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Indefinite Harmonic Theory and Unitary Representations

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Wilfried Schmid's lecture ended with an indication of our uniform construction of (possibly) singular representations of semisimple and reductive Lie groups. The positive energy representations and Gupta-Bleuler triples, described by Chris Fronsdal, typify our situation, though the correspondence is not yet exact. Here I will describe the construction in some detail and try to indicate the direction this work is now taking. Complete details of much of this can be found in the paper, *Singular unitary representations and indefinite harmonic theory*, November 1981, by Rawnsley, Schmid and myself.

Some notation is needed before I can begin. G will be a reductive Lie group, assumed connected to avoid technicalities, such as a unitary group $U(k, l)$ or the universal cover $SO(4,2)^{\sim}$ of the conformal group. Lower case gothic letters denote complexified Lie algebras. Thus $\mathfrak{g}_0$ is the real Lie algebra of G and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$. So, if H is a Lie subgroup of G we also have subalgebras $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and $\mathfrak{h} \subset \mathfrak{g}$. A "grammar" of this sort is useful because we deal with many subgroups of G.

Let me remind you of the "tempered" or "regular" or "Harish-Chandra" series of representations of G, in order to indicate the "location" of the representations I will be constructing. Let B be a Cartan subgroup of G. In other words, $\mathfrak{b}_0$ is a subalgebra of $\mathfrak{g}_0$ which is maximal for the property that

$\mathfrak{b}_0$ is abelian and $\mathrm{ad}(\mathfrak{b})$ is diagonalizable, and
$B = \{g \in G, \mathrm{Ad}(g)\xi = \xi \text{ for every } \xi \in \mathfrak{b}_0\}$.

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Then $B = T \times A$ where T is the compactly embedded (compact modulo the center of G) part and A is the split part ($\mathrm{ad}(\mathfrak{a}_0)$ has all eigenvalues real). Further, there are subgroups $P = MAN$, called cuspidal parabolic subgroups, such that

$MA = M \times A$ is a reductive subgroup of G,
T is a compactly embedded Cartan subgroup of M,
N is the nilpotent radical of P.

M is specified because $MA = \{g \in G\colon \mathrm{Ad}(g)a = a \text{ for all } a \in A\}$. N is not unique, but the result does not depend on the choice of N. Now one considers

μ: (relative) discrete series representation of M,
α: element of $\mathfrak{a}_0^*$, i.e. $e^{i\alpha}$ is a unitary character on A.

That gives a unitary representation of P on the space of μ,

$$(\mu \otimes e^{i\alpha})(man) = e^{i\alpha}(a)\mu(m)$$

and thus defines the unitarily induced representation

$$\pi(\mu, \alpha) = \operatorname*{Ind}_{P \uparrow G}(\mu \otimes e^{i\alpha})$$

of G. There are about a half dozen "standard" names and notations for these representations. I call the set of all $\pi(\mu, \alpha)$, constructed from B, the "B-series". If B is as noncompact as possible, it usually is called the "principal series". When B is as compact as possible it usually is called the "fundamental series". If B is compactly embedded, then $A = 1$, $M = G = P$, and we have the (relative) discrete series.

One can try to get other representations by letting the parameters μ, α go "out of range". If $\alpha \in \mathfrak{a}^*$ but $\alpha \notin \mathfrak{a}_0^*$, one still has $\pi(\mu, \alpha)$, but it is not unitary. Sometimes it can be unitarized when α satisfies a technical condition. That gives a "complementary" series. I will avoid those. If μ is a continued discrete series representation of M of some sort, one still has a unitary representation $\pi(\mu, \alpha)$. The uniform geometric construction, which I will describe, gives some continued discrete series representations (as well as the usual discrete series), and thus gives the sort of singular $\pi(\mu, \alpha)$ obtained by letting μ go singular.

Here is the setting for the geometric construction. Let H be the centralizer of some torus subgroup of G, and let ψ be a unitary representation of H. The space G/H has a number of structures as complex manifolds; they come from embeddings of G/H as open G-orbits in a certain compact complex manifold $G_{\mathbf{C}}/H_{\mathbf{C}}Q_-$. On the Lie algebra level, $\mathfrak{h}$ is the reductive part of a parabolic subalgebra $\mathfrak{h} + \mathfrak{q}_-$ of $\mathfrak{g}$, $\mathfrak{q}_+ = \bar{\mathfrak{q}}_-$ represents the holomorphic tangent space, and necessarily $\mathfrak{h}_0$ contains a CSA $\mathfrak{b}_0$ of $\mathfrak{g}_0$ such that B is as compact as possible. Now ψ defines a holomorphic vector bundle $\mathbf{V} \to G/H$, fiber $V =$ representation space of ψ. We look at the representation of G on L_2 cohomologies of $\mathbf{V} \to G/H$.

All this is classical when H is compactly embedded in G. There, the statement is mostly contained in the Kostant-Langlands Conjecture of the 1960s, which was refined and proved by Schmid in the 1970s. The interest here will be when H is not compactly embedded in G.

Classically, with H compact modulo the center of G, there is a (positive definite) G-invariant hermitian metric on G/H, so we have a positive definite pointwise inner product $\langle \varphi(x), \varphi'(x)\rangle$ of $\mathbf{V}$-valued differential forms, thus a global linear product

$$\langle \varphi, \varphi' \rangle_{G/H} = \int_{G/H} \langle \varphi(x), \varphi'(x)\rangle \, d(xH),$$

and genuine Hilbert spaces

$$L_2^q(G/H, \mathbf{V}) = \{\mathbf{V}\text{-valued } (0,q)\text{-forms } \omega \text{ on } G/H\colon \langle \omega, \omega \rangle_{G/H} < \infty\}$$

on which G acts by unitary representations. Let $\bar\partial^*$ denote the formal adjoint of $\bar\partial$ on $\mathbf{V}$-valued forms. Then the Kodaira-Hodge-Laplace operator is

$\square$: closure of $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on $L_2^q(G/H, \mathbf{V})$ from the dense subspaces consisting of C_c^∞ forms.

The kernel

$$\mathcal{H}_2^q(G/H, \mathbf{V}) = \{\omega \in L_2^q(G/H, \mathbf{V})\colon \square\omega = 0\}$$

is the space of harmonic forms. It is closed in L_2^q, so G acts on it by a unitary representation. Note that compactness of H is crucial at the very start of this construction.

The "compactly embedded" condition on H forces $\dim V < \infty$, so, in particular, ψ has a highest weight, say λ. Let ρ be half the sum of the positive roots. The Kostant-Langlands Conjecture is as follows.

1. If $\lambda + \rho$ is orthogonal to some root of G then $\mathcal{H}_2^q(G/H, \mathbf{V}) = 0$ for all q.

2. Suppose that $\lambda + \rho$ is not orthogonal to any root of G. A root is called "compact" if it is a root of the maximal compact subgroup, "noncompact" otherwise. Let

$$\begin{aligned} q(\lambda + \rho) = {} & (\text{number of compact positive roots } \alpha \text{ with } (\lambda + \rho, \alpha) < 0) \\ & + (\text{number of noncompact positive roots } \beta \\ & \qquad \text{with } (\lambda + \rho, \beta) > 0). \end{aligned}$$

Then $\mathcal{H}_2^q(G/H, \mathbf{V}) = 0$ for $q \neq q(\lambda + \rho)$, and G acts on $\mathcal{H}_2^{q(\lambda+\rho)}(G/H, \mathbf{V})$ irreducibly by the discrete series representation with Harish-Chandra parameter $\lambda + \rho$.

In general, there is no good relation between the harmonic L_2 spaces $\mathcal{H}_2^q(G/H, \mathbf{V})$ and ordinary Dolbeault cohomology $H^q(G/H, \mathbf{V})$, though there is of course a natural map

$$\mathcal{H}_2^q(G/H, \mathbf{V}) \to H^q(G/H, \mathbf{V}) \quad \text{by } \omega \mapsto [\omega]$$

which simply sends a harmonic form to its Dolbeault class. In the course of his work that resulted in proving the Kostant-Langlands Conjecture stated above, W. Schmid also showed that

3. If $(\lambda + \rho, \alpha) < 0$ for every noncompact positive root, i.e. if the bundle $\mathbf{V} \to G/H$ is negative, then $\mathcal{H}_2^{q(\lambda+\rho)}(G/H, \mathbf{V}) \to H^{q(\lambda+\rho)}(G/H, \mathbf{V})$ is an isomorphism on the subspaces of vectors that have finite expansion under the maximal compact subgroup K of G.

The hypotheses of (3) can always be arranged by suitable (depending on λ) choice of the complex structure, and then $q(\lambda + \rho)$ is equal to

$s = \dim_{\mathbf{C}} K/H$, dimension of the maximal compact complex submanifold K/H of G/H.

The resulting isomorphism $\mathcal{H}_2^s(G/H, \mathbf{V})_K \to H^s(G/H, \mathbf{V})_K$ of Harish-Chandra modules is useful in a number of contexts. In our situation with H noncompact it leads to a K-type analysis of $\mathcal{H}_2^s(G/H, \mathbf{V})$.

We want to carry this work over to the case where H is not necessarily compact. The first major problem is to define the appropriate analogues of the spaces $\mathcal{H}_2^q(G/H, \mathbf{V})$. Neither "harmonic" or "square integrable" has a completely obvious definition here, where the only invariant hermitian metrics are indefinite. Some technical tricks, using the fibration I will describe below, allow us to define an auxiliary positive definite hermitian metric on G/H, which is not G-invariant except in the rather special case, described above, where H is compact. But this auxiliary metric only suffers bounded distortion under any element of G. So, if we use it to define the $L_2^q(G/H, \mathbf{V})$, then G acts on that Hilbert spaces by bounded linear transformations. Then we say that a form

$\omega \in L_2^q(G/H, \mathbf{V})$ is harmonic if $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$ where $\bar\partial^*$ is the formal adjoint of $\bar\partial$ relative to the G-invariant indefinite-hermitian metric on G/H

and harmonic forms are understood as distribution solutions to the hyperbolic system $\bar\partial\omega = 0$, $\bar\partial^*\omega = 0$. Thus, we have a closed G-invariant subspace of $L_2^q(G/H, \mathbf{V})$,

$$\mathcal{H}_2^q(G/H, \mathbf{V}) = \begin{cases} \text{all } \mathbf{V}\text{-valued } (0, q)\text{-forms on } G/H \text{ that are } L_2 \\ \text{relative to the auxiliary positive definite metric and} \\ \text{harmonic relative to the invariant indefinite metric.} \end{cases}$$

It is a Hilbert space on which G acts continuously by bounded linear operators. Since Dolbeault cohomology can be done with distribution forms just as well as with smooth forms, we still have the canonical maps $\mathcal{H}_2^q(G/H, \mathbf{V}) \to H^q(G/H, \mathbf{V})$.

The invariant indefinite metric defines a hermitian inner product $\langle\ ,\ \rangle_{G/H}$ just as in the classical setting. Let $s = \dim_{\mathbf{C}} K/K \cap H$, dimension of the maximal compact subvariety. Calculating in a special case, Rawnsley and I were amazed to see that, for negative $\mathbf{V}$,

(a) $(-1)^s \langle\ ,\ \rangle_{G/H}$ is positive semidefinite on $\mathcal{H}_2^s(G/H, \mathbf{V})$,

(b) the canonical map $\mathcal{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V})$ is surjective on the level of K-finite vectors,

(c) the kernel of $\mathcal{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V})$ coincides with the kernel of $\langle\ ,\ \rangle_{G/H}$ on $\mathcal{H}_2^s(G/H, \mathbf{V})$.

Thus, in the special case, passage to the quotient defined a unitary representation

π_V: action of G on $\mathcal{H}_2^s(G/H, \mathbf{V})/(\text{kernel of } \langle\ ,\ \rangle_{G/H})$

which unitarized the Fréchet representation of G on $H^s(G/H, \mathbf{V})$. These representations π_V were irreducible and singular in the sense described earlier.

Schmid and I now discovered that (a), (b) and (c) hold in some generality, and for essentially geometrical reasons. There is a C^∞ fibration

$\pi\colon G/H \to K/L$ with structure group $L = K \cap H$.

When G/H is indefinite-hermitian symmetric, i.e. is a semisimple symmetric space with invariant complex structure, the base and fiber are complex manifolds. The projection need not be holomorphic, but still we find an analogue of the Leray spectral sequence and show that

$$H^q(G/H, \mathbf{V})_K \cong H_d^q\big(K/L, \mathbf{H}^0(\text{fiber}, \mathbf{V})\big)_K$$

as a K-module. Here d is the $\bar\partial$ operator of K/L modified by a term that measures the failure of $\pi\colon G/H \to K/L$ to be holomorphic. This, some estimates and some tensoring arguments lead to a complete analysis of the global character, the K-character and the K-spectrum of the Dolbeault cohomologies $H^q(G/H, \mathbf{V})$.

When $\pi\colon G/H \to K/L$ is holomorphic, the K-decomposition $H^s(G/H, \mathbf{V})_K = H^s(K/L, \mathbf{H}^0(\text{fiber}, \mathbf{V}))_K$ can be done on the level of harmonic forms, even L_2-harmonic forms for negative $\mathbf{V}$. Thus, when $\mathbf{V} \to G/H$ is negative, we obtain (a), (b) and (c). Since we understand the Dolbeault space $H^s(G/H, \mathbf{V})$, we then have complete character and spectral information on the unitary representations π_V. That is the content of the Rawnsley-Schmid-Wolf paper which I mentioned at the beginning of this talk.

Schmid and I are now trying to get rid of various restrictions, e.g. that $\pi\colon G/H \to K/L$ be holomorphic, or even that $\operatorname{rank} K = \operatorname{rank} G$. So far, we have made some progress in obtaining the fundamental series representations of G in this way. To be precise we have that now on the Dolbeault level, modulo correctness of a result of Schmid that has not yet been written down in complete detail. This, incidently, completes the proof that Zuckerman's derived modules are the same as the corresponding Dolbeault cohomologies, and we are now investigating the L_2 properties of certain harmonic representatives.

The unitary representations π_V, for $G/H \to K/L$ holomorphic, are highest weight representations. So their duals π_V^* are lowest weight representations, that is, positive energy representations. The scheme for the indecomposable G-module $\mathcal{H}_2^s(G/H, \mathbf{V})$ given by

$$\mathcal{H}_2^s(G/H,\mathbf{V})/\left(\ker\langle\ ,\ \rangle_{G/H}\right) \rightsquigarrow \left(\ker\langle\ ,\ \rangle_{G/H}\right) \rightsquigarrow 0$$

has formal similarity to the scheme

$$(\text{transverse photons}) \rightsquigarrow (\text{longitudinal photons}) \rightsquigarrow 0$$

of Gupta-Bleuler quantization. This appears to be tied to our definition of harmonic. More generally, if one defines a generalized harmonic space

$$\tilde{\mathcal{H}}_2^s(G/H,\mathbf{V}) = \left\{\omega \in L_2^s(G/H,\mathbf{V}) \colon (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)^N\omega = 0 \text{ for } N \gg 0\right\},$$

then the scheme of quotients of

$$\tilde{\mathcal{H}}_2^s(G/H,\mathbf{V}) \supset \mathcal{H}_2^s(G/H,\mathbf{V}) \supset \left(\ker\langle\ ,\ \rangle_{G/H}\right)$$

has strong formal similarity to the Gupta-Bleuler triples of C. Fronsdal's talk: singletons, de Sitter electrodynamics, conformal QED and possibly conformal gravity. The intriguing fact here is that, in Fronsdal's setting, an inspection of the table printed with his lecture shows that in each case he has the formal analogue of

$$\tilde{\mathcal{H}}_2^s(G/H,\mathbf{V})/\mathcal{H}_2^s(G/H,\mathbf{V}) \cong \left(\ker\langle\ ,\ \rangle_{G/H}\right)$$

and in each case his module is an irreducible positive energy module. It will be interesting to understand this from the viewpoint of our L_2 harmonic forms.

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