^\infty(G/S, \sigma)$. *Write* $t \in T'$ *as* $t = st_\theta$ *where* $s \in S$ *and* $t_\theta = \exp(-\mathrm{i}\theta^{y^{-1}} H_\alpha^*)$ *with* $0 < |\theta| < \pi$. *Then*

$$F_f^T(t) = \sum_{\lambda \in L_\sigma} \epsilon(\lambda)\overline{\xi_\lambda(t)}\Theta(T\colon \lambda)(f)$$

$$+\left(\frac{\mathrm{i}}{4}\right) \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{\chi(s)} \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f) K(\chi\colon \mu\colon t_\theta)\, \mathrm{d}\mu.$$

Here $K(\chi\colon \mu\colon t_\theta) = (2/\|\alpha\|)\, \mathrm{e}^{-\mu\theta} k(\chi\colon \mu\colon C(t_\theta))$ *where* $k(\chi\colon \mu\colon C(t_\theta)) =$

$$-\left[\exp(\epsilon\pi(\mu + \mathrm{i}p)) + 1\right]/\sinh \pi(\mu + \mathrm{i}p) \;\; \textit{and}\; p = 2\langle {}^y\delta, \alpha\rangle/\langle \alpha, \alpha\rangle$$

if $\delta \in L$ *is chosen so that* $\xi_\delta|_{Z(\mathfrak{a})} = \chi \cdot \xi_\rho$.

PROOF: Using Lemma 3.12,

$$I(1\colon t_1\colon \chi\colon \mu) = \overline{\xi_\delta(t_1)} S(\mu\colon \delta\colon t_1)$$

$$\equiv \frac{-2}{\|\alpha\|}(\pi \mathrm{i})\, \mathrm{e}^{-\mu\theta} \frac{\exp(\epsilon\pi(\mu + \mathrm{i}p))}{\sinh \pi(\mu - \mathrm{i}p)}.$$

Also $I(\gamma_\alpha\colon t_1\colon \chi\colon \mu) = \overline{\xi_\delta(t_1)}\; S(\mu\colon \delta\colon \gamma_\alpha^{-1}t_1)$. Now $\gamma_\alpha^{-1}t_1(T_1 \cap S) = \exp(-\mathrm{i}(\theta - \epsilon\pi)^{y^{-1}} H_\alpha^*)(T_1 \cap S)$ where $0 < |\theta - \epsilon\pi| < \pi$. Thus by Lemma 3.12,

$$\overline{\xi_\delta(t_1)} S\left(\mu\colon \delta\colon \gamma_\alpha^{-1}t_1\right) \equiv \frac{-2}{\|\alpha\|}(\pi \mathrm{i}) \frac{\mathrm{e}^{-\mu\theta}}{\sinh \pi(\mu + \mathrm{i}p)}$$

The lemma now follows from (3.9) and (3.11).

QED

THEOREM 3.14: *Let* $f \in C_c^\infty(G/S, \sigma)$. *Then*

$$f(1) = \frac{1}{2\pi} \sum_{\lambda \in L_\sigma} |\langle \beta, \lambda\rangle| \Theta(T\colon \lambda)(f)$$

$$+ \frac{1}{4\pi\|\alpha\|} \sum_{\chi \in Z(\mathfrak{a})_\sigma^\wedge} \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f)$$

$$\times |\langle \alpha, \mu\rangle p_\alpha(\chi\colon \mu)|\, \mathrm{d}\mu$$

where

$$p_\alpha(\chi:\ \mu)=\frac{\sinh \pi\mu_\alpha}{\cosh \pi\mu_\alpha+\left\{\dfrac{\chi(\gamma_\alpha)+\chi(\gamma_\alpha^{-1})}{2}\right\}},\quad \mu_\alpha=\frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}.$$

Here $\{\beta\}=\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})$ *and* $\{\alpha\}=\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{h}_{\mathbf{C}})$.

PROOF: We know by (2.15) that $f(1)=(-1/2\pi)\lim_{t\to 1}\{H_\beta\cdot F_f^T(t)\}$. Now $\lim_{t\to 1}H_\beta\cdot\overline{\xi_\lambda(t)}=-\langle\beta,\lambda\rangle$ and so the coefficient of $\Theta(T:\ \lambda)(f)$ in the Plancherel formula is $(1/2\pi)\epsilon(\lambda)\langle\beta,\lambda\rangle=(1/2\pi)|\langle\beta,\lambda\rangle|$.

Now $\overline{\chi(s)}\ K(\chi:\ \mu:\ t_\theta)$ is a constant multiple of $\overline{\chi(s)}\ \mathrm{e}^{-\mu\theta}=\overline{\chi(s)}$ $\times\xi_\tau(t_\theta)$, $\tau=-\mathrm{i}^{y^{-1}}\mu$, and $\lim_{st_\theta\to 1}H_\beta\cdot\overline{\chi(s)}\xi_\tau(t_\theta)=\langle\tau,\beta\rangle=-\mathrm{i}\langle\mu,\alpha\rangle$. Thus the coefficient of $\Theta(H:\ \chi:\ \mu)(f)$ in the Plancherel theorem is

$$\frac{\langle\mu,\alpha\rangle}{4\pi\|\alpha\|}\frac{\exp(\epsilon\pi(\mu+\mathrm{i}p)+1)}{\sinh\pi(\mu+\mathrm{i}p)}=\frac{\epsilon\langle\mu,\alpha\rangle}{4\pi\|\alpha\|}\frac{(\mathrm{e}^{\epsilon\pi\mu}-\mathrm{e}^{-\epsilon\pi\mathrm{i}p})}{\cosh\pi\mu-\cos\pi p}$$

$$\equiv\frac{\langle\mu,\alpha\rangle}{4\pi\|\alpha\|}\frac{\sinh\pi\mu}{\cosh\pi\mu-\cos\pi p}.$$

Thus we see that this coefficient is actually independent of the direction through which $st_\theta\to 1$. Finally $\mathrm{e}^{\mathrm{i}\pi p}=\overline{\xi_\delta(\gamma_\alpha)}=\chi\cdot\overline{\xi_\rho(\gamma_\alpha)}$ by assumption. Further, $\xi_\rho(\gamma_\alpha)=\xi_\rho(\gamma_\alpha^{-1})=-1$. Thus $\cos\pi p=-\frac{1}{2}[\chi(\gamma_\alpha)+\chi(\gamma_\alpha^{-1})]$ which completes the proof.

QED

B. The relative Plancherel Theorem for $\tilde{\mathrm{S}}\mathrm{L}(2:\mathbb{R})$

We now change notation and denote

$$\tilde{p}:\ G\to \mathrm{SL}(2;\mathbb{R})\ \text{universal covering and}\ Z=Z_G,\ \text{the center of}\ G. \tag{3.15}$$

A unitary character $\zeta\in\hat{Z}$ specifies the Hilbert space $L_2(G/Z,\zeta)=\{f:\ G\to\mathbb{C}$ measurable $|\, f(xz)=\zeta(z)^{-1}f(x)$ and $|f|\in L_2(G/Z)\}$ and its C_c^∞ analog, $C_c^\infty(G/Z,\zeta)=\{f\in C^\infty(G)\,|\, f(xz)=\zeta(z)^{-1}f(x)$ and $|f|\in C_c(G/Z)\}$. The direct integral decomposition

$$L_2(G)=\int_{\hat{Z}}L_2(G/Z,\zeta)\,\mathrm{d}\zeta \tag{3.16}$$

corresponds to

$$\hat{G} = \bigcup_{\zeta \in \hat{Z}} \hat{G}_\zeta \quad \text{where } \hat{G}_\zeta = \{[\pi] \in \hat{G} \mid \pi(xz) = \zeta(z)\pi(x)\}. \tag{3.17}$$

This is effected on $C_c^\infty(G)$ by

$$f(x) = \int_{\zeta \in \hat{Z}} f_\zeta(x)\, d\zeta \tag{3.18}$$

where $f_\zeta \in C_c^\infty(G/Z, \zeta)$ is given by $f_\zeta(x) = \sum_{z \in Z} f(xz)\zeta(z)$.

As before, $S = \{e^{i\theta}\colon \theta \in \mathbb{R}\}$ is the circle group. By $1 \in \hat{S}$ we mean the unitary character $1(e^{i\theta}) = e^{i\theta}$. Observe:

LEMMA 3.19: *Define* $G[\zeta] = \{G \times S\}/\{(z, \zeta(z)^{-1})\colon z \in Z\}$ *and define* p: $G \to G[\zeta]$ *by* $p(x) = \{(xz, \zeta(z)^{-1})\colon z \in Z\}$. *Then* p *is a Lie group homomorphism and it induces an isomorphism of* PSL(2; $\mathbb{R}$) $= G/Z$ *onto* $G[\zeta]/S$.

In effect this construction replaces Z *by* S.

We define the two-fold covering q_ζ: $\tilde{G}[\zeta] \to G[\zeta]$ *as in* (1.8). *The Plancherel formula, Theorem* 3.14, *is available for* $\tilde{G}[\zeta]$. *Suppose* $\sigma \in (\tilde{S})^{\hat{}}$ *is trivial on* $\tilde{S} \cap F$, *where* $F = \ker q_\zeta$, *and induces the identity character* $1 \in \hat{S} = (\tilde{S}/\tilde{S} \cap F)^{\hat{}}$. *We are going to push the Plancherel formula* (3.14) *from* $C_c^\infty(\tilde{G}[\zeta]/\tilde{S}, \sigma)$ *down to* $C_c^\infty(G[\zeta]/S, 1)$ *and then pull it back to* $C_c^\infty(G/Z, \zeta)$. *For the first step we use the correspondence of* (1.10) *and the normalization of Haar measures in* (1.11) *to obtain*:

LEMMA 3.20: *Let* $f \in C_c^\infty(G[\zeta]/S, 1)$. *Then*

$$f(1) = \frac{1}{\pi} \sum_{\lambda \in L_1} |\langle \beta, \lambda \rangle| \Theta(T\colon \lambda)(f)$$

$$+ \frac{1}{2\pi \|\alpha\|} \sum_{\chi \in Z(\mathfrak{a}[\zeta])_1^{\hat{}}} \int_{-\infty}^{\infty} \Theta(H\colon \chi\colon \mu)(f)$$

$$\times |\langle \alpha, \mu \rangle p_\alpha(\chi\colon \mu)|\, d\mu$$

where

$$p_\alpha(\chi : \mu) = \frac{\sinh \pi\mu_\alpha}{\cosh \pi\mu_\alpha + \left\{\dfrac{\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})}{2}\right\}}, \quad \mu_\alpha = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

The second step requires some preparation.

LEMMA 3.21: *We have a bijection* $\hat{p}\colon G[\zeta]_1^{\wedge} \to \hat{G}_\zeta$ *given by* $\hat{p}[\psi] = [\psi \circ p]$ *and a Hilbert space isometry* $p^*\colon L_2(G[\zeta]/S, 1) \to L_2(G/Z, \zeta)$ *given by* $p^*\varphi = \varphi \circ p$.

As usual write Θ_π, Θ_ψ *for the distribution characters of unitary equivalence classes* $[\pi] \in \hat{G}$ *and* $[\psi] \in G[\zeta]^{\wedge}$. *If* $[\pi] \in \hat{G}_\zeta$ *and* $f_\zeta \in C_c^\infty(G/Z, \zeta)$ *then* $\Theta_\pi(f_\zeta)$ *is understood to mean* $\int_{G/Z} f_\zeta(x)\Theta_\pi(x)\mathrm{d}(xZ)$, *which is well defined because* $f_\zeta(xz) = \zeta(z)^{-1} f_\zeta(x)$ *and* $\Theta_\pi(xz) = \zeta(z)\Theta_\pi(x)$. *In fact,*

LEMMA 3.22: *If* $f \in C_c^\infty(G)$, $f = \int_{\zeta \in \hat{Z}} f_\zeta \mathrm{d}\zeta$ *as in* (3.18), *and if* $[\pi] \in \hat{G}_{\zeta'}$, *then* $\Theta_\pi(f) = \Theta_\pi(f_{\zeta'})$.

PROOF:

$$\Theta_\pi(f) = \int_G f(x)\Theta_\pi(x)\,\mathrm{d}x = \int_{G/Z} \sum_{z \in Z} f(xz)\Theta_\pi(xz)\,\mathrm{d}(xZ)$$

$$= \int_{G/Z} \sum_{z \in Z} f(xz)\zeta'(z)\Theta_\pi(x)\,\mathrm{d}(xZ)$$

$$= \int_{G/Z} f_{\zeta'}(x)\Theta_\pi(x)\,\mathrm{d}(xZ) = \Theta_\pi(f_{\zeta'}),$$

all sums being absolutely convergent.

QED

Since $C_c^\infty(G[\zeta]/S, 1) \subset C_c^\infty(G[\zeta])$, there appears to be some ambiguity as to what we mean by $\Theta_\psi(\varphi)$ for $[\psi] \in G[\zeta]_1^{\wedge}$ and $\varphi \in C_c^\infty(G[\zeta]/S, 1)$. But the two interpretations,

$$\int_{G[\zeta]} \varphi(x)\Theta_\psi(x)\,\mathrm{d}x \text{ and } \int_{G[\zeta]/S} \varphi(x)\Theta_\psi(x)\,\mathrm{d}(xS)$$

are equal because we normalize measures so that

$$\int_{G[\zeta]} \Phi(x)\,\mathrm{d}x = \int_{G[\zeta]/S} \left\{ \int_S \Phi(xs)\,ds \right\} \mathrm{d}(xS)$$

and so that the volume of S is one.

LEMMA 3.23: *If* $[\psi] \in G[\zeta]_1^{\wedge}$ *and* $\varphi \in C_c^\infty(G[\zeta]/S, 1)$, *then* $\Theta_{\hat{p}\psi}(p^*\varphi) = \Theta_\psi(\varphi)$.

PROOF:

$$\Theta_{\hat{p}\psi}(p^*\varphi) = \int_{G/Z} (p^*\varphi)(x)\Theta_{\hat{p}\psi}(x)\,\mathrm{d}(xZ)$$

$$= \int_{G/Z} \varphi(p(x))\Theta_{\psi}(p(x))\,\mathrm{d}(xZ)$$

$$= \int_{G[\zeta]/S} \varphi(y)\Theta_{\psi}(y)\,\mathrm{d}(yS) = \Theta_{\psi}(\varphi).$$

QED

In order to apply these general lemmas we have to follow the parameterization of representations through $\hat{p}$.

Let H be the Cartan subgroup of G such that $\tilde{p}(H)$ is the noncompact Cartan subgroup of PSL(2, $\mathbb{R}$), so $H[\zeta] = \{H \times S\}/\{(z, \zeta(z)^{-1}): z \in Z\}$ is the noncompact Cartan subgroup of $G[\zeta]$. As before, $H = Z(\mathfrak{a})A$ where $Z(\mathfrak{a})$ in this case is the center of G and A is the split component. If $\chi \in Z(\mathfrak{a})\hat{}$ and $\mu \in \mathfrak{a}^*$ then $\pi(H: \chi: \mu)$ denotes the corresponding principal series representation and $\Theta(H: \chi: \mu)$ is its character.

Let T be the Cartan subgroup of G such that $\tilde{p}(T)$ is the compact Cartan subgroup of PSL(2; $\mathbb{R}$), so $T[\zeta] = \{T \times S\}/\{(z, \zeta(z)^{-1}): z \in Z\}$ is the compact Cartan subgroup of $G[\zeta]$. Then $L = it^*$ parameterizes $\hat{T}$, and we write $\pi(T: \lambda)$ for the relative discrete series representation of G corresponding to $\lambda \in L$ and $\Theta(T: \lambda)$ for its character.

$\Theta(H: \chi: \mu)$ is given, just as before, by (3.3) and (3.4). Similarly, $\Theta(T: \lambda)$ is given by (3.5) and (3.6).

As in Lemma 3.21, composition with p: $G \to G[\zeta]$ defines bijections $\hat{p}$: $Z(\mathfrak{a}[\zeta])\hat{}_1 \to Z(\mathfrak{a})\hat{}_\zeta$ and $\hat{p}$: $T[\zeta]\hat{}_1 \to \hat{T}_\zeta$, the latter given by $\hat{p}(\lambda) = \lambda$. Thus we have

LEMMA 3.24: *If* $\pi(H[\zeta]: \chi: \mu) \in G[\zeta]\hat{}_1$ *then* $\hat{p}\pi(H[\zeta]: \chi: \mu) = \pi(H: \chi \circ p: \mu)$. *If* $\pi(T[\zeta]: \lambda) \in G[\zeta]\hat{}_1$ *then* $\hat{p}\pi(T[\zeta]: \lambda) = \pi(T: \lambda)$.

Lemmas 3.21, 3.23 *and* 3.24 *are exactly what we need to re-write Lemma* 3.20 *as the relative Plancherel theorem for G. It remains only to verify that* $\chi(\gamma_\alpha)$ *has the same value in both settings. For that, since p*: $G \to G[\zeta]$ *is the identity on the semisimple part of the Lie algebra, we let* $\chi \in Z(\mathfrak{a}[\zeta])\hat{}_1$ *and compute* $\chi(\gamma_\alpha^{G[\zeta]}) = \chi(\exp_{G[\zeta]}\pi Z_\alpha) = \chi(\exp_{G[\zeta]}(p\pi Z_\alpha)) = \chi(p \cdot \exp_G \pi Z_\alpha) = (\hat{p}\chi)(\exp_G \pi Z_\alpha) = (\hat{p}\chi)(\gamma_\alpha^G)$. *Thus, finally,* THEOREM 3.25: *Let* $\zeta \in \hat{Z}$ *and* $f \in C_c^\infty(G/Z, \zeta)$. *Then*

$$f(1) = \frac{1}{\pi} \sum_{\lambda \in L_\zeta} |\langle \beta, \lambda \rangle| \Theta(T : \lambda)(f)$$

$$+ \frac{1}{2\pi \|\alpha\|} \sum_{\chi \in Z(\mathfrak{a})^\wedge_\zeta} \int_{-\infty}^{\infty} \Theta(H: \chi: \mu)(f)$$

$$\times |\langle \alpha, \mu \rangle p_\alpha(\chi : \mu)| \, \mathrm{d}\mu$$

where

$$p_\alpha(\chi : \mu) = \frac{\sinh \pi\mu_\alpha}{\cosh \pi\mu_\alpha + \left\{ \dfrac{\chi(\gamma_\alpha) + \chi(\gamma_\alpha)^{-1}}{2} \right\}} \quad \text{with } \mu_\alpha = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

Here $\{\beta\} = \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *and* $\{\alpha\} = \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

C. The global Plancherel Theorem for $\tilde{\mathrm{S}}\mathrm{L}(2; \mathbb{R})$

Let $f = \int_{\hat{Z}} f_\zeta d\zeta \in C_c^\infty(G)$, apply Theorem 3.25 to each f_ζ, and sum over $\zeta \in \hat{Z}$:

$$f(1) = \frac{1}{\pi} \int_{\hat{Z}} \sum_{\lambda \in L_\zeta} |\langle \beta, \lambda \rangle| \theta(T: \lambda)(f_\zeta) \, \mathrm{d}\zeta + \frac{1}{2\pi \|\alpha\|} \int_{\hat{Z}} \sum_{\chi \in Z(\mathfrak{a})^\wedge_\zeta}$$

$$\times \int_{-\infty}^{\infty} \theta(H: \chi: \mu)(f_\zeta) \, |\langle \alpha, \mu \rangle p_\alpha(\chi, \mu)| \, \mathrm{d}\mu \mathrm{d}\zeta.$$

In view of Lemma 3.22 we can replace f_ζ by f in the above. Then, in the first term, $\Sigma_{\lambda \in L_\zeta}$ sums over a translate of the lattice L_1 in L and $\int_{\hat{Z}} \mathrm{d}\zeta$ sums over the set of all such translates, so together they amount to $\int_L \mathrm{d}\lambda$. And similarly in the second term $\Sigma_{\chi \in Z(\mathfrak{a})^\wedge_\zeta}$ and $\int_{\hat{Z}} \mathrm{d}\zeta$ combine to give $\int_{Z(\mathfrak{a})^\wedge} \mathrm{d}\chi$. So the formula reduces to that of

THEOREM 3.26: *If* $f \in C_c^\infty(G)$ *then*

$$f(1) = \frac{1}{\pi} \int_L |\langle \beta, \lambda \rangle| \theta(T : \lambda)(f) \, \mathrm{d}\lambda$$

$$+ \frac{1}{2\pi \|\alpha\|} \int_{Z(\mathfrak{a})^\wedge} \int_{-\infty}^{\infty} \theta(H: \chi: \mu)(f)$$

$$\times |\langle \alpha, \mu \rangle p_\alpha(\chi, \mu)| \, \mathrm{d}\mu \, \mathrm{d}\chi$$

with p_α *given just as in Theorem* 3.25.

§4. The Plancherel formula in the key case

In this section we derive the Plancherel formula for Lie groups G satisfying (1.2). Much of the analysis in this section is a direct modification of results in [7a,b,c,d,e] for linear groups. Specifically, analogues of Lemmas 4.1, 4.2, and 4.3 appear in [7a], of 4.4, 4.5, 4.10, 4.12, 4.17, and 4.18 in [7b,c] and of 4.7, 4.8 in [7d]. Notation is as in §2.

We want a Fourier inversion formula of the form (0.3) for F_f^H when H is fundamental. However, as in (3.11) for the SL(2, $\mathbb{R}$) case, in order to obtain this formula we need Fourier inversion formulas for F_f^J, J non-fundamental. Thus we initially take $H = TA$ to be an arbitrary θ-stable Cartan subgroup of G. Decompose $h \in H'$ as $h = z_0 t a_0$ where $z_0 \in Z(\mathfrak{a})$, $t \in T^0$, and $a_0 \in A$. Fix $\sigma \in \hat{S}$ and suppose $f \in C_c^\infty(G/S, \sigma)$.

LEMMA 4.1:

$$F_f^H(z_0 t a_0) = \frac{(-1)^{r_I(H)+q_H}}{\operatorname{vol}(T_0)(2\pi)^{d(A)}} \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{(\chi \cdot \xi_\rho)(z_0)} \sum_{\lambda \in L_\chi} \epsilon(\lambda) \overline{\xi_\lambda(t)}$$

$$\times \int_{\mathfrak{a}^*} \Theta(H\colon \chi\colon \lambda\colon \mu)(f) a_0^{-i\mu}\, d\mu + I_f^H(z_0 t a_0)$$

where

$$I_f^H(z_0 t a_0) = \frac{(-1)^{r_I(H)+q_H+1}}{\operatorname{vol} T_0 (2\pi)^{d(A)}} \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{(\chi \cdot \xi_\rho)(z_0)} \sum_{\lambda \in L_\chi} \epsilon(\lambda) \overline{\xi_\lambda(t)}$$

$$\times \int_{\mathfrak{a}^*} a_0^{-i\mu} \left\{ \sum_{J \in \operatorname{Car}'(G,H)} [W(G, J_0)]^{-1} \right.$$

$$\left. \times \int_J \epsilon_R(j) \overline{\Delta(j)} F_f^J(j) \Theta(H\colon \chi\colon \lambda\colon \mu)(j)\, dj \right\} d\mu.$$

Here Car′(G, H) = Car(G, H)$\setminus${H}.

PROOF: Using the Weyl integral formula and the fact that $\Theta(H\colon \chi\colon \lambda\colon \mu)$ is supported on conjugates of $J \in \operatorname{Car}(G, H)$,

$$\Theta(H\colon \chi\colon \lambda\colon \mu)(f) = \sum_{J \in \operatorname{Car}(G,H)} [W(G, J_0)]^{-1}$$

$$\times \int_J \epsilon_R(j) \overline{\Delta(j)} F_f^J(j) \Theta(H\colon \chi\colon \lambda\colon \mu)(j)\, dj.$$

Look at the term corresponding to $J = H$. Formulas (2.10), (2.9), (2.1) describe the value of the character $\Theta(H\colon \chi\colon \lambda\colon \mu)(h)$ in terms of sums over $W(G, H_0)$, $M/M^\dagger$, and $W(M^0, T^0)$ respectively. Note that representatives of $M/M^\dagger$ can be chosen to normalize H. Thus in each case we can change variables in the integration and use the invariance properties (2.16), (2.17) of the remaining terms as in Lemma 4.1 of [7a]. Thus

$$[W(G, H_0)]^{-1}\int_H \epsilon_R(h)\overline{\Delta(h)}F_f^H(h)\Theta(H\colon \chi\colon \lambda\colon \mu)(h)\,\mathrm{d}h$$

$$= \frac{(-1)^{r_I(H)+q_H}\epsilon(\lambda)[M/M^\dagger][W(M^0, T^0)]}{[W(M, T_0)]}$$

$$\times \int_H \epsilon_R(h)\frac{\Delta_G(h)}{|\Delta_G(h)|}\frac{|\Delta_L(h)|}{\Delta_L(h)}F_f^H(h)(\chi\otimes\xi_\lambda\otimes \mathrm{e}^{\mathrm{i}\mu})(h)\,\mathrm{d}h.$$

But using (2.19), $\epsilon_R(h)\Delta_G(h)|\Delta_L(h)|/|\Delta_G(h)|\Delta_L(h) = \xi_\rho(z)$ if $h = zta$. Thus the integral over H is $\hat{F}_f^H(\chi\cdot\xi_\rho\colon \lambda\colon \mu)$. Note that the consistency condition required for (χ, λ) to determine a discrete series character of M implies that $(\chi\cdot\xi_\rho, \lambda)$ give a character of T. Finally, $[M/M^\dagger] \times [W(M^0, T^0)][W(M, T_0)]^{-1} = [W(M, T)][W(M, T_0)]^{-1} =$ vol $T/\mathrm{vol}(T_0)$. The lemma now follows from using the inversion formula (2.13) for $F_f^H(z_0ta_0)$ together with the fact that

$$\sum_{\chi\in Z(\mathfrak{a})^*_\sigma}\overline{\chi\cdot\xi_\rho(z_0)}\sum_{\lambda\in L_\chi}\epsilon(\lambda)\overline{\xi_\lambda(t)}\int_{\mathfrak{a}^*}\Theta(H\colon \chi\colon \lambda\colon \mu)(f)a_0^{-\mathrm{i}\mu}\,\mathrm{d}\mu$$

converges absolutely so that the sum can be rearranged. [6a]

QED

In Lemma 4.1 we have started the Fourier inversion process. If the Cartan subgroup H is maximally split, $\mathrm{Car}'(G, H) = \emptyset$ so there is no remainder term. This is what happened in (3.8). In the general case we must analyze the remainder terms using character formulas as in (3.10). This is done in the following lemma.

Lemma 4.2:

$$(-1)^{q_H}\epsilon(\lambda)\sum_{J\in\mathrm{Car}'(G,H)}[W(G, J_0)]^{-1}$$

$$\times\int_J \epsilon_R(j)\overline{\Delta(j)}F_f^J(j)\Theta(H\colon \chi\colon \lambda\colon \mu)(j)\,\mathrm{d}j$$

$$= \sum_{J \in \mathrm{Car}'(L,H)} \frac{(-1)^{r_I(J)}[W_R(\mathfrak{l}, \mathfrak{j})][M/M^\dagger]}{[W(L, J_0)][Z(\mathfrak{a}_1) \cap Z(\mathfrak{a}) J_K^0/S]}$$

$$\times \sum_{\gamma \in \Gamma(\mathfrak{a}_1)} d_\gamma^{-1} \sum_{w \in W(M^0, T^0)} \det w$$

$$\times \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma) A} F_f^J(\gamma z j_K a_1 a)(\chi \cdot \xi_\rho)(z) a^{i\mu} \xi_{w\lambda}(\gamma j_K)$$

$$\times c(\gamma\colon w\lambda\colon a_1)\, \mathrm{d}(z j_K a_1 a)$$

where $A_1 = J_p \cap M$, $\Gamma(\mathfrak{a}_1)$ is a set of coset representatives for $(Z(\mathfrak{a}_1) \cap T^0)/S$, d_γ is the number of elements in $\Gamma(\mathfrak{a}_1)$ conjugate to γ via $W_R(\mathfrak{l}, \mathfrak{j})$, and for $\gamma \in \Gamma(\mathfrak{a}_1)$, $\lambda \in L$, and $a_1 \in A_1^+(\gamma) = \{a_1 \in A_1 \mid \alpha(\log a_1) > 0 \text{ for } \alpha \in \Phi_R^+(\gamma)\}$

$$c(\gamma\colon \lambda\colon a_1) = \sum_{s \in W_R(\gamma)/W_K(\gamma)} \det s\; c(s\colon \lambda\colon \Phi_R^+(\gamma)) \exp(s^\gamma \lambda(\log a_1)).$$

Car′(L, H) denotes a complete set of θ-stable representatives for L-conjugacy classes of Cartan subgroups of L, excluding H.

PROOF: Formula (2.10) gives $\Theta(H\colon \chi\colon \lambda\colon \mu)(j)$, $j \in J'$, as a sum over Cartan subgroups $J_1, \ldots, J_k$ of L which are conjugate to J in G. This sum can be eliminated if we replace the sum over $\mathrm{Car}'(G, H)$ by a sum over $\mathrm{Car}'(L, H)$. For $J \in \mathrm{Car}'(L, H)$, the sums over $W(G, J_0)$ and $M/M^\dagger$ in the character formulas (2.10) and (2.9) can be eliminated by changes of variables in the integration. Thus

$$(-1)^{q_H} \epsilon(\lambda) \sum_{J \in \mathrm{Car}'(G,H)} [W(G, J_0)]^{-1}$$

$$\times \int_J \epsilon_R(j) \overline{\Delta(j)} F_f^J(j) \Theta(H\colon \chi\colon \lambda\colon \mu)(j)\, \mathrm{d}j$$

$$= (-1)^{q_H} \epsilon(\lambda) \sum_{J \in \mathrm{Car}'(L,H)} \frac{(-1)^{r_I(J)}[M/M^\dagger]}{[W(L, J_0)]}$$

$$\times \int_{J \cap L^\dagger} \epsilon_R(j) \Delta_G(j) F_f^J(j) \,|\Delta_G(j)|^{-1}$$

$$\times |\Delta_L(j)| (\chi \otimes \Theta_\lambda \otimes e^{i\mu})(j)\, \mathrm{d}j.$$

Fix $J \in \mathrm{Car}'(L, H)$ and write $J \cap L^{\dagger} = Z(\mathfrak{a}_1)Z(\mathfrak{a})J_K^0 A_1 A$. For $\gamma \in Z(\mathfrak{a}_1)$ write $I(\gamma) =$

$$(-1)^{q_H}\epsilon(\lambda)\int_{Z(\mathfrak{a})J_K^0 A_1 A} \epsilon_R(\gamma j)\Delta_G(\gamma j)F_f^J(\gamma j)\,|\Delta_G(\gamma j)|^{-1}$$

$$\times |\Delta_L(\gamma j)|\big(\chi \otimes \Theta_\lambda \otimes \mathrm{e}^{\mathrm{i}\mu}\big)(\gamma j)\,\mathrm{d}j.$$

Let $w \in W(M^0, (M^0 \cap J)_0)$. Then w normalizes $Z(\mathfrak{a}_1)$, J_K^0, and A_1, and centralizes $Z(\mathfrak{a})$ and A. Thus $(\chi \otimes \Theta_\lambda \otimes \mathrm{e}^{\mathrm{i}\mu})(w(\gamma j)) = (\chi \otimes \Theta_\lambda \otimes \mathrm{e}^{\mathrm{i}\mu})(\gamma j)$. Using the change of variables $j \to wj$, $j \in Z(\mathfrak{a})J_K^0 A_1 A$ we see that $I(w\gamma) = I(\gamma)$, $\gamma \in Z(\mathfrak{a}_1)$. $I(\gamma)$ depends only on the coset of γ in $Z(\mathfrak{a}_1)/S$ since $S \subseteq Z(\mathfrak{a})J_K^0$. $W_R(\mathfrak{l}, \mathfrak{j})$ acts on $Z(\mathfrak{a}_1)/S$ and the action of any element can be achieved by an element of $W(M^0, (M^0 \cap J)_0)$. Thus

$$(-1)^{q_H}\epsilon(\lambda)\int_{J \cap L^{\dagger}} \epsilon_R(j)\Delta_G(j)F_f^J(j)\,|\Delta_G(j)|^{-1}\,|\Delta_L(j)|$$

$$\times\big(\chi \otimes \Theta_\lambda \otimes \mathrm{e}^{\mathrm{i}\mu}\big)(j)\,\mathrm{d}j$$

$$= \big[Z(\mathfrak{a}_1) \cap Z(\mathfrak{a})J_K^0/S\big]^{-1} \sum_{\gamma \in Z(\mathfrak{a}_1)/S} I(\gamma)$$

and

$$\sum_{\gamma \in Z(\mathfrak{a}_1)/S} I(\gamma) = \big[W_R(\mathfrak{l}, \mathfrak{j})\big] \sum_{\gamma \in \Gamma(\mathfrak{a}_1)} d_\gamma^{-1}\big[W_R(\gamma)\big]^{-1} I(\gamma)$$

since every element of $Z(\mathfrak{a}_1)/S$ is conjugate via $W_R(\mathfrak{l}, \mathfrak{j})$ to an element of $(Z(\mathfrak{a}_1) \cap T^0)/S$, the number of elements of $Z(\mathfrak{a}_1)/S$ in the orbit $W_R(\mathfrak{l}, \mathfrak{j}) \cdot \gamma$ is $[W_R(\mathfrak{l}, \mathfrak{j})][W_R(\gamma)]^{-1}$, and d_γ is the number of elements in $(Z(\mathfrak{a}_1) \cap T^0)/S$ conjugate to γ via $W_R(\mathfrak{l}, \mathfrak{j})$. Now fix $\gamma \in Z(\mathfrak{a}_1) \cap T^0$. Then, using (2.2),

$$I(\gamma) = \int_{Z(\mathfrak{a})J_K^0 A_1 A} \epsilon_R(\gamma j)\Delta_G(\gamma j)F_f^J(\gamma j)\,|\Delta_G(\gamma j)|^{-1}$$

$$\times |\Delta_L(\gamma j)|\,\Delta_L(\gamma j)^{-1}\chi(z)a^{\mathrm{i}\mu}$$

$$\times\Bigg\{\sum_{w \in W(M^0, T^0)} \det w\ \xi_{w\lambda}(\gamma j_K)$$

$$\times \sum_{s \in W_R(\gamma)/W_K(\gamma)} \det s\ c\big(s\colon w\lambda\colon \Phi_R^+(\gamma a_1)\big)$$

$$\times \exp\big(s^y(w\lambda)(\log a_1)\big)\Bigg\}\,\mathrm{d}j$$

where $j \in Z(\mathfrak{a})J_K^0 A_1 A$ is decomposed as $j = zj_K a_1 a$. Any element of $W_R(\gamma)$ can be represented by an element of $W(M^0, (M^0 \cap J)_0)$ which centralizes γ, z, j_K, and a. Further, $c(s\colon w\lambda\colon \Phi_R^+(s(\gamma a_1))) = c(1\colon w\lambda\colon \Phi_R^+(\gamma a_1))$ and $\Delta_L(s(\gamma j)) = \det s\, \Delta_L(\gamma j)$, $s \in W_R(\gamma)$. Thus we can eliminate the sum over $W_R(\gamma)$ by changing variables in the integral and use (2.20) to write

$$\begin{aligned}
&[W_R(\gamma)]^{-1} I(\gamma) \\
&= [W_K(\gamma)]^{-1} \int_{Z(\mathfrak{a})J_K^0 A_1 A} \epsilon_R^L(\gamma j) F_f^J(\gamma j)(\chi \cdot \xi_\rho)(z) a^{i\mu} \\
&\times \sum_{w \in W(M^0, T^0)} \det w\, \xi_{w\lambda}(\gamma j_K) c(1\colon w\lambda\colon \Phi_R^+(\gamma a_1)) \\
&\times \exp({}^{y}(w\lambda)(\log a_1))\, dj.
\end{aligned}$$

Now write $Z(\mathfrak{a})J_K^0 A_1 A = \bigcup_{s \in W_R(\gamma)} sZ(\mathfrak{a})J_K^0 A_1^+(\gamma)A$ and use the fact that $\epsilon_R^L(\gamma sj) F_f^J(\gamma sj) = \det s F_f^J(\gamma j)$, $s \in W_R(\gamma)$, to obtain

$$\begin{aligned}
[W_R(\gamma)]^{-1} I(\gamma) = &\sum_{w \in W(M^0, T^0)} \det w \\
&\times \int_{Z(\mathfrak{a})J_K^0 A_1^+(\gamma)A} F_f^J(\gamma j)(\chi \cdot \xi_\rho)(z) a^{i\mu} \\
&\times \xi_{w\lambda}(\gamma j_K) \sum_{s \in W_R(\gamma)/W_K(\gamma)} \det s\, c(s\colon w\lambda\colon \Phi_R^+(\gamma)) \\
&\times \exp(s^{y}(w\lambda)(\log a_1))\, dj.
\end{aligned}$$

QED

We now combine the formula from (4.2) with the formulas given by (4.1) for F_f^J, $J \in \mathrm{Car}'(L, H)$, to continue the Fourier inversion process as in (3.11).

LEMMA 4.3: *There is a dense open subset $H^* \subseteq H'$ so that for $z_0 t a_0 \in H^*$,*

$$\begin{aligned}
I_f^H(z_0 t a_0) = &\frac{(-1)^{r_I(H)+1}[M/M^\dagger]}{\mathrm{vol}\, T_0 (2\pi)^{d(A)}} \\
&\times \sum_{J \in \mathrm{Car}'(L,H)} \frac{[W_R(\mathfrak{l}, \mathfrak{j})](-1)^{q_J}}{[W(L, J)](4\pi)^{d(A_1)}[J_K\colon J_K \cap M^\dagger]} \\
&\times \sum_{\gamma \in \Gamma(\mathfrak{a}_1)} d_\gamma^{-1} \sum_{w \in W(M^0, T^0)} \det w
\end{aligned}$$

$$\times \sum_{\eta \in Z(\mathfrak{j}_p)^*_\sigma} \sum_{j=1}^{k_\gamma} \overline{(\eta_j \cdot \xi_\rho)(z_0)}$$

$$\times \sum_{\tau \in L_\eta} \epsilon(\tau)\overline{\xi_\tau(j(w))} \int_{\mathfrak{a}^*} a_0^{-i\mu}$$

$$\times \int_{\mathfrak{a}_1^*} \Theta(J:\ \eta:\ \tau:\ \mu \otimes \nu)(f) I(\gamma:\ t_1(w):\ \eta_j:\ \nu)\, d\mu\, d\nu$$

$$+ \lim_{N\to\infty} \sum_{J \in \mathrm{Car}'(L,H)} I_{f,N}^{H,J}(z_0 t a_0).$$

Notation is as follows. For $\gamma \in \Gamma(\mathfrak{a}_1)$, *let* $Z_1(\gamma)$ *be the subgroup of* $Z(\mathfrak{j}_p)$ *generated by* $Z(\mathfrak{a})$, γ, *and* $Z_0(\mathfrak{a}_1) = \{\gamma' \in Z(\mathfrak{a}_1) \cap T^0 \mid \xi_\alpha(\gamma') = 1$ *for all* $\alpha \in \Phi_R(\mathfrak{l}, \mathfrak{j})\}$. *For* $\eta \in Z(\mathfrak{j}_p)^*_\sigma$, $\eta = \eta_1 + \ldots + \eta_{k_\gamma}$ *is the decomposition of* $\eta|_{Z_1(\gamma)}$ *into irreducible characters. For each* η_j, *pick* $\lambda_j \in L$, *the weight lattice of* T^0, *satisfying* $\xi_{\lambda_j}|_{Z_1(\gamma)\cap T^0} = (1/\deg \eta_j)\eta_j \cdot \xi_\rho|_{Z_1(\gamma)\cap T^0}$. *Write* $T_1 = \exp({}^{y^{-1}}\mathfrak{i}(\mathfrak{a}_1)) \subseteq T^0$. *Let* L_R *denote the weight lattice of* $T_1/T_1 \cap S$. *Then for any* $t_1 \in T_1$ *we define*

$$I(\gamma:\ t_1:\ \eta_j:\ \nu) = \overline{\xi_{\lambda_j}(t_1)} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma^{-1} t_1)}$$

$$\times \int_{A_1^+(\gamma)} a_1^{-i\nu} c(\gamma:\ \lambda + \lambda_j:\ a_1)\, da_1.$$

For $w \in W(M^0, T^0)$, $j(w) \in J_K^0$ *and* $t_1(w) \in T_1$ *are defined so that* $wt = j(w)t_1(w)$. *Finally,*

$$I_{f,N}^{H,J}(z_0 t a_0) = \frac{(-1)^{r_I(H)+r_I(J)+1}[M/M^\dagger]}{\mathrm{vol}\, T_0[W(L,\ J_0)]}$$

$$\times \frac{[W_R(\mathfrak{l},\ \mathfrak{j})]}{[Z(\mathfrak{a}_1) \cap Z(\mathfrak{a}) J_K^0/S]} \sum_{w \in W(M^0, T^0)} \det w$$

$$\times \sum_{\gamma \in \Gamma(\mathfrak{a}_1)} d_\gamma^{-1} \sum_{\chi \in Z(\mathfrak{a})^*_\sigma} \overline{(\chi \cdot \xi_\rho)(z_0)}$$

$$\times \sum_{\lambda \in w^{-1} L_\chi^N} \overline{\xi_\lambda(wt)} \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma)} I_f^J(\gamma z j_K a_1 a_0)$$

$$\times (\chi \cdot \xi_\rho)(z) \xi_\lambda(\gamma j_K) c(\gamma:\ \lambda:\ a_1) d(z j_K)\, da_1$$

where I_f^J *is defined as in Lemma* (4.1) *if* J *replaces* H.

PROOF: Recall that the sum over L_χ in $I_f^H(z_0 t a_0)$ does not converge absolutely. Define finite subsets L_χ^N of L_χ as in [7a] so that $\Sigma_{\lambda \in L_\chi} = \lim_{N \to \infty} \Sigma_{\lambda \in L_\chi^N}$. For fixed $J \in \text{Car}'(L, H)$, $\gamma \in \Gamma(\mathfrak{a}_1)$, and $w \in W(M^0, T^0)$, look at

$$I(N:\gamma) = \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{\chi \cdot \xi_\rho(z_0)} \sum_{\lambda \in L_\chi^N} \overline{\xi_\lambda(t)}$$

$$\times \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma) A} F_f^J(\gamma z j_K a_1 a)(\chi \cdot \xi_\rho)(z) a^{i\mu} \xi_{w\lambda}(\gamma j_K)$$

$$\times c(\gamma \colon w\lambda \colon a_1)\, \mathrm{d} j$$

$$= \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{\chi \cdot \xi_\rho(z_0)} \sum_{\lambda \in w^{-1} L_\chi^N} \overline{\xi_\lambda(wt)}$$

$$\times \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma) A} F_f^J(\gamma z j_K a_1 a) \chi \cdot \xi_\rho(z) a^{i\mu} \xi_\lambda(\gamma j_K)$$

$$\times c(\gamma \colon \lambda \colon a_1)\, \mathrm{d} j.$$

Now using Lemma 4.1 with J replacing H we know that

$$\int_A F_f^J(\gamma z j_K a_1 a) a^{i\mu} da = \frac{(-1)^{r_I(J) + q_J}}{\text{vol}(J_K)_0 (2\pi)^{d(A_1)}} \sum_{\eta \in Z(\mathfrak{j}_p)_\sigma^*} \overline{(\eta \cdot \xi_\rho)(\gamma z)}$$

$$\times \sum_{\tau \in L_\eta} \epsilon(\tau) \overline{\xi_\tau(j_K)} \int_{\mathfrak{a}_1^*} \Theta(J \colon \eta \colon \tau \colon \mu \otimes \nu)(f)$$

$$\times a_1^{-i\nu}\, \mathrm{d}\nu + \int_A I_f^J(\gamma z j_K a_1 a) a^{i\mu}\, \mathrm{d} a$$

where the sums and integral in the first term are absolutely convergent. Thus we can write $I(N:\gamma) = I_1(N:\gamma) + I_2(N:\gamma)$ where

$$I_1(N:\gamma)$$

$$= \frac{(-1)^{r_I(J) + q_J}}{\text{vol}(J_K)_0 (2\pi)^{d(A_1)}} \sum_{\eta \in Z(\mathfrak{j}_p)_\sigma^*} \sum_{\tau \in L_\eta} \epsilon(\tau) \int_{\mathfrak{a}_1^*} \Theta(J \colon \eta \colon \tau \colon \mu \otimes \nu)(f)$$

$$\times \sum_{\chi \in Z(\mathfrak{a})_\sigma^*} \overline{\chi \cdot \xi_\rho(z_0)} \sum_{\lambda \in w^{-1} L_\chi^N} \overline{\xi_\lambda(wt)}$$

$$\times \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma)} \overline{(\eta \cdot \xi_\rho)(\gamma z) \xi_\tau(j_K)} a_1^{-i\nu} \chi \cdot \xi_\rho(z) \xi_\lambda(\gamma j_K)$$

$$\times c(\gamma \colon \lambda \colon a_1)\, \mathrm{d}(z j_K)\, \mathrm{d} a_1\, \mathrm{d}\nu$$

and

$$I_2(N:\gamma)=\sum_{\chi\in Z(\mathfrak{a})^*_\sigma}\overline{\chi\cdot\xi_\rho(z_0)}\sum_{\lambda\in w^{-1}L^N_\chi}\overline{\xi_\lambda(wt)}$$

$$\times\int_{Z(\mathfrak{a})J^0_K A_1^+(\gamma)A} I^J_f(\gamma z j_K a_1 a)\chi\cdot\xi_\rho(z)a^{i\mu}\xi_\lambda(\gamma j_K)$$

$$\times c(\gamma\colon \lambda\colon a_1)\,\mathrm{d}j.$$

Decompose $\eta\in Z(\mathfrak{j}_p)^*$ as $\eta_1+\cdots+\eta_k$ with respect to $Z_1(\gamma)$. Each $\eta_j|_{Z(\mathfrak{a})}$ is still irreducible since γ and $Z_0(\mathfrak{a}_1)$ are central in $Z_1(\gamma)$. Let $\xi_j(\gamma)=(1/\deg\eta_j)\cdot\eta_j(\gamma)$. Then

$$\int_{Z(\mathfrak{a})J^0_K}\overline{\eta\cdot\xi_\rho(\gamma z)\xi_\tau(j_K)}\chi\cdot\xi_\rho(z)\xi_\lambda(\gamma j_K)\,\mathrm{d}(zj_K)$$

$$=\sum_{i=1}^{k}\overline{(\xi_i\cdot\xi_\rho)(\gamma)}\xi_\lambda(\gamma)$$

$$\times\int_{Z(\mathfrak{a})J^0_K}\overline{(\eta_i\cdot\xi_\rho)(z)\xi_\tau(j_K)}\chi\cdot\xi_\rho(z)\xi_\lambda(j_K)\,\mathrm{d}(zj_K)$$

$$=\sum_{i=1}^{k}\overline{\xi_i\cdot\xi_\rho(\gamma)}\xi_\lambda(\gamma)\times\begin{cases}\operatorname{vol}(Z(\mathfrak{a})J^0_K) & \text{if } \chi=\eta_i \text{ and } \tau=\lambda|_{\mathfrak{j}_K},\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$I_1(N:\gamma)=\frac{(-1)^{r_I(J)+q_J}\operatorname{vol}(Z(\mathfrak{a})J^0_K)}{\operatorname{vol}(J_K)_0(2\pi)^{d(A_1)}}\sum_{\eta\in Z(\mathfrak{j}_p)^*_\sigma}\sum_{j=1}^{k}\overline{(\eta_j\cdot\xi_\rho)(\gamma z_0)}$$

$$\times\sum_{\tau\in L_\eta}\epsilon(\tau)\overline{\xi_\tau(j(w))}\int_{\mathfrak{a}_1^*}\Theta(J\colon\eta\colon\tau\colon\mu\otimes\nu)(f)$$

$$\times\sum_{\{\lambda\in w^{-1}L^N_{\eta_j}|\lambda|_{\mathfrak{j}_K}=\tau\}}\overline{\xi_\lambda(\gamma^{-1}t_1(w))}$$

$$\times\int_{A_1^+(\gamma)}a_1^{-i\nu}c(\gamma\colon\lambda\colon a_1)\,\mathrm{d}a_1.$$

For $1\leqslant j\leqslant k$, look at $\{\lambda\in L_{\eta_j}|\lambda|\mathfrak{j}_K=\tau\}$. Now $\overline{\xi_\lambda(\gamma^{-1}t_1(w))}$ and $c(\gamma\colon\lambda\colon a_1)$ only depend on the restriction of λ to $\mathfrak{t}_1$. But $T_1\cap(Z(\mathfrak{a})\cap T^0)J^0_K=Z(\mathfrak{a}_1)\cap Z(\mathfrak{a})J^0_K\subseteq Z_0(\mathfrak{a}_1)\cap T^0\subseteq Z_1(\gamma)\cap T^0$. Fix $\lambda_j\in L$

satisfying $\xi_{\lambda_j}|_{Z_1(\gamma)\cap T^0} = (1/\deg \eta_j)\eta_j \cdot \xi_\rho|_{Z_1(\gamma)\cap T^0}$. For such λ_j we will have

$$\overline{(\eta_j \cdot \xi_\rho)(\gamma z_0)\xi_{\lambda_j}(\gamma^{-1}t_1(w))} = \overline{(\eta_j \cdot \xi_\rho)(z_0)\xi_{\lambda_j}(t_1(w))}.$$

Write $L_0 = \{\lambda \in L \mid \lambda|_{\mathfrak{i}_K} = 0 \text{ and } \xi_\lambda|_{Z(\mathfrak{a})\cap T^0} = 1\}$. Then

$$\sum_{\{\lambda \in L_{\eta_j} \mid \lambda|_{\mathfrak{i}_K} = \tau\}} \overline{\xi_\lambda(\gamma^{-1}t_1(w))}c(\gamma: \lambda: a_1)$$
$$= \overline{\xi_{\lambda_j}(\gamma^{-1}t_1(w))} \sum_{\lambda \in L_0} \overline{\xi_\lambda(\gamma^{-1}t_1(w))}c(\gamma: \lambda_j + \lambda: a_1).$$

But as in [7a], $|\sum_{\lambda \in L_0 \cap w^{-1}L^N} \overline{\xi_\lambda(\gamma^{-1}t_1(w))}c(\gamma: \lambda_j + \lambda: a_1)|$ is bounded independent of N, uniformly for $a_1 \in A_1^+(\gamma)$ and for $t_1(w)$ in compact subsets of T_1^*, a dense open subset of T_1. Thus we can take the $\lim_{N\to\infty}$ inside the sums and integrals.

Now $L_0 \cong (T_1/T_1 \cap Z(\mathfrak{a})J_K^0)\hat{}$. We want to enlarge the lattice to $L_R \cong (T_1/T_1 \cap S)\hat{}$. Write Γ_1 for a set of coset representatives for the finite abelian group $T_1 \cap Z(\mathfrak{a})J_K^0/T_1 \cap S$. Then

$$\sum_{\lambda \in L_0} \overline{\xi_\lambda(\gamma^{-1}t_1(w))}c(\gamma: \lambda_j + \lambda: a_1)$$
$$= [\Gamma_1]^{-1} \sum_{\gamma_1 \in \Gamma_1} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma_1\gamma^{-1}t_1(w))}c(\gamma: \lambda_j + \lambda: a_1).$$

Up until now we have worked with a fixed

$$\gamma \in \Gamma(\mathfrak{a}_1) = Z(\mathfrak{a}_1) \cap T^0/S = \bigcup_{\gamma \in Z(\mathfrak{a}_1)\cap T^0/Z_0(\mathfrak{a}_1)} \gamma Z_0(\mathfrak{a}_1)/S.$$

We will now sum over a coset $\gamma Z_0(\mathfrak{a}_1)/S$. For any $\gamma' \in Z_0(\mathfrak{a}_1)$, $d_{\gamma\gamma'} = d_{\gamma'}$ and $\Phi_R(\gamma\gamma') = \Phi_R(\gamma)$ so that $A_1^+(\gamma\gamma') = A_1^+(\gamma)$ and $c(\gamma\gamma': \lambda: a_1) = c(\gamma: \lambda: a_1)$ for any $\lambda \in L$. Finally, since $Z_0(\mathfrak{a}_1) \subseteq Z_1(\gamma)$, $Z_1(\gamma\gamma') = Z_1(\gamma)$ so that the decomposition $\eta = \eta_1 + \ldots + \eta_k$ is independent of $\gamma' \in Z_0(\mathfrak{a}_1)$. Thus

$$\lim_{N\to\infty} \sum_{\gamma' \in Z_0(\mathfrak{a}_1)/S} I_1(N: \gamma\gamma')$$
$$= \frac{(-1)^{r_I(J)+q_J}\operatorname{vol}(Z(\mathfrak{a})J_K^0)}{\operatorname{vol}(J_K)_0(2\pi)^{d(A_1)}}$$
$$\times \sum_{\eta \in Z(\mathfrak{i}_p)_\sigma^*} \sum_{j=1}^{k} \overline{(\eta_j \cdot \xi_\rho)(z_0)} \sum_{\tau \in L_\eta} \epsilon(\tau)\overline{\xi_\tau(j(w))}$$

$$\times \int_{\mathfrak{a}_1^*} \Theta(J\colon \eta\colon \tau\colon \mu \otimes \nu)(f) \overline{\xi_{\lambda_J}(t_1(w))} [\Gamma_1]^{-1}$$

$$\times \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma' \in Z_0(\mathfrak{a}_1)/S} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma_1 \gamma'^{-1} \gamma^{-1} t_1(w))}$$

$$\times \int_{A_1^+(\gamma)} a_1^{-i\nu} c(\gamma\colon \lambda + \lambda_J\colon a_1) \, \mathrm{d}a_1 \, \mathrm{d}\nu.$$

Since $\Gamma_1 \subseteq Z_0(\mathfrak{a}_1)/S$, we can make the change of variables $\gamma' \to \gamma' \gamma_1$, $\gamma_1 \in \Gamma_1$, to see that the $[\Gamma_1]^{-1}$ and the sum over Γ_1 cancel.

To complete the proof of the lemma we make the following observations. For $J \in \mathrm{Car}'(L, H)$, if $J_K^\dagger = J_K \cap M^\dagger$, we have

$$\frac{\mathrm{vol}\big(Z(\mathfrak{a}) J_K^0\big)}{[W(L, J_0)] \ \mathrm{vol}(J_K)_0 \big[Z(\mathfrak{a}_1) \cap Z(\mathfrak{a}) J_K^0/S\big]}$$

$$= \frac{\mathrm{vol}\big(J_K^\dagger\big)}{[W(L, J)] \ \mathrm{vol}\ J_K \big[Z(\mathfrak{a}_1)/S\big]}$$

$$= \big[W(L, J)\big]^{-1} \big[J_K : J_K^\dagger\big]^{-1} 2^{-d(A_1)}.$$

Also, for $a_0 \in A$,

$$\frac{1}{(2\pi)^{d(A)}} \int_{\mathfrak{a}^*} a_0^{-i\mu} \int_A I_f^J(\gamma z j_K a_1 a) a^{i\mu} \, \mathrm{d}a \, \mathrm{d}\mu = I_f^J(\gamma z j_K a_1 a_0).$$

QED

We now analyze the "second order" remainder terms $I_f^{H,J}$.

LEMMA 4.4:

$$\lim_{N \to \infty} \sum_{J \in \mathrm{Car}'(L,H)} I_{f,N}^{H,J}(z_0 t a_0)$$

$$= \frac{(-1)^{r_l(H)} [M/M^\dagger]}{\mathrm{vol}\ T_0 (2\pi)^{d(A)}}$$

$$\times \sum_{J \in \mathrm{Car}'(L,H)} \frac{[W_R(\mathfrak{l}, \mathfrak{j})][W(M_J, J_K)]}{[W(L, J)]\big[J_K : J_K \cap M^\dagger\big](4\pi)^{d(A_1)}}$$

$$\times \sum_{B \in \mathrm{Car}'(L_J, J)} \frac{[W_R(\mathfrak{l}_J, \mathfrak{b})](-1)^{q_B}}{[W(L_J, B)][B_K : B_K \cap M_J^\dagger](4\pi)^{d(A_2)}}$$

$$\times \sum_{w \in W(M^0, T^0)} \det w \sum_{\gamma_1 \in \Gamma(\mathfrak{a}_1)} d_{\gamma_1}^{-1} \sum_{\gamma_2 \in \Gamma(\mathfrak{a}_2)} d_{\gamma_2}^{-1}$$

$$\times \sum_{\eta \in Z(\mathfrak{b}_p)_\sigma^*} \sum_{i=1}^{k_2} \sum_{j=1}^{k_1} \overline{\eta_{ij}(z_0)} \sum_{\tau \in L_\eta} \epsilon(\tau) \overline{\xi_\tau(b(w))}$$

$$\times \int_{\mathfrak{a}^* \oplus \mathfrak{a}_1^* \oplus \mathfrak{a}_2^*} \Theta(B\colon \eta\colon \tau\colon \mu \otimes \nu_1 \otimes \nu_2)(f)$$

$$\times a_0^{-i\mu} I(\gamma_1\colon t_1(w)\colon \eta_{ij}\colon \nu_1)$$

$$\times I(\gamma_2 : t_2(w)\colon \eta_i\colon \nu_2) d\mu d\nu_1 d\nu_2 + \lim_{N \to \infty} \sum_{J,B} I_{f,N}^{H,J,B}(z_0 t a_0).$$

Notation is as follows. For $J \in \mathrm{Car}'(L, H)$, write $L_J = Z_G(J_p) = M_J J_p$. For $B \in \mathrm{Car}'(L_J, J)$, write $\mathfrak{a}_2 = \mathfrak{b}_p \cap \mathfrak{m}_J$, $\Gamma(\mathfrak{a}_2)$ a set of coset representatives for $Z(\mathfrak{a}_2) \cap J_K^0/S$. For $\gamma_2 \in \Gamma(\mathfrak{a}_2)$, d_{γ_2} is the number of elements of $\Gamma(\mathfrak{a}_2)$ conjugate to γ_2 via $W_R(\mathfrak{l}_J, \mathfrak{b})$ and $Z_2(\gamma_2)$ is the subgroup of $Z(b_p)$ generated by $Z(j_p)$, γ_2, and $Z_0(\mathfrak{a}_2) = \{\gamma' \in Z(\mathfrak{a}_2) \cap J_K^0 \mid \xi_\alpha(\gamma') = 1$ for all $\alpha \in \Phi_R(\mathfrak{l}_J, \mathfrak{b})\}$. For $\eta \in Z(\mathfrak{b}_p)^*$, $\eta = \eta_1 + \dots + \eta_{k_2}$ is the decomposition of $\eta \mid_{Z_2(\gamma_2)}$ into irreducible characters. For $1 \leqslant i \leqslant k_2$, $\eta_i = \eta_{i1} + \dots + \eta_{ik_1}$ is the decomposition of $\eta_i \mid_{Z_1(\gamma_1)}$ into irreducible characters. For

$$t_2 \in T_2 = \exp\left({}^{y^{-1}} i\mathfrak{a}_2\right),\ I(\gamma_2\colon t_2\colon \eta_i\colon \nu_2)$$

$$= \overline{\xi_{\lambda_i}(t_2)} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma_2^{-1} t_2)} \int_{A_2^+(\gamma_2)} a_2^{-i\nu_2} c(\gamma_2\colon \lambda + \lambda_i\colon a_2)\, da_2$$

where $L_R \cong T_2/T_2 \cap S$ and λ_i in the weight lattice of J_K^0 satisfies

$$\xi_{\lambda_i} \mid_{Z_2(\gamma_2) \cap J_K^0} = \frac{1}{\deg \eta_i} \eta_i \cdot \xi_\rho \mid_{Z_2(\gamma_2) \cap J_K^0}$$

The formula for $I_{f,N}^{H,J,B}(z_0 t a_0)$ is left to the reader.

PROOF: We will first evaluate $I_f^J(\gamma_1 z j_K a_1 a_0)$ using (4.3) with J replacing H. Rather than using the completely simplified version stated in the lemma we use an intermediate stage from the proof to write

$$I_f^J(\gamma_1 z j_K a_1 a_0)$$

$$= \frac{(-1)^{r_I(J)+1}[M_J/M_J^\dagger]}{\operatorname{vol}(J_K)_0(2\pi)^{d(A)+d(A_1)}}$$

$$\times \sum_{B \in \operatorname{Car}'(L_J, J)} \frac{[W_R(\mathfrak{l}_J, \mathfrak{b})](-1)^{q_B}}{[W(L_J, B)](4\pi)^{d(A_2)}[B_K : B_K \cap M_J^\dagger]}$$

$$\times \sum_{\gamma_2 \in \Gamma(\mathfrak{a}_2)} d_{\gamma_2}^{-1} \sum_{v \in W(M_J^0, J_K^0)} \det v \, \tilde{I}_{f,\gamma_2}^{J,B}(\gamma_1 z(v j_K) a_1 a_0)$$

$$+ \lim_{N \to \infty} \sum_{B \in \operatorname{Car}'(L_J, J)} I_{f,N}^{J,B}(\gamma_1 z j_K a_1 a_0)$$

where

$$\tilde{I}_{f,\gamma_2}^{J,B}(\gamma_1 z j_K a_1 a_0)$$

$$= \sum_{\eta \in Z(\mathfrak{b}_p)_\sigma^*} \sum_{i=1}^{k_2} \overline{(\eta_i \cdot \xi_\rho)(\gamma_2 \gamma_1 z)} \sum_{\tau \in L_\eta} \epsilon(\tau)$$

$$\times \int_{\mathfrak{a}^* \oplus \mathfrak{a}_1^* \oplus \mathfrak{a}_2^*} a_0^{-i\mu} a_1^{-i\nu_1} \Theta(B : \eta : \tau : \mu \otimes \nu_1 \otimes \nu_2)(f)$$

$$\times \sum_{\{\lambda_0 \in L_{\eta_i} \mid \lambda_0|_{\mathfrak{b}_K} = \tau\}} \overline{\xi_{\lambda_0}(\gamma_2^{-1} j_K)}$$

$$\times \int_{A_2^+(\gamma_2)} a_2^{-i\nu_2} c(\gamma_2 : \lambda_0 : a_2) \, \mathrm{d}a_2 \, \mathrm{d}\mu \, \mathrm{d}\nu_1 \, \mathrm{d}\nu_2$$

where the sum over L_η converges absolutely. Now

$$\sum_{w \in W(M^0, T^0)} \det w \sum_{\lambda \in w^{-1} L_\chi^N} \overline{\xi_\lambda(wt)}$$

$$\times \int_{Z(\mathfrak{a}) J_K^0 A_1^+(\gamma)} \Bigg\{ \sum_{v \in W(M_J^0, J_K^0)} \det v \, \tilde{I}_{f,\gamma_2}^{J,B}(\gamma_1 z(v j_K) a_1 a_0)$$

$$\times (\chi \cdot \xi_\rho)(z) \xi_\lambda(\gamma_1 j_K) c(\gamma_1 : \lambda : a_1) \Bigg\} \, \mathrm{d}(z j_K) \, \mathrm{d}a_1$$

$$= \sum_{v\in W(M_J^0,\, J_K^0)} \sum_{w\in W(M^0,\, T^0)} \det w \sum_{\lambda\in w^{-1}L_\chi^N} \overline{\xi_\lambda(wt)}$$

$$\times \int_{Z(\mathfrak{a})J_K^0A_1^+(\gamma_1)} \tilde{I}_{f,\gamma_2}^{J,B}(\gamma_1 z j_K a_1 a_0)\chi\cdot\xi_\rho(z)\xi_\lambda(\gamma_1 j_K)$$

$$\times c\big(\gamma_1\colon v^{-1}\lambda\colon a_1\big)\,\mathrm{d}(zj_K)\,\mathrm{d}a_1$$

using the changes of variables $j_K \to v^{-1}j_K$, $w \to v^{-1}w$, and $\lambda \to v^{-1}\lambda$. But $c(\gamma_1\colon v^{-1}\lambda\colon a_1)$ depends only on $v^{-1}\lambda\,|_{\mathfrak{t}_1} = \lambda\,|_{\mathfrak{t}_1}$. Thus the sum over $W(M_J^0, J_K^0)$ contributes only $[W(M_J^0, J_K^0)]$. This is combined with $[M_J/M_J^\dagger]$ to give $[W(M_J, J_K)]$.

We must evaluate

$$\sum_{\gamma_1\in\Gamma(\mathfrak{a}_1)} d_{\gamma_1}^{-1} \sum_{\gamma_2\in\Gamma(\mathfrak{a}_2)} d_{\gamma_2}^{-1} \sum_{\chi\in Z(\mathfrak{a})_\sigma^*} \overline{(\chi\cdot\xi_\rho)(z_0)} \sum_{\lambda\in w^{-1}L_\chi^N} \overline{\xi_\lambda(wt)}$$

$$\times \sum_{\eta\in Z(\mathfrak{b}_p)_\sigma^*} \sum_{i=1}^{k_2} \sum_{\tau\in L_\eta} \epsilon(\tau)$$

$$\times \int_{\mathfrak{a}^*\oplus\mathfrak{a}_1^*\oplus\mathfrak{a}_2^*} \Theta(B\colon \eta\colon \tau\colon \mu\otimes\nu_1\otimes\nu_2)(f)\,a_0^{-i\mu}\,\mathrm{d}\mu$$

$$\times \sum_{\{\lambda_0\in L_{\eta_i}\,|\,\lambda_0|_{\mathfrak{b}_K}=\tau\}} \overline{\xi_{\lambda_0}\big(\gamma_2^{-1}\big)} \int_{A_2^+(\gamma_2)} a_2^{-i\nu_2}c(\gamma_2\colon \lambda_0\colon a_2)\,\mathrm{d}a_2\,\mathrm{d}\nu_2$$

$$\times \int_{Z(\mathfrak{a})J_K^0A^+(\gamma_1)} \chi\cdot\xi_\rho(z)\xi_\lambda(\gamma_1 j_K)$$

$$\times c(\gamma_1\colon \lambda\colon a_1)\overline{(\eta_i\cdot\xi_\rho)(\gamma_2\gamma_1 z)}a_1^{-i\nu_1}\overline{\xi_{\lambda_0}(j_K)}\mathrm{d}(zj_K)\,\mathrm{d}a_1\,\mathrm{d}\nu_1$$

$$= \mathrm{vol}\big(Z(\mathfrak{a})J_K^0\big)$$

$$\times \sum_{\gamma_1\in\Gamma(\mathfrak{a}_1)} d_{\gamma_1}^{-1} \sum_{\gamma_2\in\Gamma(\mathfrak{a}_2)} d_{\gamma_2}^{-1} \sum_{\eta\in Z(\mathfrak{b}_p)_\sigma^*} \sum_{i=1}^{k_2} \sum_{j=1}^{k_1} \overline{(\xi_\rho\cdot\eta_{ij})(\gamma_2\gamma_1 z_0)}$$

$$\times \sum_{\tau\in L_\eta} \epsilon(\tau) \int_{\mathfrak{a}^*\oplus\mathfrak{a}_1^*\oplus\mathfrak{a}_2^*} a_0^{-i\mu}\Theta(B\colon \eta\colon \tau\colon \mu\otimes\nu_1\otimes\nu_2)(f)\,\mathrm{d}\mu$$

$$\times \sum_{\{\lambda_0\in L_{\eta_i}\,|\,\lambda_0|_{\mathfrak{b}_K}=\tau\}} \overline{\xi_{\lambda_0}\big(\gamma_2^{-1}j(w)\big)} \int_{A_2^+(\gamma_2)} a_2^{-i\nu_2}c(\gamma_2\colon \lambda_0\colon a_2)\,\mathrm{d}a\,\mathrm{d}\nu_2$$

$$\times \sum_{\{\lambda\in w^{-1}L_{\eta_{ij}}^N\,|\,\lambda|_{\mathfrak{t}_K}=\lambda_0\}} \overline{\xi_\lambda\big(\gamma_1^{-1}t_1(w)\big)}$$

$$\times \int_{A_1^+(\gamma_1)} c(\gamma_1 \colon \lambda \colon a_1) a_1^{-i\nu_1} \, da_1 \, d\nu_1 .$$

As in (4.3) we see that the $\lim_{N \to \infty}$ can be taken inside the sums and integrals and the sum over $\{\lambda \in L_{\eta_{i,j}} \mid \lambda\mid_{\mathfrak{t}_K} = \lambda_0\}$ can be replaced by

$$\overline{\xi_{\lambda_{i,j}}(\gamma_1^{-1} t_1(w))} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma_1^{-1} t_1(w))} \int_{A_1^+(\gamma_1)} c(\gamma_1 : \lambda + \lambda_{i,j} : a_1) a_1^{-i\nu_1} \, da_1$$

where $\lambda_{i,j} \in L$ satisfies only

$$\xi_{\lambda_{i,j}} \mid_{Z_1(\gamma_1) \cap T^0} = \frac{1}{\deg \eta_{i,j}} \eta_{i,j} \cdot \xi_\rho \mid_{Z_1(\gamma_1) \cap T^0} .$$

Similarly the sum over $\{\lambda_0 \in L_{\eta_i} \mid \lambda_0 \mid_{\mathfrak{b}_K} = \tau\}$ can be replaced by

$$\overline{\xi_\tau(b(w)) \xi_{\lambda_i}(\gamma_2^{-1} t_2(w))} \sum_{\lambda \in L_R} \overline{\xi_\lambda(\gamma_2^{-1} t_2(w))}$$

$$\times \int_{A_2^+(\gamma_2)} a_2^{-i\nu_2} c(\gamma_2 \colon \lambda + \lambda_i \colon a_2) \, da_2 .$$

The lemma now follows from observing that

$$\overline{\xi_\rho \cdot \eta_{i,j}(\gamma_2 \gamma_1 z_0)} \; \overline{\xi_{\lambda_{i,j}}(\gamma_1^{-1} t_1(w))} \; \overline{\xi_{\lambda_i}(\gamma_2^{-1} t_2(w))}$$

$$= \overline{(\xi_\rho \cdot \eta_{i,j})(z_0)} \; \overline{\xi_{\lambda_{i,j}}(t_1(w))} \; \overline{\xi_{\lambda_i}(t_2(w))}$$

and combining constants.

QED

We now organize all remainder terms as follows.

LEMMA 4.5: *Let H be a fundamental Cartan subgroup of G. There is a dense open subset $H^* \subseteq H'$ so that for $ta \in H^*$,*

$$F_f^H(ta) = \frac{(-1)^{r_I(H)} [M/M^\dagger]}{(2\pi)^{d(A)}} \sum_{w \in W(M^0, T^0)} \det w$$

$$\times \sum_{J \in \mathrm{Car}(L)} \frac{(-1)^{q_J}}{[W(M_J, J_K)] (4\pi)^{d(J_p \cap M)}}$$

$$\times \sum_{\eta \in Z(\mathfrak{j}_p)^*_\sigma} \sum_{j=1}^{k} \sum_{\tau \in L_\eta} \epsilon(\tau) \int_{\mathfrak{a}^*} a^{-i\mu}$$

$$\times \int_{(\mathfrak{j}_{M,p})^*} \Theta(J\colon \eta\colon \tau\colon \mu \otimes \nu)(f)$$

$$\times \sum_{S \in S(J)} (-1)^l I(S\colon \eta_j\colon \tau\colon \nu\colon wt)\, d\mu\, d\nu.$$

For $J \in \mathrm{Car}(L)$, $S(J)$ *is the set of all sequences* $J_0 = H,\ J_1, \ldots, J_l = J$ *where for* $0 \leqslant i \leqslant l$, $L_i = C_G(J_{i,p})$ *and* $J_i \in \mathrm{Car}'(L_{i-1}, J_{i-1})$ *for* $i \geqslant 1$. *Write* $\mathfrak{j}_{i,p} = \mathfrak{j}_{i-1,p} \oplus \mathfrak{a}_i$, $1 \leqslant i \leqslant l$ *and let* $T_i = \exp({}^{y^{-1}} i\mathfrak{a}_i)$. *For* $t \in T$, *write* $t = t_0 t_1 \ldots t_l$ *where* $t_0 \in J_K^0$, $t_i \in T_i$, $1 \leqslant i \leqslant l$.

For $\eta \in Z(\mathfrak{j}_p)^*_\sigma$, $\eta = \eta_1 + \ldots + \eta_k$ *is the decomposition into irreducible characters of* $\eta\,|_{(Z(\mathfrak{j}_p) \cap T)}$. *For H fundamental this is an abelian group so that the* η_j *are all one-dimensional. For* $1 \leqslant j \leqslant k$, $I(S\colon \eta_j\colon \tau\colon \nu\colon t) =$

$$\overline{\xi_\tau(t_0)} \prod_{i=1}^{l} w(\mathfrak{l}_{i-1}, \mathfrak{j}_i) \sum_{\gamma_i \in Z(\mathfrak{a}_i) \cap (J_{i-1})^0_K / S} d_{\gamma_i}^{-1} I(\gamma_i\colon t_i\colon \eta_j\colon \nu_i)$$

where $\nu_i = \nu\,|_{\mathfrak{a}_i}$ *and* $I(\gamma_i\colon t_i\colon \eta_j\colon \nu_i)$ *is defined using* $\lambda_j \in L$ *satisfying*

$$\xi_{\lambda_j}\,|_{Z(\mathfrak{j}_p) \cap T} = \eta_j \cdot \xi_\rho\,|_{Z(\mathfrak{j}_p) \cap T}.$$

For any $L = L_i$, $0 \leqslant i \leqslant l-1$ *and* $J \in \mathrm{Car}(L)$,

$$w(\mathfrak{l}, \mathfrak{j}) = \frac{[W_R(\mathfrak{l}, \mathfrak{j})][W(M_J, J_K)]}{[W(L, J)][J_K : J_K \cap M^\dagger]}.$$

PROOF: Note that for H fundamental, T is connected and vol $T = 1$. The lemma comes from combining (4.1), (4.3), and (4.4) and iterating the procedure until all remainder terms are exhausted. This is a finite process since for any Cartan subgroup J of G, if $B \in \mathrm{Car}'(L_J, J)$, then $\dim(B_p) > \dim(J_p)$. Rather than decomposing $\eta \in Z(\mathfrak{j}_p)^*_\sigma$ in stages as in (4.4), we have decomposed with respect to the abelian group $Z(\mathfrak{j}_p) \cap T$ which contains all the subgroups $Z_i(\gamma_i)$, $1 \leqslant i \leqslant l$.

QED

We now go back to the situation in which H is an arbitrary Cartan subgroup of G and $J \in \mathrm{Car}'(L, H)$. We want to simplify and evaluate the terms $I(\gamma\colon t_1\colon \eta_j\colon \nu)$ which appear in (4.3).

For $z \in Z(\mathfrak{a})$, $j_K \in J_K^0$, $t_1 \in T_1$, and $a \in A$, look at $I_\gamma(zj_Kt_1a) =$

$$\sum_{\eta \in Z(\mathfrak{j}_p)_\sigma^*} \sum_{j=1}^{k} \overline{(\eta_j \cdot \xi_\rho)(z)} \sum_{\tau \in L_\eta} \epsilon(\tau)\overline{\xi_\tau(j_K)}$$

$$\times \int_{\mathfrak{a}^* \oplus \mathfrak{a}_1^*} \Theta(J\colon \eta\colon \tau\colon \mu \otimes \nu)(f) a^{-i\mu} I(\gamma : t_1 : \eta_j : \nu)\, d\mu\ d\nu.$$

Let $v \in W_R(\gamma) \subseteq W(M^0, (J \cap M^0)_0) \subseteq W(G, J_0)$. Then

$$\Theta(J\colon \eta\colon \tau\colon \mu \otimes \nu)(f) = \Theta(J\colon v\eta\colon v\tau\colon v\mu \otimes v\nu)(f).$$

Since v acts trivially on J_K^0 and A we have $v\tau = \tau$ and $v\mu = \mu$. Also, v acts trivially on $Z_1(\gamma)$ so that $v\eta\,|_{Z_1(\gamma)} = \eta\,|_{Z_1(\gamma)} = \eta_1 + \ldots + \eta_k$. Thus

$$I_\gamma(zj_Kt_1a) = \sum_{\eta \in Z(\mathfrak{j}_p)_\sigma^*} \sum_{j=1}^{k} \overline{(\eta_j \cdot \xi_\rho)(z)} \sum_{\tau \in L_\eta} \epsilon(\tau)\overline{\xi_\tau(j_K)}$$

$$\times \int_{\mathfrak{a}^* \oplus \mathfrak{a}_1^*} \Theta(J\colon \eta\colon \tau\colon \mu \otimes \nu)(f) a^{-i\mu}$$

$$\times I(\gamma\colon t_1\colon \eta_j\colon v\nu)\, d\mu\ d\nu.$$

Thus we see that it is harmless to replace ν by $v\nu$, $v \in W_R(\gamma)$, in $I(\gamma\colon t_1\colon \eta_j\colon \nu)$. We will write $I(\gamma\colon t_1\colon \eta_j\colon \nu) \equiv I(\gamma\colon t_1\colon \eta_j\colon v\nu)$, $v \in W_R(\gamma)$.

Fix $\gamma \in \Gamma(\mathfrak{a}_1)$ and write $\Phi = \Phi_R(\gamma)$, $W = W_R(\gamma)$, $W_K = W_K(\gamma)$, $A_1^+(\gamma) = \exp(\mathfrak{a}_1^+)$. Since L_R is the weight lattice of $T_1/T_1 \cap S$, $\xi_\tau(\gamma^{-1}t_1)$, $\tau \in L_R$, depends only on the coset $\pi(\gamma^{-1}t_1)$ of $\gamma^{-1}t_1$ in G/S.

We will fix $1 \leqslant j \leqslant k$ and write $I(\gamma\colon t_1\colon \eta_j\colon \nu) = \overline{\xi_\delta(t)}$ $\times I(\pi(\gamma^{-1}t_1)\colon \delta\colon \nu\colon \Phi^+)$ where $\delta \in L$ satisfies $\xi_\delta\,|_{Z_1(\gamma) \cap T^0} = (1/\deg \eta_j)$ $\times \xi_\rho \cdot \eta_j\,|_{Z_1(\gamma) \cap T^0}$ and for $h \in T_1/T_1 \cap S$,

$$I(h\colon \delta\colon \nu\colon \Phi^+)$$

$$= [W_K]^{-1} \sum_{v \in W} \det v \sum_{\tau \in L_R} c(v\colon \tau + \delta\colon \Phi^+)\overline{\xi_\tau(h)}$$

$$\times \int_{\mathfrak{a}_1^+} \exp(v^\gamma(\tau + \delta) - i\nu)(H)\, dH.$$

This is the same expression that was evaluated in the linear case except for the shift by δ. As in the linear case the discrete series character formulas (2.5) and (2.8) will be used to reduce the computation of $I(h\colon \delta\colon \nu\colon \Phi^+)$ to computations involving root systems of type A_1 or B_2. In evaluating $I(h\colon \delta\colon \nu\colon \Phi^+)$ we can work entirely in $G/S = G_1$. Via the

Cayley transform we can identify $T_1/T_1 \cap S$ with $\overline{T}_1 = \exp(\mathrm{i}\mathfrak{a}_1) \subseteq G_{1,\mathbf{C}}$ and not distinguish between linear functionals on $\mathfrak{t}_1$ and on $\mathfrak{a}_1$.

Define Λ_1 as in (2.7). For $\lambda \in \Lambda_1$, let $\varphi(\lambda)$ be a two-structure for $\Phi(\lambda)$. Write $\varphi(\lambda) = \varphi_1 \cup \ldots \cup \varphi_s$ where the φ_i, $1 \leqslant i \leqslant s$, are simple roots systems of type A_1 or B_2. Let $\varphi_i^+ = \varphi_i \cap \Phi^+$. For $1 \leqslant i \leqslant s$, let $\mathfrak{a}(\varphi_i) = \sum_{\alpha \in \varphi_i} \mathbb{R} \mathrm{H}_\alpha$, $\mathfrak{a}(\varphi_i^+) = \{H \in \mathfrak{a}(\varphi_i) \mid \alpha(H) > 0 \text{ for all } \alpha \in \varphi_i^+\}$, and write $T_{\varphi_i} = \exp(\mathrm{i}\mathfrak{a}(\varphi_i))$. Then $\overline{T}_1 = \prod_{i=1}^s T_{\varphi_i}$ but the product need not be direct. Let $E(\varphi(\lambda)) = E(\lambda) = \{(\gamma_1, \ldots, \gamma_s) \mid \gamma_i \in T_{\varphi_i}, \prod_{i=1}^s \gamma_i = 1\}$. Then if we decompose $h \in \overline{T}_1$ as $h = h_1 \ldots h_s$, $h_i \in T_{\varphi_i}$, the components $(h_1, \ldots, h_s)$ are only determined up to multiplication by an element of $E(\lambda)$. For $\nu \in \mathfrak{a}_1^*$, let $\nu_i = \nu \mid_{\mathfrak{a}(\varphi_i)}$, $\delta_i = \delta \mid_{\mathfrak{a}(\varphi_i)}$. Define

$$P(\varphi^+(\lambda)\colon \nu\colon \delta\colon h) = \sum_{(\gamma_1, \ldots, \gamma_s) \in E(\lambda)} \prod_{i=1}^{s} S(\varphi_i^+\colon \nu_i\colon \delta_i\colon \gamma_i h_i) \quad (4.6)$$

where for $1 \leqslant i \leqslant s$, $S(\varphi_i^+\colon \nu_i\colon \delta_i\colon h_i) =$

$$[W(\varphi_i)]^{-1} \sum_{w \in W(\varphi_i)} \det w \sum_{\tau \in L(\varphi_i)} \bar{c}(w(\tau + \delta_i) : \varphi_i^+) \overline{\xi_\tau(h_i)}$$

$$\times \int_{\mathfrak{a}(\varphi_i^+)} \exp(w(\tau + \delta_i - \mathrm{i}\nu_i)(H))\, \mathrm{d}H.$$

Note that for φ_i^+ of type A_1 this is the same as $S(\nu_i\colon \delta_i\colon h_i)$ in (3.12) since $c(w\colon \tau\colon \varphi_i^+) = \frac{1}{2}\bar{c}(w\tau\colon \varphi_i^+)$.

LEMMA 4.7: *$I(h\colon \delta\colon \nu\colon \Phi^+) \equiv \sum_{\lambda \in \Lambda_1} c(\lambda)^{-1} \sum_{v \in W} \det v\, \kappa_\lambda(v)$ $\times P(\varphi^+(\lambda)\colon \nu\colon v\delta\colon vh)$ where*

$$c(\lambda) = [W_K] 2^n [W_1(\lambda : \Phi^+)] \epsilon(\lambda : \Phi^+) [W_1(\varphi(\lambda) : \Phi^+(\lambda))]$$

$$\times \epsilon(\varphi(\lambda) : \Phi^+(\lambda)) [E(\lambda)].$$

PROOF: Using (2.8),

$$I(h\colon \delta\colon \nu\colon \Phi^+) = [W_K]^{-1} 2^{-n} \sum_{\lambda \in \Lambda_1} [W_1(\lambda\colon \Phi^+)]^{-1} \epsilon(\lambda\colon \Phi^+)$$

$$\times I(h\colon \delta\colon \nu\colon \lambda)$$

where $I(h\colon \delta\colon \nu\colon \lambda) =$

$$\sum_{s\in W_0(\lambda:\Phi^+)} \det s \sum_{v\in W} \det v\ \kappa_\lambda(s^{-1}v) \sum_{\tau\in L_R} \bar{c}\big(s^{-1}v(\tau+\delta)\colon \Phi^+(\lambda)\big)$$

$$\times\overline{\xi_\tau(h)}\int_{\mathfrak{a}_1^+} \exp\big(v(\tau+\delta)-\mathrm{i}\nu\big)(H)\,\mathrm{d}H$$

$$= \sum_{v\in W} \det v\ \kappa_\lambda(v) \sum_{\tau\in L_R} \bar{c}\big(v(\tau+\delta)\colon \Phi^+(\lambda)\big)\overline{\xi_\tau(h)}$$

$$\times \sum_{s\in W_0(\lambda:\Phi^+)} \int_{s^{-1}\mathfrak{a}_1^+} \exp\big(v(\tau+\delta)-\mathrm{i}s^{-1}\nu\big)(H)\,\mathrm{d}H.$$

But we can replace ν by $s\nu$, $s\in W_0(\lambda\colon \Phi^+)$. Note also that

$$\bigcup_{s\in W_0(\lambda:\Phi^+)} s^{-1}\mathfrak{a}_1^+$$

$$= \mathfrak{a}\big(\Phi^+(\lambda)\big) = \big\{H\in\mathfrak{a}_1 \mid \alpha(H)>0 \text{ for all } \alpha\in\Phi^+(\lambda)\big\}.$$

Thus

$$I(h\colon \delta\colon \nu\colon \lambda) \equiv \sum_{v\in W} \det v\ \kappa_\lambda(v) \sum_{\tau\in L_R} \bar{c}\big(v(\tau+\delta)\colon \Phi^+(\lambda)\big)\overline{\xi_\tau(h)}$$

$$\times\int_{\mathfrak{a}(\Phi^+(\lambda))} \exp\big(v(\tau+\delta)-\mathrm{i}\nu\big)(H)\,\mathrm{d}H.$$

We let $\varphi=\varphi(\lambda)$ be a two-structure for $\Phi(\lambda)$ and use (2.5) to write

$$I(h\colon \delta\colon \nu\colon \lambda)$$

$$= \epsilon\big(\varphi : \Phi^+(\lambda)\big)\big[W_1\big(\varphi : \Phi^+(\lambda)\big)\big]^{-1} \sum_{s\in W_0(\varphi:\Phi^+(\lambda))} \det s$$

$$\times \sum_{v\in W} \det v\ \kappa_\lambda(v) \sum_{\tau\in L_R} \bar{c}\big(s^{-1}v(\tau+\delta) : \varphi^+\big)$$

$$\times\overline{\xi_\tau(h)}\int_{\mathfrak{a}(\Phi^+(\lambda))} \exp\big(v(\tau+\delta)-\mathrm{i}\nu\big)(H)\,\mathrm{d}H$$

$$= \epsilon\big(\varphi : \Phi^+(\lambda)\big)\big[W_1\big(\varphi : \Phi^+(\lambda)\big)\big]^{-1} \sum_{v\in W} \det v\ \kappa_\lambda(v)$$

$$\times \sum_{\tau\in L_R} \bar{c}\big(v(\tau+\delta)\colon \varphi^+\big)\overline{\xi_\tau(h)}$$

$$\times \sum_{s\in W_0(\varphi:\Phi^+(\lambda))} \int_{s^{-1}\mathfrak{a}(\Phi^+(\lambda))} \exp\big(v(\tau+\delta)-\mathrm{i}s^{-1}\nu\big)(H)\,\mathrm{d}H$$

since for all $s \in W_0(\varphi\colon \Phi^+(\lambda)) \subseteq W(\Phi(\lambda))$, $\kappa_\lambda(sv) = \kappa_\lambda(v)$, $v \in W$. Now as before we can replace ν by $s\nu$ and note that

$$\bigcup_{s \in W_0(\varphi : \Phi^+(\lambda))} s^{-1}\mathfrak{a}(\Phi^+(\lambda)) = \mathfrak{a}(\varphi^+)$$

$$= \{ H \in \mathfrak{a}_1 \mid \alpha(H) > 0 \text{ for all } \alpha \in \varphi^+ \}.$$

Thus

$$\begin{aligned} I(h\colon \delta\colon \nu\colon \lambda) &= \epsilon(\varphi\colon \Phi^+(\lambda))[W_1(\varphi\colon \Phi^+(\lambda))]^{-1} \sum_{v \in W} \det v\ \kappa_\lambda(v) \\ &\quad \times \sum_{\tau \in L_R} \overline{\xi_\tau(vh)}\bar{c}(\tau + v\delta\colon \varphi^+) \\ &\quad \times \int_{\mathfrak{a}(\varphi^+)} \exp(\tau + v\delta - i\nu)(H)\, dH. \end{aligned}$$

But now we can enlarge the lattice L_R to $L(\varphi)$ by using the fact that for $\lambda = \lambda_1 + \ldots + \lambda_s \in L(\varphi)$, $\lambda_i \in L(\varphi_i)$, $i \leqslant i \leqslant s$,

$$\sum_{(\gamma_1, \ldots, \gamma_s) \in E(\lambda)} \prod_{i=1}^{s} \xi_{\lambda_i}(\gamma_i) = \begin{cases} [E(\lambda)] & \text{if } \lambda \in L_R \\ 0 & \text{if } \lambda \notin L_R. \end{cases}$$

Thus

$$\begin{aligned} I(h\colon \delta\colon \nu\colon \lambda) &= [E(\lambda)]^{-1}\epsilon(\varphi : \Phi^+(\lambda))[W_1(\varphi : \Phi^+(\lambda))]^{-1} \sum_{v \in W} \det v\ \kappa_\lambda(v) \\ &\quad \times \sum_{(\gamma_1, \ldots, \gamma_s) \in E(\lambda)} \prod_{i=1}^{s} \left\{ \sum_{\tau \in L(\varphi_i)} \bar{c}(\tau + (v\delta)_i : \varphi_i^+) \right. \\ &\quad \left. \times \overline{\xi_\tau(\gamma_i(vh)_i)} \int_{\mathfrak{a}(\varphi_i^+)} \exp(\tau + (v\delta)_i - i\nu)(H)\, dH \right\} \\ &= [W(\varphi)]^{-1}\epsilon(\varphi\colon \Phi^+(\lambda))[W_1(\varphi\colon \Phi^+(\lambda))]^{-1}[E(\lambda)]^{-1} \\ &\quad \times \sum_{w \in W(\varphi)} \det w \sum_{v \in W} \det v\ \kappa_\lambda(v) \end{aligned}$$

$$\times \sum_{(\gamma_1, \ldots, \gamma_s) \in E(\lambda)} \prod_{i=1}^{s} \left\{ \sum_{\tau \in L(\varphi_i)} \bar{c}(w(\tau + (v\delta)_i) : \varphi_i^+) \right.$$

$$\left. \times \overline{\xi_\tau(\gamma_i(vh)_i)} \int_{\mathfrak{a}(\varphi_i^+)} \exp(w(\tau + (v\delta)_i) - \mathrm{i}\nu)(H)\, \mathrm{d}H \right\}$$

where to obtain the last formula we make the change of variables $v \to wv$, $\tau \to w\tau$, $v \in W$, $\tau \in L(\varphi_i)$, $1 \leqslant i \leqslant s$, $w \in W(\varphi)$. But this last expression is

$$\equiv [E(\lambda)]^{-1} \epsilon(\varphi : \Phi^+(\lambda))[W_1(\varphi : \Phi^+(\lambda))]^{-1}$$

$$\times \sum_{v \in W} \det v\, \kappa_\lambda(v) P(\varphi^+(\lambda) : \nu : v\delta : vh).$$

QED

The next step is to eliminate the sums over Λ_1 and W from (4.7).

Fix $\psi = \varphi_1 \cup \ldots \cup \varphi_s \in T(\Phi)$ so that ${}^yR \subseteq \psi$ where R is the set of strongly orthogonal non-compact roots of $(\mathfrak{m}, \mathfrak{t})$ used to form the Cayley transform Ad y: $\mathfrak{t}_{\mathbf{C}} \to (\mathfrak{j} \cap \mathfrak{m})_{\mathbf{C}}$. Fix $\varphi = \varphi_i$, $1 \leqslant i \leqslant s$, ν, $\delta \in \mathfrak{a}(\varphi)^*$, $h \in T_\varphi$. If φ is of type A_1, define $T(\varphi^+\colon \nu\colon \delta\colon h) = S(\varphi^+\colon \nu\colon \delta\colon h)$. If φ is of type B_2, let $\varphi_i^+ = \{\beta_i\}$, $i = 1, 2$, where $\{\beta_1, \beta_2\}$ is the set of short positive roots of φ and β_2 is simple. Set

$$\epsilon(\varphi^+) = \begin{cases} 1 & \text{if } y^{-1}\beta_2 \text{ is compact;} \\ -1 & \text{if } y^{-1}\beta_1 \text{ is compact.} \end{cases}$$

Decompose $h \in T_\varphi$ as $h = h_1 h_2$ where $h_i \in T_{\varphi_i}$, $i = 1, 2$. This decomposition is not unique since $\gamma_0 = \exp(\pi \mathrm{i} H^*_{\beta_1}) = \exp(\pi \mathrm{i} H^*_{\beta_2}) \in T_{\varphi_1} \cap T_{\varphi_2}$. Define

$$T(\varphi^+\colon \nu\colon \delta\colon h) = S(\varphi^+\colon \nu\colon \delta\colon h) + \tfrac{1}{4}\epsilon(\varphi^+) S(\varphi_s^+ : \nu\colon \delta\colon h)$$

where

$$S(\varphi_s^+ : \nu\colon \delta\colon h)$$

$$= S(\varphi_1^+ : \nu_1\colon \delta_1\colon h_1) S(\varphi_2^+ : \nu_2\colon \delta_2\colon h_2) + S(\varphi_1^+ : \nu_1\colon \delta_1\colon \gamma_0 h_1)$$

$$\times S(\varphi_2^+ : \nu_2\colon \delta_2\colon \gamma_0 h_2)$$

and

$$\nu_i = \nu|_{\mathfrak{a}(\varphi_i)}, \ \delta_i = \delta|_{\mathfrak{a}(\varphi_i)}, \quad i = 1, 2.$$

LEMMA 4.8:

$$I(h: \delta: \nu: \Phi^+) \equiv [W_1(\Phi: \psi^+)]^{-1} \epsilon(\psi: \Phi^+)[W_K]^{-1}[L(\psi): L_R]^{-1} \times \sum_{v \in W_K} \det v \, Q(\psi^+: \nu: v\delta: vh)$$

where

$$Q(\psi^+: \nu: \delta: h) = \sum_{(\gamma_1, \ldots, \gamma_s) \in E(\psi)} \prod_{i=1}^{s} T(\varphi_i^+: \nu_i: \delta_i: \gamma_i h_i).$$

As before $E(\psi) = \{(\gamma_1, \ldots, \gamma_s) \mid \gamma_i \in T_{\varphi_i}, \prod_{i=1}^{s} \gamma_i = 1\}$.

PROOF: From the definition of $P(\varphi^+(\lambda): \nu: \delta: h)$, $\lambda \in \Lambda_1$, it is clear that for all $w \in W(\varphi(\lambda))$, $P(\varphi^+(\lambda): \nu: w\delta: wh) \equiv \det w \; P(\varphi^+(\lambda): \nu: \delta: h)$ and for all $w \in W_1(\Phi : \varphi^+(\lambda))$, $P(\varphi^+(\lambda): \nu: w\delta: wh) \equiv P(\varphi^+(\lambda): \nu: \delta: h)$.

As in [7d], Φ^+ and Λ_1 can be chosen so that for φ_i^+ any simple factor of type B_2 of ψ, $\epsilon(\varphi_i^+) = 1$ and so that for all $\lambda \in \Lambda_1$:
(i) $\epsilon(\lambda: \Phi^+) = 1$;
(ii) $\varphi(\lambda) = \psi \cap \Phi(\lambda) \in T(\Phi(\lambda))$ and $\epsilon(\varphi(\lambda): \Phi^+(\lambda)) = \epsilon(\psi: \Phi^+)$;
(iii) $W_1(\Phi: \Phi^+(\lambda)) \subseteq W_1(\Phi: \psi^+)$;
(iv) λ is a sum of long roots of $\Phi(\lambda)$.

Fix $\varphi \subseteq \psi$ so that $\Lambda_1(\varphi) = \{\lambda \in \Lambda_1 \mid \varphi \in T(\Phi(\lambda))\} \neq \emptyset$. Let $U(\varphi) = S(\varphi) W_1(\Phi: \varphi^+)$ where $S(\varphi)$ is the subgroup of $W(\varphi)$ generated by reflections in long roots of φ. Then

$$\sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1} \sum_{v \in W} \det v \, \kappa_\lambda(v) P(\varphi^+(\lambda): \nu: v\delta: vh)$$

$$\equiv \sum_{v \in U(\varphi) \backslash W / W_K} \det v \left\{ \sum_{w \in W_K} \det w \, P(\varphi^+: \nu: vw\delta: vwh) \right\}$$

$$\times \frac{[S(\varphi)]}{[U(\varphi) \cap W_K]} \sum_{\lambda \in \Lambda_1(\varphi)} c(\lambda)^{-1} \sum_{u \in W_1(\Phi : \varphi^+)} \det u \, \kappa_\lambda(uv).$$

The term $c(\varphi: v) = ([S(\varphi)]/[U(\varphi) \cap W_K]) \sum_{\lambda \in \Lambda_1(\varphi)} \times c(\lambda)^{-1} \sum_{u \in W_1(\Phi : \varphi^+)} \det u \, \kappa_\lambda(uv)$ is analyzed in Lemma 4.9 of [7d], where the following statements are proved. First, if $v \notin U(\varphi) W_K$, $c(\varphi: v) = 0$ for all φ. In the case that $v = 1$ represents the identity double coset we have the following formulas for $c(\varphi) = c(\varphi : 1)$.

(i) If $\varphi = \psi$, then $c(\psi) = [W_K]^{-1} \epsilon(\psi: \Phi^+)[W_1(\Phi: \psi^+)]^{-1} \times [L(\psi): L_R]^{-1}$.

(ii) Suppose $\varphi \subsetneq \psi$ and Φ is simple of type B_{2k}, $k \geqslant 1$, or F_{2k}, $k = 2$. Then ψ is of type B_2^k and φ is of type $B_2^{k-1} \times B_1^2$ and $c(\varphi) = (k/4)c(\psi)$.

(iii) Suppose $\varphi \subsetneq \psi$ and Φ is simple of type C_n where $n = 2k$ or $2k+1$. Then ψ is of type C_2^k or $C_2^k \times C_1$ and φ is of type $A_1^{2p} \times C_2^{k-p}$ or $A^{2p} \times C_2^{k-p} \times C_1$ for some $0 < p \leqslant k$. In this case $c(\varphi) = \binom{k}{p} 4^{-p} c(\psi)$.

Now if φ has only one root length or is of type G_2 we are done since $\Lambda_1(\psi) = \Lambda_1$ and $Q(\psi^+ \colon \nu \colon \delta \colon h) = P(\psi^+ \colon \nu \colon \delta \colon h)$. In the remaining cases Q has extra terms which don't occur in P. Note that for $v \in W_K$, $Q(\psi^+ \colon \nu \colon v\delta \colon vh) \equiv Q(v^{-1}\psi^+ \colon \nu \colon \delta \colon h)$.

Case 1: Suppose Φ is simple of type B_{2k+1}, $k \geqslant 1$. Then $\psi = \varphi_1 \cup \ldots \cup \varphi_k \cup \varphi_{k+1}$ where for $1 \leqslant i \leqslant k$, φ_i is of type B_2, and φ_{k+1} is of type B_1. Here $\Lambda_1(\psi) = \Lambda_1$ and any term in $Q(\psi \colon \nu \colon \delta \colon h)$ which contains a factor of the form

$$S(\varphi_{i,s}^+ \colon \nu_i \colon \delta_i \colon h_i) S(\varphi_{k+1}^+ \colon \nu_i \colon \delta_i \colon h_i),$$

$1 \leqslant i \leqslant k$, cancels out in the sum over W_K because there is an element $v \in W_K$ with det $v = -1$ such that $v^{-1}(\varphi_{i,s}^+ \cup \varphi_{k+1}^+) = \varphi_{i,s}^+ \cup \varphi_{k+1}^+$, $v^{-1}\varphi_j^+ = \varphi_j^+$, $j \neq i$, $k+1$. Thus

$$\sum_{v \in W_K} \det v\, Q(\psi^+ \colon \nu \colon v\delta \colon vh) \equiv \sum_{v \in W_K} \det v\, P(\psi^+ \colon \nu \colon v\delta \colon vh).$$

Case 2: Suppose Φ is simple of type B_{2k}, $k \geqslant 1$, or F_{2k}, $k = 2$. Then $\Lambda_1 = \Lambda_1(\psi) \cup \Lambda_1(\varphi)$ where $\varphi \subsetneq \psi$ is as in (ii) above. Write $\psi = \varphi_1 \cup \ldots \cup \varphi_k$ where each φ_i is of type B_2. Then there is $1 \leqslant j \leqslant k$ so that $P(\varphi^+ \colon \nu \colon \delta \colon h) =$

$$\sum_{(\gamma_1, \ldots, \gamma_k) \in E(\psi)} \left[\prod_{\substack{i=1 \\ i \neq j}}^{k} S(\varphi_i^+ : \nu_i : \delta_i : \gamma_i h_i) \right] S(\varphi_{j,s}^+ : \nu_j : \delta_j : \gamma_j h_j).$$

The k possible φ's which could have been chosen are all conjugate by elements of determinant one in W_K. Further, any term in $Q(\psi^+ \colon \nu \colon \delta \colon h)$ which contains a factor of the form

$$S(\varphi_{i,s}^+ \colon \nu_i \colon \delta_i \colon h_i) S(\varphi_{j,s}^+ \colon \nu_j \colon \delta_j \colon h_j),$$

$1 \leqslant i \neq j \leqslant k$, cancels out in the sum over W_K as in Case 1. Thus $\sum_{v \in W_K}$ det v[P(ψ^+: ν: v δ: vh) + (k/4)P(φ^+: ν: vδ: vh)] $\equiv \sum_{\mathrm{v} \in \mathrm{W_K}}$ det v Q(ψ^+: ν: vδ: vh).

Case 3: Suppose Φ is of type C_n, $n=2k$ or $2k+1$. Then $\Lambda_1 = \bigcup_{p=o}^{k} \Lambda_1(\varphi(p))$ where $\varphi(0)=\psi=\varphi_1 \cup \ldots \cup \varphi_{k+1}$ where for $1 \leqslant i \leqslant k$, φ_i is of type C_2 and φ_{k+1} is of type C_1 if $n=2k+1$ and $\varphi_{k+1}=\emptyset$ if $n=2k$. For $0 \leqslant p \leqslant k$ there is a subset P of $\{1, \ldots, k\}$ with p elements so that $P(\varphi(p)^+ \colon \nu \colon \delta \colon h)=$

$$\sum_{(\gamma_1, \ldots, \gamma_{k+1}) \in E(\psi)} \prod_{i \notin P} S(\varphi_i^+ : \nu_i : \delta_i : \gamma_i h_i) \prod_{i \in P} S(\varphi_{i,s}^+ : \nu_i : \delta_i : \gamma_i h_i).$$

There are $\binom{k}{p}$ such choices of $\varphi(p)$, all conjugate by elements of determinant one in W_K. Thus

$$\begin{aligned} &\sum_{v \in W_K} \det v \sum_{p=0}^{k} \binom{k}{p} 4^{-p} P(\varphi(p)^+ \colon \nu \colon v\delta \colon vh) \\ &\quad \equiv \sum_{v \in W_K} \det v \, Q(\psi^+ \colon \nu \colon v\delta \colon vh). \end{aligned}$$

This concludes the proof for Φ^+ chosen as in [7d]. Now suppose Φ^+ is replaced by $u\Phi^+$, $u \in W$. Then

$$\begin{aligned} I(h \colon \delta \colon \nu \colon u\Phi^+) = [W_K]^{-1} &\sum_{v \in W} \det(uv) \\ &\times \sum_{\tau \in L_R} c(uv \colon \tau + \delta \colon u\Phi^+) \overline{\xi_\tau(h)} \\ &\times \int_{\mathfrak{a}(u\Phi^+)} \exp(uv(\tau+\delta) - \mathrm{i}\nu)(H) \, \mathrm{d}H \\ \equiv \det u \, I(h \colon \delta \colon \nu \colon \Phi^+). \end{aligned}$$

Now $u = ws^{-1}$ where $w \in W(\psi)$ and $s\psi^+ \subseteq \Phi^+$. Then $\epsilon(\psi \colon u\Phi^+) = \epsilon(u^{-1}\psi \colon \Phi^+) = \epsilon(s\psi \colon \Phi^+) = \det s \, \epsilon(\psi \colon \Phi^+)$. Further $\psi \cap u\Phi^+ = w\psi^+$ and $P(w\psi^+ \colon \nu \colon \delta \colon h) \equiv \det w \, P(\psi^+ : \nu : \delta : h)$. If ψ has a simple factor φ of type B_2, and if $w \in W(\varphi)$, write $w = w_1 w_2$ where $w_2 \in W(\varphi_s)$, $w\varphi_s^+ = w_2\varphi_s^+$. Then

$$S(w\varphi_s^+ : \nu \colon \delta \colon h) = S(w_2\varphi_s^+ : \nu \colon \delta \colon h) = \det w_2 \; S(\varphi_s^+ : \nu \colon \delta \colon h)$$

and $\epsilon(w\varphi^+) = \det \mathrm{w}_1 \, \epsilon(\varphi^+)$ since $\epsilon(w_2\varphi^+) = \epsilon(\varphi^+) = 1$ for all $w_2 \in W(\varphi_s)$ and either w_1 is trivial so that $\epsilon(w_1 w_2 \varphi^+) = \epsilon(w_2\varphi^+) = 1$ and $\det w_1 = 1$ or w_1 is the reflection which interchanges the compact and non-compact short roots so that $\epsilon(w_1 w_2 \varphi^+) = -1 = \det w_1$. Thus $T(w\varphi^+ \colon \nu \colon \delta \colon h) = \det w \; T(\varphi^+ \colon \nu \colon \delta \colon h)$ for $w \in W(\varphi)$. Since every $w \in W(\psi)$ is a product of elements from the $W(\varphi_i)$, $1 \leqslant i \leqslant s$, we see that $Q(w\psi^+ \colon \nu \colon \delta \colon h) \equiv \det w \; Q(\psi^+ \colon \nu \colon \delta \colon h)$.

QED

Finally, we eliminate the sum over W_K in (4.8).

LEMMA 4.9:

$$\sum_{w\in W(M^0,T^0)} \det w\, \overline{\xi_\tau(j(w))}I(\gamma\colon t_1(w)\colon \eta_j\colon \nu)$$

$$\equiv [W_1(\Phi:\psi^+)]^{-1}\epsilon(\psi\colon \Phi^+)[L(\psi)\colon L_R]^{-1}$$

$$\times \sum_{w\in W(M^0,T^0)} \det w\, \overline{\xi_\tau(j(w))}\,\overline{\xi_\delta(t_1(w))}$$

$$\times Q(\psi^+\colon \nu\colon \delta\colon \gamma^{-1}t_1(w)).$$

PROOF: Using (4.8) we have

$$\sum_{w\in W(M^0,T^0)} \det w\, \overline{\xi_\tau(j(w))}I(\gamma\colon t_1(w)\colon \eta_j\colon \nu)$$

$$\equiv [W_1(\Phi\colon \psi^+)]^{-1}\epsilon(\psi\colon \Phi^+)[L(\psi)\colon L_R]^{-1}$$

$$\times \sum_{w\in W(M^0,T^0)} \det w\, \overline{\xi_\tau(j(w))}\,\overline{\xi_\delta(t_1(w))}[W_K]^{-1}$$

$$\times \sum_{v\in {}^{\nu^{-1}}W_K} \det v\, Q(\psi^+\colon \nu\colon v\delta\colon v\pi(\gamma^{-1}t_1(w))).$$

Now ${}^{\nu^{-1}}W_K \subseteq W(M^0, T^0)$. For $v\in {}^{\nu^{-1}}W_K$, take $w\to v^{-1}w$, $w\in W(M^0, T^0)$. Then $v^{-1}wt = v^{-1}j(w)t_1(w) = j(w)(v^{-1}t_1(w))$. Thus we have

$$[W_1(\Phi\colon \psi^+)]^{-1}\epsilon(\psi\colon \Phi^+)[L(\psi)\colon L_R]^{-1} \sum_{v\in {}^{\nu^{-1}}W_K}\ \sum_{w\in W(M^0,T^0)} \det w$$

$$\times [W_K]^{-1}\overline{\xi_\tau(j(w))\xi_{v\delta}(t_1(w))}Q(\psi^+\colon \nu\colon v\delta\colon \pi(\gamma^{-1}t_1(w))).$$

But for $v\in {}^{\nu^{-1}}W_K$, $v\delta\in L$ and $v\delta-\delta|_{\mathfrak{s}}=0$ so that $v\delta-\delta|\mathfrak{t}_1$ gives a character of $T_1/T_1\cap S$. Thus $v\delta|_{\mathfrak{t}_1}=\delta|_{\mathfrak{t}_1}+\lambda_0$, $\lambda_0\in L_R$. But for any simple root system φ of type A_1 or B_2, $\lambda_0\in L(\varphi)$, $h\in T_\varphi$, ν, $\delta\in\mathfrak{a}(\varphi)^*$,

$$\overline{\xi_{\lambda_0}(h)}S(\varphi^+\colon \nu\colon \delta+\lambda_0\colon h) = [W(\varphi)]^{-1}$$

$$\times \sum_{w\in W(\varphi)} \det w \sum_{\tau\in L(\varphi)} \bar{c}(w(\tau+\delta+\lambda_0)\colon \varphi^+)$$

$$\times \overline{\xi_{\tau+\lambda_0}(h)}\int_{\mathfrak{a}(\varphi^+)} \exp(w(\tau+\delta+\lambda_0-i\nu)(H))\,\mathrm{d}H$$

$$= S(\varphi^+\colon \nu\colon \delta\colon h).$$

Thus for

$$v \in {}^{y^{-1}}W_K, \overline{\xi_{v\delta}(t_1(w))}Q(\psi^+: \nu: v\delta: \pi(\gamma^{-1}t_1(w)))$$

$$= \xi_{v\delta}(\gamma)\overline{\xi_{v\delta}(\gamma^{-1}t_1(w))}Q(\psi^+: \nu: v\delta: \pi(\gamma^{-1}t_1(w)))$$

$$= \xi_{v\delta}(\gamma)\overline{\xi_\delta(\gamma^{-1}t_1(w))}\ \overline{\xi_{\lambda_0}(\pi(\gamma^{-1}t_1(w))}$$

$$\times Q(\psi^+: \nu: \delta+\lambda_0: \pi(\gamma^{-1}t_1(w)))$$

$$= \overline{\xi_\delta(t_1(w))}Q(\psi^+: \nu: \delta: \pi(\gamma^{-1}t_1(w)))$$

since $\xi_{v\delta}(\gamma) = \xi_\delta(\gamma)$, $v \in W = W_R(\gamma)$.

QED

We now go back to the notation of (4.5).

LEMMA 4.10: *Assume that H is fundamental. Then for $J \in \mathrm{Car}(L)$,*

$$\frac{(-1)^{r_l(H)}}{(4\pi)^{d(J_p\cap M)}[W(M_J, J_K)]} \sum_{w \in W(M^0,T^0)} \det w$$

$$\times \sum_{S \in S(J)} (-1)^l I(S: \eta_j: \tau: \nu: wt)$$

$$\equiv \frac{(-1)^{r_l(J)}}{[W(L, J)][J_K : J_K \cap M^\dagger]}\left(\frac{i}{2}\right)^{d(J_p\cap M)}$$

$$\times \sum_{w \in W(M^0,T^0)} \det w\, K(J: \eta_j: \tau: \nu: wt)$$

where $K(J: \eta_j: \tau: \nu: t) = (\epsilon(\psi: \Phi^+)[T(\Phi)]/[L(\psi): L(\Phi)])\overline{\xi_\tau(t_0)}\prod_{l=1}^{s} K(\varphi_l^+: \eta_j: \nu_l: t_l)$. Here $\psi = \varphi_1 \cup \ldots \cup \varphi_s$ is a two-structure for $\Phi = \Phi_R(\mathfrak{l}, \mathfrak{j})$, chosen so that ${}^yR_J \subseteq \psi$. Decompose $t \in T'$ as $t = t_0 t_1 \ldots t_s$ where $t_0 \in J_K^0$ and $t_i \in T_i = \exp({}^{y^{-1}}i\mathfrak{a}(\varphi_i))$. Since $S \subseteq J_K^0$ we can decompose carefully, pushing extra factors of $T_i \cap S$ into J_K^0, so that if φ_i^+ is of type $A_1 = \{\alpha\}$, $t_i = \exp(-i\theta^{y^{-1}}H_\alpha^*)$ with $0 < |\theta| < \pi$ and if φ_i^+ is of type B_2 with long roots α_1, α_2, $t_i = \exp(-i\theta_1\, {}^{y^{-1}}H_{\alpha_1}^* - \theta_2\, {}^{y^{-1}}H_{\alpha_2}^*)$ with $0 < |\theta_1| \neq |\theta_2| < \pi$. Then $K(\varphi_i^+: \eta_j: \nu_i: t_i)$ is defined as in (3.13) if φ_i is of type A_1 and as in (5.3) if φ_l is of B_2.

PROOF: Note that

$$I(S: \eta_j: \tau: \nu: t)$$

$$= \overline{\xi_\tau(t_0)}\; \overline{\xi_\delta(t_1 \ldots t_l)} \prod_{\iota=1}^{l} w(\mathfrak{l}_{\iota-1}, \mathfrak{j}_\iota)$$

$$\times \sum_{\gamma_\iota \in Z(\mathfrak{a}_\iota) \cap (J_{\iota-1})_K^0 / S} d_{\gamma_\iota}^{-1} I\big(\pi(\gamma_\iota^{-1} t_\iota): \delta: \nu_\iota: \Phi_R^+(\gamma_\iota)\big)$$

where the first terms $\overline{\xi_\tau(t_0)}\; \overline{\xi_\delta(t_1 \ldots t_l)}$ are independent of $S \in S(J)$ and the remaining terms are defined on G/S and are the same as in the linear case except for the shifts. Thus the combinatorial problem of adding up all $S \in S(J)$ is exactly the same as for the linear case since all the constants are the same and the lattice of Cartan subgroups is the same. Evaluating the factors $S(\varphi_\iota^+: \nu_\iota: \delta_\iota: h_\iota)$ for root systems of type A_1 and B_2 is done in (3.12) and (5.1) respectively. Adapting the $K(\varphi_\iota^+: \eta_j: \nu_\iota: t_\iota)$ factors from the linear case to include the shifts is done in (3.13) and (5.3).

QED

THEOREM 4.11: *Let $f \in C_c^\infty(G/S, \sigma)$ and let $h = ta \in H'$ where H is a fundamental Cartan subgroup of G. Then*

$$F_f^H(ta) = \frac{[M/M^\dagger]}{(2\pi)^{d(A)}} \sum_{J \in \mathrm{Car}(L)} \frac{(-1)^{r_I(J)+q_J}}{[W(L, J)][J_K : J_K \cap M^\dagger]}$$

$$\times \left(\frac{i}{2}\right)^{d(J_P \cap M)} \sum_{\eta \in Z(\mathfrak{j}_p)_\sigma^*} \sum_{j=1}^{k} \sum_{\tau \in L_\eta} \epsilon(\tau)$$

$$\times \int_{\mathfrak{a}^*} a^{-i\mu} \int_{(\mathfrak{l}_{M,p})^*} \Theta(J: \eta: \tau: \mu \otimes \nu)(f)$$

$$\times \sum_{w \in W(M^0, T^0)} \det w\, K\big(J: \eta_j: \tau: \nu: wt\big)\, d\mu\, d\nu.$$

Here $\eta = \eta_1 + \ldots + \eta_k$ is the decomposition of $\eta\,|_{Z(\mathfrak{l}_p) \cap T}$ into irreducible characters and the $K(J: \eta_j: \tau: \nu: t)$ are defined as in (4.10).

PROOF: Combine (4.5) and (4.10). The restriction that $ta \in H^*$ can be relaxed to $ta \in H'$ since both sides of the equation are continuous functions on H'.

QED

We now have the Fourier inversion formula for $F_f^H(h)$, $h \in H'$ fundamental. The next step is to differentiate and evaluate at $h = 1$.

LEMMA 4.12:

$$f(1) = \frac{[W(M, T)](-1)^q}{(2\pi)^{r+p}} \sum_{J \in \operatorname{Car}(L)} d_J$$

$$\times \sum_{\chi \in Z(\mathfrak{j}_p)_\sigma^{\hat{}}} \sum_{\tau \in L_\chi} \epsilon(\tau) \int_{\mathfrak{j}_p^*} \Theta(J : \chi : \tau : \nu)(f)$$

$$\times \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + \mathrm{i}\nu \rangle \operatorname{tr}\left\{ \prod_{\alpha \in \psi^+} p_\alpha(\chi : \nu) \right\} \mathrm{d}\nu .$$

Here for $\alpha \in \psi^+$ $p_\alpha(\chi : \nu)$ is the square matrix of size $k = \deg(\chi)$ given by

$$p_\alpha(\chi : \nu) = \sinh \pi\nu_\alpha \cdot I_k \Big\{ \cosh \pi\nu_\alpha \cdot I_k + \xi_{\rho - \rho(\psi^+)}(\gamma_\alpha)$$

$$\times (1/2)\left[\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})\right]\Big\}^{-1}$$

where $\nu_\alpha = 2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle$, $\gamma_\alpha = \exp(\pi Z_\alpha)$, and $\rho = \rho(\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}}))$. The constants are defined by

$$r = \left[\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})\right], \; p = \operatorname{rank} G - \operatorname{rank} K,$$

$$q = \tfrac{1}{2}\{\dim G/K - \operatorname{rank} G + \operatorname{rank} K\},$$

and

$$d_J = (i)^{d(J_p \cap M)} \frac{[T(\Phi_R)] \epsilon(\psi : \Phi_R^+)(-1)^{q_J}}{[W(L, J)][J_K : J_K \cap M^\dagger][L(\psi) : L(\Phi_R)]}$$

$$\times \prod_{\alpha \in R_J} 1/\|\alpha\|$$

where $\Phi_R = \Phi_R(\mathfrak{l}, \mathfrak{j})$ and $\psi \in T(\Phi_R)$ satisfies ${}^y R_J \subseteq \psi$.

PROOF: As in (2.13) we have $f(1) = ((-1)^q/(2\pi)^r) \lim_{(ta) \to 1} F_f^H(ta; D)$ where $D = \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})} H_\alpha$. Fix $\eta \in Z(\mathfrak{j}_p)_\sigma^*$, $\tau \in L_\eta$, $\nu \in \mathfrak{j}_{M,p}^*$, $\mu \in \mathfrak{a}^*$. Let $R_J = \{\alpha_1, \ldots, \alpha_l\}$. Then if $t = t_0 \exp(-\mathrm{i}\Sigma_{i=1}^l \theta_i H_{\alpha_i}^*)$ where $t_0 \in J_K^0$ and $|\theta_i| < \pi$ for $1 \leqslant i \leqslant l$, $a^{-\mathrm{i}\mu} K(J\colon \eta_J\colon \tau\colon \nu\colon t)$ is a constant multiple depending only on the chamber of t, of

$$\overline{\xi_\tau(t_0)} \exp\left(- \sum_{i=1}^{l} \nu_i \theta_i \right) a^{-\mathrm{i}\mu} = \xi_{-\lambda}(ta)$$

where $\lambda = \tau + {}^{y^{-1}}(i\nu) + i\mu \in \mathfrak{h}_{\mathbb{C}}^*$. Thus for each $w \in W(M^0, T^0)$, $a^{-i\mu}K(J\colon \eta_j\colon \tau\colon \nu\colon wt)$ is a constant multiple of $\xi_{-\lambda}(wta) = \xi_{-w^{-1}\lambda}(ta)$. Now for $\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $H_\alpha \cdot \xi_{-w^{-1}\lambda}(ta) = \langle \alpha, -w^{-1}\lambda\rangle \xi_{-w^{-1}\lambda}(ta)$. Thus

$$\lim_{(ta)\to 1} D\xi_{-w^{-1}\lambda}(ta) = \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\,\mathfrak{h}_{\mathbb{C}})} \langle \alpha, -w^{-1}\lambda\rangle$$

$$= (-1)^r \det w \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\,\mathfrak{h}_{\mathbb{C}})} \langle \alpha, \tau + {}^{y}(i\nu) + i\mu\rangle$$

$$= (-1)^r \det w \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\,\mathfrak{j}_{\mathbb{C}})} \langle \alpha, \tau + i\nu + i\mu\rangle, \text{ so that}$$

$$\lim_{(ta)\to 1} D\cdot \sum_{w\in W(M^0,T^0)} \det w\, a^{-i\mu}K\big(J\colon \eta_j\colon \tau\colon \nu\colon wt\big)$$

$$= \prod_{\alpha\in R_J} \frac{2}{\|\alpha\|} \frac{(-1)^r\epsilon(\psi:\Phi_R^+)[T(\Phi_R)]}{[L(\psi)\colon L(\Phi_R)]}$$

$$\times \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\,\mathfrak{j}_{\mathbb{C}})} \langle \alpha, \tau + i\nu + i\mu\rangle$$

$$\times \sum_{w\in W(M^0,T^0)} \prod_{i=1}^{s} k\big(\varphi_i^+ : \eta_j\colon \nu_i\colon C_i((wt)_i)\big).$$

But for all $v \in W(\varphi_i)$, $k(\varphi_i^+ : (v\eta)_j\colon v\nu_i\colon C_i^+) \equiv \det v\, k(\varphi_i^+ : \eta_j\colon \nu_i\colon C_i^+)$, and it is shown for φ_i of type A_1 in (3.14) and for φ_i of type B_2 in (5.4) and (5.5) that $[W(\varphi_i)]^{-1}\sum_{v\in W(\varphi_i)} \det v\, k(\varphi_i^+ : (\nu\eta)_j\colon v\nu_i\colon C_i^+)$ is independent of the chamber C_i^+ and is equal to

$$(-1)^{[\varphi_i^+]} \prod_{\alpha\in\varphi_i^+} p_\alpha(\eta_j\colon \nu)$$

where

$$p_\alpha(\eta_j\colon \nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha + \left[\dfrac{\eta_j(\gamma_\alpha) + \eta_j(\gamma_\alpha^{-1})}{2}\right]\xi_{\rho-\rho(\psi^+)}(\gamma_\alpha)}.$$

Note that for $\varphi_i^+ = \{\alpha\}$ of type A_1 we would have as in (3.14),

$$k\big(\varphi_i^+ : \eta_j\colon \nu_i\colon C_i^+\big) \equiv \frac{-\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha - \xi_\rho\dfrac{(\gamma_\alpha)}{2}\big[\eta_j(\gamma_\alpha) + \eta_j(\gamma_\alpha^{-1})\big]}.$$

But in this case $\xi_{\rho(\psi^+)}(\gamma_\alpha)=\xi_{\alpha/2}(\gamma_\alpha)=-1$ so that $k(\varphi_i^+ : \eta_j : \nu_i : C_i^+)\equiv -p_\alpha(\eta_j : \nu_i)$. Thus we have

$$f(1)=\frac{[W(M^0,T^0)][M/M^\dagger](-1)^{r+q}}{(2\pi)^{r+p}}$$

$$\times \sum_{J\in \mathrm{Car}(L)} \frac{(-1)^{r_I(J)+q_J}\epsilon(\psi:\Phi_R^+)[T(\Phi_R)](-1)^{[\psi^+]}}{[W(L,J)][J_K:J_K^+][L(\psi):L(\Phi_R)]}$$

$$\times\left(\frac{i}{2}\right)^{d(J_p\cap M)}\prod_{\alpha\in R_J}\frac{2}{\|\alpha\|}\sum_{\eta\in Z(\mathfrak{j}_p)^*_\sigma}\sum_{\tau\in L_\eta}\epsilon(\tau)\int_{\mathfrak{j}_p^*}\Theta(J:\eta:\tau:\nu)(f)$$

$$\times\prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\,\mathfrak{j}_{\mathbf{C}})}\langle\alpha,\tau+i\nu\rangle\sum_{i=1}^{k}\left\{\prod_{\alpha\in\psi^+}p_a(\eta_j:\nu)\right\}\mathrm{d}\nu.$$

Suppose $\eta=\mathrm{tr}\,\chi$, $\chi\in Z(\mathfrak{j})^\wedge_\sigma$. Since $\eta=\eta_1+\ldots+\eta_k$ is the decomposition of η with respect to the abelian group $Z(\mathfrak{j}_p)\cap T$ which contains all the elements γ_α, $\alpha\in\psi^+$,

$$\sum_{j=1}^{k}\prod_{\alpha\in\psi^+}p_\alpha(\eta_j:\nu)=\mathrm{tr}\left\{\prod_{\alpha\in\psi^+}p_\alpha(\chi:\nu)\right\}.$$

The lemma now follows from noting that $[W(M^0,T^0)][M/M^\dagger]=[W(M,T)]$, $(-1)^r=(-1)^{r_I(J)}(-1)^{[\psi^+]}$, and $d(J_p\cap M)=[R_J]$.

QED

We now have a Plancherel formula for G. The next step is to rewrite it in a form inspired by Harish-Chandra's product formula for the Plancherel measure. That is, we want to replace the term $\mathrm{tr}\prod_{\alpha\in\psi^+}p_\alpha(\chi:\nu)$ by $\prod_{\alpha\in\Phi_R^+(\mathfrak{g},\mathfrak{j})}[(\deg\chi)^{-1}\,\mathrm{tr}\,p_\alpha(\chi:\nu)]$. In order to do this we first need to define $p_\alpha(\chi:\nu)$ for arbitrary $\alpha\in\Phi_R^+(\mathfrak{g},\mathfrak{j})$ and see that $(\deg\chi)^{-1}\,\mathrm{tr}\,p_\alpha(\chi:\nu)$ is the Harish-Chandra factor given in [6d].

Recall that for $\alpha\in\psi^+$, $p_\alpha(\chi:\nu)$ involves a factor of $\xi_{\rho-\rho(\psi^+)}(\gamma_\alpha)$ where $\rho=\rho(\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}}))$ and $\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})$ was chosen so that $\beta\in\Phi_{CPX}^+$ implies that $\bar{\beta}\in\Phi_{CPX}^+$. Now for an appropriate choice of Φ_R^+ and ψ we have, as in the linear case [7b], that $\xi_{\rho(\psi^+)}(\gamma_\alpha)=\xi_{\rho_R}(\gamma_\alpha)$ for all $\alpha\in\psi^+$ where $\rho_R=\rho(\Phi_R^+(\mathfrak{g},\mathfrak{j}))$.

LEMMA 4.13: *Let* $\Phi_R^+=\Phi_R(\mathfrak{g},\mathfrak{j})\cap\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})$. *Then* $\xi_{\rho-\rho_R}(\gamma_\alpha)=-\xi_{\rho_\alpha}(\gamma_\alpha)$ *for all* $\alpha\in\Phi_R^+$ *where* $\rho=\rho(\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}}))$, $\rho_R=\rho(\Phi_R^+)$, *and* $\rho_\alpha=\rho(\Phi_\alpha^+)$ *where* $\Phi_\alpha^+=\{\beta\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})\mid\beta\mid\mathfrak{j}_p=k\alpha\}$.

PROOF: Since $\xi_{\rho-\rho_R}|_S=\xi_{\rho_\alpha}|_S=1$ we may as well work in G/S where

$\gamma_\alpha = \exp(\pi i H_\alpha^*)$. We will first show that for any $w \in W(G, J)$, $\xi_{w(\rho-\rho_R)}(\gamma_\alpha) = \xi_{\rho-\rho_R}(\gamma_\alpha)$. This is because $\rho - \rho_R - w(\rho - \rho_R)$ is a sum of complex and imaginary roots satisfying $\beta > 0$ and $w^{-1}\beta < 0$. For β imaginary, $(\beta, \alpha) = 0$ so that $\xi_\beta(\gamma_\alpha) = 1$ for $\alpha \in \Phi_R^+$. If β is complex and satisfies $\beta > 0$ and $w^{-1}\beta < 0$, then we also have $\bar{\beta} > 0$, $w^{-1}\bar{\beta} = \overline{w^{-1}\beta} < 0$. But $\langle \beta, \alpha \rangle = \langle \bar{\beta}, \alpha \rangle$ so that $\xi_\beta(\gamma_\alpha) = \xi_{\bar{\beta}}(\gamma_\alpha) = \pm 1$ and $\xi_{\beta+\bar{\beta}}(\gamma_\alpha) = 1$. Thus $\xi_{\rho-\rho_R-w(\rho-\rho_R)}(\gamma_\alpha) = 1$. Thus changing the ordering Φ^+ on Φ by an element of $W(G, J)$ does not change the value of $\xi_{\rho-\rho_R}(\gamma_\alpha)$. Now fix $\alpha \in \Phi_R^+$. We can modify the ordering on Φ by the action of an element of $N_G(J_p)/Z_G(J_p)$ so that α or $\frac{1}{2}\alpha$ is a simple root for the set of restricted roots $\Sigma^+ = \{\beta|_{\mathfrak{j}_p} \mid \beta \in \Phi^+\}$. Then for $\delta \in \Sigma^+$, $\delta \neq k\alpha$, $s_\alpha\delta \in \Sigma^+$. But $\langle \delta + s_\alpha\delta, \alpha \rangle = 0$ so that $\xi_\rho(\gamma_\alpha) = \xi_{\rho_\alpha}(\gamma_\alpha)$. Also, for $\beta \in \Phi_R^+$, $\beta \neq \alpha$, $s_\alpha\beta \in \Phi_R^+$. Thus $\xi_{\rho_R}(\gamma_\alpha) = \xi_{\alpha/2}(\gamma_\alpha) = -1$. Thus $\xi_{\rho-\rho_R}(\gamma_\alpha) = -\xi_{\rho_\alpha}(\gamma_\alpha)$.

QED

We now define, for $\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})$,

$$p_\alpha(\chi: \nu) = \sinh \pi\nu_\alpha \cdot I_k \times \left\{ \cosh \pi\nu_\alpha \cdot I_k - \frac{\xi_{\rho_\alpha}(\gamma_\alpha)}{2}\left[\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})\right] \right\}^{-1} \quad (4.14)$$

where $\nu_\alpha = 2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle$, $k = \deg \chi$, and ρ_α is defined as in (4.13).

We will now derive some facts about the elements $\gamma_\alpha = \exp(\pi Z_\alpha) \in G$ which are well-known in the linear case.

LEMMA 4.15: *Suppose* $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{j})$, $w \in W(G, J_0)$. *Then* $w \cdot \gamma_\alpha = \gamma_{w\alpha}^{\pm 1}$.

PROOF: Recall $Z_\alpha = X_\alpha - Y_\alpha$ where $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ are chosen so that $\theta(Z_\alpha) = Z_\alpha$, θ the Cartan involution, and $[X_\alpha, Y_\alpha] = H_\alpha^*$. For $w \in W(G, J_0)$, $wX_\alpha \in \mathfrak{g}_{w\alpha}$ and $wY_\alpha \in \mathfrak{g}_{-w\alpha}$ so that $wZ_\alpha = cX_{w\alpha} - c'Y_{w\alpha}$. But $w = \mathrm{Ad}\, k$ for some $k \in K$ so that wZ_α is θ-stable. Thus $c = c'$ and $wZ_\alpha = cZ_{w\alpha}$. But $[wX_\alpha, wY_\alpha] = wH_\alpha^* = H_{w\alpha}^*$. Thus $c^2[X_{w\alpha}, Y_{w\alpha}] = H_{w\alpha}^*$ so that $c^2 = 1$ and $wZ_\alpha = \pm Z_{w\alpha}$.

QED

The proof of Lemma 4.16 below is due to Michel Duflo and is considerably shorter than our original proof.

LEMMA 4.16: *For all* $\chi \in Z(\mathfrak{j}_p)^\wedge$ *and* $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{j})$, $p_\alpha(\chi: \nu)$ *is a scalar matrix.*

PROOF: Fix $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{j})$ and $\chi \in Z(\mathfrak{j}_p)^\wedge$. We must show that $\chi(\gamma_\alpha) +$

$\chi(\gamma_\alpha^{-1})$ is a scalar matrix. For any $z \in Z(\mathfrak{j}_p)$, $z\gamma_\alpha z^{-1} = \gamma_\alpha^{\pm 1}$ by (4.15) since the adjoint action of z is trivial on the roots. Thus the cyclic subgroup C generated by γ_α is normal in $Z(\mathfrak{j}_p)$. Let $\chi|_C = \chi_1 + \ldots + \chi_k$ be the decomposition of χ into irreducible (one-dimensional) representations of C and let $\xi = \chi_1(\gamma_\alpha)$. Then for each $2 \leqslant i \leqslant k$, $\chi_i(\gamma_\alpha) = \chi_1(z_i\gamma_\alpha z_i^{-1})$ for some $z_i \in Z(\mathfrak{j}_p)$. Thus $\chi_i(\gamma_\alpha) = \chi_1(\gamma_\alpha^{\pm 1}) = \xi^{\pm 1}$, and $\chi_i(\gamma_\alpha) + \chi_i(\gamma_\alpha^{-1}) = \xi + \xi^{-1}$ for all $1 \leqslant i \leqslant k$.

QED

LEMMA 4.17: *Write* $\Phi = \Phi_R(\mathfrak{g}, \mathfrak{j})$. *Then for any* $\chi \in Z(\mathfrak{j}_p)^{\wedge}$,

$$\sum_{\varphi \in T(\Phi)} \epsilon(\varphi : \Phi^+) \prod_{\alpha \in \varphi^+} \bar{p}_\alpha(\chi : \nu) = \prod_{\alpha \in \Phi^+} \bar{p}_\alpha(\chi : \nu)$$

where for $\alpha \in \Phi^+$, $\bar{p}_\alpha(\chi : \nu) = (\deg \chi)^{-1}$ tr $p_\alpha(\chi : \nu)$; $p_\alpha(\chi : \nu)$ *defined as in* (4.14).

PROOF: We will reduce the proof to the linear case [7b,e]. Let $\tilde{G}_1$ denote the universal covering group of $G_1 = G/S$.

Case 1: If $\tilde{G}_1 = G_1$ the result follows directly from the linear case.

Case 2: Suppose $\tilde{G}_1$ is a two-fold covering of G_1. We may as well assume that Φ is simple. Then if $\alpha, \beta \in \Phi$ have the same length, $\beta = w\alpha$ for some $w \in W(\Phi) \subseteq W(G, J_0)$ so that $w\gamma_\alpha = \gamma_\beta^{\pm 1}$. Thus γ_α and γ_β have the same order, either two or four.

(i) Suppose all roots of Φ have the same length. Then all γ_α, $\alpha \in \Phi$, have the same order and the same square. For $\chi \in Z(\mathfrak{j}_p)^{\wedge}$ if $\chi(\gamma_\alpha^2) = I$ for all α, then χ is one-dimensional with $\chi(\gamma_\alpha) = \pm 1$. The result follows from the linear case. If $\chi(\gamma_\alpha^2) = -I$ for all α, then

$$\bar{p}_\alpha(\chi : \nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha} = \frac{\sinh \pi 2\nu_\alpha}{\cosh \pi 2\nu_\alpha + 1} = \bar{p}_\alpha(\xi_{\rho - \rho_R} : 2\nu),$$

and so the result again follows from the linear case.

(ii) Suppose Φ is of type B_n. Write $\Phi = \Phi_s \cap \Phi_l$ where Φ_s, Φ_l denote the sets of short and long roots of Φ respectively. Now $\Phi_s \subseteq \varphi$ for all $\varphi \in T(\Phi)$. Thus

$$\begin{aligned} &\sum_{\varphi \in T(\Phi)} \epsilon(\varphi : \Phi^+) \prod_{\alpha \in \varphi^+} \bar{p}_\alpha(\chi : \nu) \\ &= \prod_{\alpha \in \Phi_s^+} \bar{p}_\alpha(\chi : \nu) \sum_{\varphi \in T(\Phi)} \epsilon(\varphi : \Phi^+) \prod_{\alpha \in \varphi^+ \cap \Phi_l} \bar{p}_\alpha(\chi : \nu) \\ &= \prod_{\alpha \in \Phi_s^+} \bar{p}_\alpha(\chi : \nu) \prod_{\alpha \in \Phi_l^+} \bar{p}_\alpha(\chi : \nu) \end{aligned}$$

using the argument from (i) for the long roots.

(iii) Suppose Φ is of type F_4. Let φ_1 be any collection of 4 orthogonal short positive roots of Φ. Then $\{\varphi \in T(\Phi) \mid \varphi_1 \subseteq \varphi\} = T(\Phi_1)$ where $\Phi_1^+ = \varphi_1 \cup \Phi_l^+$ is a root system of type B_4. By the argument used in (ii),

$$\sum_{\varphi \in T(\Phi_1)} \epsilon(\varphi\colon \Phi^+) \prod_{\alpha \in \varphi^+} \bar{p}_\alpha(\chi\colon \nu) = \epsilon(\varphi_1) \prod_{\alpha \in \Phi_1^+} \bar{p}_\alpha(\chi\colon \nu).$$

All root systems Φ_1 of this type have the long roots of Φ in common. Thus

$$\sum_{\varphi_1} \epsilon(\varphi_1) \prod_{\alpha \in \Phi_1^+} \bar{p}_\alpha(\chi\colon \nu) = \prod_{\alpha \in \Phi_l^+} \bar{p}_\alpha(\chi\colon \nu) \sum_{\varphi_1} \epsilon(\varphi_1) \prod_{\alpha \in \varphi_1} \bar{p}_\alpha(\chi\colon \nu).$$

But the different choices of φ_1 can be identified with the different two-structures for D_4, and so by (i)

$$\sum_{\varphi_1} \epsilon(\varphi_1) \prod_{\alpha \in \varphi_1} \bar{p}_\alpha(\chi\colon \nu) = \prod_{\alpha \in \Phi_s^+} \bar{p}_\alpha(\chi\colon \nu).$$

(iv) Suppose Φ is of type G_2. Then if α is a long positive root and β is the unique short positive root orthogonal to it, $\pi(\gamma_\alpha) = \pi(\gamma_\beta) \in G_1$. Thus $\gamma_\alpha = \gamma_\beta \gamma_0$ where γ_0 is central in $Z(\mathfrak{j}_p)$ and satisfies $\gamma_0^2 = 1$. Thus they have the same order so as in (i) the result follows from the linear case.

Case 3: If G_1 is of Hermitian type, then $Z(\mathfrak{j}_p)$ is abelian. In every case except when Φ is of type C_n, the only two-structure for Φ is Φ itself so that there is nothing to prove. Suppose Φ is of type C_n. Then $\Phi_l \subseteq \varphi$ for all $\varphi \in T(\Phi)$. Thus

$$\sum_{\varphi \in T(\Phi)} \epsilon(\varphi\colon \Phi^+) \prod_{\alpha \in \varphi^+} \bar{p}_\alpha(\chi\colon \nu)$$

$$= \prod_{\alpha \in \Phi_l^+} \bar{p}_\alpha(\chi\colon \nu) \sum_{\varphi \in T(\Phi)} \epsilon(\varphi\colon \Phi^+) \prod_{\alpha \in \varphi^+ \cap \Phi_s} \bar{p}_\alpha(\chi\colon \nu).$$

Let $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$ denote the universal covering group of $\mathrm{Sp}(2, \mathbb{R})$. Any short root $\beta \in \Phi$ can be included in a root system $\varphi_\beta \subseteq \Phi$ of type C_2. This determines an injection $\mathfrak{sp}(2, \mathbb{R}) \hookrightarrow \mathfrak{g}$ and hence a homomorphism $q\colon \widetilde{\mathrm{Sp}}(2, \mathbb{R}) \to G$. Then using (1.5c), $\gamma_\beta^2 = q(\gamma_{\beta_i}^2) = 1$, $i = 1, 2$. Thus for every $\alpha \in \Phi_s$ we have $\gamma_\alpha^2 = 1$ so that $\chi(\gamma_\alpha) = \pm 1$ and again by the linear case

$$\sum_{\varphi \in T(\Phi)} \epsilon(\varphi\colon \Phi^+) \prod_{\alpha \in \varphi^+ \cap \Phi_s} \bar{p}_\alpha(\chi\colon \nu) = \prod_{\alpha \in \Phi_s^+} \bar{p}_\alpha(\chi\colon \nu).$$

QED

We are now ready to state and prove the final version of the Plancherel Theorem for groups satisfying (1.2). For convenience we repeat many definitions.

THEOREM 4.18: *Let* $f \in C_c^\infty(G/S, \sigma)$. *Then*

$$f(1) = \frac{[W(G, H)]}{(2\pi)^{r+p}} \sum_{J \in \mathrm{Car}(G)} c_J^{-1} \sum_{\chi \in Z(\mathfrak{i}_p)_\sigma^\wedge} \deg \chi$$

$$\times \sum_{\tau \in L'_\chi} \int_{\mathfrak{i}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f)$$

$$\times \left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi\colon \nu) \right| d\nu.$$

Here H *is a fundamental Cartan subgroup of* G, $r = [\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})]$, *and* $p = \mathrm{rank}\ G - \mathrm{rank}\ K$. *Further, for* $J \in \mathrm{Car}(G)$,

$$c_J = [W(G, J)][J_K\colon J_K \cap M_H^\dagger][L(\psi)\colon L(\Phi_R)] \prod_{\alpha \in R_J} \|\alpha\| \cdot$$

where R_J *is the set of strongly orthogonal non-compact roots of* $(\mathfrak{g}, \mathfrak{h})$ *which determines* J. *Finally,* ψ *is a two-structure for* $\Phi_R = \Phi_R(\mathfrak{l}_H, \mathfrak{j})$. *The following table give values for* $[L(\psi)\colon L(\Phi_R)]$ *when* Φ_R *is simple. In general* $[L(\psi)\colon L(\Phi_R)]$ *would be the product of the values for the simple factors of* Φ_R.

Φ_R	A_1	B_{2n}	B_{2n+1}	C_{2n}	C_{2n+1}	D_{2n}	E_7	E_8	F_4	G_2
$[L(\psi)\colon L(\Phi_R)]$	1	2^{n-1}	2^n	1	1	2^{n-1}	2^3	2^4	2	2

For $\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})$, $\bar{p}_\alpha(\chi\colon \nu) = (\deg \chi)^{-1}\ \mathrm{tr}\ p_\alpha(\chi\colon \nu)$ *where if* $k = \deg \chi$ *and* I_k *denotes the identity matrix of size* k,

$$p_\alpha(\chi\colon \nu) = \sinh \pi\nu_\alpha \cdot I_k$$

$$\times \left\{ \cosh \pi\nu_\alpha \cdot I_k - \frac{\xi_{\rho_\alpha}(\gamma_\alpha)}{2} [\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})] \right\}^{-1}.$$

Here $\nu_\alpha = 2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle$, $\gamma_\alpha = \exp(\pi Z_\alpha)$, *and* $\rho_\alpha = \rho(\Phi_\alpha^+)$ *where* $\Phi_\alpha^+ = \{\beta \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}}) \mid \beta|_{\mathfrak{i}_p}$ *is a multiple of* $\alpha\}$.

PROOF: Write $\Phi = \Phi_R(\mathfrak{g}, \mathfrak{j})$. Then $T(\Phi) = \{w\psi \mid w \in W_0\}$ where $W_0 = \{w \in W_R(\mathfrak{g}, \mathfrak{j}) \mid w\psi^+ \subseteq \Phi^+\}$. Since $W_0 \subseteq W(G, J_0)$ it is possible by changing variables and using the invariance of $\Theta(J\colon \chi\colon \tau\colon \nu)(f)$ under the action of $W(G, J_0)$ to obtain

$$\epsilon(\psi\colon \Phi_R^+) \sum_{\chi\in Z(\mathfrak{j}_p)_\sigma^{\wedge}} \deg\chi \sum_{\tau\in L_\chi} \epsilon(\tau)$$

$$\times \int_{\mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f) \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu\rangle$$

$$\times \prod_{\alpha\in\psi^+} \bar{p}_\alpha(\chi\colon \nu)\, d\nu$$

$$= [W_0]^{-1} \sum_{w\in W_0} \epsilon(\psi\colon \Phi_R^+) \sum_\chi \deg(w^{-1}\chi) \sum_\tau \epsilon(w^{-1}\tau)$$

$$\times \int_{\mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f) \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})} \langle \alpha, w^{-1}(\tau + i\nu)\rangle$$

$$\times \sum_{\alpha\in\psi^+} \bar{p}_\alpha(w^{-1}\chi\colon w^{-1}\nu)\, d\nu.$$

But $\deg(w^{-1}\chi) = \deg\chi$, $\epsilon(w^{-1}\tau) = \epsilon(\tau)$ since $w \in W_R(\mathfrak{g}, \mathfrak{j})$, and

$$\prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})} \langle \alpha, w^{-1}(\tau + i\nu)\rangle = \det w \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu\rangle.$$

Finally, $p_\alpha(w^{-1}\chi\colon w^{-1}\nu) = p_{w\alpha}(\chi\colon \nu)$ since $(w^{-1}\nu)_\alpha = \nu_{w\alpha}$ and

$$(w^{-1}\chi)(\gamma_\alpha) + (w^{-1}\chi)(\gamma_\alpha^{-1}) = \chi(w\gamma_\alpha) + \chi(w\gamma_\alpha^{-1})$$

$$= \chi(\gamma_{w\alpha}) + \chi(\gamma_{w\alpha}^{-1})$$

by (4.15), and $(-1)^{\rho_{w\alpha}} = (-1)^{\rho_\alpha}$ using (4.13). Thus the above expression is equal to

$$[T(\Phi)]^{-1} \sum_{\varphi\in T(\Phi)} \epsilon(\varphi\colon \Phi^+) \sum_\chi \deg\chi \sum_\tau \epsilon(\tau) \int_{\mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f)$$

$$\times \prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbf{C}},\mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu\rangle \prod_{\alpha\in\varphi^+} \bar{p}_\alpha(\chi\colon \nu)\, d\nu$$

since for $w \in W_0$, $\epsilon(w\psi\colon \Phi_R^+) = \det w\, \epsilon(\psi\colon \Phi_R^+)$. Now use (4.17). The constants are simplified as follows. In (4.12) we are summing over $J \in \mathrm{Car}(L)$. However, if $J_1, J_2 \in \mathrm{Car}(L)$ are conjugate in G, we see that they make the same contribution to the Plancherel formula. Thus we replace the sum over Car(L) by a sum over Car(G) with an extra constant, $c(L, J)$, the number of elements of Car(L) with are conjugate to J in G. Then

$$\frac{c(L, J)[W(M, T)][T(\Phi_R)]}{[W(L, J)][T(\Phi)]} = \frac{[W(G, H)]}{[W(G, J)]} \text{ and}$$

$$(-1)^{q+q_J}(i)^{d(J_p \cap M)}\epsilon(\tau) \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi : \nu)$$

$$= \left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi : \nu) \right|.$$

QED

§5. Sp(2, ℝ) calculations

In this section we will make all the computations which were omitted from §4 involving root systems of type B_2. Thus the general set up is that, as in §4, H is a Cartan subgroup of G, $J \in \mathrm{Car}'(L, H)$, and φ is a root system of type B_2 contained in $\Phi_R(\mathfrak{l}, \mathfrak{j})$. Write $\mathfrak{a} = \Sigma_{\alpha \in \varphi} \mathbb{R} H_\alpha \subseteq \mathfrak{j}_p$. We will write $\varphi^+ = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ where α_1 and α_2 are long roots, $\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2)$, $\beta_2 = \frac{1}{2}(\alpha_1 - \alpha_2)$, and ${}^{y^{-1}}\beta_2$ is a compact roots of $(\mathfrak{g}, \mathfrak{h})$. Thus here $\epsilon(\varphi^+) = 1$.

The first two lemmas involve computations taking place in $G/S \subseteq G_{1,\mathbf{C}}$. We will write $h = \exp(-i\theta H^*_{\alpha_1} - i\psi H^*_{\alpha_2}) \in \exp(i\mathfrak{a}) \subseteq G_{1,\mathbf{C}}$. We can assume $0 < |\theta| \neq |\psi| < \pi$. Write

$$\epsilon_i = \begin{cases} +1 & \text{if } 0 < \theta_i < \pi,\ 1 \leqslant i \leqslant 4, \\ -1 & \text{if } -\pi < \theta_i < 0 \end{cases}$$

where $\theta_1 = \theta$, $\theta_2 = \psi$, $\theta_3 = (\theta + \psi)/2$, $\theta_4 = (\theta - \psi)/2$. For $\nu \in \mathfrak{a}^*$, write $\nu_i = 2\langle \nu, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$. For $\delta \in L$, write $p_i = 2\langle {}^y\delta, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$. If $\tau \in W(M^0, T^0)$ is the reflection corresponding to ${}^{y^{-1}}\beta_2$, $\delta - \tau\delta \in L$ and $\delta - \tau\delta|_{\mathfrak{z}} = 0$ so that ${}^y(\delta - \tau\delta)$ is a weight for φ. Since ${}^y\tau$ permutes α_1 and α_2 we see that this implies that $p_2 = p_1 + n$, $n \in Z$.

LEMMA 5.1:

$$S(\varphi^+ : \nu : \delta : h) \equiv \frac{2(\pi i)^2}{\|\alpha_1\|^2} e^{-(\nu_1 + ip_1)\theta - (\nu_2 + ip_2)\psi}$$

$$\times \left\{ \frac{\exp(\epsilon_4 \pi(\nu_1 + ip_1) + \epsilon_2 \pi(\nu_1 + \nu_2 + 2ip_1))}{\sinh \pi(\nu_1 + ip_1) \sin \pi(\nu_1 + \nu_2 + 2ip_1)} \right.$$

$$\left. - \frac{\exp(\epsilon_3 \pi(\nu_2 + ip_1) + \epsilon_1 \pi(\nu_1 - \nu_2))}{\sinh \pi(\nu_2 + ip_1) \sinh \pi(\nu_1 - \nu_2)} \right\}$$

PROOF: By definition (4.6)

$$S(\varphi^+:\nu:\delta:h)$$
$$=\tfrac{1}{8}\sum_{s\in W(\varphi)}\det s\sum_{\tau\in L(\varphi)}\overline{\xi_\tau(h)}\bar{c}(s(\tau+\delta):\varphi^+)$$
$$\times\int_{\mathfrak{a}^+(\varphi)}\exp(s(\tau+\delta-i\nu)(H))\,dH.$$

Now $L(\varphi)=\{n_1\Lambda_1+n_2\Lambda_2\mid n_i\in Z\}$ where $2\langle\Lambda_i,\alpha_j\rangle/\langle\alpha_j,\alpha_j\rangle=\delta_{ij}$, $i,\ j=1,2$, and $\mathfrak{a}^+(\varphi)=\{xH^*_{\alpha_1}+yH^*_{\alpha_2}\mid x>y>0\}$. As in [7b] $dH=k\,dx\,dy$ where $k=4/\|\alpha_1\|\,\|\alpha_2\|=4/\|\alpha_1\|^2$ and

$$\bar{c}(\tau+\delta:\varphi^+)=\begin{cases}4 & \text{if } \tau+\delta\in F_1\cup F_2\\ 0 & \text{otherwise.}\end{cases}$$

Here $F_1=\{\lambda\in\mathfrak{a}^*\mid\lambda_2<\lambda_1<0\}$ and $F_2=\{\lambda\in\mathfrak{a}^*\mid 0<\lambda_2<-\lambda_1\}$, where for $\lambda\in\mathfrak{a}^*$, $\lambda_i=2\langle\lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle$, $i=1,2$.

Write $r=n_1+p_1$, $s=n_2+p_2$. Then $\exp(\tau+\delta-i\nu)(H)=\exp(x(r-i\nu_1)+y(s-i\nu_2))$ and

$$\int_{\mathfrak{a}^+(\varphi)}\exp(\tau+\delta-i\nu)(H)\,dH=\frac{k}{(r-i\nu_1)(r+s-i\nu_1-i\nu_2)}.$$

Thus

$$S(\varphi^+:\nu:\delta:h)$$
$$=\tfrac{4}{8}\xi_\delta(h)\sum_{s\in W(\varphi)}\det s\sum_{\tau+\delta\in s^{-1}F_1\cup s^{-1}F_2}\overline{\xi_{\tau+\delta}(h)}$$
$$\times\int_{\mathfrak{a}^+(\varphi)}\exp(s(\tau+\delta-i\nu))(H)\,dH$$
$$=\xi_\delta(h)\cdot k/2\times\Bigg\{\sum_{\substack{s<r<0\\0<s<-r}}e^{ir\theta+is\psi}_{\substack{-s<-r<0\\0<-s<r}}/(r-i\nu_1)(r+s-i\nu_1-i\nu_2)$$
$$-\sum_{\substack{s<-r<0\\0<s<r}}e^{ir\theta+is\psi}_{\substack{-s<r<0\\0<-s<-r}}/(r-i\nu_1)(r-s-i\nu_1+i\nu_2)$$
$$-\sum_{\substack{r<s<0\\0<r<-s}}e^{ir\theta+is\psi}_{\substack{-r<-s<0\\0<-r<s}}/(s-i\nu_2)(r+s-i\nu_1-i\nu_2)$$

$$- \sum_{\substack{r<-s<0 \\ 0<r<s}} \underset{\substack{-r<s<0 \\ 0<-r<-s}}{e^{ir\theta+is\psi}} /(s-i\nu_2)(r-s-i\nu_1+i\nu_2)\Bigg\}$$

$$= \frac{k}{2}\xi_\delta(h) \sum_{\substack{r\neq 0, s\neq 0 \\ |r|\neq|s|}} \left\{ \frac{e^{ir\theta+is\psi}}{(r-i\nu_1)(r+s-i\nu_1-i\nu_2)} \right.$$

$$\left. - \frac{e^{ir\theta+is\psi}}{(s-i\nu_2)(r-s-i\nu_1+i\nu_2)} \right\}.$$

Now consider the corresponding sum over any of the singular elements of the lattice, for example the sum over $r=0$. This would be

$$\sum_s \frac{e^{is\psi}}{(-i\nu_1)(s-i\nu_1-i\nu_2)} + \frac{e^{is\psi}}{(s-i\nu_2)(s+i\nu_1-i\nu_2)}$$

$$= \sum_s e^{is\psi} \left\{ \frac{-1}{(i\nu_1)(s-i\nu_1-i\nu_2)} - \frac{1}{(i\nu_1)(s+i\nu_1-i\nu_2)} \right.$$

$$\left. + \frac{1}{(s-i\nu_2)(i\nu_1)} \right\} \equiv 0$$

since it is anti-symmetric under the Weyl group element which takes ν_1 to $-\nu_1$ and leaves ν_2 fixed. Similarly the sums over $s=0$ and $r=\pm s$ are anti-symmetric under the Weyl group action and so can be added in without changing the value of the expression inside the Fourier inversion formula. Thus

$$S(\varphi^+ : \nu : \delta : h) \equiv \frac{k}{2} \sum_{n,m} e^{in\theta+im\psi}$$

$$\times \left\{ \frac{1}{(n+p_1-i\nu_1)(n+m+p_1+p_2-i\nu_1-i\nu_2)} \right.$$

$$\left. - \frac{1}{(m+p_2-i\nu_2)(n-m+p_1-p_2-i\nu_1+i\nu_2)} \right\}$$

$$= \frac{k}{2}(\pi i)^2 \left\{ \frac{e^{-(\nu_1+ip_1)(\theta-\psi-\epsilon_4\pi)}\, e^{-(\nu_1+\nu_2+ip_1+ip_2)(\psi-\epsilon_2\pi)}}{\sinh\pi(\nu_1+ip_1)\sinh\pi(\nu_1+\nu_2+ip_1+ip_2)} \right.$$

$$\left. - \frac{e^{-(\nu_2+ip_2)(\theta+\psi-\epsilon_3\pi)}\, e^{-(\nu_1-\nu_2+ip_1-ip_2)(\theta-\epsilon_1\pi)}}{\sinh\pi(\nu_2+ip_2)\sinh\pi(\nu_1-\nu_2+ip_1-ip_2)} \right\}.$$

The lemma is obtained by factoring out the exponentials involving θ and ψ, recalling that $p_2 = p_1 + n$, $n \in Z$, and noting that the factors of $(-1)^n$ resulting from evaluating all terms of the form $\sinh \pi(a + \mathrm{i}n) = (-1)^n \sinh \pi a$ and $\mathrm{e}^{\pm\pi(a+\mathrm{i}n)} = (-1)^n \mathrm{e}^{\pm\pi a}$ cancel out.

QED

LEMMA 5.2:

$$S(\varphi_s^+ : \nu : \delta : h) \equiv \frac{4(\pi i)^2}{\|\beta_1\|^2} \mathrm{e}^{-(\nu_1 + \mathrm{i}p_1)\theta - (\nu_2 + \mathrm{i}p_2)\psi}$$

$$\times \frac{\exp(\epsilon_3 \pi(\nu_1 + \nu_2 + 2\mathrm{i}p_1) + \epsilon_4 \pi(\nu_1 - \nu_2)) + 1}{\sinh \pi(\nu_1 + \nu_2 + 2\mathrm{i}\mathrm{p}_1) \sinh \pi(\nu_1 - \nu_2)}.$$

PROOF: Recall that as in (4.8)

$$\begin{aligned} &S(\varphi_s^+ : \nu : \delta : h) \\ &\quad = S(\beta_1 : \nu : \delta_1 : h_1) S(\beta_2 : \nu : \delta_2 : h_2) + S(\beta_1 : \nu : \delta_1 : \gamma_0 h_1) \\ &\quad\quad \times S(\beta_2 : \nu : \delta_2 : \gamma_0 h_2) \end{aligned}$$

where

$$h_1 = \exp\left(-\mathrm{i}\left(\frac{\theta + \psi}{2}\right) H^*_{\beta_1}\right), \quad h_2 = \exp\left(-\mathrm{i}\left(\frac{\theta - \psi}{2}\right) H^*_{\beta_2}\right)$$

and

$$\gamma_0 = \exp(\mp \mathrm{i}\pi H^*_{\beta_1}) = \exp(\mp \mathrm{i}\pi H^*_{\beta_2}).$$

Since $2\langle \nu, \beta_1\rangle / \langle \beta_1, \beta_1\rangle = \nu_1 + \nu_2$, $2\langle \nu, \beta_2\rangle / \langle \beta_2, \beta_2\rangle = \nu_1 - \nu_2$, $2\langle {}^{\nu}\delta, \beta_1\rangle / \langle \beta_1, \beta_1\rangle = \mathrm{p}_1 + \mathrm{p}_2$ and $2\langle {}^{\nu}\delta, \beta_2\rangle / \langle \beta_2, \beta_2\rangle = p_1 - p_2$, we have using Lemma 3.12, writing $\Theta_3 = (\Theta + \psi)/2$, $\Theta_4 = (\Theta - 4)/2$,

$$\begin{aligned} &S(\beta_1 : \nu : \delta : h_1) S(\beta_2 : \nu : \delta : h_2) \\ &\quad \equiv \frac{4(\pi i)^2}{\|\beta_1\| \, \|\beta_2\|} \frac{\mathrm{e}^{-(\nu_1 + \nu_2 + \mathrm{i}p_1 + \mathrm{i}p_2)(\theta_3 - \epsilon_3\pi)}}{\sinh \pi(\nu_1 + \nu_2 + \mathrm{i}p_1 + \mathrm{i}p_2)} \\ &\quad\quad \times \frac{\mathrm{e}^{-(\nu_1 - \nu_2 + \mathrm{i}p_1 - \mathrm{i}p_2)(\theta_4 - \epsilon_4\pi)}}{\sinh \pi(\nu_1 - \nu_2 + \mathrm{i}p_1 - \mathrm{i}p_2)}. \end{aligned}$$

Since $\gamma_0 h_1 = \exp(-\mathrm{i}((\theta+\psi)/2 - \epsilon_3\pi))$ and $\gamma_0 h_2 = \exp(-\mathrm{i}((\theta-\psi)/2 - \epsilon_4\pi))$ we see that

$$S(\beta_1\colon \nu\colon \delta\colon \gamma_0 h_1) S(\beta_2\colon \nu\colon \delta\colon \gamma_0 h_2)$$

$$\equiv \frac{4(\pi \mathrm{i})^2}{\|\beta_1\|\,\|\beta_2\|} \frac{e^{-(\nu_1+\nu_2+\mathrm{i}p_1+\mathrm{i}p_1)\theta_3}}{\sinh \pi(\nu_1+\nu_2+\mathrm{i}p_1+\mathrm{i}p_2)}$$

$$\times \frac{e^{-(\nu_1-\nu_2+\mathrm{i}p_1-\mathrm{i}p_2)\theta_4}}{\sinh \pi(\nu_1-\nu_2+\mathrm{i}p_1-\mathrm{i}p_2)}.$$

The lemma follows from factoring out the exponentials involving θ and ψ and noting that $p_2 = p_1 + n$, $n \in Z$.

QED

Lemma 5.3: *Suppose* $t = \exp(-\mathrm{i}\theta^{y^{-1}} H^*_{\alpha_1} - \mathrm{i}\psi^{y^{-1}} H^*_{\alpha_2})$ *satisfies* $0 < |\theta| \neq |\psi| < \pi$. *Then*

$$K(\varphi^+\colon \eta_j\colon \nu\colon t) = \frac{4}{\|\alpha_1\|^2} e^{-\nu_1\theta - \nu_2\psi} k(\varphi^+\colon \eta_j\colon \nu\colon C(t))$$

where

$$k(\varphi^+\colon \eta_j\colon \nu\colon C(t))$$

$$= \Big\{\Big[-\exp(\epsilon_4\pi(\nu_1+\mathrm{i}p) + \epsilon_2\pi(\nu_1+\nu_2+2\mathrm{i}p))$$

$$-\exp(\epsilon_4\pi(\nu_1+\mathrm{i}p))\overline{(\eta_j\cdot\xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]$$

$$\times\big[\sinh\pi(\nu_1+\mathrm{i}p)\sinh\pi(\nu_1+\nu_2+2\mathrm{i}p)\big]^{-1}$$

$$+\Big[\exp(\epsilon_3\pi(\nu_2+\mathrm{i}p) + \epsilon_1\pi(\nu_1-\nu_2))$$

$$+\exp(\epsilon_3\pi(\nu_2+\mathrm{i}p))\overline{(\eta_j\cdot\xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]$$

$$\times\big[\sinh\pi(\nu_2+\mathrm{i}p)\sinh\pi(\nu_1-\nu_2)\big]^{-1}$$

$$+\Big[\epsilon(\varphi^+)\Big[\exp(\epsilon_3\pi(\nu_1+\nu_2+2\mathrm{i}p)) + \overline{(\eta_j\cdot\xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]$$

$$\times\Big[\exp(\epsilon_4\pi(\nu_1-\nu_2) + \overline{\eta_j\cdot\xi_\rho(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]\Big]$$

$$\times\left[\sinh \pi(\nu_1+\nu_2+2\mathrm{i}p)\sinh \pi(\nu_1-\nu_2)\right]^{-1}$$

$$+\Big[\exp(\epsilon_2\pi(\nu_2+\mathrm{i}p))$$

$$+\exp(\epsilon_1\pi(\nu_1+\mathrm{i}p))\overline{(\xi_\rho\cdot\eta_j)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]$$

$$\times\left[\sinh \pi(\nu_1+\mathrm{i}p)\sinh \pi(\nu_2+\mathrm{i}p)\right]^{-1}\Big\}.$$

Here p *satisfies* $\exp(\mathrm{i}\pi p)=\overline{(\eta_j\cdot\xi_\rho)(\gamma_{\alpha_1})}$, $\gamma_{\alpha_i}=\exp(-\mathrm{i}\pi^{y^{-1}}H^*_{\alpha_i})=\exp(\pi Z_{\alpha_i})$, $i=1, 2$.

PROOF: Using the linear case [7d] we know that

$$K(\varphi^+: \eta_j: \nu: t)$$

$$=\overline{\xi_{\delta_j}(t)}\frac{1}{(\pi\mathrm{i})^2}\Big\{-2S(\varphi^+: \nu: \delta_j: h)-2S(\varphi^+: \nu: \delta_j: \gamma_1^{-1}\gamma_2^{-1}h)$$

$$+\tfrac{1}{2}\epsilon(\varphi^+)S(\varphi_s^+: \nu: \delta_j: h)+\tfrac{1}{2}\epsilon(\varphi^+)S(\varphi_s^+: \nu: \delta_j: \gamma_1^{-1}\gamma_2^{-1}h)$$

$$+S(\varphi_\ell^+: \nu: \delta_j: \gamma_1^{-1}h)+S(\varphi_l^+: \nu: \delta_j: \gamma_2^{-1}h)\Big\}$$

where $h=\exp(-\mathrm{i}\theta H^*_{\alpha_1}-\mathrm{i}\psi H^*_{\alpha_2})$, $\gamma_1, \gamma_2\in Z(\mathfrak{a})/S$ satisfy $\gamma_i S=\gamma_{\alpha_i}S$, $i=1, 2$, and δ_j satisfies $\xi_{\delta_j}(\gamma)=\eta_j\cdot\xi_\rho(\gamma)$, $\gamma\in Z(\mathfrak{a})\cap T^0$. The first and third terms above can be evaluated directly using Lemmas 5.1 and 5.3. In the notation of Lemmas 5.1 and 5.2 $\overline{\xi_{\delta_j}(t)}=\mathrm{e}^{\mathrm{i}p_1\theta+\mathrm{i}p_2\psi}$ which cancels the $\mathrm{e}^{-\mathrm{i}p_1\theta-\mathrm{i}p_2\psi}$ occurring in the formulas for $S(\varphi^+: \nu: \delta_j: h)$ and $S(\varphi_s^+: \nu: \delta_j: h)$. In the third term, $1/(2\|\beta_1\|^2)=1/(\|\alpha_1\|^2)$. To evaluate the second and fourth terms we write

$$\gamma_1^{-1}\gamma_2^{-1}h=\exp\left(-\mathrm{i}(\theta-\epsilon_1\pi)H^*_{\alpha_1}-\mathrm{i}(\psi-\epsilon_2\pi)H^*_{\alpha_2}\right)$$

and we use this expression in Lemmas 5.1 and 5.2 to obtain:

$$S(\varphi^+: \nu: \delta_j: \gamma_1^{-1}\gamma_2^{-1}h)=\frac{2(\pi\mathrm{i})^2}{\|\alpha_1\|^2}\mathrm{e}^{-(\nu_1+\mathrm{i}p_1)\theta-(\nu_2+\mathrm{i}p_2)\psi}$$

$$\times\left\{\frac{\exp(\epsilon_4\pi(\nu_1+\mathrm{i}p_1))}{\sinh \pi(\nu_1+\mathrm{i}p_1)\sinh \pi(\nu_1+\nu_2+\mathrm{i}p_1+\mathrm{i}p_2)}\right.$$

$$\left.-\frac{\exp(\epsilon_3\pi(\nu_2+\mathrm{i}p_2))}{\sinh \pi(\nu_2+\mathrm{i}p_2)\sinh \pi(\nu_1-\nu_2+\mathrm{i}p_1-\mathrm{i}p_2)}\right\}$$

and

$$S\left(\varphi_s^+ : \nu : \delta_j : \gamma_1^{-1}\gamma_2^{-1}h\right)$$
$$= \left[\left(4(\pi\mathrm{i})^2\right)\left(\|\beta_1\|^2\right)^{-1}\right]\left[\mathrm{e}^{-(\nu_1+\mathrm{i}p_1)\theta-(\nu_2+\mathrm{i}p_2)\psi}\right.$$
$$\times\left\{\exp\left(\epsilon_4\pi(\nu_1-\nu_2+\mathrm{i}p_1-\mathrm{i}p_2)\right)+\exp\left(\epsilon_3\pi(\nu_1+\nu_2+\mathrm{i}p_1+\mathrm{i}p_2)\right)\right\}\Big]$$
$$\times\left[\sinh\pi(\nu_1+\nu_2+ip_1+ip_2)\sinh\pi(\nu_1-\nu_2+\mathrm{i}p_1-\mathrm{i}p_2)\right]^{-1}$$

In this case if we write $p_2 = p_1 + n$, $n \in Z$, $(-1)^n$ factors out of each term. But

$$(-1)^n = \mathrm{e}^{\mathrm{i}\pi p_1}\,\mathrm{e}^{-\mathrm{i}\pi p_2} = \overline{\xi_{\delta_j}\left(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2}\right)} = \overline{\left(\eta_j\cdot\xi_\rho\right)\left(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2}\right)}.$$

Note $[\eta_j\cdot\xi_\rho(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})]^2 = 1$ because using (1.5c) $\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2}$ is an element of order two.

Finally, by definition

$$S\left(\varphi_l^+ : \nu : \delta_j : h\right) = S\left(\alpha_1 : \nu_1 : \delta_j : h_1\right)S\left(\alpha_2 : \nu_2 : \delta_j : h_2\right)$$

where

$$h_1 = \exp\left(-\mathrm{i}\theta H_{\alpha_1}^*\right), \quad h_2 = \exp\left(-\mathrm{i}\psi H_{\alpha_2}^*\right).$$

Thus using (3.12),

$$S\left(\varphi_l^+ : \nu : \delta_j : \gamma_1^{-1}h\right) + S\left(\varphi_l^+ : \nu : \delta_j : \gamma_2^{-1}h\right)$$
$$= \left[\left(4(\pi\mathrm{i})^2\right)\left(\|\alpha_1\|\,\|\alpha_2\|\right)^{-1}\right]$$
$$\times\left[\mathrm{e}^{-(\nu_1+\mathrm{i}p_1)\theta-(\nu_2+\mathrm{i}p_2)(\psi-\epsilon_2\pi)}+\mathrm{e}^{-(\nu_1+\mathrm{i}p_1)(\theta-\epsilon_1\pi)-(\nu_2+\mathrm{i}p_2)\psi}\right]$$
$$\times\left[\sinh\pi(\nu_1+\mathrm{i}p_1)\sinh\pi(\nu_2+\mathrm{i}p_2)\right]^{-1}$$
$$= \left[\left(4(\pi\mathrm{i})^2\right)\left(\|\alpha_1\|^2\right)^{-1}\right]\mathrm{e}^{-(\nu_1+\mathrm{i}p_1)\theta-(\nu_2+\mathrm{i}p_2)\psi}$$
$$\times\left\{\left[\exp\left(\epsilon_2\pi(\nu_2+\mathrm{i}p_2)\right)+\exp\left(\epsilon_1\pi(\nu_1+\mathrm{i}p_1)\right)\right]\right\}$$
$$\times\left\{\sinh\pi(\nu_1+\mathrm{i}p_1)\sinh\pi(\nu_2+\mathrm{i}p_2)\right\}^{-1}.$$

Here, again writing $p_2 = p_1 + n$, the $(-1)^n$ terms cancel in the first summand, but not in the second.

QED

LEMMA 5.4: *$\Sigma_{v \in W(\varphi)} \det v\, k(\varphi^+ : (v\eta)_j : v\nu : C^+)$ is independent of the chamber C^+ where $k(\varphi^+ : \eta_j : \nu : C^+)$ is defined as in* (5.3).

PROOF: For any $\eta \in Z(\mathfrak{j}_p)^*$, $k(\varphi^+ : \eta_j : \nu : C^+)$ depends only on $\eta|_{Z(\varphi)}$ where $Z(\varphi)$ is the abelian subgroup of $Z(\mathfrak{j}_p)$ generated by γ_{α_1} and γ_{α_2}, $\gamma_{\alpha_i} = \exp(\pi Z_{\alpha_i})$. The reflections s_i in s_{α_i}, $i = 1, 2$, are given by conjugation by $\exp(\pi/2\ Z_{\alpha_i})$ and so clearly centralize $Z(\varphi)$. Thus for $v \in W(\varphi_l)$, $v\eta_j|_{Z(\varphi)} = \eta_j|_{Z(\varphi)}$. Thus we can average any term in $k(\varphi^+ : \eta_j : \nu : C^+)$ over the group generated by s_1 and s_2. For example, averaging over $\nu_2 \to -\nu_2$,

$$\frac{\exp(\epsilon_2 \pi(\nu_2 + ip))}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)}$$

$$\equiv \frac{1}{2}\left\{ \frac{\exp(\epsilon_2 \pi(\nu_2 + ip))}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)} \right.$$

$$\left. - \frac{\exp(\epsilon_2 \pi(-\nu_2 + ip))}{\sinh \pi(\nu_1 + ip) \sinh \pi(-\nu_2 + ip)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\cosh \pi(\nu_2 + ip)}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)} \right.$$

$$\left. + \frac{\cosh \pi(\nu_2 - ip)}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)} \right\}$$

$$\equiv \frac{\cosh \pi(\nu_2 + ip)}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)}$$

is independent of ϵ_2. Similarly averaging over $\nu_1 \to -\nu_1$ we see that

$$\frac{\exp(\epsilon_1 \pi(\nu_1 + ip))}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)}$$

$$\equiv \frac{\cosh \pi(\nu_1 + ip)}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)}.$$

This shows that the last term in $k(\varphi^+ : \eta_j : \nu : C^+)$ is independent of C^+ when averaged over $W(\varphi_l)$. For the first three terms we also want to average over s_{β_2}, the permutation (12) which interchanges α_1 and α_2.

Now $\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2} = \exp(\pi Z_{\beta_1}) = \exp(\pi Z_{\beta_2})$ and so is centralized by the reflection s_{β_2} which can be represented by conjugation by $\exp(\pi/2\ Z_{\beta_2})$.

Further, the first three terms can be expressed in terms of even powers $e^{\epsilon\pi 2ip} = \overline{\eta_j \cdot \xi_\rho(\gamma_{\alpha_1}^{\pm 2})}$ and $\gamma_{\alpha_1}^{\pm 2} \in S$ is centralized by $W(\varphi)$. Thus in these terms we can average ν over the full group $W(\varphi)$. A messy computation which can be done by computer yields the following formula.

$$
\begin{aligned}
&k(\varphi^+ \colon \eta_j \colon \nu \colon C^+) \\
&\equiv k(\varphi^+ \colon \eta_j \colon \nu) \\
&= \Big[-\cosh \pi(\nu_1 + ip) \cosh \pi(\nu_1 + \nu_2 + 2ip) \\
&\qquad - \overline{(\eta_j \cdot \xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})} \cosh \pi(\nu_1 + ip)\Big] \\
&\quad \times [\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_1 + \nu_2 + 2ip)]^{-1} \\
&\quad + \Big[\cosh \pi(\nu_2 + ip) \cosh \pi(\nu_1 - \nu_2) \\
&\qquad + \overline{(\eta_j \cdot \xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})} \cosh \pi(\nu_2 + ip)\Big] \\
&\quad \times [\sinh \pi(\nu_2 + ip) \sinh \pi(\nu_1 - \nu_2)]^{-1} \\
&\quad + \Big\{\Big[\cosh \pi(\nu_1 + \nu_2 + 2ip) + \overline{\eta_j \cdot \xi_\rho(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big] \\
&\qquad \times \Big[\cosh \pi(\nu_1 - \nu_2) + \overline{(\eta_j \cdot \xi_\rho)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})}\Big]\Big\} \\
&\quad \times \{\sinh \pi(\nu_1 + \nu_2 + 2ip) \sinh \pi(\nu_1 - \nu_2)\}^{-1} \\
&\quad + \frac{\cosh \pi(\nu_2 + ip) + \overline{(\xi_\rho \cdot \eta_j)(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})} \cosh \pi(\nu_1 + ip)}{\sinh \pi(\nu_1 + ip) \sinh \pi(\nu_2 + ip)}.
\end{aligned} \tag{5.5}
$$

QED

LEMMA 5.6: *$k(\varphi^+ \colon \eta_j \colon \nu) \equiv \prod_{\alpha \in \varphi^+} p_\alpha(\eta_j \colon \nu)$ where*

$$
p_\alpha(\eta_j \colon \nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha + \left[\dfrac{\eta_j(\gamma_\alpha) + \eta_j(\gamma_\alpha^{-1})}{2}\right] \xi_{\rho - \rho(\varphi^+)}(\gamma_\alpha)}
$$

for all $\alpha \in \varphi^+$.

PROOF: A computation done exactly as in the linear case but with $\nu_i + \mathrm{i}p$ replacing ν_i, $i = 1, 2$, shows that

$$\begin{aligned}
&k(\varphi^+ : \eta_j : \nu)\\
&\quad\equiv \{[\cosh \pi(\nu_1 + \mathrm{i}p) - 1][\cosh \pi(\nu_2 + \mathrm{i}p) - \epsilon]\\
&\qquad\quad \times [\cosh \pi(\nu_1 + \nu_2 + 2\mathrm{i}p) - \epsilon][\cosh \pi(\nu_1 - \nu_2) - \epsilon]\}\\
&\qquad \times \{\sinh \pi(\nu_1 + \mathrm{i}p) \sinh \pi(\nu_2 + \mathrm{i}p)\\
&\qquad\qquad \times \sinh \pi(\nu_1 + \nu_2 + 2\mathrm{i}p) \sinh \pi(\nu_1 - \nu_2)\}^{-1}\\
&\qquad + \frac{[\cosh \pi(\nu_1 + \nu_2 + 2\mathrm{i}p) - \epsilon][\cosh \pi(\nu_1 - \nu_2) - \epsilon]}{\sinh \pi(\nu_1 + \nu_2 + 2\mathrm{i}p) \sinh \pi(\nu_1 - \nu_2)}
\end{aligned}$$

where $\epsilon = [\xi_{\rho - \rho(\varphi^+)} \cdot \eta_j](\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2}) = \pm 1$.
Now for $\epsilon = \pm 1$,

$$\frac{\cosh \pi(\nu + \mathrm{i}p) - \epsilon}{\sinh \pi(\nu + \mathrm{i}p)} = \frac{\sinh \pi\nu + \mathrm{i}\epsilon \sin \pi p}{\cosh \pi\nu + \varepsilon \cos \pi p}.$$

Thus

$$\begin{aligned}
&k(\varphi^+ : \eta_j : \nu)\\
&\quad\equiv [(\sinh \pi\nu_1 + \mathrm{i} \sin \pi p)(\sinh \pi\nu_2 + \mathrm{i}\epsilon \sin \pi p)\\
&\qquad\quad \times (\sinh \pi(\nu_1 + \nu_2) + \mathrm{i}\epsilon \sin 2\pi p) \sinh \pi(\nu_1 - \nu_2)]\\
&\quad \times [(\cosh \pi\nu_1 + \cos \pi \mathrm{p})(\cosh \pi\nu_2 + \epsilon \cos \pi p)\\
&\quad \times (\cosh \pi(\nu_1 + \nu_2) + \epsilon \cos 2\pi p)(\cosh \pi(\nu_1 - \nu_2) + \epsilon)]^{-1}\\
&\quad + \frac{(\sinh \pi(\nu_1 + \nu_2) + \mathrm{i}\epsilon \sin 2p\pi) \sinh \pi(\nu_1 - \nu_2)}{(\cosh \pi(\nu_1 + \nu_2) + \epsilon \cos 2p\pi)(\cosh \pi(\nu_1 - \nu_2) + \epsilon)}.
\end{aligned}$$

If we average over the Weyl group element $s_1 s_2$: $(\nu_1, \nu_2) \to (-\nu_1, -\nu_2)$, the denominators and terms involving $\sin(p\pi)$ or $\sin(2p\pi)$ are all invariant, but each $\sinh(\pi\nu_\alpha)$ term changes sign. Thus the only terms in the numerators which survive are those involving an even number of terms of the form $\sinh \pi\nu_\alpha$, $\alpha \in \varphi^+$. Finally, averaging over s_2: $(\nu_1, \nu_2) \to (\nu_1, -\nu_2)$ we obtain

$$k(\varphi^+ : \eta_j : \nu) \equiv \frac{1}{2}\left\{\frac{N_1}{D_1} + \frac{N_2}{D_2} + \frac{N_3}{D_3} - \frac{N_4}{D_4}\right\} \text{ where}$$

$$\begin{aligned} N_1 = {} & \sinh \pi\nu_1 \sinh \pi\nu_2 \sinh \pi(\nu_1 - \nu_2) \sinh \pi(\nu_1 + \nu_2) \\ & - \sinh \pi\nu_1 \sinh \pi(\nu_1 - \nu_2) \sin(p\pi) \sin(2p\pi) \\ & - \epsilon \sinh \pi\nu_2 \sinh \pi(\nu_1 - \nu_2) \sin(p\pi) \sin(2p\pi) \\ & - \epsilon \sinh \pi(\nu_1 + \nu_2) \sinh \pi(\nu_1 - \nu_2) \sin^2(p\pi), \end{aligned}$$

$$\begin{aligned} D_1 = {} & (\cosh \pi\nu_1 + \cos \pi p)(\cosh \pi\nu_2 + \epsilon \cos \pi p) \\ & \times (\cosh \pi(\nu_1 + \nu_2) + \epsilon \cos 2\pi p)(\cosh \pi(\nu_1 - \nu_2) + \epsilon), \end{aligned}$$

$$\begin{aligned} N_2 = {} & \sinh \pi\nu_1 \sinh \pi\nu_2 \sinh \pi(\nu_1 - \nu_2) \sinh \pi(\nu_1 + \nu_2) \\ & + \sinh \pi\nu_1 \sinh \pi(\nu_1 + \nu_2) \sin(p\pi) \sin(2p\pi) \\ & - \epsilon \sinh \pi\nu_2 \sinh \pi(\nu_1 + \nu_2) \sin(p\pi) \sin(2p\pi) \\ & + \epsilon \sinh \pi(\nu_1 - \nu_2) \sinh \pi(\nu_1 + \nu_2) \sin^2(p\pi), \end{aligned}$$

$$\begin{aligned} D_2 = {} & (\cosh \pi\nu_1 + \cos \pi p)(\cosh \pi\nu_2 + \epsilon \cos \pi p) \\ & \times (\cosh \pi(\nu_1 - \nu_2) + \epsilon \cos 2\pi p)(\cosh \pi(\nu_1 + \nu_2) + \epsilon), \end{aligned}$$

$$N_3 = N_4 = \sinh \pi(\nu_1 + \nu_2) \sinh \pi(\nu_1 - \nu_2),$$

$$D_3 = (\cosh \pi(\nu_1 + \nu_2) + \varepsilon \cos 2\pi p)(\cosh \pi(\nu_1 - \nu_2) + \epsilon), \text{ and}$$

$$D_4 = (\cosh \pi(\nu_1 - \nu_2) + \epsilon \cos 2\pi p)(\cosh \pi(\nu_1 + \nu_2) + \epsilon).$$

This last expression is equal to

$$\begin{aligned} & [\sinh \pi\nu_1 \sinh \pi\nu_2 \sinh \pi(\nu_1 - \nu_2) \sinh \pi(\nu_1 + \nu_2)] \\ & \quad \times [(\mathit{cosh}\ \pi\nu_1 + \cos \pi p)(\mathit{cosh}\ \pi\nu_2 + \epsilon \cos \pi p) \\ & \qquad \times (\mathit{cosh}\ \pi(\nu_1 - \nu_2) + \epsilon)(\mathit{cosh}\ \pi(\nu_1 + \nu_2) + \epsilon)]^{-1}. \end{aligned}$$

This equality is easiest to prove by thinking of both expressions as meromorphic functions of a complex variable ν_1. Then show by comparing zeroes and poles that the first divided by the second is holomorphic on all of $\mathbb{C}$. But it is easily shown to be a bounded function, hence constant. Finally by looking at limiting values or by comparing residues at a pole we see that the quotient is identically one. The computation can also be carried out directly using a computer.

Now recall that

$$e^{\pi i p} = \overline{\eta_J \cdot \xi_\rho(\gamma_{\alpha_1})} = \overline{\eta_J \cdot \xi_{\rho-\rho(\varphi^+)}(\gamma_{\alpha_1})}$$

since $\xi_{\rho(\varphi^+)}(\gamma_{\alpha_1}) = 1$ and $\epsilon = \eta_J \cdot \xi_{\rho-\rho(\varphi^+)}(\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2})$. Thus

$$\cos p\pi = \frac{\xi_{\rho-\rho(\varphi^+)}(\gamma_{\alpha_1})}{2}\left(\eta_J(\gamma_{\alpha_1}) + \eta_J\left(\gamma_{\alpha_1}^{-1}\right)\right)$$

and

$$\epsilon \cos p\pi = \frac{\xi_{\rho-\rho(\varphi^+)}(\gamma_{\alpha_2})}{2}\left(\eta_J(\gamma_{\alpha_2}) + \eta_J(\gamma_{\alpha_2})^{-1}\right).$$

Further $\gamma_{\alpha_1}^{-1}\gamma_{\alpha_2} = \gamma_{\beta_1} = \gamma_{\beta_2}$ has order two by (1.5c) so that

$$\epsilon = \frac{\xi_{\rho-\rho(\varphi^+)}(\gamma_{\beta_i})}{2}\left(\eta_J(\gamma_{\beta_i}) + \eta_J(\gamma_{\beta_i})^{-1}\right).$$

QED

§6. The general Plancherel formula

Let G be a reductive group in our general class (1.1). Recall the notation (1.6)–(1.17). In particular assume Z enlarged if necessary so that $Z \cap G^0 = Z_{G^0}$. The group $\tilde{G}[\zeta]$ belongs to the special class (1.2), so its Plancherel Theorem is given by Theorem 4.18. Restrict attention to the characters $\sigma \in (\tilde{S})\hat{}$ which are trivial on $\tilde{S} \cap (\text{kernel } \tilde{G}[\zeta] \to G[\zeta])$. Since the central circle subgroup $S \subset G[\zeta]$ is the image of $\tilde{S} \subset \tilde{G}[\zeta]$ under the covering, they induce unitary characters $\sigma \in \hat{S}$, and all of $\hat{S}$ is obtained that way. Thus, by combining Lemma 1.8 and Theorem 4.18, and using the correspondence of Proposition 1.10, we have the Plancherel theorem for $G[\zeta]$:

LEMMA 6.1: *Let $\sigma \in \hat{S}$ and $f \in C_c^\infty(G[\zeta]/S, \sigma)$. Then*

$$f(1) = c \cdot \frac{[W(G[\zeta], H)]}{(2\pi)^{r+p}} \sum_{J \in \operatorname{Car} G[\zeta]} c_J^{-1}$$

$$\times \sum_{\chi \in Z_{M_J[\zeta]}(M_J[\zeta]^0)\hat{}_\sigma} \deg(\chi) \sum_{\tau \in L'_\chi} \int_{\mathfrak{j}_p^*} \Theta(J: \chi: \tau: \nu)(f)$$

$$\times \left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi: \nu) \right| d\nu$$

where c is the multiplicity (which is independent of ζ, in fact is equal to the order of the fundamental group $\pi_1(\overline{G}_{1\mathbb{C}})$) of the cover q_ζ: $\tilde{G}[\zeta] \to G[\zeta]$ of Lemma 1.8, *and with other notation as in Theorem* 4.18.

Fix $\zeta \in \hat{Z}$. The correspondence between Cartan subgroups of ZG^0 and $G[\zeta]$, of Proposition 1.10, *is*

$$J \leftrightarrow J[\zeta] = \{S \times J\}/\left\{\left(\zeta(z)^{-1}, z\right): z \in Z\right\}.$$

As in §3*B, with ZG^0 in place of $\tilde{S}L(2;\mathbb{R})$,*

$$p: ZG^0 \to G[\zeta] \quad \text{by } p(x) = \left\{\left(\zeta(z)^{-1}, xz\right): z \in Z\right\}$$

is a Lie group homomorphism that induces $ZG^0/Z \cong G[\zeta]/S$, induces a bijection

$$\hat{p}: G[\zeta]_1^\wedge \to (ZG^0)_\zeta^\wedge \quad \text{by } \hat{p}[\psi] = [\psi \circ p]$$

where $1 \in \hat{S}$ is given by $1(e^{ix}) = e^{ix}$, and induces a Hilbert space isometry

$$p^*: L_2(G[\zeta]/S, 1) \to L_2\left(ZG^0/Z, \zeta\right) \quad \text{by } p^*f = f \circ p.$$

In the notation following (2.9),

$$\hat{p}[\pi(J[\zeta]: \chi: p_*\tau: p_*\nu)] = [\pi(J: \chi \circ p: \tau: \nu)]$$

so, as in Lemma 3.23,

$$\Theta(J: \chi \circ p: \tau: \nu)(p^*\varphi) = \Theta(J[\zeta], \chi: p_*\tau: p_*\nu)(\varphi).$$

Now, exactly as for Theorem 3.25, *we can rewrite Lemma* 6.1 *as*

THEOREM 6.2: (*Relative Plancherel Formula*). *Let $\zeta \in \hat{Z}$ and $f_\zeta \in C_c^\infty(ZG^0/Z, \zeta)$. Then*

$$f_\zeta(1) = c \cdot \frac{\left[W(ZG^0, H)\right]}{(2\pi)^{r+p}} \sum_{J \in \mathrm{Car}(ZG^0)} c_J^{-1} \sum_{\chi \in (Z_{M_J}(M_J^0) \cap ZG^0)_\zeta^\wedge} \deg(\chi)$$

$$\times \sum_{\tau \in L'_\chi} \int_{\mathfrak{j}_p^*} \Theta(J: \chi: \tau: \nu)(f_\zeta)$$

$$\times \left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi: \nu) \right| d\nu$$

with c as in Lemma 6.1 *and other notation as in Theorem* 4.18.

Now we integrate the formula of Theorem 6.2. *First normalize the measure* $d\chi^0$ *on* $\{Z_{M_J}(M_J^0)\cap ZG^0\}^{\hat{}}$ *by*

$$\int_{\{Z_{M_J}(M_J^0)\cap ZG^0\}^{\hat{}}}\Phi(\chi^0)\,\mathrm{d}\chi^0$$

$$=\int_{\hat{Z}}\sum_{\chi\in\{Z_{M_J}(M_J^0)\cap ZG^0\}_\zeta^{\hat{}}}\Phi(\chi)\,\mathrm{d}\zeta. \tag{6.3}$$

This is counting measure in case $Z_{M_J}(M_J^0)$ *is compact. Now, as in* §3C, *we have*

LEMMA 6.4: *If* $f\in C_c^\infty(ZG^0)$ *and* $Z\cap G^0=Z_{G^0}$ *then*

$$f(1)=c\big[Z_G(G^0)/Z\big]\frac{[W(ZG^0,\,H)]}{(2\pi)^{r+p}}$$

$$\times\sum_{J\in\mathrm{Car}(ZG^0)}c_J^{-1}\int_{\chi\in\{Z_{M_J}(M_J^0)\cap ZG^0\}^{\hat{}}}\deg(\chi)$$

$$\times\sum_{\tau\in L_\chi'}\int_{\nu\in\mathfrak{j}_p^*}\Theta(J:\,\chi:\,\tau:\,\nu)(f)$$

$$\times\left|\prod_{\alpha\in\Phi^+(\mathfrak{g}_\mathbb{C},\mathfrak{j}_\mathbb{C})}\langle\alpha,\,\tau+\mathrm{i}\nu\rangle\prod_{\alpha\in\Phi_R^+(\mathfrak{g},\mathfrak{j})}\bar{p}_\alpha(\chi:\,\nu)\right|\mathrm{d}\nu\,\mathrm{d}\chi$$

with notation as in (1.17), *in Lemma* 6.1, *and in Theorem* 4.18.

We need a series of remarks in order to expand Lemma 6.4 *from* ZG^0 *to* G. *The point is that we must do something like induction by stages, where the stage vary with* $J\in\mathrm{Car}(G)$.

LEMMA 6.5: *If* $J\in\mathrm{Car}(G)$, *then there is a set* $\{x_1,\dots,x_r\}$ (*depending on* J) *of coset representatives of* G *modulo* ZG^0 *such that each* $x_i\in K$ *and each* $\mathrm{Ad}(x_i)J=J$.

PROOF: As noted at the end of §1, [17, Remark 4.2.4] shows that the G-conjugacy classes of Cartan subgroups of G are the same as the G^0-conjugacy classes. If $x\in G$ now $\mathrm{Ad}(x)J$ is G^0-conjugate to J. That gives a system $\{x_1,\dots,x_r\}$ of coset representative of G modulo ZG^0 which is contained in the normalizer $N_G(J)$. Just as K meets every component of G, so $K\cap N_G(J)$ meets every component of $N_G(J)$, so we may choose the x_i in K.

QED

LEMMA 6.6: *If* $J \in \mathrm{Car}(G)$ *then* $M_J^\dagger G^0$ *and* $M_J G^0$ *are normal subgroups of* G.

PROOF: Choose $\{x_\iota\}$ for J as in Lemma 6.5. Then $\mathrm{Ad}(x_\iota)J_p = J_p$, so $\mathrm{Ad}(x_\iota)$ preserves M_J, thus M_J^0, and thus also $M_J^\dagger = Z_{M_J}(M_J^0)M_J^0$. And of course $Z \subset M_J^\dagger \subset M_J^\dagger G^0 \subset M_J G^0$.

QED

LEMMA 6.7: *If* $J \in \mathrm{Car}(G)$ *then there exist* $\{y_1, \ldots, y_s\} \subset Z_{M_J}(M_J^0) \cap K$ *coset representatives of* $M_J^\dagger G^0$ *modulo* ZG^0, *and* $\{z_1, \ldots, z_t\} \subset N_G(J) \cap K$ *coset representatives of* G *modulo* $M_J^\dagger G^0$, *such that* $\{x_\iota\} = \{z_u y_v\}$ *satisfies Lemma* 6.5 *for* J.

PROOF: The y_v, initially those x_i inside $M_J^\dagger G^0$, can be taken in $M_J^\dagger$ and then in $K \cap Z_{M_J}(M_J^0)$. Then take z_u of the form $x_\iota y_v^{-1}$.

QED

LEMMA 6.8: *Let* $J \in \mathrm{Car}(G)$ *and* $\{x_1, \ldots, x_r\}$ *as in Lemma* 6.5. *Then conjugation of* G *by* x_ι *does not change the value of the weighting factor*

$$\left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi; \nu) \right|$$

in the relative Plancherel formula of Theorem 6.2.

PROOF: Let $x \in \{x_1, \ldots, x_r\}$. Conjugation by x sends $\langle \alpha, \tau + i\nu \rangle$ to $\langle \alpha, \tau^x + i\nu^x \rangle = \langle \alpha^{x^{-1}}, \tau + i\nu \rangle$, hence multiplies $\prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle$ by $\det(w)$ where $w \in W(G, J)$ is represented by x. That doesn't change the absolute value. Similarly, $\mathrm{Ad}(x)$ sends real roots to real roots because $x \in K$, and thus sends

$$\begin{aligned} k \cdot \bar{p}_\alpha(\chi: \nu) = \mathrm{trace}\big[&\sinh \pi\nu_\alpha \cdot I_k \\ &\times \left\{\cosh \pi\nu_\alpha \cdot I_k - \tfrac{1}{2}\xi_{\rho_\alpha}(\gamma_\alpha)\left[\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})\right]\right\}^{-1}\big], \end{aligned}$$

$k = \deg \chi$, to $k \cdot \bar{p}_\alpha(\chi^x: \nu^x) = k \cdot \bar{p}_\alpha \mathrm{x}^{-1}(\chi: \nu)$, hence multiplies $\prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi: \nu)$ by ± 1, which doesn't change the absolute value.

QED

We will use superscript ^{0} for representations of $Z_{M_J}(M_J^0) \cap ZG^0$, of ZG^0, and of their characters. Similarly we will use superscript † for representations and characters of $M_J^\dagger G^0$.

LEMMA 6.9: *Let* $J \in \mathrm{Car}(G)$, π_τ *a relative discrete series representation of*

M_J^0, *and* $\chi^0 \in \{Z_{M_J}(M_J^0) \cap ZG^0\}^\wedge$ *consistent with* π_τ. *Define integers* n_b *and representations* $\chi_b \in Z_{M_J}(M_J^0)^\wedge$ *by*

$$\mathrm{Ind}_{M_J^\dagger \cap ZG^0}^{M_J^\dagger}\left(\chi^0 \otimes \pi_\tau\right) = \sum n_b \chi_b \otimes \pi_\tau.$$

Then

$$\mathrm{Ind}_{ZG^0}^G \pi\left(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu\right)^0 = \sum n_b \pi(J \colon \chi_b \colon \tau \colon \nu).$$

PROOF: This is induction by stages using invariance of $\nu \in \mathfrak{j}_p^*$ and of τ under conjugation from $Z_{M_J}(M_J^0)$:

$$\begin{aligned}
\mathrm{LHS} &= \mathrm{Ind}_{ZG^0}^G\ \mathrm{Ind}_{M_J J_p N_J \cap ZG^0}^{ZG^0}\ \mathrm{Ind}_{M_J^\dagger J_p N_J \cap ZG^0}^{M_J J_p N_J \cap ZG^0}\left(\chi^0 \otimes \pi_\tau \otimes e^{i\nu}\right) \\
&= \mathrm{Ind}_{(M_J^\dagger \cap ZG^0) J_p N_J}^G\left(\chi^0 \otimes \pi_\tau \otimes e^{i\nu}\right) \\
&= \mathrm{Ind}_{M_J J_p N_J}^G\ \mathrm{Ind}_{M_J^\dagger J_p N_J}^{M_J J_p N_J}\ \mathrm{Ind}_{(M_J^\dagger \cap ZG^0) J_p N_J}^{M_J^\dagger J_p N_J}\left(\chi^0 \otimes \pi_\tau \otimes e^{i\nu}\right) \\
&= \mathrm{Ind}_{M_J J_p N_J}^G\ \mathrm{Ind}_{M_J^\dagger J_p N_J}^{M_J J_p N_J}\left\{\mathrm{Ind}_{M_J^\dagger \cap ZG^0}^{M_J^\dagger}\left(\chi^0 \otimes \pi_\tau\right) \otimes e^{i\nu}\right\} \\
&= \mathrm{RHS}.
\end{aligned}$$

QED

LEMMA 6.10: *Let* $J \in \mathrm{Car}(G)$. *Then* $J \subset M_J^\dagger G^0$ *and*
$\mathrm{Ind}_{M_J^\dagger G^0}^G \pi(J \colon \chi \colon \tau \colon \nu)^\dagger = \pi(J \colon \chi \colon \tau \colon \nu)$.

PROOF: If $x \in J_K$ then $\mathrm{Ad}(x)$ is inner on M_J^0 so $x \in M_J^\dagger$. Now induction by stages:

$$\begin{aligned}
\mathrm{LHS} &= \mathrm{Ind}_{M_J^\dagger G^0}^G\ \mathrm{Ind}_{M_J^\dagger J_p N_J}^{M_J^\dagger G^0}\left(\chi \otimes \pi_\tau \otimes e^{i\nu}\right) \\
&= \mathrm{Ind}_{M_J J_p N_J}^G\ \mathrm{Ind}_{M_J^\dagger J_p N_J}^{M_J J_p N_J}\left(\chi \otimes \pi_\tau \otimes e^{i\nu}\right) \\
&= \mathrm{Ind}_{M_J J_p N_J}^G\left\{\mathrm{Ind}_{M_J^\dagger}^{M_J}\left(\chi \otimes \pi_\tau\right) \otimes e^{i\nu}\right\} \\
&= \mathrm{RHS}.
\end{aligned}$$

QED

We now combine Lemmas 6.4 through 6.10. Let $f \in C_c^\infty(G)$. Then

$$\left[G/ZG^0\right] f(1) = \sum_{i=1}^{r} \left(f|_{ZG^0}\right)^{x_i}(1)$$

where $\{x_1, \ldots, x_r\}$ is any set of coset representatives of G modulo ZG^0. Using Lemma 6.4,

$$(f|_{ZG^0})^{x_t}(1) = \sum_{J \in \mathrm{Car}(G)} \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)^{\widehat{}}} \deg(\chi^0)$$

$$\times \sum_{\tau \in L'_{\chi^0}} \int_{\mathfrak{j}_p^*} \Theta(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)^0\big((f|_{ZG^0})^{x_t}\big)$$

$$\times m(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)\, \mathrm{d}\nu$$

where

$$m(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)$$

$$= \frac{c \cdot [W(ZG^0, H \cap ZG^0)]}{(2\pi)^{r+p}} c_{J \cap ZG^0}^{-1} \Big| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + \mathrm{i}\nu \rangle$$

$$\times \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi^0, \nu) \Big|.$$

Notice that $\Theta(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)^0((f|_{ZG^0})^{x_t})$ is not changed when x is replaced by any $x_t' \in x_t ZG^0$, by invariance of the distribution $\Theta(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)^0$. Thus we may interchange $\Sigma_{t=1}^r$ and $\Sigma_{J \in \mathrm{Car}(G)}$, and in the terms for a given J we may assume $\{x_1, \ldots, x_r\}$ of the form $\{z_u y_v\}$ as in Lemma 6.7. Now

LEMMA 6.11: *If $f \in C_c^\infty(G)$ then*

$$[G/ZG^0] f(1)$$

$$= \sum_{J \in \mathrm{Car}(G)} \sum_{u,v} \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)^{\widehat{}}} \sum_{\tau \in L'_{\chi^0}} \deg(\chi^0)$$

$$\times \int_{\mathfrak{j}_p^*} \Theta(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)^0\big((f|_{ZG^0})^{z_u y_v}\big)$$

$$\times m(J \cap ZG^0 \colon \chi^0 \colon \tau \colon \nu)\, \mathrm{d}\nu\, \mathrm{d}\chi^0.$$

In order to lift the integration from $(Z_J(M_J^0) \cap ZG^0)^{\widehat{}}$ to $Z_{M_J}(M_J^0)^{\widehat{}}$ in Lemma 6.11, we will need

LEMMA 6.12: *There is a unique positive Radon measure* $\mathrm{d}\chi$ *on* $Z_{M_J}(M_J^0)^{\widehat{}}$ *such that*

$$\int_{Z_{M_J}(M_J^0)^{\widehat{}}} \Phi(\chi)\deg(\chi)\,\mathrm{d}\chi = \int_{\{Z_{M_J}(M_J^0)\cap ZG^0\}^{\widehat{}}} \sum_{\chi\in Z_{M_J}(M_J^0)^{\widehat{}}} \Phi(\chi)$$

$$\times \operatorname{mult}(\chi^0, \chi|)\deg(\chi^0)\,\mathrm{d}\chi^0 \qquad (6.13)$$

where $\mathrm{d}\chi^0$ *is normalized as in* (6.3) *and* $x|$ *denotes restriction to* $Z_{M_J}(M_J^0)\cap ZG^0$. *The measure* $\mathrm{d}\chi$ *is independent of choice of* Z *such that* $Z\cap G^0 = Z_{G^0}$. *If* $Z_{M_J}(M_J^0)$ *is compact then* $\mathrm{d}\chi$ *is counting measure.*

PROOF: The right hand side of the formula (6.13) defining $\mathrm{d}\chi$ is a positive Radon measure because, given χ^0, $\operatorname{mult}(\chi^0, \chi|) = 0$ for all but finitely many χ. For independence of Z it suffices to consider a finite index subgroup $Z_1 \subset Z$ such that $Z_1 \cap G^0 = Z_{G^0}$ and Z_1 is normal in G. Denote

$$B = Z_{M_J}(M_J^0), \quad B_1 = B\cap Z_1G^0, \quad B_2 = B\cap ZG^0$$

and write χ, χ_1 and χ_2 for their respective representations. Then the right hand side of (6.13) is

$$\int_{\hat{B}_2} \sum_{\chi\in\hat{B}} \Phi(\chi)\operatorname{mult}(\chi_2, \chi|)\deg(\chi_2)\,\mathrm{d}\chi_2$$

$$= \int_{\hat{Z}} \sum_{\chi_2\in(B_2)^{\widehat{}}_{\zeta}} \sum_{\chi\in\hat{B}} \Phi(\chi)\operatorname{mult}(\chi_2, \chi|)\deg(\chi_2)\,\mathrm{d}\zeta$$

where we use (6.3). Now break up $\int_{\hat{Z}}$ and obtain

$$\int_{\hat{Z}_1} \sum_{\zeta\in\hat{Z}_{\zeta_1}} \sum_{\chi_2\in(B_2)^{\widehat{}}_{\zeta}} \sum_{\chi\in\hat{B}} \Phi(\chi)\operatorname{mult}(\chi_2, \chi|)\deg(\chi_2)\,\mathrm{d}\zeta_1$$

$$= \int_{\hat{Z}_1} \sum_{\chi_2\in(B_2)^{\widehat{}}_{\zeta_1}} \sum_{\chi\in\hat{B}} \Phi(\chi)\operatorname{mult}(\chi_2, \chi|)\deg(\chi_2)\,\mathrm{d}\zeta_1$$

$$= \int_{\hat{Z}_1} \sum_{\chi_1\in(B_1)^{\widehat{}}_{\zeta_1}} \sum_{\chi_2\in(B_2)^{\widehat{}}_{\zeta_1}} \sum_{\chi\in\hat{B}}$$

$$\times \Phi(\chi)\operatorname{mult}(\chi_2, \chi|)\operatorname{mult}(\chi_1, \chi_2|)\deg(\chi_1)\,\mathrm{d}\zeta_1$$

$$= \int_{\hat{Z}_1} \sum_{\chi_1 \in (B_1)\hat{}_{\zeta_1}} \sum_{\chi \in \hat{B}} \Phi(\chi) \operatorname{mult}(\chi_1, \chi|) \deg(\chi_1)\, d\zeta_1$$

$$= \int_{\hat{B}_1} \sum_{\chi \in \hat{B}} \Phi(\chi) \operatorname{mult}(\chi_1, \chi|) \deg(\chi_1)\, d\chi_1$$

where at the last step we again used (6.3). But this is just the right hand side of (6.13) with Z_1 in place of Z. That proves independence of $d\chi$ from the choice of Z. The statement on counting measure is clear.

QED

Compute

$$\sum_{v=1}^{s} \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)\hat{}} \sum_{\tau \in L'_{\chi^0}} \deg(\chi^0)$$

$$\times \Theta(J \cap ZG^0 : \chi^0 : \tau : \nu)^0 ((f|_{ZG^0})^{z_u y_v})$$

$$\times m(J \cap ZG^0 : \chi^0 : \tau : \nu)\, d\chi^0$$

$$= \sum_{v=1}^{s} \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)\hat{}} \sum_{\tau \in L'_{\chi^0}} \deg(\chi^0)$$

$$\times \Theta\left(J \cap ZG^0 : (\chi^0)^{y_v^{-1}} : \tau^{y_v^{-1}} : \nu^{y_v^{-1}}\right)^0 ((f|_{ZG^0})^{z_u})$$

$$\times m(J \cap ZG^0 : \chi^0 : \tau : \nu)\, d\chi^0$$

$$= \sum_{v=1}^{s} \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)\hat{}} \sum_{\tau \in L'_{\chi^0}} \deg(\chi^0)$$

$$\times \Theta\left(J \cap ZG^0 : (\chi^0)^{y_v^{-1}} : \tau : \nu\right)^0 ((f|_{ZG^0})^{z_u})$$

$$\times m(J \cap ZG^0 : \chi^0 : \tau : \nu)\, d\chi^0$$

because the y_v centralize $\mathfrak{j}$

$$= \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)\hat{}} \sum_{\tau \in L'_{\chi^0}} \deg(\chi^0)$$

$$\times \Theta\left(\operatorname{Ind}_{ZG^0}^{M_J^\dagger G^0}\left(\pi(J \cap ZG^0 : \chi^0 : \tau : \nu)^0\right)\right) ((f|_{ZG^0})^{z_u})$$

$$\times m(J \cap ZG^0 : \chi^0 : \tau : \nu)\, d\chi^0$$

by Lemma 6.10 applied to induction from ZG^0 to $M_J^\dagger G^0$

$$= \int_{\chi^0 \in (Z_{M_J}(M_J^0) \cap ZG^0)^\frown} \sum_{\chi \in Z_{M_J}(M_J^0)^\frown} \sum_{\tau \in L'_\chi} \deg(\chi^0)$$

$$\times \operatorname{mult}\left(\chi^0 \text{ in } \chi \mid Z_{M_J}\left(M_J^0\right) \cap ZG^0\right)$$

$$\times \Theta(J\colon \chi\colon \tau\colon \nu)^\dagger\left(\left(f \mid_{M_J^\dagger G^0}\right)^{z_u}\right)$$

$$\times m(J \cap ZG^0\colon \chi^0\colon \tau\colon \nu)\, \mathrm{d}\chi^0$$

by Frobenius Reciprocity, because $L'_\chi = L'_{\chi^0}$ here, and because the induced character is supported in ZG^0

$$= \int_{\chi \in Z_{M_J}(M_J^0)^\frown} \sum_{\tau \in L'_\chi} \deg(\chi) \Theta(J\colon \chi\colon \tau\colon \nu)^\dagger\left(\left(f \mid M_J^\dagger G^0\right)^{z_u}\right)$$

$$\times m(J\colon \chi\colon \nu\colon \tau)\, \mathrm{d}\chi$$

by Lemma 6.12. Here $m(J\colon \chi\colon \tau\colon \nu)$ is defined in Lemma 6.14 below. Substituting this into Lemma 6.11 we have

LEMMA 6.14: *If $f \in C_c^\infty(G)$ then*

$$[G/ZG^0]\, f(1) = \sum_{J \in \operatorname{Car}(G)} \int_{\chi \in Z_{M_J}(M_J^0)^\frown} \deg(\chi) \sum_{\tau \in L'_\chi}$$

$$\times \sum_{u=1}^{t} \int_{\mathfrak{i}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)^\dagger\left(\left(f \mid_{M_J^\dagger G^0}\right)^{z_u}\right)$$

$$\times m(J\colon \chi\colon \tau\colon \nu)\, \mathrm{d}\nu\, \mathrm{d}\chi$$

where $m(J\colon \chi\colon \tau\colon \nu)$ denotes any $m(J \cap ZG^0\colon \chi^0\colon \tau\colon \nu)$ such that χ^0 is a summand of $x \mid Z_{M_J}(M_J^0) \cap ZG^0$.

Since $M_J^\dagger G^0$ is normal in G, Lemma 6.10, *with the observation that the resulting character is supported in $M_J^\dagger G^0$, gives us*

$$\sum_{u=1}^{t} \Theta(J\colon \chi\colon \tau\colon \nu)^\dagger\left(\left(f \mid_{M_J^\dagger G^0}\right)^{z_u}\right) = \Theta(J\colon \chi\colon \tau\colon \nu)\left(f \mid_{M_J^\dagger G^0}\right)$$

$$= \Theta(J\colon \chi\colon \tau\colon \nu)(f).$$

Thus we can rewrite Lemma 6.14 *as*

LEMMA 6.15: *If* $f \in C_c^\infty(G)$ *then*

$$[G/ZG^0]\, f(1) = \sum_{J \in \mathrm{Car}(G)} \int_{\chi \in Z_{M_J}(M_J^0)^\curlyvee} \deg(\chi) \times \sum_{\tau \in L'_\chi} \int_{\mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f) \times m(J\colon \chi\colon \tau\colon \nu)\, d\nu\, d\chi.$$

Now $\chi|_{Z_{M_J}(M_J^0) \cap ZG^0} = \sum (\chi^0)^{y_v^{-1}}$ *where* χ^0 *is any summand of the restriction. As* y_v *centralizes* $\mathfrak{j}$, $y_v \in J$ *so* $\gamma_\alpha^{y_v} = \gamma_\alpha^{\pm 1}$ *by* (4.15). *Thus*

$$(\chi^0)^{y_v^{-1}}(\gamma_\alpha) + (\chi^0)^{y_v^{-1}}(\gamma_\alpha^{-1}) = \chi^0(\gamma_\alpha) + \chi^0(\gamma_\alpha^{-1}),$$

so $p_\alpha(\chi\colon \nu)$ (*defined in Lemma* 6.16 *just below*) *is a direct sum of copies of the scalar matrix* $p_\alpha(\chi^0\colon \nu)$. *Thus, as in Lemma* 4.16,

LEMMA 6.16: *Let* $\chi \in Z_{M_J}(M_J^0)^\curlyvee$, $k = \deg \chi$, $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{j})$ *and* $\nu \in \mathfrak{j}_p^*$. *Then*

$$p_\alpha(\chi\colon \nu) = \sinh \pi\nu_\alpha \cdot I_k \times \left\{\cosh \pi\nu_\alpha \cdot I_k - \tfrac{1}{2}\xi_{\rho_\alpha}(\gamma_\alpha)\left[\chi(\gamma_\alpha) + \chi(\gamma_\alpha)^{-1}\right]\right\}^{-1}$$

is a scalar matrix, equal to $\bar{p}_\alpha(\chi^0\colon \nu) \cdot I_k$ *for any irreducible summand* χ^0 *of*

$$\chi|_{Z_{M_J}(M_J^0) \cap ZG^0}.$$

Finally, combine Lemmas 6.15 *and* 6.16 *using* $[Z_G(G^0)/Z]/[G/ZG^0] = 1/[G/Z_G(G^0)G^0]$. *The result is*

THEOREM 6.17: (*Global Plancherel Formula*). *Let G be a reductive group in the general class* (1.1).

If $f \in C_c^\infty(G)$, *then*

$$f(1) = c \cdot \frac{[W(G^0, H \cap G^0)]}{[G/Z_G(G^0)G^0](2\pi)^{r+p}} \sum_{J \in \mathrm{Car}(G)} c_{J \cap G^0}^{-1} \times \int_{\chi \in Z_{M_J}(M_J^0)^\curlyvee} \deg(\chi) \sum_{\tau \in L'_\chi} \int_{\nu \in \mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f) \times \left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi\colon \nu) \right| d\nu\, d\chi$$

where $c = [\pi_1(\bar{G}_{1\mathbb{C}})]$ *as in Lemma* 6.1, *where* $\bar{p}_\alpha(\chi\colon \nu) = (\deg \chi)^{-1} \times \operatorname{tr} p_\alpha(\chi\colon \nu)$ *as in Lemma* 6.16, *where*

$$c_{J \cap G^0} = \left[W(G^0, J \cap G^0)\right] \cdot \left[J \cap K^0 / J \cap K^0 \cap M_H^\dagger\right] \times \left[L(\psi)\colon L(\Phi_R)\right] \cdot \prod_{\alpha \in R_J} \|\alpha\|,$$

and where H is a fundamental Cartan subgroup.

See Theorem 4.18 for further explanation of notation.

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