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Classes of unitarizable derived functor modules

(semisimple Lie group/unitary representation/continuous cohomology/coherent continuation)

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ABSTRACT For a real semisimple Lie group G, the description of the unitary dual remains an elusive question. One of the difficulties has been the lack of technique for constructing unitary representations. Unitary induction from parabolic subgroups of G yields unitary representations by the very definition of these representations. However, not all unitary irreducible representations of G are obtained by this type of induction. In addition, we need derived functor parabolic induction [cf. Vogan, D. (1981) *Representations of Real Reductive Lie Groups* (Birkhäuser, Boston)] to describe all irreducible representations of G. For this second type of induction, the obvious analogues from parabolic subgroup induction regarding unitarity are false. In this announcement, we describe a setting where derived functor parabolic induction yields unitary representations of G. These results include proofs of unitarity for some of the representations conjectured to be unitary by Vogan and Zuckerman [(1983) *Invent. Math.*, in press] and also proofs of unitarity for some which lie outside the domain described in those conjectures.

1. Introduction

The main results are described in *Section 4* following the introduction of θ-stable parabolic subalgebras and the associated generalized Verma modules (*Section 2*) and a brief description of the derived functors introduced by Zuckerman (*Section 3*). These results include a proof of unitarity of Zuckerman's modules when the parabolic subalgebra has the property that the compact and noncompact parts of the nilradical commute with each other. These parabolic subalgebras will be called quasi-abelian.

The remaining three sections describe applications of the results in *Section 4*. In *Section 5*, for each orthogonal group $SO(p, q)$ with $p + q$ even, we define a unique representation that is multiplicity free as a $\mathfrak{k}$-module and has $\mathfrak{k}$-highest weights lying along a single line. We call these representations ladder representations of $SO(p, q)$. It is a result of Howe and of Vogan (1) that no such ladder representations exist if $p + q$ is odd and $p, q \geq 4$.

In *Section 6*, we observe that the simply connected covering group of $SL(2n, \mathbf{R})$ has a unique maximal parabolic subalgebra that is θ-stable. Applying the results of *Section 4* to this parabolic, we obtain a set of unitary representations. These include the family of unitary representations $I(k)$, $k \in \mathbf{N}$ of Speh (2), which she constructed by analyzing certain poles of Eisenstein series associated to automorphic forms. When we apply reduction techniques to subgroups (3), these modules $I(k)$ prove Zuckerman's conjecture for $SL(n, \mathbf{R})$. The results of *Section 4* prove, in a similar way, Zuckerman's conjecture for $SU^*(2n)$.

In ref. 4, Wallach studied the analytic continuation of the holomorphic discrete series representations having a one-dimensional cyclic $\mathfrak{k}$-module. In *Section 7*, we apply the results of *Section 4* and those of Jantzen (5) to prove analogous results for certain discrete series representations of $\mathfrak{g}$ with $(\mathfrak{g}, \mathfrak{k})$ not Hermitian symmetric. The results here prove unitarity for certain coherent continuations of discrete series representations out of the Borel de Siebenthal Weyl chamber. These results are given in Appendix 1.

There is a vast literature on various techniques for proving the unitarity of certain representations of G. Vogan and Zuckerman (3) have shown that all representations "having" nonzero continuous cohomology are Zuckerman representations. Thus these representations are a particularly important class of representations and their unitarity has been investigated many times (cf. refs. 2, 6–10). To date, the main success has been in the cases where the representations are of holomorphic type (6, 9, 11, 12), where they can be related to Howe's theory of dual pairs (13), or where they can be related to automorphic forms (2).

Interesting classes of unitary representations have been constructed recently by Flensted-Jensen (14) and Schlichtkrull (15). These representations are obtained by analytic methods by decomposing $L^2(G/H)$ for H the fixed points of an involution of G.

Unitary representations have been constructed in many cases by geometric methods. In particular, the recent work of Rawnsley *et al.* (7) develops a theory of L^2 cohomology based on harmonic forms for indefinite Kaehler metrics to unitarize representations on Dolbeault cohomology in a number of cases.

The direct algebraic approach to proving unitarizability for $\mathfrak{g}$-modules other than highest weight modules has been used in only a few cases: Parthasarathy's work on the discrete series (16), Vogan's work on representations associated to the minimal coadjoint orbit (1), and Enright's work comparing representations of Hermitian symmetric pairs and complex Lie groups (17). It is this direct approach that yields the results described here.

2. θ-stable parabolic subalgebras

Let G be a connected, simply connected semisimple Lie group and let K be a maximal connected subgroup of G whose image in G/center G is compact. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras of G and K. Denote by θ the Cartan involution of $\mathfrak{g}_0$ giving the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Choose a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and let $\mathfrak{h}_0$ be the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. Then $\mathfrak{h}_0$ is a fundamental Cartan subalgebra of $\mathfrak{g}_0$. Let the complexification of a space be denoted by deleting the subscript 0. This gives $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ and Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{h}$.

Let $\mathfrak{q}$ denote a θ-stable parabolic subalgebra of $\mathfrak{g}$ with decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$. Assume that $\mathfrak{h}$ is contained in $\mathfrak{m}$. Since $\mathfrak{q}$ is θ-stable, so is $\mathfrak{u}$. Also, since $\mathfrak{t}$ is fixed by θ, so is $\mathfrak{h}$ and then so is $\mathfrak{m}$. For any θ-stable vector space, let subscripts c and n denote the $+1$ and -1 eigenspaces

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for θ. For example, $\mathfrak{u}_c = \mathfrak{u} \cap \mathfrak{k}$, $\mathfrak{u}_n = \mathfrak{u} \cap \mathfrak{p}$, giving the decomposition $\mathfrak{u} = \mathfrak{u}_c \oplus \mathfrak{u}_n$ and, similarly, $\mathfrak{m} = \mathfrak{m}_c \oplus \mathfrak{m}_n$. Most of the results described here involve a special class of θ-stable parabolic subalgebras $\mathfrak{q}$ defined by the property: $[\mathfrak{u}_c, \mathfrak{u}_n] = 0$. We call these θ-stable parabolic subalgebras *quasi-abelian*.

For any ad($\mathfrak{h}$) stable subspace of $\mathfrak{g}$, let $\Delta(E)$ denote the roots that occur in the root-space decomposition of E. If E is ad($\mathfrak{t}$)-stable but not ad($\mathfrak{h}$)-stable, let $\Delta(E)$ denote the nonzero $\mathfrak{t}$-weights that occur in E. Let $\Delta = \Delta(\mathfrak{g})$ be the set of roots of $\mathfrak{g}$ and fix a θ-stable positive system of roots Δ^+. For any ad($\mathfrak{h}$)-stable E, let $\Delta^+(E) = \Delta^+ \cap \Delta(E)$. Let $\mathfrak{b}$ be the Borel subalgebra corresponding to Δ^+; i.e., $\mathfrak{b} = \mathfrak{h} \oplus \Sigma_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Then, $\mathfrak{b}$ is θ-stable and $\mathfrak{b}_c = \mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. Let $\Delta^+(\mathfrak{k})$ be the positive system of $\Delta(\mathfrak{k})$ corresponding to $\mathfrak{b}_c$. If E is ad($\mathfrak{t}$)-stable, put $\Delta^+(E) = \Delta^+(\mathfrak{k}) \cap \Delta(E)$.

If $\lambda \in \mathfrak{h}^*$ is a $\Delta^+(\mathfrak{m})$-dominant integral and $\mu \in \mathfrak{t}^*$ is $\Delta^+(\mathfrak{m}_c)$-dominant integral, let $F(\lambda)$ and $F(\mu)$ denote the irreducible finite-dimensional $\mathfrak{m}$ and $\mathfrak{m}_c$ modules having highest weights λ and μ. Define generalized Verma modules by

$$N(\lambda) = U(\mathfrak{g}) \underset{U(\mathfrak{q})}{\otimes} F(\lambda), \qquad N_c(\mu) = U(\mathfrak{k}) \underset{U(\mathfrak{q}_c)}{\otimes} F(\mu).$$

Let $L(\lambda)$ and $L_c(\mu)$ denote the unique irreducible quotients of $N(\lambda)$ and $N_c(\mu)$. There are many especially interesting cases where $\mathfrak{t} = \mathfrak{h}$ and, for this reason, we include the subscript c to distinguish $\mathfrak{g}$- and $\mathfrak{k}$-modules. For convenience, we single out two Weyl chambers associated to these generalized Verma modules. Let $\mathfrak{C}$ (resp $\mathfrak{C}_c$) be the closed Weyl chamber in $\mathfrak{h}^*$ (resp $\mathfrak{t}^*$) corresponding to the positive system $\Delta^+(\mathfrak{m}) \cup -\Delta(\mathfrak{u})$ [resp $\Delta^+(\mathfrak{m}_c) \cup -\Delta(\mathfrak{u}_c)$]. Let ρ (resp ρ_c) be half the sum of elements in Δ^+ [resp $\Delta^+(\mathfrak{k})$]. These Weyl chambers are distinguished by the following property:

$$\text{If } \lambda + \rho \in \mathfrak{C} \text{ (resp } \mu + \rho_c \in \mathfrak{C}_c\text{)}, \text{ then } N(\lambda) \text{ [resp } N_c(\mu)] \text{ is irreducible.} \qquad [2.1]$$

Let $\mathfrak{u}^-$ (resp $\mathfrak{u}_c^-$) be the sum of the root spaces $\mathfrak{g}_\alpha$ (resp $\mathfrak{k}_\beta$) with $-\alpha \in \Delta(\mathfrak{u})$ [resp $-\beta \in \Delta(\mathfrak{u}_c)$].

3. Derived functors of the $\mathfrak{k}$-finite functor

Here, we briefly describe the derived functors introduced by Zuckerman (13, 18). Let $\mathfrak{C}(\mathfrak{g},\mathfrak{m}_c)$ be the category of $\mathfrak{g}$-modules that as $\mathfrak{m}_c$-modules are $U(\mathfrak{m}_c)$-locally finite and completely reducible. For a $\mathfrak{g}$-module A in $\mathfrak{C}(\mathfrak{g},\mathfrak{m}_c)$ define ΓA to be the subspace of $U(\mathfrak{k})$-locally finite vectors. Γ is called the $\mathfrak{k}$-finite submodule functor. There are enough injective objects in this category to construct injective resolutions; $0 \to A \to I^* \to \cdots$. Now define $\Gamma^i A$ to the ith cohomology group of the complex: $0 \to \Gamma I^0 \to \Gamma I^1 \to \cdots \to \Gamma I^j \to \cdots$. The Γ^i are the right derived functors of Γ on $\mathfrak{C}(\mathfrak{g},\mathfrak{m}_c)$. For all i, $\Gamma^i A$ is a $U(\mathfrak{k})$-locally finite $\mathfrak{g}$-module. If it has finite $\mathfrak{k}$-multiplicities, it is infinitesimally equivalent to an admissible representation of G.

These right derived functors are quite computable on the category of $\mathfrak{k}$-modules that are $U(\mathfrak{m}_c)$-locally finite and completely reducible and, in fact, possess a duality property (18). These results combine to give the following vanishing theorem of Zuckerman.

THEOREM 3.1. *Let* $s = \dim \mathfrak{u}_c$. (i) *Assume that the generalized Verma module* $N(\lambda)$ *is irreducible. Then* $\Gamma^i N(\lambda)$ *is zero if* $i \neq s$. (ii) *Assume that* $L(\lambda)$ *is free as a module over* $U(\mathfrak{u}_c^-)$. *Then* $\Gamma^i L(\lambda)$ *is zero if* $i \neq s$.

We note that (*i*) follows from (*ii*).

In the next section, we study the unitarity of the $\mathfrak{g}$-modules $\Gamma^s N(\lambda)$ and $\Gamma^s L(\lambda)$.

4. The main results

A $U(\mathfrak{k})$-locally finite $\mathfrak{g}$-module will be called *unitarizable* (with respect to $\mathfrak{g}_0$) if it is infinitesimally equivalent to a unitary representation of G. The results of this section describe various sufficient conditions that imply that $\Gamma^s N(\lambda)$ and $\Gamma^s L(\lambda)$ are unitarizable.

Let $M \subset G$ be the connected subgroup of G with complexified Lie algebra equal to $\mathfrak{m}$. We call the $\mathfrak{m}$-module $F(\lambda)$ *unitarizable* if $F(\lambda)$ is a unitary representation of the simply connected covering group of M. If we write $\mathfrak{m}$ as a sum of simple ideals $\mathfrak{m} = \Sigma\, \mathfrak{m}_i$, then $F(\lambda)$ is isomorphic to a tensor product $F(\lambda) \simeq \bigotimes_i F_i$, where F_i is a finite-dimensional representation of $\mathfrak{m}_i$. Since all finite-dimensional representations of compact groups are unitarizable, $F(\lambda)$ is unitarizable precisely when F_i is a unitary one-dimensional representation for all i with $\mathfrak{m}_i$ not contained in $\mathfrak{k}$.

If $\mathfrak{k}$ is contained in $\mathfrak{m}$, then, since $\mathfrak{k}_0$ is a maximal subalgebra of $\mathfrak{g}_0$, $\mathfrak{k} = \mathfrak{m}$ and $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a Hermitian symmetric pair. In this case, $s = 0$ and $\Gamma^s N(\lambda) = N(\lambda)$ and we are considering the holomorphic discrete series and its analytic continuation. The unitarity results are known here (12) and, so, *we will assume throughout* that $\mathfrak{k} \cap \mathfrak{m} \neq \mathfrak{k}$.

Recall from *Section 2* the definition of quasi-abelian $\mathfrak{q}$. The first main result is the following:

THEOREM 4.1. *Assume that* $\mathfrak{q}$ *is quasi-abelian,* $F(\lambda)$ *is a unitarizable finite-dimensional* $\mathfrak{m}$*-module and* $\lambda + \rho$ *satisfies the inequalities*

$$Re\langle \lambda + \rho, \alpha \rangle \leq 0 \text{ for all } \alpha \in \Delta(\mathfrak{u}).$$

Then, $\Gamma^s N(\lambda)$ *is either zero or unitarizable.*

This result is implied by the following more general formulation.

THEOREM 4.2. *Let* $\xi \in \mathfrak{h}^*$ *and suppose that* $F(\xi)$ *is a one-dimensional unitarizable* $\mathfrak{m}$-module. *Assume that* (i) $\langle \xi, \alpha \rangle < 0$ *for all* $\alpha \in \Delta(\mathfrak{u})$ *and* (ii) $N(\lambda + t\xi)$ *is irreducible for* $t \geq 0$. *Also assume that* $\mathfrak{q}$ *is quasi-abelian,* $F(\lambda)$ *is a unitarizable finite-dimensional* $\mathfrak{m}$*-module, and* $\lambda|_{\mathfrak{t}} + \rho_c \in \mathfrak{C}_c$. *Then,* $\Gamma^s N(\lambda)$ *is either zero or unitarizable.*

A sharper result is proved if $[\mathfrak{u}_c, \mathfrak{u}] = 0$.

THEOREM 4.3. *Let* ξ *be as in* Theorem 4.2 *with* (i) *and* (ii) *satisfied. Assume that* $[\mathfrak{u}_c, \mathfrak{u}] = 0$ *and* $F(\lambda)$ *is a unitarizable finite-dimensional* $\mathfrak{m}$*-module. Then,* $\Gamma^s N(\lambda)$ *is either zero or unitarizable.*

This result can be rephrased loosely by saying that unitarity is preserved at least as far as the first reduction point for $N(\lambda + t\xi)$, $t \in \mathbf{R}$.

The sharpest results are available in the case where the nilradical $\mathfrak{u}$ is abelian. In this case, there is a real form $\mathfrak{g}'$ of $\mathfrak{g}$ with compactly embedded subalgebra $\mathfrak{k}'$ such that $(\mathfrak{g}', \mathfrak{k}')$ is a Hermitian symmetric pair and $\mathfrak{m}$ is the complexification of $\mathfrak{k}'$. Let G' be the simply connected, connected Lie group with Lie algebra $\mathfrak{g}'$. Since $N(\lambda)$ is $U(\mathfrak{m})$-locally finite, $N(\lambda)$ and $L(\lambda)$ are infinitesimally equivalent to representations of G'. If the representation of G' is unitarizable, we say that the corresponding $\mathfrak{g}$-module is unitarizable for G'. The unitarizable highest weight modules of $\mathfrak{g}$ are known (12) and, so, the following result yields a class of especially interesting unitarizable representations for G.

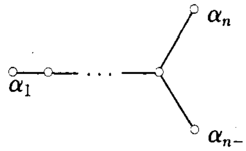
THEOREM 4.4. *Assume that the nilradical* $\mathfrak{u}$ *is abelian and* $F(\lambda)$ *is a unitarizable* $\mathfrak{m}$*-module. Assume that* $L(\lambda)$ *is a free module over* $U(\mathfrak{u}_c^-)$ *and is unitarizable for* G'. *Then,* $\Gamma^s L(\lambda)$ *is either zero or unitarizable for* G.

Note that, if $N(\lambda) = L(\lambda)$, then $L(\lambda)$ is free over $U(\mathfrak{u}_c^-)$. Also, if $\mathfrak{q}$ is quasi-abelian and $\lambda|_{\mathfrak{t}} + \rho_c \in \mathfrak{C}_c$, then $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module and, so, $L(\lambda)$ is free over $U(\mathfrak{u}_c^-)$.

5. Ladder representations for orthogonal groups $SO(p, q)$ with $p + q$ even

In this section, we describe the first application of the results of *Section 4*. We define a distinguished unitary representation for each orthogonal group $SO(p, q)$ with $p + q$ even. We call these representations ladder representations because their $\mathfrak{k}$-types have highest weights that lie on a single line in $\mathfrak{t}^*$.

Consider the Dynkin diagram D_n



with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n-1)$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. Let $\mathfrak{q}$ be the θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u}$ with $\Delta^+(\mathfrak{m})$ having simple roots $\alpha_2, \ldots, \alpha_n$. Assume that $n \ge 4$ and l is an integer, $2 \le l \le n-2$. Let all α_i be compact roots except α_l, which is noncompact. The corresponding real form of $\mathfrak{g}$ is $SO(2l, 2n - 2l)$. In the ε_i coordinates, let $\lambda = (z, 0, 0, \ldots, 0)$, $z \in \mathbf{R}$. From ref. 4, we know that $N(\lambda)$ is irreducible and unitary for G' (cf. *Theorem 4.4*) if and only if $z < -n + 2$. At $z = -n + 2$, $N(\lambda)$ is reducible. For convenience, set $\lambda = (-n + 2, 0, \ldots, 0)$, $L = L(\lambda)$, and $N = N(\lambda)$.

N is completely reducible as a $\mathfrak{k}$-module:

$$N \simeq \bigoplus_{a,b \in \mathbf{N}} N_c(-n + 2 - 2a - b, 0 \ldots 0, b, 0 \ldots 0),$$

with b occurring as the $l{+}1$st coordinate. In this case, L is the quotient of N by $N(\lambda - 2\varepsilon_1)$ and, so, we obtain the $\mathfrak{k}$-decomposition of L

$$L \simeq \bigoplus_{b \in \mathbf{N}} N_c(-n + 2 - \mathrm{b}, 0 \ldots 0, b, 0 \ldots 0).$$

Applying Γ^s to L gives the desired unitary representation by *Theorem 4.4*. In fact, the result is the same for $SO(p, q)$ with p and q either both even or both odd. For p and q odd, α_{n-1} and α_n are complex instead of compact.

THEOREM 5.1. *Let* $2n = p + q$, $l = [p/2]$ *and assume that* $n \ge 4$ *and* p *and* q *are positive integers. Let* $G = SO(p, q)$. *Then,* $X = \Gamma^s L$ *is a unitarizable* $\mathfrak{g}$*-module. Moreover,* X *is a multiplicity-free* $\mathfrak{k}$*-module whose highest weights are the elements*

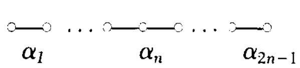
$$(n - p + j, 0, \ldots, 0, j, 0 \ldots 0), \qquad j \in N$$

with $n - p + j \ge 0$. *Here,* j *occurs as the* $l{+}1$st *coordinate.*

6. The Speh representations and their analogues for the covering group of $SL(2n, \mathrm{R})$

Here we describe an alternative proof of unitarity for the Speh representations as well as a proof of unitarity for their coherent continuation. By using *Theorem 4.4*, we compare representations of $SU(n, n)$ with representations of $SL(2n, \mathbf{R})$.

Consider the Dynkin diagram for A_{2n-1}



with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le 2n-1)$. Let θ be the Cartan involution of $SL(2n, \mathbf{R})$ that flips the diagram: $\theta\alpha_i = \alpha_{2n-i}$ $(1 \le i \le 2n-1)$. In this case, α_n is a noncompact root and the other α_i are complex. Let $\Delta^+(\mathfrak{m})$ be the positive root system with simple roots $\{\alpha_i \mid 1 \le i \le 2n-1, i \ne n\}$. Define for $1 \le i \le n$, $2\gamma_i = \alpha_i + \cdots + \alpha_{2n-i}$ restricted to $\mathfrak{t}$. Then the γ_i span $\mathfrak{t}^*$, and

$$\Delta^+(\mathfrak{f}) = \{\gamma_i \pm \gamma_j \mid 1 \le i < j \le n\},$$

$$\Delta(\mathfrak{u}) = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \le n < j \le 2n\},$$

$$\Delta(\mathfrak{u}_c) = \{\gamma_i + \gamma_j \mid 1 \le i < j \le n\},$$

$$\Delta(\mathfrak{u}_n) = \{\gamma_i + \gamma_j \mid 1 \le i \le j \le n\}.$$

Let $\lambda = z(^1/_2, \ldots, {}^1/_2, -{}^1/_2, \ldots, -{}^1/_2)$ with n $^1/_2$'s and n $-{}^1/_2$'s. Then, $\lambda|_{\mathfrak{t}} = z(\gamma_1 + \cdots + \gamma_n)$. Let G and G' be the simply connected covering groups of $SL(2n, \mathbf{R})$ and $SU(n, n)$. Then, from ref. 4, $N(\lambda)$ is irreducible and unitary for G' if and only if $z < -n + 1$. Let $X(\lambda) = \Gamma^s N(\lambda)$.

THEOREM 6.1. *For any half integer* $z < -n + 1$, $X(\lambda)$ *is a unitarizable representation of* G. *Moreover,* $X(\lambda)$ *is multiplicity free as a* $\mathfrak{k}$*-module and the highest weights are precisely those elements of the form*

$$(-z - n + 1)(\gamma_1 + \cdots + \gamma_{n-1} \pm \gamma_n) + 2a_1\gamma_1 + 2a_2\gamma_2 + \cdots \pm 2a_n\gamma_n,\ a_i \in \mathbf{N},\ a_1 \ge \cdots \ge a_n.$$

Here, + *(resp* − *) is used if* n *is even (resp odd).*

In this case, $\lambda + \rho$ lies in the canonical chamber $\mathfrak{C}$ for $z \le -2n + 1$. In ref. 2, Speh has constructed a family of representation that she denotes by $I(k)$, $k \in \mathbf{N}$. These modules are isomorphic to $X(\lambda)$ for $z = -k - n$. The half integral but nonintegral values of z correspond to representations that are defined only on the simply connected covering group of $SL(2n, \mathbf{R})$.

If $z = -n + 1$, then $\lambda|_{\mathfrak{t}} + \rho_c \in \mathfrak{C}_c$ and, so, by *Theorem 4.4*, we obtain the following:

THEOREM 6.2. *Let* $z = -n + 1$ *and* $L = L(\lambda)$. *Then* $X = \Gamma^s L$ *is a unitarizable representation of* $SL(2n, \mathbf{R})$. *Moreover, it is multiplicity free as a* $\mathfrak{k}$*-module with highest weights* $2a_1\gamma_1 + \cdots + 2a_{n-1}\gamma_{n-1} + 0 \cdot \gamma_n$,

$$a_i \in N,\ a_1 \ge a_2 \cdots \ge a_{n-1}.$$

The $\mathfrak{k}$-module structure described above follows from properties of Γ^s and results in ref. 19.

There is a similar series of representations for the group $SU^*(2n)$. In this case, the Dynkin diagram is as above except that the pure imaginary roots are all compact instead of noncompact.

7. Coherent continuation of Borel deSiebenthal discrete series representations

In ref. 4, Wallach described the analytic continuation of the holomorphic discrete series representations having a one-dimensional cyclic $\mathfrak{k}$-module. In this section, we apply *Theorem 4.3* to prove analogous results for certain discrete series representations of $\mathfrak{g}$ with $(\mathfrak{g}, \mathfrak{k})$ not Hermitian symmetric.

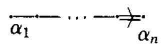
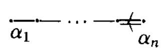
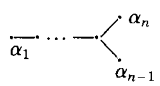
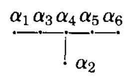
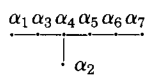
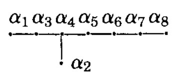
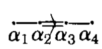
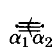
Let the notation be as in earlier sections. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of Δ^+. Assume that all α_i are compact except $\alpha = \alpha_l$, which is noncompact. Let $\Delta^+(\mathfrak{m})$ have simple roots α_i, all $i \ne l$ and *assume* that the coefficient of α in the expansion of the maximal root as a sum of simple roots is 2. Then $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u}$ is a maximal θ-stable parabolic subalgebra with $\mathfrak{u} = \mathfrak{u}_c \oplus \mathfrak{u}_n$. $\Delta(\mathfrak{u}_n)$ [resp $\Delta(\mathfrak{u}_c)$] is the set of roots β whose coefficient of α in the expansion of β as a sum of simple roots is 1 (resp 2). In this case, weight vectors in $[\mathfrak{u}, \mathfrak{u}_c]$ would have a weight with coefficient of α greater than 2. So $[\mathfrak{u}, \mathfrak{u}_c] = 0$ and *Theorem 4.3* applies in this setting.

Let $\zeta \in \mathfrak{h}^*$ be orthogonal to $\Delta(\mathfrak{m})$ and normalized by $2\langle\alpha, \zeta\rangle/\langle\alpha, \alpha\rangle = 1$. Consider the line $z\zeta$, $z \in \mathbf{R}$, and let λ_0 be the unique point on this line such that $\lambda_0 + \rho$ lies on a wall of $\mathfrak{C}$. Let $\lambda = \lambda_0 + z\zeta$. We consider the modules $N(\lambda)$ for various values of z. Let a be the smallest value of z with $N(\lambda)$ reducible. We call this the *first reduction point*. By *Theorem 4.3* we obtain the following:

THEOREM 7.1. *For* $z < a$, $N(\lambda)$ *is completely reducible as a* $\mathfrak{k}$*-module and if* λ *is* $\Delta(\mathfrak{k})$*-integral*, $X(\lambda) = \Gamma^s N(\lambda)$ *is a unitary representation of* G. *Moreover, for* $z < 0$ *and* λ $\Delta(\mathfrak{k})$*-integral*, $X(\lambda)$ *is a discrete series representation of* G.

Using the results of Jantzen (5), we can determine the value a above. The results are summarized in Appendices 1 and 2.

Appendix 1. First reduction point

Root system	Diagram	Complementary simple root α_l	First reduction point $z = a$
B_n		$2 \le l \le n-1$	$n - \left[\frac{l}{2}\right]$
		$l = n$	$2\left[\frac{n}{2}\right] + 1$
C_n		$2 \le l \le n-1$	$n - \left[\frac{l}{2}\right]$
		$l = 1$	$2n - 1$
D_n		$2 \le l \le n-2$	$n - 1 - \left[\frac{l}{2}\right]$
E_6		3 or 5	5
		2	11/2
E_7		1	17/2
		2	7
		6	6
E_8		1	23/2
		8	29/2
F_4		1	4
		4	5
G_2		2	4/3

Appendix 2. $\mathfrak{k}$-integrality

Root system	Condition for $\Delta(\mathfrak{k})$ integrality	Number of unitary $\Gamma^s N(\lambda)$ $(0 \le z < a)$	Unitarity of $\Gamma^s L(\lambda)$ at $z = a$
B_n	$2z \in \mathbf{Z}$	$2\left(n - \left[\frac{l}{2}\right]\right)$	Yes if $\left[\frac{3l+1}{2}\right] \le n$
	$z \in \mathbf{Z}$	$2\left[\frac{n}{2}\right] + 1$	? otherwise

Appendix 2. (*Continued*)

Root system	Condition for $\Delta(\mathfrak{k})$ integrality	Number of unitary $\Gamma^s N(\lambda)$ $(0 \le z < a)$	Unitarity of $\Gamma^s L(\lambda)$ at $z = a$
C_n	$z \in \mathbf{Z}$	$n - \left[\frac{l}{2}\right]$	Yes
	$z \in \mathbf{Z}$	$2n - 1$	Yes
D_n	$2z \in \mathbf{Z}$	$2\left(n - 1 - \left[\frac{l}{2}\right]\right)$	Yes if $\left[\frac{3l+1}{2}\right] \le n$? otherwise
E_6	$2z \in \mathbf{Z}$	10 11	Yes
E_7	$2z \in \mathbf{Z}$	17 14 12	Yes
E_8	$2z \in \mathbf{Z}$	23 29	Yes
F_4	$2z \in \mathbf{Z}$	8	Yes
	$z \in \mathbf{Z}$	5	?
G_2	$2z \in \mathbf{Z}$	3	$\Gamma^s L(a) = 0$

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