COMPLETENESS OF POINCARÉ SERIES FOR AUTOMORPHIC FORMS ASSOCIATED TO THE INTEGRABLE DISCRETE SERIES

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§1. Introduction

A couple of years ago, Wolf [11] studied the Poincaré series operator $\mathcal{P}$ for a homogeneous holomorphic vector bundle $\mathbb{E} \to D$ over a flag domain $D = G/V$ and an arbitrary discrete subgroup $\Gamma \subset G$. He showed that if $\mathbb{E} \to D$ is nondegenerate (see below), and if G acts on the square integrable cohomology space $H_2^s(D; \mathbb{E})$ by an integrable discrete series representation, where s is the complex dimension of the maximal compact subvariety K/V in D, then every Γ-automorphic L_p cohomology class $\psi \in H_p^s(\Gamma \backslash D; \mathbb{E})$, $1 \leqslant p \leqslant \infty$, is represented by a Poincaré series

$$\psi = \mathcal{P}(\phi) = \sum_{\gamma \in \Gamma} \gamma^* \phi \quad \text{with} \quad \phi \in H_p^s(D; \mathbb{E}) . \tag{1.1}$$

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The purpose of this note is to shift the context from L_p bundle-valued harmonic forms over flag domains G/V to eigenspaces of the Casimir operator of G on L_p sections of bundles over symmetric spaces G/K. This lets us drop the nondegeneracy condtions of [11], where it was used to ensure that every K-finite element of $H_2^s(D; \mathbb{E})$ is in $H_1^s(D; \mathbb{E})$, using [10]. This allowed sharp estimates on the L_p behavior, $1 \leqslant p \leqslant \infty$, of the reproducing kernel for $H_2^s(D; \mathbb{E})$ inside the space of all $\mathbb{E}$-valued square integrable $(0, s)$-forms on D. These estimates replaced the explicit calculations of Bers [3, 4] and Ahlfors [1, 2] for classical automorphic forms over the unit disc. Here, over G/K, those sharp L_p estimates are obtained very easily. Once we have them, everything proceeds as in [11].

For simplicity we work with the case of a connected semisimple Lie group G in this paper. However, all the results go through in the same way for the larger class of reductive Lie groups considered in [12].

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§2. The Bundles

G is a connected real semisimple Lie group, θ is a Cartan involution of G, and $K = G^\theta$ is its fixed point set. So K is the Ad_G^{-1} image of a maximal compact subgroup of $\mathrm{Ad}(G)$. We assume that rank K = rank G, so G has relative discrete series representations, and we choose a Cartan subgroup $T \subset K$ of G. Write

$$(2.1)\quad \begin{cases} \mathfrak{g}_0, \ \mathfrak{k}_0, \ \mathfrak{t}_0 & \text{Lie algebras of } G, K, T; \\ \mathfrak{g}, \ \mathfrak{k}, \ \mathfrak{t} & : \text{complexifications of } \mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0; \\ \Phi, \ \Phi_K, \ \Phi_{G/K} & : \mathfrak{t}\text{-roots of } \mathfrak{g}, \ \mathfrak{t}\text{-roots of } \mathfrak{k}, \ \Phi \setminus \Phi_K . \end{cases}$$

If necessary, replace by G a double covering group so that in some hence every) positive root system Φ^+, $\rho = \frac{1}{2}\sum_{\Phi^+}\alpha$ exponentiates to a character on T. G has center $Z \subset T \subset K \subset G$.

Let $\lambda \in \mathfrak{t}^*$ be $\mathfrak{g}$-regular and K-integral. Denote the corresponding (Harish Chandra parameterization) relative discrete series representation of G by π_λ, its class by $[\pi_\lambda] \in \hat{G}$. Then

$$(2.2)\quad \Phi^+ = \{\alpha \in \Phi: (\lambda, \alpha) > 0\}, \quad \Phi_K^+ = \Phi_K \cap \Phi^+, \quad \Phi_{G/K}^+ = \Phi_{G/K} \cap \Phi^+$$

are the positive roots systems with which we work. Denote

$$(2.3)\qquad \rho = \frac{1}{2}\sum_{\Phi^+}\alpha, \quad \rho_K = \frac{1}{2}\sum_{\Phi_K^+}\alpha, \quad \rho_{G/K} = \frac{1}{2}\sum_{\Phi_{G/K}^+}\alpha .$$

Then $\lambda - \rho_K + \rho_{G/K}$ is Φ_K^+-dominant and K-integral. Denote the irreducible representation of K with that highest weight by

$$(2.4) \qquad \tau = \tau_{\lambda-\rho_K+\rho_{G/K}} , \quad \text{representation space} \quad E = E_{\lambda-\rho_K+\rho_{G/K}} .$$

It is (see Schmid [7] or Wallach [9]) the lowest K-type of π_λ, in the sense that all others have highest weights obtained by adding elements of $\Phi^+_{G/K}$ to $\lambda - \rho_K + \rho_{G/K}$, and it has multiplicity 1 in π_λ.

Consider the associated homogeneous hermitian C^∞ vector bundle

$$(2.5) \qquad \mathbb{E} = G \times_K E \to X = G/K , \quad \text{typical fibre} \quad E .$$

Denote its space of L_p sections by

$$(2.6) \quad L_p(X, \mathbb{E}) = \{f: G \to E: f(gk) = \tau(k)^{-1}f(g), \ \|f(\cdot)\| \in L_p(X)\} .$$

Let Ω be the Casimir element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, $\widetilde{\Omega}$ its closure as operator on $L_p(X, \mathbb{E})$. Then, for $1 \leqslant p \leqslant \infty$,

$$(2.7) \qquad H_p(X, \mathbb{E}) = \{f \in L_p(X, \mathbb{E}): \widetilde{\Omega}f = (\|\lambda\|^2 - \|\rho\|^2)f\}$$

is a closed subspace of $L_p(X, \mathbb{E})$ on which G acts continuously and isometrically. In particular G acts on $H_2(X, \mathbb{E})$ by a unitary representation. The point is that (Hotta [6])

(2.8) the representation of G on $H_2(X, \mathbb{E})$ is equivalent to π_λ.

If $[\pi] \in \hat{G}$ and H is the representation space of π, we have the <u>coefficients</u>

(2.9) $\quad f_{u,v}: G \to \mathbb{C}$ by $f_{u,v}(x) = (u, \pi(x)v)_H$ for $u, v \in H$.

We recall that $[\pi]$ is said to be L_p if its K-finite coefficients satisfy $|f_{u,v}| \in L_p(G/Z)$ where Z is the center of G. So $[\pi]$ is in the relative discrete series if it is L_2, is in the <u>integrable relative discrete series</u> if it is L_1. The Trombi-Varadarajan-Hecht-Schmid condition for a relative discrete series class $[\pi_\lambda]$ to be integrable is ([5], [8])

(2.10) $$|\langle\lambda, \gamma\rangle| > \frac{1}{2} \sum_{\alpha\in\Phi^+} |\langle\alpha, \gamma\rangle| \quad \text{for all } \gamma \in \Phi^+_{G/K}.$$

Finally, we note that

(2.11)

if $[\pi_\lambda]$ is L_p and $f \in H_2(X, \mathbb{E})$ is K-finite then $\|f\| \in L_p(G/Z)$.

For $f \in H \otimes H^* \otimes E$, $H = H_{\pi_\lambda}$, and $H \otimes H^*$ is the L_2-closure of the space of coefficients of π_λ. We are assuming f left-K-finite. $f(gk) = \tau(k)^{-1}f(g)$ forces it to be right K-finite. That proves (2.11).

§3. The Reproducing Kernel

The K-type (τ, E) occurs with multiplicity 1 in π_λ. Denote the isometric K-equivariant inclusion which is adjoint to evaluation at $1 \in G$, by

$$\text{(3.1)} \qquad i: E \to \mathcal{E} = \mathcal{H}_2(X, \mathbb{E})_\tau .$$

Denote also

$$\text{(3.2)} \qquad e: L_2(X, \mathbb{E}) \to \mathcal{E}, \text{ orthogonal projection, and}$$

$$\text{(3.3)} \qquad \Xi: G \to GL(E) \text{ by } \Xi(x)v = i^{-1}\cdot e\cdot\pi_\lambda(x)\cdot i(v) .$$

We are going to prove that

$$\text{(3.4)} \quad K_\lambda(x, y) = d_\lambda \cdot \text{trace } \Xi(y^{-1}x) , \quad d_\lambda = \text{formal degree of } [\pi_\lambda] ,$$

is the reproducing kernel for $\mathcal{H}_2(X, \mathbb{E})$ inside $L_2(X, \mathbb{E})$.

Let $\{v_1, \ldots, v_\ell\}$ be an orthonormal basis of E and $\{\phi_j = i(v_j)\}$ the corresponding orthonormal basis of $\mathcal{E}$. We have K-finite coefficients of π_λ,

$$\text{(3.5)} \qquad f_j = f_{\phi_j,\phi_j}: x \mapsto (\phi_j, \pi_\lambda(x)\phi_j) ,$$

and we note from the definition (3.4) that

$$K_\lambda(x, y) = d_\lambda \sum_{j=1}^{\ell} (\pi_\lambda(y^{-1}x)\phi_j, \phi_j) = d_\lambda \sum_{j=1}^{\ell} f_j(x^{-1}y) . \tag{3.6}$$

We assert that $i: E \to H_2(X, \mathbb{E})$ is given by

$$i(v)(x) = d_\lambda^{1/2} \cdot \Xi(x^{-1})v \quad \text{for} \quad v \in E , \quad x \in G . \tag{3.7}$$

To see this, let $\phi: G \to E$ be given by $\phi_v(x) = \Xi(x^{-1})v$. Then $\phi_v(xk) = \Xi(k^{-1}x^{-1})v = i^{-1}\cdot e\cdot\pi_\lambda(k)^{-1}\cdot\pi_\lambda(x^{-1})\cdot i(v) = i^{-1}\cdot\pi_\lambda(k)^{-1}\cdot e\cdot\pi_\lambda(x^{-1})\cdot i(v)$ $= \tau(k)^{-1}\cdot i^{-1}\cdot e\cdot\pi_\lambda(x^{-1})\cdot v = \tau(k)^{-1}\phi_v(x)$ for $k \in K$, so ϕ is a section of $\mathbb{E} \to X$. Also, writing H for $H_2(X, \mathbb{E})$,

$$\begin{aligned}
\int_X \|\phi_v(x)\|_E^2 d(xK) &= \int_X \|e\cdot\pi_\lambda(x^{-1})\cdot i(v)\|_H^2 d(xK) \\
&= \int_X \Big\| \sum_{j=1}^{\ell} (\pi_\lambda(x^{-1})\cdot i(v), \phi_j)_H \phi_j \Big\|_H^2 d(xK) \\
&= \int_X \sum_{j=1}^{\ell} |(i(v), \pi_\lambda(x)\phi_j)_H|^2 d(xK) \\
&= \sum_{j=1}^{\ell} \|f_{i(v),\phi_j}\|_{L_2(G/Z)}^2 = d_\lambda^{-1}\|i(v)\|_H^2 = d_\lambda^{-1}\|v\|_E^2
\end{aligned}$$

so $\phi_v \in L_2(X, \mathbb{E})$. Further,

$$\begin{aligned}
\Omega\phi_v(x) &= i^{-1}\cdot e\cdot d\pi_\lambda(\Omega)\cdot\pi_\lambda(x^{-1})\cdot i(v) \\
&= (\|\lambda\|^2 - \|\rho\|^2) i^{-1}\cdot e\cdot\pi_\lambda(x^{-1})\cdot i(v) = (\|\lambda\|^2 - \|\rho\|^2)\phi_v(x)
\end{aligned}$$

so $\phi_v \in H_2(X, \mathbb{E})$. Finally, if $k \in K$ then

$$\phi_{\tau(k)v}(x) = \Xi(x^{-1})\tau(k)v = \Xi((k^{-1}x)^{-1})v$$
$$= \phi_v(k^{-1}x) = (\pi_\lambda(k)\phi_v)(x)$$

so $v \mapsto \phi_v$ is K-equivariant. As $\|\phi_v\|_{H}^2 = d_\lambda^{-1}\|v\|_E^2$, now $v \mapsto d_\lambda^{1/2}\phi_v$ coincides with $i: E \to \mathcal{E}$ up to multiplication by a scalar of absolute value 1. Now (3.7) follows by our choice of i as adjoint to evaluation $\mathcal{E} \to E$ at $1 \in G$.

<u>3.8. Theorem</u>. If $f \in L_2(X, \mathbb{E})$, then its orthogonal projection to $H_2(X, \mathbb{E})$ is given by an absolutely convergent integral

$$Hf(x) = \int_{G/Z} K_\lambda(x, y)f(y)d(yZ) \tag{3.9}$$

<u>Proof</u>. Let $\zeta \in \hat{Z}$ such that $\tau(kz) = \zeta(z)\tau(k)$ for $k \in K$ and $z \in Z$. Then the action $\tilde{\pi}$ of G on $L_2(X, \mathbb{E})$ satisfies

$$\tilde{\pi}(z)f(x) = f(z^{-1}x) = f(xz^{-1}) = \tau(z)\cdot f(x) = \zeta(z)f(x) ,$$

so $\tilde{\pi}$ and its subrepresentation π_λ have central character ζ. Now $K_\lambda(x, yz) = \zeta(z)^{-1}K_\lambda(x, y)$, so $K_\lambda(x, yz)f(yz) = K_\lambda(x, y)f(y)$, and the integrand in (3.9) is well defined.

In (3.5) we have $|f_j| \in L_2(G/Z)$, so (3.6) shows that $|K_\lambda(x, y)|$ is in $L_2(G/Z)$ for each variable separately. Thus the integral (3.9) converges absolutely. Now compute

$$\begin{aligned} Hf(xk) &= \int_{G/Z} d_\lambda \cdot \text{trace } \Xi(y^{-1}xk)f(y)d(yZ) \\ &= \int_{G/Z} d_\lambda \cdot \text{trace } \Xi(ky^{-1}x)f(y)d(yZ) \\ &= \int_{G/Z} d_\lambda \cdot \text{trace } \Xi(y^{-1}x)f(yk)d(yZ) \\ &= \int_{G/Z} d_\lambda \cdot \text{trace } \Xi(y^{-1}x)\cdot\tau(k)^{-1}f(y)d(yZ) \\ &= \tau(k)^{-1}Hf(x) . \end{aligned}$$

Thus Hf is a well defined section of $\mathbb{E} \to G/K$.

Denote $\langle u, v \otimes a\rangle_{\mathcal{H}} = (u, v)_{\mathcal{H}}a$ for $u, v \in \mathcal{H}$ and $a \in E$. If $f \in L_2(X, \mathbb{E})$ and $u, v \in \mathcal{H}$ then

$$f_{u,v} \otimes f\colon x \mapsto (u, \pi(x)v)_{\mathcal{H}}f(x)$$

is integrable over G/Z . That defines a map $\Pi(f)\colon \mathcal{H} \to \mathcal{H} \otimes E$ by

$$\langle u, \Pi(f)v\rangle_{\mathcal{H}} = \int_{G/Z} (u, \pi(x)v)_{\mathcal{H}}f(x)d(xZ) .$$

As $\|f_{u,v}\|_{L_2(G/Z)} = d_\lambda^{-1/2}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$, now

$$|\langle u, \Pi(f)v\rangle_{\mathcal{H}}| \leqslant d_\lambda^{-1/2}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|f\|_{L_2(X)} ,$$

so the operator norm

$$\|\Pi(f)\| \leq d_\lambda^{-1/2} \|f\|_{L_2(X)} .$$

The calculation just above, will allow us to estimate

$$\|Hf\|^2_{L_2(X)} = \int_X \|Hf(x)\|^2_E d(xK) = \int_{G/Z} \|Hf(x)\|^2_E d(xZ)$$

$$= \int_{G/Z} \| \int_{G/Z} d_\lambda \sum_{j=1}^{\ell} f_j(x^{-1}y) f(y) d(yZ) \|^2_E d(xZ) .$$

For the inner integral, note

$$\int_{G/Z} f_j(x^{-1}y) f(y) d(yZ) = \int_{G/Z} (\pi(x)\phi_j, \pi(y)\phi_j)_H f(y) d(yZ)$$

$$= \langle \pi(x)\phi_j, \Pi(f)\phi_j \rangle_H .$$

Thus

$$\|Hf\|^2_{L_2(X)} = \int_{G/Z} \|d_\lambda \sum_{j=1}^{\ell} \langle \pi(x)\phi_j, \Pi(f)\phi_j \rangle_H \|^2_E d(xZ)$$

$$= d_\lambda^2 \int_{G/Z} \sum_{j,k} (\langle \pi(x)\phi_j, \Pi(f)\phi_j \rangle_H, \langle \pi(x)\phi_k, \Pi(f)\phi_k \rangle_H)_E d(xZ)$$

$$= d_\lambda^2 \sum_{j,k} d_\lambda^{-1} (\phi_j, \phi_k)_H (\Pi(f)\phi_k, \Pi(f)\phi_j)_{H \otimes E}$$

$$= d_\lambda \sum_j \|\Pi(f)\phi_j\|^2_{H \otimes E}$$

$$\leq d_\lambda \cdot \dim E \cdot \|\Pi(f)\|^2$$

$$\leq (\dim E) \|f\|^2_{L_2(X)} .$$

Thus $Hf \in L^2(X, \mathbb{E})$ with $\|H\| \leq (\dim E)^{1/2}$. Finally,

$$(\widetilde{\Omega}\cdot Hf)(x) = \int_{G/Z} d_\lambda \cdot \mathrm{trace}(\widetilde{\Omega}_x \Xi(y^{-1}x)) f(y) d(yZ)$$

$$= \int_{G/Z} d_\lambda \cdot d\pi_\lambda(\Omega) \mathrm{trace}\, \Xi(y^{-1}x) f(y) d(yZ)$$

$$= (\|\lambda\|^2 - \|\rho\|^2) Hf(x) \ ,$$

showing $Hf \in H_2(X, \mathbb{E})$. Now the integral operator (3.9) is a bounded operator $H\colon L_2(X, \mathbb{E}) \to H_2(X, \mathbb{E})$.

Next, we prove that H is a projection. It is hermitian because

$$(3.10) \quad \begin{cases} (Hf, f')_{L_2(X)} \\ = \int_{G/Z} \left(\int_{G/Z} d_\lambda \cdot \sum_{j=1}^{\ell} f_j(x^{-1}y) f(y) d(yZ), f'(x) \right)_E d(xZ) \\ = \int_{G/Z} \int_{G/Z} d_\lambda \cdot \sum_{j=1}^{\ell} f_j(x^{-1}y)(f(y), f'(x))_E d(yZ) d(xZ) \\ = \int_{G/Z} \left(f(y), \int_{G/Z} d_\lambda \cdot \sum_{j=1}^{\ell} f_j(y^{-1}x) f'(x) d(xZ) \right)_E d(yZ) \\ = (f, Hf')_{L_2(X)} \ . \end{cases}$$

And H is idempotent because

$$H^2f(g) = \int_{G/Z} K_\lambda(g, x) \int_{G/Z} K_\lambda(x, y)f(y)d(yZ)d(xZ)$$

$$= \int_{G/Z}\int_{G/Z} d_\lambda^2 \sum_{j,k} f_j(g^{-1}x)f_k(x^{-1}y)f(y)d(yZ)d(xZ)$$

$$= \int_{G/Z} d_\lambda^2 \sum_{j,k} \left\{ \int_{G/Z} f_{\widetilde{\pi}(g)\phi_j,\phi_j}(x) f_{\widetilde{\pi}(y)\phi_k,\phi_k}(x)d(xZ)\right\} f(y)d(yZ)$$

$$= \int_{G/Z} d_\lambda \cdot \sum_{j,k} (\widetilde{\pi}(g)\phi_j, \widetilde{\pi}(y)\phi_k)\overline{(\phi_j, \phi_k)}f(y)d(yZ)$$

$$= \int_{G/Z} d_\lambda \sum_{j=1}^{\ell} f_i(g^{-1}y)f(y)d(yZ) = Hf(g) .$$

The projection H is G-equivariant because $(\pi_\lambda(g)\cdot Hf)(x) = Hf(g^{-1}x)$ $= \int_{G/Z} K_\lambda(g^{-1}x, y)f(y)d(yZ) = \int_{G/Z} K_\lambda(g^{-1}x, g^{-1}y)f(g^{-1}y)d(yZ)$ $= \int_{G/Z} K_\lambda(x, y)f(g^{-1}y)d(yZ) = H(\widetilde{\pi}(g)f)(x)$. Thus its range is a G-invariant subspace of the G-irreducible space $\mathcal{H}_2(X, \mathbb{E})$. Let $v \in E$ and $\phi_v = d_\lambda^{-1/2} i(v)\colon x \mapsto \Xi(x^{-1})v$ as in (3.7) and its proof. Then

$$\phi_v(x) = i^{-1}\cdot e\cdot\pi_\lambda(x^{-1})\cdot i(v) = \sum_{j=1}^{\ell} (\pi_\lambda(x^{-1})\cdot i(v), \phi_j)_{\mathcal{H}} v_j .$$

Thus

$$H\phi_v(x) = \int_{G/Z} d_\lambda \cdot \sum_{j=1}^{\ell} f_{\phi_j,\phi_j}(x^{-1}y) \sum_{k=1}^{\ell} f_{i(v),\phi_k}(y)v_k d(yZ)$$

$$= d_\lambda \sum_{j,k} (f_{i(v),\phi_k} * f_{\phi_j,\phi_j})(x)v_k \quad \text{convolution over } G/Z$$

$$= \sum_{j,k} (i(v), \phi_j)_{\mathcal{H}} f_{\phi_j,\phi_k}(x)v_k$$

$$= \sum_{j,k} (v, v_j)_E(\pi_\lambda(x^{-1})\phi_j, \phi_k)v_k$$

$$= \sum_j (v, v_j)\Xi(x^{-1})v_j$$

$$= \Xi(x^{-1})v = \phi_v(x) \ .$$

In particular $H \neq 0$. This completes the proof of Theorem 3.8.

q.e.d.

§4. Projection to $\mathcal{H}_p(X, \mathbb{E})$

We now assume that the relative discrete series representation π_λ of G on $\mathcal{H}_2(X, \mathbb{E})$ is integrable, i.e., that λ satisfies (2.10).

4.1. Lemma. $|K_\lambda(x, y)|$ is in $L_p(G/Z)$ in each variable, for $1 \leqslant p \leqslant \infty$, with L_p norms $\|K_\lambda(x, \cdot)\|_{L_p(G/Z)} = \|K_\lambda(\cdot, y)\|_{L_p(G/Z)}$ independent of $x, y \in G$.

Proof. By (3.6), $x \mapsto |K_\lambda(x, 1)|$ is a finite sume of K-finite coefficients of π_λ, hence is $L_p(G/Z)$ for $1 \leqslant p \leqslant \infty$. If $y \in G$ and $1 \leqslant p < \infty$ then

$$\|K_\lambda(\cdot, y)\|^p_{L_p(G/Z)} = \int_{G/Z} |K_\lambda(x, y)|^p d(xZ)$$

$$= \int_{G/Z} |K_\lambda(y^{-1}x, 1)|^p d(xZ) = \|K_\lambda(\cdot, 1)\|^p_{L_p(G/Z)} < \infty$$

and

$$\|K_\lambda(\cdot, y)\|_{L_\infty(G/Z)} = \text{ess sup}_{x \in G} |K_\lambda(x, y)|$$

$$= \text{ess sup}_{x \in G} |K_\lambda(y^{-1}x, 1)| = \|K_\lambda(\cdot, 1)\|_{L_\infty(G/Z)} < \infty .$$

The same argument works in the other variable. q.e.d.

Now define a constant $b = b(G, K, \lambda)$ by

$$(4.2) \qquad b = \|K_\lambda(x, \cdot)\|_{L_1(G/Z)} = \|K_\lambda(\cdot, y)\|_{L_1(G/Z)} .$$

4.3. Theorem. Assume $[\pi_\lambda]$ integrable. Let $1 \leqslant p \leqslant \infty$. If $f \in L_p(X, \mathbb{E})$, then

$$(4.4) \qquad Hf(x) = \int_{G/Z} K_\lambda(x, y)f(y)d(yZ)$$

converges absolutely to an element of $H_p(X, \mathbb{E})$. Furthermore, $H: L_p(X, \mathbb{E}) \to H_p(X, \mathbb{E})$ has norm $\|H\| \leqslant b$, and if $\phi \in H_p(X, \mathbb{E})$ then $H\phi = \phi$.

Proof. Convergence and the bound on H are clear for $p = \infty$:

$$\begin{aligned}\|Hf\|_\infty &\leqslant \text{ess sup}_{x\in G} \int_{G/Z} \|K_\lambda(x, y)f(y)\|_E d(yZ) \\ &\leqslant \sup_{x\in G} \int_{G/Z} |K_\lambda(x, y)| d(yZ) \cdot \text{ess sup}_{y\in G} \|f(y)\|_E \\ &= b\|f\|_\infty .\end{aligned}$$

If f is continuous with support compact modulo K , it is in $L_\infty(X, \mathbb{E})$ so Hf converges absolutely, as just seen, and of course

$$\|Hf\|_1 = \int_{G/Z} \| \int_{G/Z} K_\lambda(x, y)f(y)d(yZ)\|_E d(xZ)$$

$$\leqslant \int_{G/Z} \int_{G/Z} |K(x, y)| \|f(y)\|_E d(yZ)d(xZ)$$

$$\leqslant b \int_{G/Z} \|f(y)\|_E d(yZ) = b\|f\|_1 .$$

Extend H to $L_p(X, \mathbb{E})$ by continuity for $1 \leqslant p < \infty$. The Riesz-Thorin Theorem gives convergence of (4.4) and shows that $H: L_p(X, \mathbb{E}) \to L_p(X, \mathbb{E})$ has norm $\|H\| \leqslant b$. Also,

$$(\widetilde{\Omega}\cdot Hf)(x) = \pi_\lambda(\Omega)Hf(x) = (\|\lambda\|^2 - \|\rho\|^2)Hf(x)$$

for $1 \leqslant p \leqslant \infty$, so in fact $H: L_p(X, \mathbb{E}) \to \mathcal{H}_p(X, \mathbb{E})$.

If $p, q \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have a sesquilinear pairing

(4.6)

$$L_p(X, \mathbb{E}) \times L_q(X, \mathbb{E}) \to \mathbb{C} \text{ by } (f, f')_X = \int_{G/Z} (f(x), f'(x))_E d(xZ) .$$

We assert that it satisfies

(4.7) $(Hf, f')_X = (f, Hf')_X$ for $f \in L_p$, $f' \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

If one of f, f' is in the space $C_c(X, \mathbb{E})$ of continuous compactly supported sections, and the other is in $C_c(X, \mathbb{E})$ or $L_\infty(X, \mathbb{E})$, then this follows by the calculation (3.10). As H is L_p ; L_q bounded, it now follows when one is in the closure of $C_c(X, \mathbb{E})$ in its

$L_r(X, \mathbb{E})$, and the other is L_∞ or also in the closure of C_c in its $L_r(X, \mathbb{E})$. This covers all cases, so we have (4.7).

Now let $f \in H_p(X, \mathbb{E})$. If $f' \in C_c(X, \mathbb{E})$ then (4.7) applies, so $((1 - H)f, f')_X = (f, (1 - H)f')_X$. Let

$$D = \tilde{\pi}(\Omega) - (\|\lambda\|^2 - \|\rho\|^2) .$$

The L_2 range of D is dense in $(1 - H)L_2(X, \mathbb{E})$. The same follows for the L_q range, $\frac{1}{p} + \frac{1}{q} = 1$. Thus $(1 - H)f' = \lim_{n\to\infty} Df_n$ in $L_q(X, \mathbb{E})$, and

$$\begin{aligned} ((1 - H)f, f')_X &= \lim_{n\to\infty}(f, Df_n)_X \\ &= \lim_{n\to\infty}(Df, f_n)_X = 0 . \end{aligned}$$

We have shown that $f = Hf$ as distribution section of $\mathbb{E} \to X$. It follows that $Hf = f$. <u>q.e.d.</u>

§5. Projection to $\mathcal{H}_p(X/\Gamma, \mathbb{E})$

Fix a discrete subgroup $\Gamma \subset G$ that acts discontinuously on $X = G/K$, i.e., such that ΓZ is closed in G. Let F be a fundamental domain for the action of Γ on X. Then we have

$$(5.1) \qquad L_p(X/\Gamma, \mathbb{E}) : \begin{cases} \text{all measurable } \Gamma\text{-invariant sections } f \\ \text{of } \mathbb{E} \to X \text{ with } \|f|_F(\cdot)\| \in L_p(F) \\ \text{and norm } \|f\|_{\Gamma,p} = \|f|_F(\cdot)\|_{L_p(F)} \end{cases}$$

and its closed subspace

$$(5.2) \qquad \mathcal{H}_p(X/\Gamma, \mathbb{E}) = \{f \in L_p(X/\Gamma, \mathbb{E}) : \widetilde{\Omega} f = (\|\lambda\|^2 - \|\rho\|^2)f\} .$$

As in (4.6), for $\frac{1}{p} + \frac{1}{q} = 1$ these Banach spaces have a nondegenerate sesquilinear pairing

(5.3)

$$L_p(X/\Gamma, \mathbb{E}) \times L_q(X/\Gamma, \mathbb{E}) \to \mathbb{C} \text{ by } (f, f')_\Gamma = \int_F (f(x), f'(x))_{\mathbb{E}} d(xK) .$$

The arguments of Wolf [11, §5] now apply without any modification. The result is

5.4. Theorem. Assume $[\pi_\lambda]$ integrable. Let $1 \leqslant p \leqslant \infty$. If $f \in L_p(X/\Gamma, \mathbb{E})$ then Hf is well defined by

$$Hf(x) = \int_{G/Z} K_\lambda(x, y) f(y) dy \quad \text{for } p = \infty ,$$

L_p limits from $C_c(X/\Gamma, \mathbb{E})$ for $1 \leqslant p < \infty$.

Furthermore,

(5.5) $$H\colon L_p(X/\Gamma, \mathbb{E}) \to \mathcal{H}_p(X/\Gamma, \mathbb{E}) \text{ with } \|H\| \leqslant b ,$$

(5.6) $$\text{if } f \in \mathcal{H}_p(X/\Gamma, \mathbb{E}) \text{ then } Hf = f , \text{ and}$$

(5.7) $$\begin{cases} \text{if } f \in L_p(X/\Gamma, \mathbb{E}) \text{ and } f' \in L_q(X/\Gamma, \mathbb{E}) \text{ with} \\ \frac{1}{p} + \frac{1}{q} = 1 , \text{ then } (Hf, f')_\Gamma = (f, Hf')_\Gamma . \end{cases}$$

In effect, if $p = \infty$ then (5.5) is a computation and (5.6) follows from Theorem 4.3. If $p = 1$, then (5.7) is proved by approximation, and (5.5) and (5.6) are extracted in the distributional sense from the case $q = \infty$. If $1 < p < \infty$ the assertions extend from C_c to L_p by Riesz-Thorin.

Proceding exactly as in Wolf [11, §6] we obtain

5.8. Theorem. Assume $[\pi_\lambda]$ integrable. Let $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the pairing (5.3) establishes a conjugate-linear isomorphism between $\mathcal{H}_q(X/\Gamma, \mathbb{E})$ and the dual space of $\mathcal{H}_p(X/\Gamma, \mathbb{E})$. If $f' \in \mathcal{H}_q(X/\Gamma, \mathbb{E})$ corresponds to the linear functional ℓ , then $b^{-1}\|f'\|_{\Gamma,q} \leqslant \|\ell\| \leqslant \|f'\|_{\Gamma,q}$.

5.9. Corollary. Let $f \in L_p(X/\Gamma, \mathbb{E})$ and $f' \in L_q(X/\Gamma, \mathbb{E})$. Then $Hf = 0$ if and only if $(f, \mathcal{H}_q(X/\Gamma, \mathbb{E}))_\Gamma = 0$, and $f' \in \mathcal{H}_q(X/\Gamma, \mathbb{E})$ if and only if $((1 - H)L_p(X/\Gamma, \mathbb{E}) , f')_\Gamma = 0$.

§6. The Poincaré Series Operator

The Poincaré series of a section f of $\mathbb{E} \to X$, relative to a discrete subgroup $\Gamma \subset G$, is defined by

$$(6.1) \qquad \mathcal{V}(f)(x) = \sum_{\gamma\in\Gamma} f(\gamma^{-1}x) \quad \text{for} \quad x \in G$$

whenever the right hand side converges in some suitable sense. In that case, $\mathcal{V}(f)$ is a Γ-invariant section of $\mathbb{E} \to X$.

For example, if $f \in L_1(X, \mathbb{E})$ then $\mathcal{V}(f)$ converges absolutely a.e. because $\int_{G/Z} \|f(x)\|_E d(xZ) = \sum_{\gamma\in\Gamma} \int_F \|f(\gamma^{-1}x)\|_E d(xK) = \|f\|_1$, $\mathcal{V}(f) \in L_1(X/\Gamma, \mathbb{E})$ with $\|\mathcal{V}(f)\|_{\Gamma,1} \leqslant \|f\|_1$ because

$$\int_F \|\mathcal{V}(f)(x)\|_E d(xK) = \int_F \Big\|\sum_{\gamma\in\Gamma} f(\gamma^{-1}x)\Big\|_E d(xK)$$

$$\leqslant \int_F \sum_{\gamma\in\Gamma} \|f(\gamma^{-1}x)\|_E d(xK) = \sum_{\gamma\in\Gamma} \int_F \|f(\gamma^{-1}x)\|_E d(xK) = \|f\|_1 ,$$

and if $f \in H_1(X, \mathbb{E})$ then $\mathcal{V}(f) \in H_1(X/\Gamma, \mathbb{E})$ because

$$\widetilde{\Omega}\cdot\mathcal{V}(f)(x) = \sum_{\gamma\in\Gamma} \widetilde{\Omega}_x \cdot f(\gamma^{-1}x) = \mathcal{V}(\Omega f)(x) = (\|\lambda\|^2 - \|\rho\|^2)\mathcal{V}(f)(x) .$$

In brief, using ellipticity of Ω on X,

$$(6.2) \quad \begin{cases} \mathcal{V}: H_1(X, \mathbb{E}) \to H_1(X/\Gamma, \mathbb{E}) \text{ with } \|\mathcal{V}\| \leqslant 1 , \text{ and} \\ \text{here each } \mathcal{V}(f) \text{ converges absolutely and uniformly on compact sets.} \end{cases}$$

If $f \in L_1(X, \mathbb{E})$ and $f' \in L_\infty(X/\Gamma, \mathbb{E})$ we compute

$$(\mathcal{P}(f), f')_\Gamma = \sum_{\gamma\in\Gamma} \int_F (f(\gamma^{-1}x), f'(x))_{\mathbb{E}} d(xK) = \sum_{\gamma\in\Gamma} \int_{\gamma F} (f(x), f'(\gamma x))_{\mathbb{E}} d(xK)$$

$$= \sum_{\gamma\in\Gamma} \int_{\gamma F} (f(x), f'(x))_{\mathbb{E}} d(xK) \quad = \int_X (f(x), f'(x))_{\mathbb{E}} d(xK)$$

$$= (f, f')_X .$$

Thus $\mathcal{P}: L_1(X, \mathbb{E}) \to L_1(X/\Gamma, \mathbb{E})$ has adjoint

$$\mathcal{P}^*: L_\infty(X/\Gamma, \mathbb{E}) \hookrightarrow L_\infty(X, \mathbb{E})$$

which is continuous inclusion of a closed subspace. This says that $\mathcal{P}: L_1(X, \mathbb{E}) \to L_1(X/\Gamma, \mathbb{E})$ is surjective. The case $p = 1$ of Theorem 5.8 lets us specialize this to H_1, as in (6.2). Thus

6.3. Proposition. The Poincaré series map $\mathcal{P}: H_1(X, \mathbb{E}) \to H_1(X/\Gamma, \mathbb{E})$ is continuous and surjective, and its adjoint is the inclusion $\mathcal{P}^*: H_\infty(X/\Gamma, \mathbb{E}) \to H_\infty(X, \mathbb{E})$.

One now continues just as in Wolf [11, §7]. $\mathcal{P}$ converges on the dense subspace $H_1(X, \mathbb{E}) \cap H_p(X, \mathbb{E})$ of $H_p(X, \mathbb{E})$ -- dense because it contains all K-finite elements -- so $\mathcal{P} \circ H$ converges on

(6.4) $$J_F = \{\chi f: f \in C_c(X/\Gamma, \mathbb{E})\}$$

where χ is the indicator function of the fundamental domain F of Γ and $C_c(X/\Gamma, \mathbb{E})$ is the space of Γ-invariant sections of $\mathbb{E} \to X$ with support compact modulo Γ from X, i.e., compact modulo ΓZ from G.

If $f \in L_1(X/\Gamma, \mathbb{E})$, then $\chi f \in L_1(X, \mathbb{E})$, so $H(\chi f) \in H_1(X, \mathbb{E})$ and thus $\mathcal{V}(H(\chi f)) \in H_1(X/\Gamma, \mathbb{E})$. If $f' \in H_\infty(X/\Gamma, \mathbb{E})$ then

$$(Hf, f')_\Gamma = (f, f')_\Gamma \quad \text{by Corollary 5.9}$$

and, using the calculation just after (6.2),

$$(\mathcal{V}H(\chi f), f')_\Gamma = (H(\chi f), f')_X = (\chi f, f')_X = (f, f')_\Gamma .$$

Thus, from Theorem 5.8 with $p = 1$,

$$\text{(6.5)} \qquad \mathcal{V}H(\chi f) = Hf \quad \text{for all} \quad f \in L_1(X/\Gamma, \mathbb{E}) .$$

In particular, if $\eta = \chi f \in J_F$ then (5.5) says $\|\mathcal{V}H(\eta)\|_{\Gamma,p} = \|\mathcal{V}H(\chi f)\|_{\Gamma,p} = \|Hf\|_{\Gamma,p} \leqslant b\|f\|_{\Gamma,p} = b\|\eta\|_p$. That is the L_p bound on $\mathcal{V}H$ in

6.6. Proposition. Let $1 \leqslant p < \infty$. If $\eta \in J_F$ then $\mathcal{V}H(\eta)$ converges absolutely, uniformly on compact subsets of X, to an element of $H_p(X/\Gamma, \mathbb{E})$, and $\|\mathcal{V}H(\eta)\|_{\Gamma,p} \leqslant b\|\eta\|_{L_p(X)}$. So $\mathcal{V}H$ extends by continuity to a linear map

$$\mathcal{V}H\colon (L_p\text{-closure of } J_F) \to H_p(X/\Gamma, \mathbb{E})$$

of norm $\leqslant b$. This extension is surjective: if $\phi \in H_p(X/\Gamma, \mathbb{E})$ then $\chi\phi$ is in the L_p-closure of J_F and $\mathcal{V}H(\chi\phi) = \phi$.

The case $p = \infty$ is slightly different. If $f \in L_\infty(X/\Gamma, \mathbb{E})$ then H is absolutely convergent on $\mathcal{V}(\chi f)$ because

$$\int_X \sum_{\gamma\in\Gamma} |K_\lambda(x, y)|\cdot\|(\chi f)(\gamma^{-1}y)\|_E d(yZ) \leqslant b\|\chi f\|_\infty$$

Since $\mathcal{V}(\chi f) = f$ now $\mathcal{V}(H(\chi f)) = H(\mathcal{V}(\chi f)) = Hf$. Thus

6.7. Proposition. If $\eta \in H(\chi\cdot L_\infty(X/\Gamma, \mathbb{E}))$ then $\mathcal{V}(\eta)$ converges absolutely to an element of $H_\infty(X/\Gamma, \mathbb{E})$. The map $\mathcal{V}\colon H(\chi\cdot L_\infty(X/\Gamma, \mathbb{E})) \to H_\infty(X/\Gamma, \mathbb{E})$ is surjective: if $\phi \in H_\infty(X/\Gamma, \mathbb{E})$ then $\mathcal{V}(H(\chi\phi)) = \phi$.

In summary, now, we have completeness of Poincaré series for the bundles $\mathbb{E} \to X$.

6.8. Theorem. Suppose that $[\pi_\lambda]$ is integrable. Let $1 \leqslant p \leqslant \infty$. Then the Poincaré series operator is defined on

$p = 1$: all of $H_1(X, \mathbb{E})$ as in (6.2);

$1 < p < \infty$: $H(\chi\cdot L_p(X/\Gamma, \mathbb{E}))$ as in Proposition 6.6;

$p = \infty$: $H(\chi\cdot L_\infty(X/\Gamma, \mathbb{E}))$ as in Proposition 6.7;

and maps that space onto $H_p(X/\Gamma, \mathbb{E})$. In fact, if $\phi \in H_p(X/\Gamma, \mathbb{E})$ then $\|H(\chi\phi)\|_p \leq b\|\phi\|_p$ and $\mathcal{V} H(\chi\phi) = \phi$.

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