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A VANISHING THEOREM FOR OPEN ORBITS ON COMPLEX FLAG MANIFOLDS

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ABSTRACT. A real reductive Lie group G acts on complex flag manifolds $G_{\mathbf{C}}/(\text{parabolic subgroup})$. The open orbits $D = G(x)$ are precisely the homogeneous complex manifolds G/H, where H is the centralizer of a torus. We prove that D is $(s+1)$-complete in the sense of Andreotti and Grauert, with $s =$ complex dimension of a maximal compact subvariety of D. Thus $H^q(D, \mathcal{F}) = 0$ for $q > s$ and any coherent sheaf $\mathcal{F} \to D$. This vanishing theorem is needed for the realization of certain unitary representations on Dolbeault cohomology groups of homogeneous vector bundles.

Real group orbits on complex flag manifolds are important in several parts of mathematics, especially in group representation theory. See [**2**–**5**], which rely on special cases of the following useful result.

THEOREM. *Let G be a connected reductive real Lie group. Let D be an open G-orbit on a complex flag manifold $X = G_{\mathbf{C}}/Q$. Let Y be a maximal compact subvariety of D and $s = \dim_{\mathbf{C}} Y$. Then D is $(s+1)$-complete in the sense of Andreotti and Grauert* [**1**]. *In particular, if $\mathcal{F} \to D$ is a coherent analytic sheaf and if $q > s$, then $H^q(D; \mathcal{F}) = 0$.*

REMARK. The argument is a variation on the proof of the special case [**3**, Theorem 4.1].

PROOF. We may, and do, assume $G_{\mathbf{C}}$ simply connected and semisimple, and $G \subset G_{\mathbf{C}}$. Now $D = G/H$, where $H = G \cap Q$, $\mathfrak{q}_{\pm}$ represent the holomorphic and antiholomorphic tangent spaces of D at $1 \cdot H$, and Q is the parabolic subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{h} + \mathfrak{q}_-$. Set $\mathfrak{q} = \mathfrak{q}_+ + \mathfrak{q}_-$ so $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$.

The real Lie algebra $\mathfrak{h}_0$ of H contains a fundamental Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$ stable under a Cartan involution θ of $\mathfrak{g}_0$. Decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mp_0$ and $\mathfrak{b}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ under θ.

The fact that D is open in X is equivalent to $\mathfrak{q}_- + \overline{\mathfrak{q}_+}$, conjugate of $\mathfrak{g}$ over $\mathfrak{g}_0$. Write $\Phi(\cdot)$ for the set of $\mathfrak{b}$-roots in the indicated subspace. The parabolic $\mathfrak{h} + \mathfrak{q}_+$ being θ-stable we have $\nu \in \mathfrak{b}^*$ in the weight lattice, so that

$$\nu \perp \Phi(\mathfrak{h}) \quad \text{and} \quad (\nu, \alpha) > 0 \quad \text{for all } \alpha \in \Phi(\mathfrak{q}_+).$$

Now set

$$\lambda = \nu + \theta(\nu): \qquad \lambda \perp \Phi(\mathfrak{h}),\ (\lambda, \alpha) > 0 \text{ for } \alpha \in \Phi(\mathfrak{q}_+),\ \lambda(\mathfrak{a}) = 0.$$

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Then λ exponentiates to a unitary character $e^\lambda \in \hat{H}$. Consider the hermitian holomorphic line bundle

$$\mathbf{L}_\lambda \to G/H \text{ associated to } G \to G/H \text{ by } e^\lambda.$$

The metric connection for its G-invariant hermitian metric h_0 has connection form λ (extended to $\mathfrak{g}$ by zero on all root spaces) and curvature form ω_0. These are related by

$$\omega_0 = 2\pi\sqrt{-1}\, d\lambda \quad \text{and} \quad \omega_0 = -\partial\overline{\partial} \log h_0.$$

If $\alpha, \beta \in \Phi(\mathfrak{q}_+)$, then
if $\overline{\beta} \neq -\alpha$: $\lambda([e_\alpha, \overline{e}_\beta]) = 0$ and $\langle e_\alpha, \overline{e}_\beta \rangle = 0$,
if $\overline{\beta} = -\alpha$: $[e_\alpha, \overline{e}_\beta] = ch_\alpha$ for some $c \neq 0$, and

$$\lambda([e_\alpha, \overline{e}_\beta]) = (\lambda, \alpha)c = (\lambda, \alpha)\langle e_\alpha, \overline{e}_\beta \rangle.$$

Let $x, y \in \mathfrak{q}_+$, say $x = \sum x_\alpha$ and $y = \sum y_\beta$, where $x_\gamma, y_\gamma \in \mathfrak{g}_\gamma$. Then

$$\begin{aligned}(\sqrt{-1}\partial\overline{\partial} \log h_0)(x, y) &= 2\pi\, d\lambda(x, \overline{y}) = \pi\lambda([x, \overline{y}]) \\ &= \pi \sum_{\alpha, \beta \in \Phi(\mathfrak{q}_+)} (\lambda, \alpha)\langle x_\alpha, \overline{y}_\beta \rangle.\end{aligned}$$

But each $(\lambda, \alpha) > 0$, and $\langle x, \overline{y} \rangle$ is positive definite on $\mathfrak{p} \cap \mathfrak{q}_+$, negative definite on $\mathfrak{k} \cap \mathfrak{q}_+$. With $n = \dim_{\mathbf{C}} X$ and $s = \dim_{\mathbf{C}} Y$ we conclude that

> the hermitian form $\sqrt{-1}\partial\overline{\partial} \log h_0$ on the holomorphic tangent bundle of D has signature: $n - s$ pluses and s minuses.

Note that e^λ extends to a holomorphic map $Q \to \mathrm{GL}(1; C)$ by triviality on $\exp(\mathfrak{q}_-)$. Thus $\mathbf{L}_\lambda \to D$ extends to a homogeneous holomorphic line bundle

$$\tilde{\mathbf{L}}_\lambda \to G_{\mathbf{C}}/Q \text{ associated to } G_{\mathbf{C}} \to G_{\mathbf{C}}/Q \text{ by } e^\lambda.$$

$\mathfrak{g}$ has a compact real form $\mathfrak{g}_u = \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0$, and $X = G_u/H_u$, where $\mathfrak{h}_u = \mathfrak{h} \cap \mathfrak{g}_u$. Note that e^λ is unitary on H_u, so $\tilde{\mathbf{L}}_\lambda$ has a G_u-invariant hermitian metric h_u, connection form λ and curvature form ω_u. The difference between h_u and h_0, and ω_u and ω_0, is that we use conjugation of $\mathfrak{g}$ over $\mathfrak{g}_u$ rather than $\mathfrak{g}_0$. Denote it $x \to \overline{\overline{x}}$ $(= \theta\overline{x})$. As above,

$$(\sqrt{-1}\partial\overline{\partial} \log h_u)(x, y) = \pi \sum_{\alpha, \beta \in \Phi(\mathfrak{q}_+)} (\lambda, \alpha)(x_\alpha, \overline{\overline{y}}_\beta),$$

and so

> the hermitian form $\sqrt{-1}\partial\overline{\partial} \log h_u$ on the holomorphic tangent bundle of D is negative definite.

The ratio $f = h_0/h_u$ is C^∞ and positive in D. We are going to show that $\phi = \log f$ is an exhaustion function for D whose Levi form has at least $n - s$ positive eigenvalues at every point of D. Then D will be $(s+1)$-complete.

Let $g \in G$. As h_0 is G-invariant, $\sqrt{-1}\partial\overline{\partial} \log h_0$ is positive definite on the $(n-s)$-dimensional subspace $\mathrm{Ad}(g)(\mathfrak{p} \cap \mathfrak{q}_+)$ of the holomorphic tangent space $\mathrm{Ad}(g)\mathfrak{q}_+$ at gH. Also, $\sqrt{-1}\partial\overline{\partial} \log h_u$ is negative definite there, because it is negative definite everywhere. Thus the Levi form

$$L(\phi) = \sqrt{-1}\partial\overline{\partial} \log h_0 - \sqrt{-1}\partial\overline{\partial} \log h_u$$

is positive definite on $\mathrm{Ad}(g)(\mp \cap \mathfrak{q}_+)$. We have just verified that $L(\phi)$ has at least $n-s$ positive eigenvalues at every point of D.

It remains, now, only to show that $\phi = \log f$ is an exhaustion function for D, i.e. that

$$\{x \in D \colon \phi(x) \leq c\} \quad \text{is compact for every } c \in \mathbf{R}.$$

For that, we need only prove that $e^{-\phi}$ has a continuous extension to X which vanishes on the topological boundary $\mathrm{bd}(D)$ of D in X. For then

$$\{x \in D \colon \phi(x) \leq c\} = \{x \in D \colon e^{-\phi(x)} \geq e^{-c}\}$$

while $e^{-c} > 0$ so that set is interior to $D \cup \mathrm{bd}(D)$, while $\{x \in X \colon e^{-\phi(x)} \geq e^{-c}\}$ is compact because X is compact.

In fact, we need only prove that $e^{-\phi}$ has a continuous extension to X that vanishes on $\mathrm{bd}(D)$ for one particular choice of λ. We will use

$$\lambda = \rho_{G/H} + \theta(\rho_{G/H}) = 2\rho_{G/H},$$

where $\rho_{G/H}$ is half the sum of the roots in $\Phi(\mathfrak{q}_+)$. Here note that

$$(\rho_{G/H}, \alpha) = 0 \quad \text{if } \alpha \in \Phi(\mathfrak{h}), \qquad > 0 \quad \text{if } \alpha \in \Phi(\mathfrak{q}_+), \qquad < 0 \quad \text{if } \alpha \in \Phi(\mathfrak{q}_-)$$

simply because $\mathfrak{h} + \mathfrak{q}_+$ is a parabolic subalgebra of $\mathfrak{g}$. Now, with λ as above,

$$\mathbf{L}_\lambda = \mathbf{K}_D^* \quad \text{and} \quad \tilde{\mathbf{L}}_\lambda = \mathbf{K}_X^* \quad \text{(dual line bundles)},$$

where $K_D \to D$ and $K_X \to X$ are the canonical line bundles. Since $e^{-\phi} = e^{-\log f} = 1/f = h_u/h_0$, and since we have

$$h_0^* \colon G\text{-invariant hermitian metric on } \mathbf{K}_D,$$

$$h_u^* \colon G_u\text{-invariant hermitian metric on } \mathbf{K}_X$$

such that $h_0 h_0^* = 1$ on D and $h_u h_u^* = 1$ on X, we have $e^{-\phi} = h_0^*/h_u^*$. So we will complete the proof by showing that h_0^*/h_u^* has a continuous extension to X that vanishes on $\mathrm{bd}(D)$.

The holomorphic cotangent bundle $\mathbf{T}_X^* \to X$ has fibre at gQ given by $\mathrm{Ad}(g)\mathfrak{q}_+^* = \mathrm{Ad}(g)\mathfrak{q}_-$. It has G_u-invariant hermitian metric given at qG by

$$F_u(\xi, \eta) = -\langle \xi, \overline{\overline{\eta}} \rangle \quad \text{for } \xi, \eta \in \mathrm{Ad}(g)\mathfrak{q}_-.$$

Similarly $\mathbf{T}_D^* = \mathbf{T}_X^*|_D$ has G-invariant indefinite-hermitian metric given at gH, $g \in G$, by

$$F_0(\xi, \eta) = -\langle \xi, \overline{\eta} \rangle \quad \text{for } \xi, \eta \in \mathrm{Ad}(g)\mathfrak{q}_-.$$

Note that, since $\mathbf{K}_D = \Lambda^n \mathbf{T}_D^*$ and $\mathbf{K}_X = \Lambda^n \mathbf{T}_X^*$, there is a real constant $c \neq 0$ such that

$$h_0^*/h_u^* = c(\text{determinant of } F_0 \text{ with respect to } F_u).$$

Thus

$$f^*(gQ) = c(\text{determinant of } F_0 \text{ on } \mathrm{Ad}(g)\mathfrak{q}_- \text{ with respect to } F_u)$$

is a C^∞ function on X that extends h_0^*/h_u^* from D to X.

It now remains only to show that f^* vanishes on $\mathrm{bd}(D)$, i.e. that the hermitian form F_0 on $\mathrm{Ad}(g)\mathfrak{q}_-$ is singular whenever $gQ \in \mathrm{bd}(D)$. If $gQ \in \mathrm{bd}(D)$, then its G-orbit on X is of positive codimension, so

$$\mathrm{Ad}(g)(\mathfrak{h} + \mathfrak{q}_-) + \overline{\mathrm{Ad}(g)(\mathfrak{h} + \mathfrak{q}_-)} \neq \mathfrak{g}.$$

Choose a θ-stable Cartan subalgebra $\tilde{\mathfrak{b}}_0$ of $\mathfrak{g}_0$ contained in $\mathrm{Ad}(g)(\mathfrak{h} \cap \mathfrak{q}_-)$. Then there is a $\tilde{\mathfrak{b}}$-root α such that $\mathfrak{g}_\alpha \subset \mathrm{Ad}(g)\mathfrak{q}_-$, but

$$\mathfrak{g}_{-\alpha} \not\subset \mathrm{Ad}(g)(\mathfrak{h} + \mathfrak{q}_-) + \overline{\mathrm{Ad}(g)(\mathfrak{h} + \mathfrak{q}_-)}.$$

Thus, if β is any $\tilde{\mathfrak{b}}$-root with $\mathfrak{g}_\beta \subset \mathrm{Ad}(g)\mathfrak{q}_-$ we have $F_0(e_\alpha, e_\beta) = \langle e_\alpha, \bar{e}_\beta \rangle$, nonzero only if $\overline{\beta} = -\alpha$, while $\overline{\beta} = -\alpha$ would imply $\mathfrak{g}_{-\alpha} \subset \overline{\mathrm{Ad}(g)\mathfrak{q}_-}$, which is false. Thus F_0 is singular on $\mathrm{Ad}(g)\mathfrak{q}_-$. That completes the proof. Q.E.D.

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