GEOMETRIC QUANTIZATION IN THE SPIRIT OF GUPTA AND BLEULER

Joseph A. Wolf

Department of Mathematics
University of California
Berkeley, CA 94720

The standard method of geometric quantization produces the tempered representations of a semisimple Lie group. Here an extension of that method is described, which produces singular unitary representations. The method has strong formal similarities with Gupta-Bleuler quantization of the transverse photon.

Just to place the context of my talk, let me remind you that a semisimple Lie group has several kinds of unitary representations. The ones that enter into the Plancherel formula for the group are the ones whose characters are tempered distributions. These are very well understood now. For each conjugacy class of Cartan subgroups H in the semisimple Lie group G, there is a series of representations with a discrete parameter associated to the compact part of H and a continuous parameter associated to the noncompact part of H. See (5) and (10).

The delicate part of this is the case where H is compact (modulo the center of G). The analysis of those representations is Harish-Chandra's famous theory of the discrete series, which I denote $\hat{G}_{disc}$. See (3), (4) and (10). In fact, both for convenience and for technical reasons, one deals with a slightly larger class of groups, the reductive Lie groups. The discrete series picture for reductive groups is summarized in Plate I. Part 1 is the heart of the matter there, Parts 2 and 3 are just some necessary technicalities.

The series for the other conjugacy classes of Cartan sub-

*Research partially supported by N.S.F. Grant MCS-7902522.

PLATE I. Discrete Series

G: reductive Lie group, identity component G^0, Lie algebra $\mathfrak{g}_0$ such that (i) $Z_G(G^0)/Z_{G^0}$ is compact,
(ii) $G/G^\dagger$ is finite, where $G^\dagger = Z_G(G^0)\cdot G^0$, and
(iii) if $x \in G$ then $Ad(x)$ is an inner automorphism on the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_R C$.

T: compactly embedded Cartan subgroup of G, i.e. $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, $T = Z_G(\mathfrak{t}_0)$, and $T/Z_G(G^0)$ is compact.

K: maximal compactly embedded subgroup of G that contains T, i.e. $Z_G(G^0) \subset K$ and $K/Z_G(G^0)$ is a maximal compact subgroup of the linear semisimple Lie group $G/Z_G(G^0)$.

W: Weyl group $W(G^0,T^0) = N_{G^0}(T)/T$.

Φ: root system of $\mathfrak{g}$ relative to $\mathfrak{t}$; Φ^+: positive roots.

ρ: half the sum of the positive roots.

G': regular elements of G, i.e. elements $x \in G$ such that the fixed point set of $Ad(x)$ is a Cartan subalgebra of $\mathfrak{g}$.

Part 1: Topological Identity Component G^0.

$\widehat{G^0}_{disc} = \{\pi^0 \in \widehat{G^0} : \pi^0$ has coefficients in $L_2(G^0/Z_{G^0})\}$ has an element π^0_λ, for every nonsingular integral $\lambda \in i\mathfrak{t}^*$, whose distribution character is given on $T \cap (G^0)'$ by the formula $\Theta(\pi^0_\lambda) = \pm\left\{\sum_{w\in W} \mathrm{sign}(w)e^{w\lambda}\right\}/\prod(e^{\alpha/2} - e^{-\alpha/2})$. The π^0_λ exhaust $\widehat{G^0}_{disc}$, $\pi^0_\lambda = \pi^0_{\lambda'}$ if and only if $\lambda \in W(\lambda')$, and π^0_λ has infinitesimal character χ_λ.

Part 2: Algebraic Identity Component $G^\dagger = Z_G(G^0)\cdot G^0$.

$\widehat{G^\dagger}_{disc} = \{\pi^\dagger_{\lambda,\psi} : \psi \in Z_G(G^0)^\wedge$ agrees with $e^{\lambda-\rho}$ on $Z_{G^0}\}$ where $\pi^\dagger_{\lambda,\psi}(zg) = \psi(z) \otimes \pi^0_\lambda(g)$ for $z \in Z_G(G^0)$ and $g \in G^0$. $\pi^\dagger_{\lambda,\psi}$ has infinitesimal character χ_λ and has distribution character $\Theta(\pi^\dagger_{\lambda,\psi})(zg) = \{\mathrm{trace}\ \psi(z)\} \times \Theta(\pi^0_\lambda)(g)$.

Part 3: The Entire Group G.

$\hat{G}_{disc} = \{\pi_{\lambda,\psi} = \mathrm{Ind}^G_{G^\dagger}(\pi^\dagger_{\lambda,\psi})\}$. $\pi_{\lambda,\psi}$ has infinitesimal character χ_λ and has distribution character $\Theta(\pi_{\lambda,\psi}) = 0$ on $G \backslash G^\dagger$,

$$\Theta(\pi_{\lambda,\psi})(zg) = \sum \Theta(\pi^\dagger_{\lambda,\psi})(x_i^{-1} zg x_i) \text{ on } G^\dagger, \text{ where } G = \cup x_i G^\dagger.$$

groups are constructed from the discrete series of certain special subgroups. Any Cartan subgroup has a natural splitting $H = T \times A$ where $T \subset K$ is the compact part and where $A = \exp(\mathfrak{a}_0)$ is a vector group that is orthogonal to K in a suitable sense: roots are pure imaginary on $\mathfrak{t}_0$ and real on $\mathfrak{a}_0$. Restrict the roots to $\mathfrak{a}_0$ to get the $\mathfrak{a}_0$-roots of $\mathfrak{g}_0$, pick a positive subsystem, let $\mathfrak{n}_0$ be the sum of the positive $\mathfrak{a}_0$-root spaces, and you have a cuspidal parabolic subgroup

$$P = MAN \quad , \qquad MA = M \times A = Z_G(A) \quad .$$

M satisfies the reductive group conditions for G on Plate I, and T is a compactly embedded Cartan subgroup of M. The corresponding series of unitary representations,

$$\left\{ \mathrm{Ind}_P^G(\eta \otimes e^{i\mu}): \ \eta \in \hat{M}_{disc} \text{ and } \mu \in \mathfrak{a}_0^* \right\} \quad ,$$

depends only on the conjugacy class of $H = T \times A$. My only point here is that the parameters of η provide a discrete parameter for this series, and μ is a continuous parameter.

What about the other unitary representations? There certainly are many non-tempered representations in general. One gets some by letting the continuous parameter go non-real inside $\mathfrak{a} = \mathfrak{a}_0 \otimes_R C$. I'll discuss the ones obtained by letting η go singular, in other words by dropping the nonsingularity condition on λ in the parametrization (Plate I) of the discrete series of M.

Now let us concentrate on the key case: continuation of the discrete series when G is connected.

Specifically, I want to describe the early stages of a uniform geometric construction of unitary representations, which includes the geometric construction of the discrete series. This represents completed joint work with John Rawnsley and Wilfried Schmid and continuing joint work with Wilfried Schmid. See (8). In the language of geometric quantization, it may lead to quantization of all elliptic co-adjoint orbits, in particular to a geometric treatment of all positive energy (= lowest weight) representations. The basic setup is given in Plate II. That picture, and the point of looking in dimension s, is motivated by the classical case, which is recalled in Plate III. In that classical case, note that (2) really is the Kostant-Langlands Conjecture, and that given a discrete series representation one can choose a positive root system Φ^+ so that the representation does occur on harmonic forms of degree s.

As the representation goes singular, its coefficients grow faster, and we cannot hope to find it on a space of square integrable forms on G/(compact) for any finite dimensional vector bundle. So it is natural to try to imitate the classical procedure

PLATE II. Setup for the Elliptic Case

G: connected relative Lie group, e.g. $U(k,\ell)$.

H: centralizer of a torus subgroup, e.g. $U(k_1,\ell_1) \times U(k_2,\ell_2)$ inside $U(k_1+k_2, \ell_1+\ell_2)$.

$\mathfrak{g}_0, \mathfrak{h}_0$: respective real Lie algebras.
$\mathfrak{g}, \mathfrak{h}$: complexified Lie algebras.

Invariant complex structures on G/H are in one-one correspondence with parabolic subalgebras $\mathfrak{h} + \mathfrak{n}_-$ of $\mathfrak{g}$ with reductive part $\mathfrak{h}$; here $\mathfrak{n}_+ = \bar{\mathfrak{n}}_-$ represents the holomorphic tangent space of G/H. Example: $G/H = U(k_1+k_2, \ell_1+\ell_2)/U(k_1,\ell_1) \times U(k_2,\ell_2)$ where $G_{\mathbb{C}} = GL(k_1+k_2+\ell_1+\ell_2; C)$ has Lie algebra given in matrix block form as indicated here. There are two invariant complex structures; interchange $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to obtain the one from the other. They are realized as open G-orbits on the complex flag manifold $G_{\mathbb{C}}/H_{\mathbb{C}}Q_-$.

$$\begin{pmatrix} \mathfrak{h} & \mathfrak{n}_+ & \mathfrak{h} & \mathfrak{n}_+ \\ \mathfrak{n}_- & \mathfrak{h} & \mathfrak{n}_+ & \mathfrak{h} \\ \mathfrak{h} & \mathfrak{n}_- & \mathfrak{h} & \mathfrak{n}_+ \\ \mathfrak{n}_- & \mathfrak{h} & \mathfrak{n}_- & \mathfrak{h} \end{pmatrix} \begin{matrix} \}\, k_1 \\ \}\, k_2 \\ \}\, \ell_1 \\ \}\, \ell_2 \end{matrix}$$
$$\quad \underbrace{}_{k_1} \quad \underbrace{}_{k_2} \quad \underbrace{}_{\ell_1} \quad \underbrace{}_{\ell_2}$$

ψ: irreducible unitary representation of H, i.e. $\psi \in \hat{H}$.
V: representation space of ψ.
$\mathbb{V} \to G/H$: associated G-homogeneous holomorphic vector bundle with invariant hermitian metric derived from V.
s: complex dimension of the maximal compact subvariety K/L, $L = K \cap H$, of G/H.

$H^p(G/H,\mathbb{V})$: Dolbeault cohomology in degree p.

$\tilde{\mathcal{H}}_2^p(G/H,\mathbb{V})$: "cohomology" from L_2 harmonic forms.

The Problem:

Define the space $\tilde{\mathcal{H}}_2^s(G/H,\mathbb{V})$ so that it is a Hilbert space, understand the unitary representation of G on $\tilde{\mathcal{H}}_2^s(G/H,\mathbb{V})$, and use that to unitarize the Fréchet representation of G on $H^s(G/H,\mathbb{V})$.

PLATE III. Classical Case.
Schmid's Solution to the Kostant-Langlands Conjecture.

G and H are as before, but with H compact. Fix a (positive definite) G-invariant hermitian metric on G/H. Choose a compact Cartan subgroup T of G which is contained in K. As before, Φ denotes the root system of $(\mathfrak{g}_0, \mathfrak{t}_0)$. Let Φ^+ denote a positive root system such that $\mathfrak{n}_+ = \sum_{\alpha \in \Phi^+ \setminus \Phi^+(\mathfrak{h})} \mathfrak{g}_\alpha$ represents the holomorphic tangent space. ψ, V and $\mathbb{V}$ are as before, with ψ irreducible, thus finite dimensional.

Define:

χ: highest weight of the irreducible representation ψ.

$L_2^p(G/H,\mathbb{V}) = \{\mathbb{V}\text{-valued } (0,p)\text{-forms } \phi : \int_{G/H} \|\phi(x)\|^2\, dx < \infty\}$

$\bar\partial^*$: formal adjoint of $\bar\partial : L_2^p(G/H,\mathbb{V}) \longrightarrow L_2^{p+1}(G/H,\mathbb{V})$.

$\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, complex Laplace-Beltrami operator.

$\mathcal{H}_2^p(G/H,\mathbb{V}) = \{\phi \in L_2^p(G/H,\mathbb{V}) : \square\phi = 0 \text{ as distribution}\}$.

Theorems:

1. If $\chi + \rho$ is singular then every $\mathcal{H}_2^p(G/H,\mathbb{V}) = 0$.
2. Suppose that $\chi + \rho$ is nonsingular and define

   $q(\chi+\rho) = \#\{\text{compact positive roots } \alpha : (\alpha, \chi+\rho) < 0\}$
   $\qquad + \#\{\text{noncompact positive roots } \beta : (\beta, \chi+\rho) > 0\}$.

   If $p \neq q(\chi+\rho)$ then $\mathcal{H}_2^p(G/H,\mathbb{V}) = 0$;
   if $p = q(\chi+\rho)$ then $\mathcal{H}_2^p(G/H,\mathbb{V}) \neq 0$, and G acts irreducibly on it by the discrete series representation $\pi_{\chi+\rho}$.
3. Suppose that $\chi+\rho$ is nonsingular and $s = q(\chi+\rho)$. Then the map $\mathcal{H}_2^s(G/H,\mathbb{V}) \to H^s(G/H,\mathbb{V})$, of a harmonic form to its Dolbeault class, is an isomorphism on the K-finite level; so $\mathcal{H}_2^s(G/H,\mathbb{V})$ unitarizes $H^s(G/H,\mathbb{V})$.

on a bundle $\mathbb{V} \to G/H$ where H need not be compact and $\mathbb{V}$ need not be finite dimensional.

What are the difficulties in imitating the classical procedure, when H is noncompact? In order to answer that I must be more specific about just what we are trying to do.

PLATE IV. Specific Program

θ: Cartan involution of G such that $\theta(H) = H$.
K: maximal compactly embedded subgroup of G given by $K = G^\theta$.
$L = H \cap K$, so K/L is a maximal compact subvariety of G/H and s is the complex dimension of K/L.
$\langle\ ,\ \rangle$: G-invariant indefinite-Kaehler metric on G/H.

Problems:

1. Define an auxiliary K-invariant G-bounded positive definite hermitian metric on G/H, and define the space $\mathcal{H}_2^s(G/H,\mathbb{V})$ of $\mathbb{V}$-valued $(0,s)$-forms on G/H that are L_2 for the positive definite metric and harmonic for the invariant metric.
2. Show that the G-invariant global hermitian form $\langle\phi,\phi'\rangle_{G/H} = \int_{G/H} \langle\phi(x),\phi'(x)\rangle\, dx$ is semidefinite on $\mathcal{H}_2^s(G/H,\mathbb{V})$.
3. Show that the natural map $\mathcal{H}_2^s(G/H,\mathbb{V}) \to H^s(G/H,\mathbb{V})$ of a harmonic form to its Dolbeault class is surjective on the K-finite level.
4. Show that the kernel of $\langle\ ,\ \rangle_{G/H}$ on $\mathcal{H}_2^s(G/H,\mathbb{V})$ coincides with the kernel of the natural map to Dolbeault cohomology.

If all this goes through:

then the action of G on $\mathcal{H}_2^s(G/H,\mathbb{V})$ induces a unitary representation of G on $\mathcal{H}^s(G/H,\mathbb{V})/(\text{kernel of } \langle\ ,\ \rangle_{G/H})$ which unitarizes the Fréchet representation of G on $H^s(G/H,\mathbb{V})$.

Note the similarity to the Gupta-Bleuler quantization scheme. This was pointed out to me separately by Flato, Fronsdal and Varadarajan after they heard about this work. The kernel corresponds to the longitudinal photons, and the quotient corresponds to the space of transverse photons. The scalar photons also have an analog here — we'll see it later.

Several conditions are necessary before a program like this can have any hope of success.

1. The notion of L_2 should be canonical and well defined. First, the auxiliary positive definite hermitian metric must be K-invariant so that we can keep track of K-types. Second, even though a general element of G distorts L_2-norm, it should be bounded on any closed space of L_2 forms. Third, the global invariant hermitian form $(\ ,\ )_{G/H}$ should be jointly continuous on any of those Hilbert spaces. Wilfried Schmid and I have carried this out in general.

2. The notion of "harmonic" must be clarified. There really are two choices,

$$(\bar\partial\phi = 0 \text{ and } \bar\partial^*\phi = 0) \quad\text{and}\quad (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\phi = 0 \ .$$

We use the first because we need to be able to compare our harmonic spaces with Dolbeault cohomology spaces. That comparison comes into the unitarization procedure itself, and we also use it to identify the resulting representations and prove that they are irreducible. See item 3 just below.

3. We must understand the Fréchet representation, say $\pi_{\mathbf{V}}$, of G on $H^s(G/H,\mathbf{V})$. Ideally this means that we should show that $\pi_{\mathbf{V}}$ is admissible, we should find its infinitesimal, distribution and K-characters, and we should work out a concrete description of its K-spectrum. Schmid and I have carried this out in a moderately general setting.

4. The indefinite-Kaehler geometry of G/H should be related to the Kaehler geometry of K/L so that we can understand what it means for a form on G/H to be harmonic. Schmid and I have done this in a somewhat restricted context.

The key to #3 and #4 is a fibration $\pi: G/H \to K/L$ and a variation on the Leray spectral sequence. See Plate V, next page. When the fibre V of $\mathbf{V} \to G/H$ is finite dimensional, this reduces analysis of the K-spectrum of $H^s(G/H,\mathbf{V})$ to an algebraic question, and when $\dim V = 1$ the algebraic question is easily answered and it shows that $H^s(G/H,\mathbf{V})$ is K-multiplicity free.

More generally, Schmid and I get character information by resolving the vector bundle $\mathbf{V} \to G/H$ and using methods of coherent continuation. The main results for the finite dimensional case are collected in Plate VI. These character formulae are completely explicit when $\mathbf{V} \to G/H$ is finite dimensional and hermitian and $\pi: G/H \to K/L$ is holomorphic; see (6).

PLATE V. The Fibration and the Spectral Sequence

The Fibration:

$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0$ where $\mathfrak{q}_0 = (\mathfrak{q}_+ + \mathfrak{q}_-) \cap \mathfrak{g}_0$

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ where $\mathfrak{p}_0$ is the (-1)-eigenspace of θ on $\mathfrak{g}_0$.

Theorem of Mostow (7):

$(k,\xi,\eta) \to k\cdot\exp(\xi)\circ\exp(\eta)$ defines a diffeomorphism of $K \times (\mathfrak{p}_0 \cap \mathfrak{q}_0) \times (\mathfrak{p}_0 \cap \mathfrak{h}_0)$ onto G.

Reformulation of Mostow's Theorem:

$\pi(k\cdot\exp(\xi)\cdot\exp(\eta)) = kL \quad (k \in K,\ \xi \in \mathfrak{p}_0 \cap \mathfrak{q}_0,\ \eta \in \mathfrak{p}_0 \cap \mathfrak{h}_0)$ defines a C^∞ fibre bundle $\pi: G/H \to K/L$ with fibre $F = \mathfrak{p}_0 \cap \mathfrak{q}_0$ and with structure group L acting on F by restriction of the adjoint representation of G.

The Spectral Sequence:

1. Suppose $\mathfrak{p}_0 \cap \mathfrak{q}_0 = \dot{\mathfrak{m}}_0 \cap \mathfrak{p}_0$ for some θ-stable subalgebra $\mathfrak{m}_0$ of $\mathfrak{g}_0$. Then F is a bounded symmetric domain holomorphically embedded in G/H, say F = M/L.

2. Further, there is a spectral sequence abutting to $H^*(G/H,\mathbf{V})$ with $E_2^{p,q} = E_\infty^{p,q} = H^p(K/L, \mathbf{H}^q(M/L,\mathbf{V}))$ on the K-finite level. If G/H is symmetric it gives

(*) $$H^s(G/H,\mathbf{V})_K = H^s_\delta(K/L, \mathbf{H}^0(M/L,\mathbf{V})_L)_K \quad \text{as a K-module,}$$

where δ is a certain first order differential operator on the bundle $\mathbf{H}^0(M/L,\mathbf{V}) \to K/L$.

3. If G/H is symmetric and $\pi: G/H \to K/L$ is holomorphic, then $\mathbf{H}^0(M/L,\mathbf{V}) \to K/L$ is holomorphic, δ reduces to its $\bar\partial$-operator, and (*) gives the K-spectrum of $H^s(G/H,\mathbf{V})$ by means of the Bott-Borel-Weil Theorem.

PLATE VI. Character Formulae: Case Rank G = rank K

If ψ is finite dimensional but not necessarily unitary:

Let C: negative Weyl chamber in it_0^*

$\Theta(C,\lambda)$: the coherent family of invariant eigendistributions on G such that, if λ is regular, then $\Theta(C,\lambda)$ is the character of the discrete series representation π_λ.

χ: heighest weight of ψ.

Then

$$\sum_{p\geqq 0} (-1)^p\, \Theta(H^p(G/H,\mathbb{V})) = \frac{(-1)^{\dim K/T}}{|W_L|} \sum_{u\in W(H)} \det(u)\Theta(C, u(\chi+\rho))$$

and

$$|W_L| \sum_{p\geqq 0} (-1)^p \Theta_K(H^p(G/H,\mathbb{V})) =$$

$$\sum_{\substack{n_i \geqq 0\\ u\in W(H)}} \frac{\sum_{w\in W(K)} \det(w)\; e^{w\left(u(\chi+\rho) - \rho + \rho_K - \Sigma n_i\beta_i\right)}}{\prod_{\alpha>0 \text{ compact}} (e^{\alpha/2} - e^{-\alpha/2})}$$

If $\dim\psi < \infty$, ψ unitary, $\pi: G/H \to K/L$ holomorphic:

Each $H^p(G/H,\mathbb{V})$ is a Harish-Chandra module of finite type, T-finite with weights bounded from above, infinitesimal character $\chi_{\chi+\rho}$. Write Σ_v for the sum over the set of all v in $W(\mathfrak{h})$ such that $v(\rho)-\rho$ is L-dominant. Then the $\Theta(C,\lambda)$ are just holomorphic characters and the above formulae reduce to

$$\sum_{p\geqq 0}(-1)^p\, \Theta(H^p(G/H,\mathbb{V})) = \frac{(-1)^{\dim K/T}}{|W_L|} \sum_{v} \det(v)\; \Theta(v(\chi+\rho)) \quad \text{and}$$

$$\sum_{p\geqq 0}(-1)^p \Theta_K(H^p(G/H,\mathbb{V})) = \sum_{v} \det(v)\Theta_K(v(\chi+\rho)) =$$

$$\sum_{v} \det(v) \sum_{n_i\geqq 0} \frac{\sum_{w\in W(K)} \det(w)\; e^{w\left(v(\chi+\rho) - \rho + \rho_K - \Sigma n_i\beta_i\right)}}{\prod_{\alpha>0 \text{ compact}} (e^{\alpha/2} - e^{-\alpha/2})}$$

We now make the working assumption that G/H is symmetric (so, in particular, rank K = rank G here), that $\mathbb{V} \to G/H$ is hermitian (i.e. ψ is unitary), and that $\pi: G/H \to K/L$ is holomorphic. Then it is easy, at least when V is finite dimensional, to follow equation (*) on Plate V, K-type by K-type, and the character theory described in Plate VI is explicit. In any case, one can follow square integrability through the spectral sequence, and the notion of "harmonic" on G/H becomes transparent. Thus, under certain negativity conditions on V which I will describe in a moment, we carry out the program described in Plate IV and produce irreducible, possibly singular, unitary representations in a uniform geometric manner.

Write $A^p(G/H,\mathbb{V})$ for the space of $\mathbb{V}$-valued C^∞ (0,p)-forms on G/H. We call a form $\phi \in A^p(G/H,\mathbb{V})$ horizontal if it is horizontal relative to $G/H \to K/L$, i.e. if $\phi(KM) \subset V \otimes \Lambda^p(\mathfrak{k}\cap\mathfrak{n}_-)^*$.

Theorem. A form $\omega \in A^s(K/L,\mathbb{H}^0(M/L,\mathbb{V}))_K$ is a harmonic on K/L if and only if the corresponding ((*) in Plate V) horizontal form $\phi \in A^s(G/H,\mathbb{V})$ is harmonic relative to the invariant indefinite Kaehler metric on G/H.

Since G/H is symmetric, H is the fixed point set of an involutive automorphism τ of G, and τ commutes with the Cartan involution θ defining K because $\theta(H) = H$. So the group M and its Lie algebra $\mathfrak{m}_0$, in #1 in the description (Plate V) of the fibration and spectral sequence, are the respective fixed point sets of $\theta\tau$ on G and $\mathfrak{g}_0$. Let $\{\gamma_i\}$ be the maximal roots of the noncompact simple factors of $\mathfrak{m}_0$. Consider the

L_2 *Condition*: if $\nu\in\hat{L}$ and $V_\nu \neq 0$ then $(\nu+\rho_M,\gamma_i) < 0$ for all i.

Theorem. If the L_2 condition holds, then every class $c \in H^s(G/H,\mathbb{V})_K$ has a unique horizontal L_2 harmonic representative.

Notes. Modulo tensor factors of V corresponding to subgroups of H that act trivially on G/H, the L_2 condition forces V to have a highest L-type. If V has a highest L-type, say χ, then the L_2 condition reduces to: $(\chi+\rho_M, \gamma_i) < 0$ for all i.

Theorem. Suppose that, whenever $\nu\in\hat{L}$ and $V_\nu \neq 0$, we have

$$(**)\qquad 2(\nu+\rho_M, \gamma_i)/(\gamma_i,\gamma_i) \leqq -1 \quad \text{for all } i.$$

Then $(-1)^s\langle\, ,\, \rangle_{G/H}$ is positive semidefinite on $\mathcal{H}_2^s(G/H,\mathbb{V})$, its null space on $\mathcal{H}_2^s(G/H,\mathbb{V})$ coincides with the kernel of the natural map $\mathcal{H}_2^s(G/H,\mathbb{V}) \to H^s(G/H,\mathbb{V})$, and that natural map is surjective on the K-finite level. In consequence, G acts on $\mathcal{H}_2^s(G/H,\mathbb{V})/(\text{kernel}\langle\, ,\, \rangle_{G/H})$ by a unitary representation, and this unitarizes the Fréchet representation of G on $H^s(G/H,\mathbb{V})$.

Notes. If G, or even just M, is a linear group, or if V is finite dimensional, then (**) reduces to the L_2 condition. In any case one can get an almost identical but slightly less geometric result, with ≤ -1 in (**) weakened to $\leq -\frac{1}{2}$.

Theorem: If V is finite dimensional, and π_V is the unitary representation constructed just above, then π_V is irreducible and its characters are explicitly described in Plate VI.

Let me indicate how this works in a very special case, the ladder representations of the conformal group. So G is the double cover of U(2,2), H is the subgroup of G that covers $U(1)\times U(1,2)$, $K/L = \{U(2)\times U(2)\}/\{U(1)\times U(1)\times U(2)\} = U(2)/U(1)\times U(1)$ is the complex projective line (Riemann sphere), $s = 1$, and by using the theorems just above with various negative holomorphic line bundles $\mathbb{V}\to G/H$ we obtain all but one of the ladder representations. See (8), Section 13. The ladder representation not obtained this way is the one with a 1-dimensional K-type. It is the very singular representation of SO(2,4) that remains irreducible on SO(1,4), and is associated to a nilpotent coadjoint orbit rather than an elliptic orbit. See (9) and (1).

Finally, let us return to quantum electrodynamics, specifically to the photon. Fronsdal and others have studied the notion Gupta-Bleuler Triple in the context of QED. That is an indecomposable but reducible representation $0 = X_0 \subset X_1 \subset X_2 \subset X_3 = X$ where X is an indefinite-unitary representation space, X_1 is a totally isotropic invariant subspace, and X_2 is the orthogonal of X_1. See (2) and (11). The scalar product on X pairs the representations X/X_2 and X_1, which turn out to be unitary in the cases described by Fronsdal. Thus, in the photon case, X/X_2 gives the scalar photon, X_2/X_1 gives the transverse photon, and X_1 gives the longitudinal photon. In the material I describe here, the setting is really different but the spirit is similar. Thus the analog of X is the weak harmonic space

$$\{\phi \in L_2^s(G/H,\mathbb{V}) : \square\phi = 0 \text{ in the sense of distributions}\},$$

the analog of X_2 is the harmonic space $\mathcal{H}_2^s(G/H,\mathbb{V})$ studied here, and the analog of X_1 consists of the $\bar\partial$-exact forms in $\mathcal{H}_2^s(G/H,\mathbb{V})$. The quotient representations I have described then correspond to the quantization of the transverse photon, the other two pieces to the scalar and longitudinal photon. But so far there is no general argument in the holomorphic setting that the invariant indefinite global hermitian form is nondegenerate on the weak harmonic space.

REFERENCES

(1) Blattner,R.J. and Wolf,J.A., "Explicit quantization of the Kepler manifold," Proc. Amer. Math. Soc., Vol. 77 (1979) pp.145-149.

(2) Fronsdal,C. "Semisimple gauge theories and conformal gravity," Lectures in Applied Mathematics, Proceedings of the 1982 Summer Seminar, American Mathematical Society, to appear.

(3) Harish-Chandra, "Discrete series for semisimple Lie groups, I," Acta Math., Vol. 113 (1965), pp.241-317.

(4) Harish-Chandra, "Discrete series for semisimple Lie groups, II," Acta Math., Vol. 116 (1966), pp.1-111.

(5) Harish-Chandra, "Harmonic analysis on semisimple Lie groups," Bull. Amer. Math. Soc., Vol. 76 (1970), pp.529-551.

(6) Hecht,H., "The characters of some representations of Harish-Chandra," Math. Ann., Vol. 219 (1976), pp.213-226.

(7) Mostow,G.D., "Some new decomposition theorems for semisimple Lie groups," Mem. Amer. Math. Soc., Vol. 14 (1955), pp.31-54.

(8) Rawnsley,J., Schmid,W. and Wolf,J.A., "Singular unitary representations and indefinite harmonic theory," Jour. Funct. Anal., Vol. 48 (1983), to appear.

(9) Sternberg,S. and Wolf,J.A., "Hermitian Lie algebras and metaplectic representations," Trans. Amer. Math. Soc., Vol. 238 (1978), pp.1-43.

(10) Wolf,J.A., "The action of a real semisimple Lie group on a complex flag manifold, II: Unitary representations on partially holomorphic cohomology spaces," Mem. Amer. Math. Soc., Vol. 138 (1974).

(11) Wolf,J.A., "Indefinite harmonic theory and unitary representations," Lectures in Applied Mathematics, Proceedings of the 1982 Summer Seminar, American Mathematical Society, to appear.