Geometric construction of singular unitary representations of reductive Lie groups

W. Schmid and J. A. Wolf

1. Introduction

Let G be a connected reductive Lie group. The representations of G involved in the decomposition of $L_2(G)$ are the $\mathrm{Ind}_P^G(\eta \otimes \alpha)$ where $P = MAN$ is a cuspidal parabolic subgroup in Langlands' decomposition, η is a discrete series representation of M, and α is a unitary character on A. The difficult part of the construction is that of the discrete series representation η. Other irreducible unitary representations of G exist, say on $L_2(G/\Gamma)$ where Γ is a discrete subgroup, and their construction usually comes about by some sort of continuation of η (as in the 'analytic continuation' of the holomorphic discrete series) or of α (as in the various 'complementary series'). The representations $\mathrm{Ind}_P^G(\eta \otimes \alpha)$ can be realized geometrically, e.g. on partially holomorphic cohomology spaces, and there the difficult part is the realization of η as the action of M on some space $\mathcal{H}_2^q(M/U, \mathbf{E})$ of L_2 harmonic forms on a flag domain M/U, with values in an appropriate vector bundle $\mathbf{E} \to M/U$. This naturally leads to the question of realizing 'continued discrete series' representations in a similar manner. That same question arises in at least three other contexts: intrinsic realization of the ladder representation of the groups $U(k, l)$, intrinsic construction of the Penrose Inner Product in twistor theory, and unitarization of Zuckerman's derived functor representations. Here we sketch a partial answer to that problem.

Let G/H be an indefinite-Kaehler semisimple symmetric space. In other words, $H = G^\tau$ fixed point set of an involutive automorphism τ of G, H contains the centre of G, and G/H has a G-invariant complex structure. Fix an irreducible unitary representation ψ of H, say on a Hilbert space V, and let $\mathbf{V} \to G/H$ denote the corresponding holomorphic vector bundle. We define spaces

$\mathcal{H}_2^q(G/H; \mathbf{V})$: L_2 harmonic $\mathbf{V}$-valued $(0, q)$-forms on G/H

on which G acts naturally, preserving an hermitian form $\langle\ ,\ \rangle$ which is the integral of the pointwise inner products of $\mathbf{V}$-valued forms. Let s be the dimension of the maximal compact subvarieties of G/H. Under a certain negativity condition on $\mathbf{V} \to G/H$, we prove

(1) $(-1)^s\langle\ ,\ \rangle$ is positive semidefinite on $\mathcal{H}_2^s(G/H, \mathbf{V})$,
(2) the nullspace of $\langle\ ,\ \rangle$ on $\mathcal{H}_2^s(G/H, \mathbf{V})$ is the kernel of the natural map $\phi \mapsto [\phi]$ to Dolbeault cohomology,
(3) the K-finite subspace (K a maximal compactly embedded subgroup of G) $\mathcal{H}_2^s(G/H, \mathbf{V})_K$ maps onto the K-finite subspace of the Dolbeault cohomology.

Thus, we have a unitary representation π_V of G on $\mathcal{H}_2^s(G/H, \mathbf{V})/(\text{kernel of } \langle\ ,\ \rangle)$, and π_V unitarizes the Dolbeault space $H^s(G/H, \mathbf{V})$. Furthermore, we work out the K-spectrum of $H^s(G/H, \mathbf{V})$, and thus are able to compute the character of π_V. In the very special case

$$G/H = U(k, l)/U(1) \times U(k-1, l) \quad \text{and} \quad \dim V = 1$$

we realize the ladder representation intrinsically. The even more special case $k = l = 2$ gives an intrinsic formulation of the Penrose Inner Product.

In 1979, Rawnsley and Wolf carried this out by direct computation for $U(2, 1)/U(1) \times U(1, 1)$ and a line bundle. They also made some progress on $U(n, 1)/U(1) \times U(n-1, 1)$. Then Schmid and Wolf reworked and extended that calculation, in Lie algebra terms, by explicitly solving for the radial part of the 'harmonic' differential equations. Now we have a simpler geometric approach, which we will try to indicate here.

2. The spaces and the fibration

G is a connected reductive Lie group and G/H is an indefinite-Kaehler semisimple symmetric space. In other words, $\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0$ under the involution τ that defines H, and $\mathfrak{h}_0$ has an element ζ such that $\operatorname{ad}(\zeta)$ is 0 on $\mathfrak{h}_0$ and has square -1 on $\mathfrak{q}_0$. Choose a Cartan involution θ of G that commutes with τ, and denote

$$K = G^\theta, \qquad M = G^{\tau\theta}, \qquad L = K \cap H = K \cap M = M \cap H.$$

Then K is a maximal compactly embedded subgroup of G, as is L in M and H. The corresponding Cartan decompositions are

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \quad \mathfrak{m}_0 = \mathfrak{l}_0 + (\mathfrak{p}_0 \cap \mathfrak{q}_0), \quad \mathfrak{h}_0 = \mathfrak{l}_0 + (\mathfrak{p}_0 \cap \mathfrak{h}_0).$$

An old result of Mostow gives a diffeomorphism $K \times (\mathfrak{p}_0 \cap \mathfrak{q}_0) \times (\mathfrak{p}_0 \cap \mathfrak{h}_0) \to G$ by $(k, \xi, \eta) \to k \cdot \exp(\xi) \cdot \exp(\eta)$. First that says $G/H = KM(x_0)$ where $x_0 = 1 \cdot H \in G/H$, and second it results in a K-equivariant fibration

$$\pi : G/H \to K/L \qquad \text{with structure group } L$$

of G/H over its maximal compact subvariety K/L. An example: if $G/H = U(1, n+1)/U(1, n)\times U(1)$, open set in complex projective space $P^{n+1}(\mathbb{C})$, then K/L is a hyperplane $U(n+1)/U(n)\times U(1) = P^n(\mathbb{C})$, so π is a fibration (in this case, holomorphic) of G/H over a hyperplane that sits inside it as a sort of equator.

We order the roots relative to a Cartan subalgebra contained in $\mathfrak{l}$, so that $\mathfrak{q} = (\mathfrak{q}_0)_{\mathbb{C}} = \mathfrak{q}_+ + \mathfrak{q}_-$ in such a manner that $\mathfrak{q}_+$, which is the sum of the positive root spaces in $\mathfrak{q}$, represents the holomorphic tangent space of G/H.

3. The spectral sequence and the K-types

Fix an irreducible unitary representation ψ of H on a Hilbert space V. Let $\mathbf{V} \to G/H$ denote the associated hermitian holomorphic homogeneous vector bundle. For simplicity, assume that $\pi : G/H \to K/L$ is holomorphic. Then there is a holomorphic vector bundle $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$, typical fibre $H^0(M/L, \mathbf{V})$ and structure group L, and an analog of the Leray Spectral Sequence, as follows

$$E_0^{p,q} = A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V})) \quad \text{and} \quad d_0 = (-1)^p \bar\partial_{M/L},$$
$$E_1^{p,q} = 0 \text{ for } q>0,\ E_1^{p,0} = A^p(K/L, \mathbf{H}^0(M/L, \mathbf{V})) \quad \text{and} \quad d_1 = \bar\partial_{K/L},$$
$$H^p(G/H, \mathbf{V}) = E_2^{p,0} = H^p(K/L, \mathbf{H}^0(M/L, \mathbf{V})).$$

This depends on detailed computations within the Lie algebras and is specific to our situation. The point is that, by means of Bott–Borel–Weil, it lets us approach an understanding of the K-spectrum of the Dolbeault cohomologies $H^p(G/H, \mathbf{V})$ in terms of the L-spectrum of V. That, in turn, gives us the K-spectrum of the unitary representation to whose construction we now turn.

4. Harmonic forms on G/H

The Killing form of $\mathfrak{g}_0$ specifies a G-invariant indefinite-Kaehler metric on G/H. As in the positive definite situation, the metric specifies a Kodaira–Hodge orthocomplementation on $\mathbf{V}$-values (p, q)-forms

$$\# : E^{p,q}(G/H, \mathbf{V}) \to E^{n-p,n-q}(G/H, \mathbf{V}^*)$$

where $n = \dim_{\mathbb{C}} G/H$ and $\mathbf{V}^* \to G/H$ is the bundle dual to $\mathbf{V} \to G/H$.

Then

$$\bar{\partial}: E^{p,q}(G/H, \mathbf{V}) \to E^{p,q+1}(G/H, \mathbf{V})$$

has formal adjoint

$$\bar{\partial}^* = -\#\bar{\partial}\#: E^{p,q+1}(G/H, \mathbf{V}) \to E^{p,q}(G/H, \mathbf{V})$$

relative to the global G-invariant, generally indefinite, hermitian form

$$\langle \phi, \phi' \rangle = \int_{G/H} \phi \bar{\wedge} \# \phi'.$$

Here $\bar{\wedge}$ denotes exterior product followed by contraction of V against V^*.

We say that $\phi \in E^{p,q}(G/H, \mathbf{V})$ is *harmonic* if $\bar{\partial}\phi = 0$ and $\bar{\partial}^*\phi = 0$. Of course this implies

$$\square\phi = 0 \quad \text{where} \quad \square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

but the converse fails because $\langle\ ,\ \rangle$ is not definite.

By computation we compare $\bar{\partial}^*$ on G/H and $\bar{\partial}^*$ on K/L. This works well for $(0, s)$-forms when π is holomorphic. The result: if $\pi: G/H \to K/L$ is holomorphic, then every K-finite class in $H^s(G/H, \mathbf{V})$ has a unique K-finite harmonic representative $\phi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^*$ such that $\phi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$.

5. Square integrability

In addition to the G-invariant indefinite Kaehler metric, G/H carries a K-invariant positive definite hermitian metric given on $\mathfrak{q}$ by $((\xi, \eta)) = -B(\xi, \theta\bar{\eta})$ where B is the Killing form. G distorts it by

$$\|dt_x(\xi_{km})\|^2 = -B(\mathrm{Ad}\,(h)\xi, \theta \cdot \mathrm{Ad}\,(h)\bar{\xi})$$

where $t_z: gH \mapsto zgH$ denotes translation on G/H, $\xi \in \mathfrak{q} = T_{x_0}(G/H)_{\mathbb{C}}$ gives $\xi_{km} = dt_k\, dt_m \xi$ at $km \cdot H$, $k \in K$, $m \in M$, $x \in G$ and $xkm \in KMh$.

A basic fact: if $x \in G$ then dt_x is uniformly bounded on the tangent spaces of G/H, with bound continuous in x. Thus the natural action of G on forms specifies continuous representations by bounded transformations on the Hilbert spaces

$L_2^{p,q}(G/H, \mathbf{V})$: completion of the space of compactly supported forms in $E^{p,q}(G/H, \mathbf{V})$ relative to the positive definite inner product.

The invariant indefinite hermitian form $\langle\ ,\ \rangle$ is continuous on $L_2^{p,q}(G/H, \mathbf{V})$.

Tracing measures through the fibration $\pi: G/H \to K/L$, when π is holomorphic one sees that the 'L_2 condition'

if $\nu \in \hat{L}$ with $V_\nu \neq 0$ then $(\nu+\rho_M, \gamma)<0$ for all roots γ of $\mathfrak{p} \cap \mathfrak{q}_+$

implies

if $\phi \in A^s(G/H, \mathbf{V})$ is K-finite and $\bar{\partial}$-closed, and satisfies $\phi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)$, then $\phi \in L_2^{0,s}(G/H, \mathbf{V})$.

Combining that with the result described in Section 4, we have

Theorem. *If $\pi: G/H \to K/L$ is holomorphic and every (nonzero) L-type V_ν satisfies $(\nu+\rho_M, \gamma)<0$ for all roots γ of $\mathfrak{p} \cap \mathfrak{q}_+$, then every K-finite class in $H^s(G/H, \mathbf{V})$ is represented uniquely by a form $\phi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^*$ such that: $\phi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$, ϕ is harmonic relative to the invariant metric, and ϕ is L_2 relative to the positive definite metric.*

In view of this it is convenient to denote the L_2 harmonic space $\mathcal{H}_2^s(G/H, \mathbf{V}) = \{\phi \in A^s(G/H, \mathbf{V})$: ϕ is harmonic and $L_2\}$ and its 'special' subspace $\mathcal{S}(G/H, \mathbf{V}) = \{\phi \in \mathcal{H}_2^s(G/H, \mathbf{V})$: $\phi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)\}$. According to the theorem just stated, the natural map of a form to its Dolbeault class gives an isomorphism $\mathcal{S}(G/H, \mathbf{V})_K \cong H^s(G/H, \mathbf{V})_K$ on the K-finite subspaces.

6. The unitary representations

The L_2 harmonic space $\mathcal{H}_2^s(G/H, \mathbf{V})$ is a G-module on which the invariant inner product is usually indefinite and the positive definite inner product is not invariant. If $\pi: G/H \to K/L$ is holomorphic and every nonzero L-type V_ν satisfies

$2(\nu+\rho, \gamma)/(\gamma, \gamma) \leq -1$ for all roots γ of $\mathfrak{p} \cap \mathfrak{q}_+$

then every element ϕ in the kernel of the map $\mathcal{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V})$ to Dolbeault cohomology is in the kernel of the invariant bilinear form $\langle\ ,\ \rangle$ on $\mathcal{H}_2^s(G/H, \mathbf{V})$. Since $(-1)^s\langle\ ,\ \rangle$ is positive definite on $\mathcal{S}(G/H, \mathbf{V})$ we can prove

Theorem. *If $\pi: G/H \to K/L$ is holomorphic and every nonzero L-type satisfies $2(\nu+\rho, \gamma)/(\gamma, \gamma) \leq -1$ for all roots γ of $\mathfrak{p} \cap \mathfrak{q}_+$, then*

(1) *$(-1)^s\langle\ ,\ \rangle$ is positive semidefinite on $\mathcal{H}_2(G/H, \mathbf{V})$ and its nullspace there is the kernel of the natural map to Dolbeault cohomology,*

(2) *the natural action of G on $\mathbf{V}$-valued forms induces a unitary representation $\pi_{\mathbf{V}}$ of G on the 'reduced L_2 harmonic space'*

$\bar{\mathcal{H}}_2^s(G/H, \mathbf{V}) = \mathcal{H}_2^s(G/H, \mathbf{V})/(\textit{kernel of } \langle\ ,\ \rangle)$, *inner product from* $(-1)^s\langle\ ,\ \rangle$, *and*

(3) π_V *unitarizes the action of G on the Dolbeault cohomology space* $H^s(G/H, \mathbf{V})$.

The key argument here, that $[\phi]=0$ inside $H^s(G/H, \mathbf{V})_K$ implies $\langle\phi, \mathcal{H}_2^s(G/H, \mathbf{V})\rangle=0$, comes down to finding K-finite forms $\eta_i \in L_2^{0,s-1}(G/H, \mathbf{V})$ such that $\bar{\partial}\eta_i \to \phi$ in L_2. That depends on the fact that the fibres of $\pi: G/H \to K/L$ are bounded symmetric domains.

There is a slightly weaker result which uses part of $\mathcal{H}_2^s(G/H, \mathbf{V})$ in case the inequalities $2(\nu+\rho, \gamma)/(\gamma, \gamma) \leq -1$ are relaxed a little.

Theorem. *If* $\pi: G/H \to K/L$ *is holomorphic and every nonzero L-type satisfies* $2(\nu+\rho_M, \gamma)/(\gamma, \gamma) < -\frac{1}{2}$ *for all roots of* $\mathfrak{p} \cap \mathfrak{q}_+$, *then*

(1) $(-1)^s\langle\ ,\ \rangle$ *is positive semidefinite on the subspace* $\mathcal{U}(\mathfrak{g}) \cdot \mathcal{S}(G/H, \mathbf{V})$ *of* $\mathcal{H}_2^s(G/H, \mathbf{V})$ *and its nullspace there is the kernel of the map to Dolbeault cohomology,*

(2) *G acts by a unitary representation* π_V *on the 'reduced special* L_2 *harmonic space'*

$$\bar{\mathcal{S}}(G/H, V) = \mathcal{U}(\mathfrak{g}) \cdot \mathcal{S}(G/H, \mathbf{V})/(\textit{kernel of } \langle\ ,\ \rangle)$$

with inner product from $(-1)^s\langle\ ,\ \rangle$,

(3) *under the stronger conditions* $2(\nu+\rho, \gamma)/(\gamma, \gamma) \leq -1$ *the two representations* π_V *agree, and*

(4) π_V *unitarizes the action of G on* $H^s(G/H, \mathbf{V})$.

If G is linear or V is finite dimensional, then ν is integral so the above inequalities coincide with the L_2 condition: $(\nu+\rho_M, \gamma)<0$ for all roots γ of $\mathfrak{p} \cap \mathfrak{q}_+$.

7. K-spectrum and characters on Dolbeault cohomology

If λ is a K-dominant-regular weight, then there is an irreducible K-module $W_{\lambda-\rho}$ of highest weight $\lambda-\rho$, and one has the corresponding homogeneous holomorphic bundle $\mathbf{W}_{\lambda-\rho} \to G/K$. The Fréchet G-module $H^0(G/K, \mathbf{W}_{\lambda-\rho})$ is in the 'continuation of the holomorphic discrete series'. Its subspace of K-finite vectors, $H^0(G/K, \mathbf{W}_{\lambda-\rho})_K$, has finite composition series, has infinitesimal character of Harish–Chandra parameter λ, has K-module structure $W_{\lambda-\rho} \otimes S(\mathfrak{p}_-)$, and is T-finite with highest weight $\lambda-\rho$. So G carries a distribution

$$\theta(\lambda): \text{global character of } G \text{ on } H^0(G/K, \mathbf{W}_{\lambda-\rho}).$$

Hecht worked out explicit formulae for those $\theta(\lambda)$. We extend their definition to all λ in the weight lattice, by

$$\theta(\lambda)=\begin{cases}0 & \text{if } \lambda \text{ is } W_K\text{-singular,}\\ \varepsilon(w)\theta(w\lambda) & \text{if } w\in W_K \text{ and } w\lambda \text{ is } K \text{ dominant regular.}\end{cases}$$

Similarly we extend

$$\theta_K(\lambda)\text{: formal character of } K \text{ on } H^0(G/K,\mathbf{W}_{\lambda-\rho})_K$$

by W_K-antisymmetry to arbitrary weights λ.

Theorem. *Let $\pi: G/H\to K/L$ be holomorphic,* $\dim V<\infty$, *and $\chi=\psi\mid L$. Then for every integer $p\geq 0$, the K-finite Dolbeault cohomology space $H^p(G/H,\mathbf{V})_K$ is a Harish-Chandra module of finite length which is T-finite with weights bounded from above and which has infinitesimal character $\chi+\rho$. Let $W'=\{v\in W_{H_{\mathbb{C}}}: v\rho-\rho$ is L-dominant$\}$. Then the global characters $\theta(H^p(G/H,\mathbf{V}))$ exist and satisfy*

$$\sum_{p\geq 0}(-1)^p\theta(H^p(G/H,\mathbf{V}))=\sum_{v\in W'}\varepsilon(v)\theta(v(\chi+\rho)).$$

The formal K-characters $\theta_K(H^p(G/H,\mathbf{V}))$ also exist; they satisfy the K-analog of the alternating sum formula above, and also

$$\sum_{p\geq 0}(-1)^p\theta_K(H^p(G/H,\mathbf{V}))$$
$$=\sum_{v\in W'}\varepsilon(v)\sum_{n_1,\ldots,n_t=0}^{\infty}\frac{\sum_{w\in W_K}\varepsilon(w)e^{w(v(\chi+\rho)-\rho+\rho_K-n_1\beta_1-\cdots-n_t\beta_t)}}{\prod_{\alpha\in\Phi(\mathfrak{k})^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

where $\{\beta_1,\ldots,\beta_t\}$ is an enumeration of the roots of $\mathfrak{p}_+$.

If one has $H^p(G/H,\mathbf{V})=0$ for $p\neq s$, then the theorem gives precise formulae for the global and K-characters of $H^s(G/H,\mathbf{V})$.

8. K-spectrum and characters of the unitary representations

Suppose that $\pi: G/H\to K/L$ is holomorphic, $\dim V<\infty$, and $\chi=\psi\mid L$ satisfies the L_2 conditions: $(\chi+\rho_M,\gamma)<0$ for all roots γ of $\mathfrak{p}\cap\mathfrak{q}_+$. From $\dim V<\infty$, χ is integral in M, so in fact we have the condition

$$2(\chi+\rho_M,\gamma)/(\gamma,\gamma)\leq -1,\qquad \text{all roots } \gamma \text{ of } \mathfrak{p}\cap\mathfrak{q}_+$$

which provides a unitary representation π_V of G as described in Section 6. The L_2 condition also allows us to verify the vanishing

$$H^p(G/H,\mathbf{V})=0 \quad \text{for} \quad p\neq s.$$

Thus the characters of π_V are given precisely by the theorem stated in Section 7, and we identify the π_V. For example, in the case $U(k, l)/U(1)\times U(k-1, l)$ with dim $V=1$, the π_V and their duals give, among other things, the ladder representations of $U(k, l)$.

Details will appear in a paper of John Rawnsley and ourselves.

W. SCHMID
Department of Mathematics
Harvard University
Cambridge, Mass. 02138
USA

J. WOLF
Department of Mathematics
University of California
Berkeley, California 94720
USA

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