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Singular Unitary Representations and Indefinite Harmonic Theory*

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Square-integrable harmonic spaces are defined and studied in a homogeneous indefinite metric setting. In the process, Dolbeault cohomologies are unitarized, and singlar unitary representations are obtained and studied.

Contents.

1. INTRODUCTION

A fundamental problem in the representation theory of a semisimple Lie group G is to describe its irreducible unitary representations. One can divide them, loosely, into two classes: those that enter the Plancherel decomposition of $L_2(G)$, and the remainder. We shall refer to the former as regular, to the latter as singular unitary representations. The regular representations are parametrized by the regular semisimple integral orbits of the adjoint group in the dual of the Lie algebra, and have geometric realizations which are related to this parametrization. Elliptic orbits, for example, correspond to discrete series representations, provided G does have a discrete series: any such

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regular elliptic integral orbit $\mathscr{O}$ can be turned into a homogeneous holomorphic manifold, and carries a distinguished homogeneous line bundle $\mathbf{L}$; the discrete series representation attached to $\mathscr{O}$ arises as a space of square-integrable harmonic $\mathbf{L}$-valued differential forms on $\mathscr{O}$ [31, 32]. In a similar fashion, the representations parametrized by nonelliptic orbits can be realized as spaces of "partially harmonic" differential forms. See [42].

Except for some groups of low rank, little is known about singular unitary representations. Guided by the analogy with the nilpotent and solvable cases, one might hope for a natural bijection between the totality of integral coadjoint orbits and the full unitary dual. Apparently no such natural bijection exists, but several pieces of evidence suggest that all semisimple integral orbits, at least, do correspond to unitary representations. In this paper we realize a special class of singular unitary representations geometrically by a method which fits the correspondence between orbits and representations, and which makes sense conjecturally for all representations attached to elliptic integral orbits.

To put our results into perspective, we now recall the realization of discrete series representations in terms of L_2-cohomology groups [31, 32]. We suppose that the semisimple Lie group G contains a compact Cartan subgroup T—this is necessary and sufficient if G is to have a nonempty discrete series. We denote the Lie algebras of G, T by $\mathfrak{g}_0$, $\mathfrak{t}_0$, and their complexifications by $\mathfrak{g}$, $\mathfrak{t}$. The choice of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, with $\mathfrak{t} \subset \mathfrak{b}$, determines an invariant complex structure on the quotient manifold G/T whose holomorphic tangent space at the identity coset corresponds to $\mathfrak{b}/\mathfrak{t}$. Every element χ of the weight lattice $\Lambda \subset i\mathfrak{t}_0^*$ lifts to a character of T, and thus defines a homogeneous line bundle $\mathbf{L}_\chi \to G/T$. Once an invariant complex structure on G/T has been fixed, $\mathbf{L}_\chi$ can be turned uniquely into a holomorphic homogeneous line bundle. Since T is compact, both G/T and $\mathbf{L}_\chi$ carry invariant Hermitian metrics. The pth L_2-cohomology group $\mathscr{H}_2^p(G/T, \mathbf{L}_\chi)$ is defined as the kernel of the Laplace–Beltrami operator on the space of square-integrable $\mathbf{L}_\chi$-valued differential forms of bidegree $(0, p)$; G acts on it unitarily, by translation. We let ρ denote one half of the trace of $\operatorname{ad} \mathfrak{t}$ on $\mathfrak{b}/\mathfrak{t}$.[1] Then $\mathscr{H}_2^p(G/T, \mathbf{L}_\chi)$ vanishes for all p whenever $\chi + \rho$ is singular. If $\chi + \rho$ is regular, $\mathscr{H}_2^p(G/T, \mathbf{L}_\chi)$ does not vanish for exactly one value of p, depending on χ and the choice of $\mathfrak{b}$. The resulting unitary representation is irreducible, belongs to the discrete series, and is an invariant of the G-orbit of $-i(\chi + \rho)$, viewed as an element of $\mathfrak{g}_0^*$ via the Killing form. We should remark that this orbit can be identified with G/T, and that the distinguished line bundle on the orbit is $\mathbf{L}_{\chi+\rho}$. The shift by ρ serves the purpose of making the representation depend only on the orbit, not on the choices of T and $\mathfrak{b}$. In

[1] In order to make integral orbits correspond to representations, one must assume that ρ lies in Λ, which can be arranged by going to a finite covering of G.

this manner, the discrete series corresponds bijectively to the set of regular integral elliptic G-orbits in $\mathfrak{g}_0^*$.

Every square-integrable harmonic form is $\bar{\partial}$-closed, and hence represents a Dolbeault cohomology class. The resulting map

$$\mathscr{H}_2^p(G/T, \mathbf{L}_\chi) \to H^p(G/T, \mathbf{L}_\chi) \tag{1.1}$$

commutes with the action of G. According to the Hodge theorem, (1.1) is an isomorphism if G/T, or equivalently G, is compact. One cannot expect the same result for a noncompact quotient G/T. Nevertheless, there is a close relationship between the L_2-cohomology and Dolbeault cohomology of $\mathbf{L}_\chi$ in an important special case: so far, the choice of $\mathfrak{b}$, which determines the complex structure, has been left open; we can fix it by requiring that $\chi + \rho$ should lie in the anti-dominant Weyl chamber, relative to $\mathfrak{b}$. In this situation, both $\mathscr{H}_2^p(G/T, \mathbf{L}_\chi)$ and $H^p(G/T, \mathbf{L}_\chi)$ vanish for all p other than s, the dimension of a maximal compact subvariety of G/T. Moreover, the mapping (1.1) is injective for $p = s$, and its image contains all vectors in $H^s(G/T, \mathbf{L}_\chi)$ which transform finitely under the action of a maximal compact subgroup K. In Harish-Chandra's terminology, $H^s(G/T, \mathbf{L}_\chi)$ is infinitesimally equivalent to $\mathscr{H}_2^s(G/T, \mathbf{L}_\chi)$. Both may be regarded as realizations of the same discrete series representation, although only the latter displays the unitary structure.

If one allows the parameter $\chi + \rho$ to wander outside the anti-dominant Weyl chamber, the Dolbeault cohomology group $H^s(G/T, \mathbf{L}_\chi)$ tends to become reducible, and cohomology turns up in other dimensions as well. In a vague sense, the composition factors are "continuations" of discrete series representations. Some representations in this class are either known or suspected to be infinitesimally equivalent to unitary representations. It is natural from several points of view to associate them to singular integral elliptic orbits in $\mathfrak{g}_0^*$. A conjecture of Zuckerman makes this precise. The conjecture involves his derived functor construction [39, 43], which is an algebraic analogue of Dolbeault cohomology on homogeneous spaces. We now describe the conjecture, translated back into geometric terms.

As homogeneous spaces, the integral elliptic orbits in $\mathfrak{g}_0^*$ can be identified with quotients of G by the centralizer of a torus. Any such quotient G/H carries invariant complex structures, corresponding to certain choices of parabolic subalgebras of $\mathfrak{g}$, and homogeneous holomorphic line bundles $\mathbf{L}_\chi$, indexed by the differential χ of the character by which H acts on the fibre at the identity coset. To each triple consisting of an invariant complex structure, a character of H, and an integer p between 0 and $\dim_{\mathbb{C}} G/H$, Zuckerman's construction assigns a Harish-Chandra module—presumably the Harish-Chandra module of K-finite vectors in $H^p(G/H, \mathbf{L}_\chi)$, although this has not yet been proved. If χ satisfies a condition similar to the anti-dominance of $\chi + \rho$ in the case of the discrete series, the modules vanish in all but one degree, equal to the dimension s of a maximal compact subvariety of G/H. The remaining module in degree s is nonzero, irreducible, and may be viewed as an invariant of the G-orbit through $-i\chi$, shifted by a quantity which depends on the complex structure. Zuckerman's conjecture predicts that it is unitary. By analogy, one should expect

$$H^p(G/H, \mathbf{L}_\chi) = 0 \qquad \text{for} \quad p \neq s, \tag{1.2a}$$

$H^s(G/H, \mathbf{L}_\chi)$ is nonzero and is infinitesimally equivalent to an irreducible unitary representation, (1.2b)

provided again χ satisfies the appropriate anti-dominance assumption.

If H is compact, $H^s(G/H, \mathbf{L}_\chi)$ turns out to be infinitesimally equivalent to a discrete series representation and can be realized unitarily as a space of square-integrable harmonic $\mathbf{L}_\chi$-valued forms on G/H. One might hope to prove (1.2) in general by relating $H^s(G/H, \mathbf{L}_\chi)$ to an L_2-cohomology group, but one quickly faces a serious obstacle: the manifold G/H has no G-invariant Hermitian metric unless H is compact. What does exist is a G-invariant, indefinite, nondegenerate metric. Although it seems unlikely at first glance that an indefinite metric can be used to produce a Hilbert space of harmonic forms, there are some encouraging precedents, such as the Bleuler–Gupta construction [4, 10] of the photon representation, and the quantization of the $U(k, l)$ action on $\mathbf{R}^{2(k+l)}$ by Blattner and Rawnsley [3].

The $\bar{\partial}$-operator on G/H has a G-invariant formal adjoint $\bar{\partial}^*$ relative to the invariant indefinite metric. We call an $\mathbf{L}_\chi$-valued differential form ω harmonic if it satisfies the two first-order equations[2]

$$\bar{\partial}\omega = 0, \qquad \bar{\partial}^*\omega = 0. \tag{1.3}$$

In a noncanonical fashion, one can manufacture a positive definite metric from the indefinite one by "reversing signs." If this is done judiciously, G preserves the space of square-integrable forms, even though it distorts the L_2 norm. We fix such a definite, non-invariant metric, and define $\mathscr{H}_2^p(G/H, \mathbf{L}_\chi)$ as the space of square-integrable $\mathbf{L}_\chi$-valued $(0, p)$-forms with measurable coefficients which satisfy Eqs. (1.3) in the sense of distributions. It is a Hilbert space on which H acts continuously. A differential form $\omega \in \mathscr{H}_2^p(G/H, \mathbf{L}_\chi)$ need not be smooth, since (1.3) is a hyperbolic system. Nevertheless, as a $\bar{\partial}$-closed distribution, ω determines a Dolbeault cohomology class: just as in the case of a definite metric, there is a natural G-invariant map

$$\mathscr{H}_2^p(G/H, \mathbf{L}_\chi) \to H^p(G/H, \mathbf{L}_\chi) \tag{1.4}$$

[2] For square-integrable forms ω on a manifold with a positive definite, complete metric, these two equations are equivalent to the Laplace–Beltrami equation $(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*) = 0$, but in the present context they are more restrictive.

from L_2-cohomology to Dolbeault cohomology. The invariant indefinite metric defines a G-invariant indefinite bounded Hermitian form $\langle \ , \ \rangle$ on $\mathscr{H}^p_2(G/H, \mathbf{L}_\chi)$. If one is optimistic, one may conjecture

the image of the map (1.4) contains all K-finite cohomology classes, (1.5a)

the kernel coincides with the radical of $\langle \, , \rangle$, (1.5b)

the induced Hermitian form on the image is positive definite. (1.5c)

In particular this would make $H^p(G/H, \mathbf{L}_\chi)$ infinitesimally equivalent to a unitary representation.

The main result of our paper is a special case of (1.5). We suppose

$$G/H \text{ is pseudo-Kähler symmetric;} \tag{1.6}$$

more precisely, we require H to be not only the centralizer of a torus, but also the group of fixed points of an involutive automorphism. Homogeneous spaces of this type can be identified with minimal semisimple orbits in $\mathfrak{g}_0^*$, and should correspond to highly singular representations. In addition to (1.6), we assume that G/K has a Hermitian symmetric structure which is compatible with the complex structure of G/H in the following sense: replacing K by a suitable conjugate, we make $H \cap K$ maximal compact in H; we want $G/H \cap K$ to carry an invariant complex structure such that

$$\text{both} \quad G/H \cap K \to G/H \quad \text{and} \quad G/H \cap K \to G/K \quad \text{are holomorphic.} \tag{1.7}$$

Under these conditions on G/H, with the usual anti-dominance hypothesis on χ, we prove (1.2) and (1.5). The representations covered by Zuckerman's conjecture all have regular infinitesimal characters. Perhaps surprisingly, a modified version of our construction produces unitary representations on L_2 harmonic spaces well beyond the range where the infinitesimal character is regular. If H has a compact simple factor, there exist homogeneous Hermitian vector bundles $\mathbf{V} \to G/H$, modelled on finite-dimensional irreducible unitary H-modules V of dimension greater than one. Since our arguments extend easily to this situation, we work with homogeneous vector bundles from the very beginning. We also prove analogues of (1.2) and (1.5) for certain infinite-dimensional Hermitian vector bundles. The resulting unitary representations are infinitesimally equivalent to Dolbeault cohomology groups of finite-dimensional vector bundles over homogeneous spaces G/H_1, with $H_1 \subset H$.

Condition (1.7) imposes a very severe restriction on the indefinite-Kähler symmetric space G/H. It has the effect of making the G-modules $H^s(G/H, \mathbf{V})$ belong to the continuation of the "holomorphic discrete series." The unitary representations in the continuation of the "holomorphic discrete series" were classified recently by algebraic arguments [7, 8]; also see [23]. In particular, the representations we exhibit are already known to be unitary. We should point out, however, that our purpose is not so much the construction of some specific unitary representation. Rather, we want to explore a general method which fits into the framework of geometric quantization and which might apply eventually to all representations attached to elliptic integral orbits in $\mathfrak{g}_0^*$. For this reason we carry our arguments as far as we can without using (1.7): we prove the vanishing theorem (1.2a), we identify the G-module $H^s(G/H, \mathbf{L}_\chi)$, and we prove that the radical of the invariant Hermitian form on $\mathscr{H}_2^s(G/H, \mathbf{L}_\chi)$ contains the kernel of the map (1.4).

Let us close the introduction with a brief guide through our paper. In Section 2 we describe a fibration of the indefinite Kähler symmetric space G/H over its maximal compact subvariety $K/H \cap K$; it has Hermitian symmetric fibres and is holomorphic precisely when G/H satisfies condition (1.7). Holomorphic or not, the fibration leads to a spectral sequence for the cohomology of any homogeneous vector bundle $\mathbf{V}$, which is the subject of Section 3. The next two sections use the spectral sequence to identify the K-spectrum and the global characters of the cohomology $H^*(G/H, \mathbf{V})$. Up to this point condition (1.7) is not needed. It assumes a crucial role in Section 6, where we write down, more or less explicitly, certain special harmonic forms—enough to represent all K-finite cohomology classes. We construct a particular positive definite, non-invariant metric in Section 7, and show that the special harmonic forms of Section 6 are square-integrable with respect to it under appropriate hypotheses on the vector bundle $\mathbf{V}$. To complete the proof of (1.5), we must prove that a square-integrable $\bar{\partial}$-exact form can be approximated in L_2 norm by $\bar{\partial}$-boundaries of square-integrable forms. We do so in Section 8, by means of an L_2 version of the spectral sequence of Section 3. Section 9 contains a variant of (1.5), which we can prove under less stringent assumptions on $\mathbf{V}$, but which suffices to unitarize the G-modules $H^s(G/H, \mathbf{V})$. The various criteria for the vanishing theorem (1.2a), for the existence of square-integrable harmonic forms, for the nonsingularity of the infinitesimal character, and for certain properties of the K-spectrum are not immediately comparable; we sort out the relationships among them in Section 10. Except for the irreducibility, which we treat in Section 11, this completes the study of our representations in the case of a finite-dimensional bundle $\mathbf{V}$. The infinite-dimensional case requires some further considerations and is the subject of Section 12. Unitary highest weight modules of the groups $U(k, l)$, and of $U(2, 2)$ in particular, have been studied extensively by mathematical physicists. To facilitate a comparison of our construction with other realizations of these representations, we give a very detailed account of our results for the indefinite unitary groups in the last section.

The work described in this paper went through several stages. Its origins and history are recounted in Appendix A. Appendix B contains completely explicit formulas for the special harmonic forms of Section 6.

2. Fibration over the Maximal Compact Subvariety

Let G be a connected reductive Lie group, τ an involutive automorphism of G, and $H=(G^\tau)^0$, the identity component of the fixed point set of τ. The Killing form of G defines an indefinite metric symmetric space structure on G/H. Choose a Cartan involution θ of G that commutes with τ. Its fixed point set $K=G^\theta$ is the Ad_G^{-1}-image of a maximal compact subgroup of $\mathrm{Ad}(G)$. The orbit $K\cdot H\cong K/K\cap H$ of K in G/H is a Riemannian symmetric space; it is our maximal compact subvariety. We are going to describe a C^∞ fibration $\pi\colon G/H\to K/K\cap H$ and analyse it thoroughly in the case where G/H is indefinite Kähler. That analysis is the geometric basis for our study of square-integrable cohomologies of homogeneous holomorphic vector bundles $\mathbf{V}\to G/H$.

Retain G, H, K, τ, and θ as above. Denote $M=G^{\theta\tau}$ and $L=K\cap H$, so $L=M^\theta=M^\tau=K^\tau=H^\theta$, maximal compactly embedded subgroup in M and in H.

Denote the respective real Lie algebras

$$\mathfrak{g}_0,\mathfrak{h}_0,\mathfrak{k}_0,\mathfrak{m}_0,\mathfrak{l}_0 \qquad \text{for} \quad G,H,K,M,L. \tag{2.1}$$

Denote the (± 1)-eigenspace Cartan decompositions

$$\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0 \quad \text{under} \quad \theta, \qquad \mathfrak{g}_0=\mathfrak{h}_0+\mathfrak{q}_0 \quad \text{under} \quad \tau. \tag{2.2}$$

Then of course

$$\begin{gathered}\mathfrak{g}_0=(\mathfrak{k}_0\cap\mathfrak{h}_0)+(\mathfrak{k}_0\cap\mathfrak{q}_0)+(\mathfrak{p}_0\cap\mathfrak{h}_0)+(\mathfrak{p}_0\cap\mathfrak{q}_0),\\ \mathfrak{l}_0=\mathfrak{k}_0\cap\mathfrak{h}_0, \qquad \mathfrak{h}_0=\mathfrak{l}_0+(\mathfrak{p}_0\cap\mathfrak{h}_0), \qquad \mathfrak{m}_0=\mathfrak{l}_0+(\mathfrak{p}_0\cap\mathfrak{q}_0).\end{gathered} \tag{2.3}$$

Complexifications will be denoted by dropping the subscript, as in $\mathfrak{g}=(\mathfrak{g}_0)_{\mathbf{C}}$.

A result of Mostow [22, Theorem 5] says that

$$a\colon K\times(\mathfrak{p}_0\cap\mathfrak{q}_0)\times(\mathfrak{p}_0\cap\mathfrak{h}_0)\to G \qquad \text{by} \quad a(k,\xi,\eta)=k\cdot\exp(\xi)\cdot\exp(\eta) \tag{2.4}$$

is a diffeomorphism. In particular, $G=KMH$. More precisely, choose

$$\begin{aligned}\mathfrak{a}_0\subset\mathfrak{p}_0\cap\mathfrak{q}_0&:\ \text{Cartan subalgebra for } (\mathfrak{m}_0,\mathfrak{l}_0),\\ \mathfrak{a}_0^+&:\ \text{positive chamber for an ordering of the}\\ &\ \mathfrak{a}_0\text{-roots of } \mathfrak{m}_0.\end{aligned} \tag{2.5}$$

Then the Cartan decomposition $M = LA^+L$ gives a decomposition

$$G = KA^+H, \qquad \text{where} \quad A^+ = \text{closure}(\exp \mathfrak{a}_0^+) \tag{2.6}$$

which is C^∞ and unique up to $(kz)ah = ka(zh)$ with $z \in Z_L(\mathfrak{a}_0)$. Flensted-Jensen also noticed this.

2.7 PROPOSITION. *Define* $\pi: G/H \to K/L$ *by* $\pi(gH) = kL$, *where* $g = k \cdot \exp(\xi) \cdot \exp(\eta)$ *as in* (2.4). *Then* π *is well defined and is given by* $\pi(kmH) = kL$ *for* $k \in K$ *and* $m \in M$. *Here* $\pi: G/H \to K/L$ *is a K-equivariant* C^∞ *fibre bundle with typical fibre* M/L *and structure group* L. *It is associated to the principal L-bundle* $K \to K/L$ *by the* (*usual left*) *action of* L *on* M/L.

Proof. Let $g' \in gH$, say $g' = gh$, and express $g = k \cdot \exp(\xi) \cdot \exp(\eta)$ as in (2.4). Then $\exp(\eta)h \in H = L \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{h}_0)$ has form $l \cdot \exp(\eta')$, and $\exp(\xi)l \in M = L \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{q}_0)$ has form $l' \cdot \exp(\xi')$, so $g' = (kl') \cdot \exp(\xi') \cdot \exp(\eta')$ as in (2.4). As $kL = (kl')L$, this shows π well defined. Similarly, in any expression $g = k_1 m_1 h_1$, express $h_1 = l_1 \cdot \exp(\eta_1)$ and $m_1 l_1 = l_2 \cdot \exp(\xi_1)$ to see $k_1 = kl_2^{-1} \in kL$, so $\pi(gH) = k_1 L$.

The K-equivariance property, $\pi(kgH) = k \cdot \pi(gH)$, is clear.

Identify $\mathfrak{p}_0 \cap \mathfrak{q}_0$ with M/L under $\xi \mapsto \exp(\xi)L$. The adjoint action of L on $\mathfrak{p}_0 \cap \mathfrak{q}_0$ goes over to the usual left action of L on M/L. The associated bundle to $K \to K/L$ for this action has total space $K \times_L (\mathfrak{p}_0 \times \mathfrak{q}_0)$, equivalence classes of pairs (k, ξ) under $[kl, \xi] = [k, \text{Ad}(l)\xi]$. Now $[k, \xi] \mapsto k \cdot \exp(\xi)H$ is a fibre space equivalence of the associated bundle with $\pi: G/H \to K/L$. Q.E.D.

From now on, we assume that G/H has a G-invariant complex structure, i.e., that

$$G/H \text{ is an indefinite Kähler semisimple symmetric space.} \tag{2.8}$$

In other words, $G = G_1 \cdots G_r \cdot Z$ and $H = H_1 \cdots H_r \cdot Z$, local direct products, where the G_i are the simple normal analytic subgroups of G and Z is the identity components of the center, and $H_i = H \cap G_i$ either equals G_i or has one-dimensional center contained in K. Here H is the identity component of the G-centralizer of its center, and $\mathfrak{h}_0$ has a central element ζ such that $\text{ad}(\zeta): \mathfrak{q}_0 \to \mathfrak{q}_0$ gives the almost-complex structure. The holomorphic and antiholomorphic tangent spaces of G/H are represented respectively by

$$\mathfrak{q}_+ = \{\xi \in \mathfrak{q}: [\zeta, \xi] = i\xi\} \qquad \text{and} \qquad \mathfrak{q}_- = \{\xi \in \mathfrak{q}: [\zeta, \xi] = -i\xi\}. \tag{2.9}$$

Let Q denote the parabolic subgroup of $\text{Ad}(G)_{\mathbf{C}}$ with Lie algebra $\text{ad}(\mathfrak{h} + \mathfrak{q}_-)$. We remark that $\text{Ad}(G)_{\mathbf{C}}/Q$ is the complex flag manifold presentation of a Hermitian symmetric space of compact type and that the G-orbit of the identity coset can be identified with $G/\tilde{H}$, where $\tilde{H}$ = normalizer of $\mathfrak{h} \oplus \mathfrak{q}_-$ in G has identity component H. See [41]. The quotient $G/\tilde{H}$ is then simply connected, hence $\tilde{H}$ is connected, and hence H is the full normalizer of $\mathfrak{h} \oplus \mathfrak{q}_-$ in G as a consequence of the existence of the complex structure. A number of other familiar facts about Hermitian symmetric spaces have exact analogs in our case. Thus, either by direct verification or by application of [41, Theorem 4.5] one sees that

H contains a θ-stable fundamental Cartan subgroup T_G of G (2.10)

and that

there is a system Φ_G^+ of positive $\mathfrak{t}_G$-roots of $\mathfrak{g}$ such that complex conjugation sends Φ_G^+ to $-\Phi_G^+$, the root system of $\mathfrak{h}$ is generated by certain simple roots, and $\mathfrak{q}_+$ is a sum of positive roots spaces. (2.11)

In particular,

$T = T_G \cap K$ is a Cartan subgroup in L, in K, and in M. (2.12)

Note that T contains the center of H.

The central element ζ of $\mathfrak{h}_0$, which gives the almost-complex structure, is in $\mathfrak{l}$. Thus

$$\mathfrak{q}_+ = (\mathfrak{k} \cap \mathfrak{q}_+) + (\mathfrak{p} \cap \mathfrak{q}_+) \quad \text{and} \quad \mathfrak{q}_- = (\mathfrak{k} \cap \mathfrak{q}_-) + (\mathfrak{p} \cap \mathfrak{q}_-). \tag{2.13}$$

It follows that the maximal compact subvariety

K/L is a complex submanifold of G/H and is a Hermitian symmetric space of compact type (2.14)

and the fibres

$\pi^{-1}(kL) = kM/L$ are complex submanifolds of G/H and are Hermitian symmetric spaces of noncompact type. (2.15)

Nevertheless, the C^∞ bundle $\pi: G/H \to K/L$ often is not holomorphic. See Proposition 2.21 below.

If $x \in G$, then $t_x: G/H \to G/H$ denotes the translation $gH \mapsto xgH$, and if $x \in K$, we also write t_x for $kK \mapsto xkL$. Tangent spaces are given by

$$\begin{aligned} &T_{gH}(G/H) = t_g\mathfrak{q}_0, && T_{gH}^{1,0}(G/H) = t_g\mathfrak{q}_+, && T_{gH}^{0,1}(G/H) = t_g\mathfrak{q}_-; \\ &T_{kL}(K/L) = t_k(\mathfrak{k}_0 \cap \mathfrak{q}_0), && T_{kL}^{1,0}(K/L) = t_k(\mathfrak{k} \cap \mathfrak{q}_+), && T_{kL}^{0,1}(K, L) = t_k(\mathfrak{k} \cap \mathfrak{q}_-). \end{aligned} \tag{2.16}$$

The bundle $G/H \to K/L$ has a K-invariant connection whose horizontal spaces are the translates

$$t_k t_m(\mathfrak{k} \cap \mathfrak{q}_0) \subset T_{kmH}(G/H), \qquad k \in K, \quad m \in M. \tag{2.17}$$

The (-1)-eigenspace of $\theta\tau$ on $\mathfrak{g}$ is $(\mathfrak{k} \cap \mathfrak{q}) + (\mathfrak{p} \cap \mathfrak{h})$. It is $\mathrm{Ad}(M)$-invariant. Denote its projections

$$p'\colon (\mathfrak{k} \cap \mathfrak{q}) + (\mathfrak{p} \cap \mathfrak{h}) \to \mathfrak{k} \cap \mathfrak{q} \qquad \text{and} \qquad p''\colon (\mathfrak{k} \cap \mathfrak{q}) + (\mathfrak{p} \cap \mathfrak{h}) \to \mathfrak{p} \cap \mathfrak{h}. \tag{2.18}$$

Now we can express the differential of π at $kmH \in G/H$ as

$$\begin{aligned} \pi_*(t_k t_m \xi) &= 0, && \text{for} \quad \xi \in \mathfrak{p} \cap \mathfrak{q}, \\ &= t_k(p' \circ \mathrm{Ad}(m)\xi), && \text{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}. \end{aligned} \tag{2.19}$$

A slight reformulation of (2.19): We can assume $m = \exp(\eta)$ with $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, so

$$\mathrm{Ad}(m)\xi = \exp(\mathrm{ad}(\eta))\xi = \cosh(\mathrm{ad}(\eta))\xi + \sinh(\mathrm{ad}(\eta))\xi.$$

Even powers of $\mathrm{ad}(\eta)$ send $\mathfrak{k} \cap \mathfrak{q}$ to $\mathfrak{k} \cap \mathfrak{q}$ and odd powers send it to $\mathfrak{p} \cap \mathfrak{h}$. Thus (2.19) can be phrased

$$\begin{aligned} \pi_*(t_k t_m \xi) &= 0, && \text{for} \quad \xi \in \mathfrak{p} \cap \mathfrak{q}, \\ &= t_k(\cosh(\mathrm{ad}(\eta))\xi), && \text{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}, \end{aligned} \tag{2.20}$$

where $m = \exp(\eta)$ with $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$.

2.21. PROPOSITION. *The following conditions are equivalent*:

(1) $\pi\colon G/H \to K/L$ *is holomorphic.*

(2) G/K *is a Hermitian symmetric space, and one can choose* G*-invariant complex structures on* G/K *and* G/L *in such a way that both* $G/L \to G/K$ *and* $G/L \to G/H$ *are holomorphic.*

(3) *If* $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, *then* $\cosh(\mathrm{ad}(\eta))(\mathfrak{k} \cap \mathfrak{q}_-) \subset \mathfrak{k} \cap \mathfrak{q}_-$.

(4) *If* $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, *then* $\mathrm{ad}(\eta)^2(\mathfrak{k} \cap \mathfrak{q}_-) \subset \mathfrak{k} \cap \mathfrak{q}_-$.

(5) $[\mathfrak{p} \cap \mathfrak{q}_+, [\mathfrak{p} \cap \mathfrak{q}_+, \mathfrak{k} \cap \mathfrak{q}_-]] = 0$.

Remark. In (2), the G-invariant complex structure corresponds to an $\mathrm{Ad}(K)$-invariant splitting $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$, with $\mathfrak{p}_+ \cap \mathfrak{q}_- = 0$.

Proof. The map π is holomorphic just when every $\pi_* T^{0,1}_{gH}(G/H) \subset T^{0,1}_{\pi(gH)}(K/L)$, so (2.20) says that (1) and (3) are equivalent, and evidently (3) and (4) are equivalent.

Let $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, say $\eta = \eta_+ + \eta_-$ with $\eta_\pm \in \mathfrak{p} \cap \mathfrak{q}_\pm$, so $\eta_- = \overline{\eta_+}$. Expand $\eta_+ = \sum \eta_\alpha$, where η_α is in the t-root space for $\alpha \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$. If ξ is in the t-root space for $\beta \in \Phi(\mathfrak{k} \cap \mathfrak{q}_-)$, then $[\mathfrak{q}_-, \mathfrak{q}_-] = 0$ gives

$$\mathrm{ad}(\eta)^2\xi = \sum_{\alpha_1, \alpha_2} \{[\eta_{\alpha_1}, [\eta_{\alpha_2}, \xi]] + [\overline{\eta_{\alpha_1}}, [\eta_{\alpha_2}, \xi]]\}$$

and

$$[\overline{\eta_{\alpha_1}}, [\eta_{\alpha_2}, \xi]] = [[\overline{\eta_{\alpha_1}}, \eta_{\alpha_2}], \xi] \in [\mathfrak{l}, \mathfrak{k} \cap \mathfrak{q}_-] \subset \mathfrak{k} \cap \mathfrak{q}_-.$$

Now, since $[\eta_{\alpha_1}, [\eta_{\alpha_2}, \xi]] \in \mathfrak{k} \cap \mathfrak{q}_+$,

$$\mathrm{ad}(\eta)^2\xi \in \mathfrak{k} \cap \mathfrak{q}_- \Leftrightarrow \sum_{\alpha_1, \alpha_2} [\eta_{\alpha_1}, [\eta_{\alpha_2}, \xi]] = 0.$$

Suppose that ξ, η_α belong to a Chevalley basis of $\mathfrak{g}$. If $\{t_\alpha\}$ are algebraically independent and $\eta' = \eta'_+ + \overline{\eta'_+}$ with $\eta'_+ = \sum t_\alpha \eta_\alpha$, then $\mathrm{ad}(\eta')^2\xi \in \mathfrak{k} \cap \mathfrak{q}_-$ if and only if each

$$t_{\alpha_1} t_{\alpha_2}\{[\eta_{\alpha_1}, [\eta_{\alpha_2}, \xi]] + [\eta_{\alpha_2}, [\eta_{\alpha_1}, \xi]]\} = 0.$$

As $[\mathfrak{q}_+, \mathfrak{q}_+] = 0$, that just says that each $[\eta_{\alpha_1}, [\eta_{\alpha_2}, \xi]] = 0$. Thus (4) and (5) are equivalent.

We next prove: if π is holomorphic, then G/K is Hermitian symmetric. For this, we may assume G simple, and need only prove that

$$\mathfrak{r}: \quad \mathfrak{k}\text{-submodule of } \mathfrak{p} \text{ generated by } \mathfrak{p} \cap \mathfrak{q}_+$$

is properly contained in $\mathfrak{p}$. Calculate with the Killing form and use (5):

$$\begin{aligned}(\mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})^2(\mathfrak{p} \cap \mathfrak{q}_+), \mathfrak{p} \cap \mathfrak{q}_+)\\ = ([\mathfrak{p} \cap \mathfrak{q}_+, [\mathfrak{p} \cap \mathfrak{q}_+, \mathfrak{k} \cap \mathfrak{q}_-]], \mathfrak{k} \cap \mathfrak{q})) = 0.\end{aligned}$$

As $\mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})^2(\mathfrak{p} \cap \mathfrak{q}_+) \subset \mathfrak{p} \cap \mathfrak{q}$, this forces $\mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})^2(\mathfrak{p} \cap \mathfrak{q}_+) \subset \mathfrak{p} \cap \mathfrak{q}_+$. Now we can prove by induction on n that

$$\mathrm{ad}(\mathfrak{k})^n(\mathfrak{p} \cap \mathfrak{q}_+) \subset (\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+).$$

In effect, that is clear for $n = 0$ and

$$\begin{aligned}&\mathrm{ad}(\mathfrak{k})\{(\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+)\}\\ &= \mathrm{ad}(\mathfrak{l})(\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{l})\,\mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+)\\ &\quad + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})^2(\mathfrak{p} \cap \mathfrak{q}_+)\\ &\subset (\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+).\end{aligned}$$

Since $(\mathfrak{p} \cap \mathfrak{q}_+) + \mathrm{ad}(\mathfrak{k} \cap \mathfrak{q})(\mathfrak{p} \cap \mathfrak{q}_+)$ is a subspace of $\mathfrak{p}$ that does not contain $\mathfrak{p} \cap \mathfrak{q}_-$, we have proved $\mathfrak{r} \subsetneq \mathfrak{p}$ and so G/K is Hermitian.

Now we can show that (1) implies (2). For as above, when π is holomorphic, G/K is Hermitian with holomorphic tangent space $\mathfrak{p}_+ = \mathrm{ad}(U(\mathfrak{k}))(\mathfrak{p} \cap \mathfrak{q}_+)$, so $\mathfrak{p}_+ \cap \mathfrak{q}_- = 0$ and we may assume that the positive root system (2.11) has $\Phi(\mathfrak{p}_+) \cup \Phi(\mathfrak{q}_+) \subset \Phi^+$. Now G/L has holomorphic tangent space $(\mathfrak{p}_+ + \mathfrak{q}_+)$, so its maps to G/K and G/H are holomorphic.

Conversely, given (2), $\mathfrak{p}_- \cap \mathfrak{q}_+ = 0$, so $[\mathfrak{p} \cap \mathfrak{q}_+, [\mathfrak{p} \cap \mathfrak{q}_+, \mathfrak{k} \cap \mathfrak{q}_-]] \subset [\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{k}]] \subset [\mathfrak{p}_+, \mathfrak{p}_+] = 0$, showing (5). Q.E.D.

2.22. Example. $G/H = U(a_1 + a_2, b_1 + b_2)/U(a_1, b_1) \times U(a_2, b_2)$. The Lie algebras are given in matrix blocks by

$$\begin{pmatrix} \mathfrak{k} \cap \mathfrak{h} & \mathfrak{p} \cap \mathfrak{h} & \mathfrak{k} \cap \mathfrak{q}_+ & \mathfrak{p} \cap \mathfrak{q}_+ \\ \mathfrak{p} \cap \mathfrak{h} & \mathfrak{k} \cap \mathfrak{h} & \mathfrak{p} \cap \mathfrak{q}_+ & \mathfrak{k} \cap \mathfrak{q}_+ \\ \mathfrak{k} \cap \mathfrak{q}_- & \mathfrak{p} \cap \mathfrak{q}_- & \mathfrak{k} \cap \mathfrak{h} & \mathfrak{p} \cap \mathfrak{h} \\ \mathfrak{p} \cap \mathfrak{q}_- & \mathfrak{k} \cap \mathfrak{q}_- & \mathfrak{p} \cap \mathfrak{h} & \mathfrak{k} \cap \mathfrak{h} \end{pmatrix} \begin{matrix} \}\, a_1 \\ \}\, b_1 \\ \}\, a_2 \\ \}\, b_2 \end{matrix}$$
$$\qquad \underbrace{\quad}_{a_1} \qquad \underbrace{\quad}_{b_1} \qquad \underbrace{\quad}_{a_2} \qquad \underbrace{\quad}_{b_2}$$

One easily checks that $[\mathfrak{p} \cap \mathfrak{q}_+, [\mathfrak{p} \cap \mathfrak{q}_+, \mathfrak{k} \cap \mathfrak{q}_-)) \neq 0$ unless at least one of a_1, a_2, b_1, b_2 vanishes. Thus $\pi: G/H \to K/L$ is holomorphic if and only if one or more of a_1, a_2, b_1, b_2 is zero. Note that G/K is a Hermitian symmetric space in this example.

The projection p' of (2.18) is the direct sum of two projections p'_+ and p'_-, given by

$$p'_\pm : (\mathfrak{k} \cap \mathfrak{q}_+) + (\mathfrak{k} \cap \mathfrak{q}_-) + (\mathfrak{p} \cap \mathfrak{h}) \to \mathfrak{k} \cap \mathfrak{q}_\pm. \tag{2.23}$$

We use them to see that π is, at least, never infinitesimally antiholomorphic.

2.24. Lemma. *The differential of $\pi: G/H \to K/L$, followed by projection to the antiholomorphic tangent space, at $kmH \in G/H$, is*

$$\begin{aligned} t_k t_m \xi &\mapsto 0, && \text{for} \quad \xi \in \mathfrak{p} \cap \mathfrak{q}, \\ &\mapsto t_k(p'_- \circ \mathrm{Ad}(m)\xi), && \text{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}. \end{aligned} \tag{2.25}$$

It restricts to a linear isomorphism of the (0, 1) *horizontal space*

$$t_k t_m(\mathfrak{k} \cap \mathfrak{q}) \cap T^{0,1}_{kmH}(G/H) = t_k t_m(\mathfrak{k} \cap \mathfrak{q}_-)$$

onto $T^{0,1}_{kL}(K/L)$.

Proof. The first assertion, (2.25), comes from (2.19). Write (ξ_1, ξ_2) for the positive definite Hermitian inner product $-B(\xi_1, \bar{\xi}_2)$ on $\mathfrak{k} \cap \mathfrak{q}$, where B is the Killing form. Each $\mathrm{ad}(\eta)^{2n}$, $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$ and $n \geqslant 0$ integer, is symmetric and positive semi-definite on $\mathfrak{k} \cap \mathfrak{p}$ with respect to $(\, , \,)$, so $p' \circ \mathrm{Ad}(\exp(\eta)) = \cosh(\mathrm{ad}(\eta))$ is symmetric and positive definite. Thus $p'_- \circ \mathrm{Ad}(\exp(\eta))$ maps $\mathfrak{k} \cap \mathfrak{q}_-$ isomorphically to itself. Q.E.D.

The inverse of (2.25) is the horizontal left operation for (0, 1) vector fields on K/L,

$$R_{km}: \quad T^{0,1}_{kL}(K/L) \to R^{0,1}_{kmH}(G/H) \qquad \text{by} \quad R_{km}(t_k \xi) = t_k t_m(\{p'_- \circ \mathrm{Ad}(m)|_{\mathfrak{k} \cap \mathfrak{q}_-}\}^{-1} \xi) \tag{2.26}$$

for $\xi \in \mathfrak{p} \cap \mathfrak{q}_-$. The deviation of $\pi: G/H \to K/L$ from holomorphicity is given by

$$S_{km}: \quad T^{0,1}_{kL}(K/L) \to T^{1,0}_{kL}(K/L) \qquad \text{by} \quad S_{km}(t_k \xi) = t_k(p'_+ \circ \mathrm{Ad}(m) \circ \{p'_- \circ \mathrm{Ad}(m)|_{\mathfrak{k} \cap \mathfrak{q}_-}\}^{-1} \xi). \tag{2.27}$$

To see that, note $p' = p'_+ + p'_-$, so (2.19) becomes

$$\pi_* R_{km}(t_k \xi) = t_k(\xi) + S_{km}(t_k \xi), \tag{2.28}$$

so π is holomorphic just when all the $S_{km} = 0$.

3. A Spectral Sequence for $\pi: G/H \to K/L$

Retain the setup and notation of Section 2 for the indefinite Kähler semisimple symmetric space G/H and its fibration $\pi: G/H \to K/L$. Fix

$$\begin{aligned} \psi &: \text{ irreducible } L\text{-finite Fréchet representation of } H, \\ V &: \text{ representation space of } \psi, \\ \mathbf{V} \to G/H &: \text{ associated homogeneous holomorphic vector bundle.} \end{aligned} \tag{3.1}$$

In general, but not always, ψ will be an irreducible unitary representation of H, in which case L-finiteness is automatic. In this section we will develop a variation on the Leray spectral sequence for the Dolbeault cohomologies $H^p(G/H, \mathbf{V})$.

Here $A^m(G/H, \mathbf{V})$ is the Fréchet space of $\mathbf{V}$-valued C^∞ $(0, m)$-forms on G/H. A form $\varphi \in A^m(G/H, \mathbf{V})$ *vanishes to order p* on the fibres of $\pi: G/H \to K/L$ if its has base degree $\geqslant p$, i.e., if locally every monomial summand of φ has p or more 1-form factors that annihilate the tangent spaces of the fibres. Equivalently, $\varphi(\eta_1, ..., \eta_m) = 0$ whenever at least $m - p + 1$ of the vector fields η_i are tangent to the fibres. Set

$$\mathscr{C}^{p,q} = \{\varphi \in A^{p+q}(G/H, \mathbf{V}) \colon \varphi \text{ vanishes to order } p\}. \tag{3.2}$$

From the definition,

$$\bar{\partial}\mathscr{C}^{p,q} \subset \mathscr{C}^{p,q+1}; \text{ so } \mathscr{F}^p = \sum_{q \geqslant 0} \mathscr{C}^{p,q} \text{ filters the complex } \{\mathscr{C}^{p,q}, \bar{\partial}\}. \tag{3.3}$$

This is a decreasing filtration, $\mathscr{F}^0 = \sum_{p,q} \mathscr{C}^{p,q}$, and $\mathscr{F}^m = 0$ for $m > \dim_{\mathbf{C}} K/L$. So it leads to a spectral sequence $\{E_r^{p,q}, d_r\}$, with

$$E_0^{p,q} = \mathscr{C}^{p,q}/\mathscr{C}^{p+1,q-1}. \tag{3.4}$$

Since the $\mathscr{C}^{p,q}$ are closed in $A^{p+q}(G/H, \mathbf{V})$, $E_0^{p,q}$ inherits the structure of Fréchet space.

3.5. Lemma. *Let $A^q(M/L, \mathbf{V})$ denote the Fréchet space of $\mathbf{V}|_{M/L}$-valued C^∞ $(0, q)$-forms on M/L. Then the image of $\bar{\partial}_{M/L} \colon A^q(M/L, \mathbf{V}) \to A^{q+1}(M/L, \mathbf{V})$ is equal to the kernel of $\bar{\partial}_{M/L} \colon A^{q+1}(M/L, \mathbf{V}) \to A^{q+2}(M/L, \mathbf{V})$. In particular,* (i) *$\bar{\partial}_{M/L}$ has closed range,* (ii) *the Dolbeault cohomologies $H^q(M/L, \mathbf{V})$ inherit Fréchet space structures, and* (iii) *$H^q(M/L, \mathbf{V}) = 0$ for $q > 0$.*

Proof. Bungart [5] extended Cartan's Theorems A and B to Fréchet space valued coherent analytic sheaves over Stein analytic spaces. This applies to the sheaf $\mathscr{O}(\mathbf{V}) \to M/L$ of germs of holomorphic sections of $\mathbf{V}|_{M/L}$ over the Stein manifold M/L. Thus the sheaf cohomology spaces $H^q(M/L, \mathscr{O}(\mathbf{V})) = 0$ for $q > 0$. We need to carry this over to Dolbeault cohomology. That is just a matter of noticing that the usual finite-dimensional argument is valid in our case. The details follow:

First, $\mathbf{V}|_{M/L} \to M/L$ is holomorphically trivial in the standard manner: an L-finite section $w \colon M \to V$ corresponds to the function $f_w \colon M/L \to V$ given by $f_w(mL) = \psi(\tilde{m}) \cdot w(m)$, where $\tilde{m} \in L_{\mathbf{C}}$ and $m \in \exp(\mathfrak{p} \cap \mathfrak{q}_+) \cdot \tilde{m} \cdot \exp(\mathfrak{p} \cap \mathfrak{q}_-)$. So we need only check that, in the standard proof of Dolbeault's theorem for the constant line bundle over M/L, we can replace any space of $\mathbf{C}$-valued forms by the corresponding space of $\mathbf{V}$-valued forms. Second, that corresponding space is obtained by injective (Schwartz ε) tensor product with V. Now the verification of extensibility of Dolbeault's theorem comes down to the question of whether exactness of $0 \to E' \to E \to E'' \to 0$ implies exactness of $0 \to E' \otimes V \to E \otimes V \to E'' \otimes V \to 0$ when the E's are all of, or a quotient of, or a closed subspace of a Fréchet space $A^m(K/L, \mathbf{C})$ and $\otimes$ is the injective tensor product. But that is known (see [5, Theorem 5.3]) because the $A^m(K/L, \mathbf{C})$, and thus the E's, are nuclear. Q.E.D.

The (continuous linear) action of L on $A^q(M/L, \mathbf{V})$ gives us

$$\mathbf{A}^q(M/L, \mathbf{V}) \to K/L: \quad \text{associated vector bundle.} \tag{3.6}$$

Similarly, the action of L on the Dolbeault cohomology $H^q(M/L, \mathbf{V})$ defines

$$\mathbf{H}^q(M/L, \mathbf{V}) \to K/L: \quad \text{associated vector bundle.} \tag{3.7}$$

These are the homogeneous vector bundles over K/L whose fibres at kL are the respective Fréchet spaces $A^q(kK/L, \mathbf{V})$ and $H^q(kM/L, \mathbf{V})$. We need them to analyse the spectral sequence.

3.8. LEMMA. *If $\varphi \in \mathscr{C}^{p,q}$, define ω_φ by*

$$\omega_\varphi(kL)(X_1, ..., X_p)(kmH)(Y_1, ..., Y_q) = \varphi(kmH)(R_{km}X_1, ..., R_{km}X_p, X_1, ..., Y_q), \tag{3.9}$$

where the X_i are (0, 1) *vector fields on K/L in a neighborhood of kL, the Y_j are* (0, 1) *vector fields on G/H tangent to the fibres in a neighborhood of kmH, and R_{km} is the* (0, 1) *horizontal lift of* (2.26). *Then $\omega_\varphi \in A^p(K/A, \mathbf{A}^q(M/L, \mathbf{V}))$, the Fréchet space of $\mathbf{A}^q(M/L, \mathbf{V})$-valued C^∞ $(0, p)$-forms on K/L. Further, $\varphi \mapsto \omega_\varphi$ induces a K-equivariant Fréchet space isomorphism of $E_0^{p,q}$ onto $A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$.*

Proof. Interpret φ as a C^∞ function $G \to V \otimes \Lambda^{p+q}(\mathfrak{q}_-)^*$ such that $\varphi(gh) = \{\psi \otimes \Lambda^{p+q}(\mathrm{Ad}^*)\}(k^{-1})\, \varphi(g)$, and ω_φ as a C^∞ function $K \times \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-) \to C^\infty(M) \otimes V \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$. In doing this,

$$\omega_\varphi(k)(\xi_1, ..., \xi_p)(m)(\eta_1, ..., \eta_q) = \varphi(km)(T_m\xi_1, ..., T_m\xi_p, \eta_1, ..., \eta_q), \tag{3.10}$$

where $k \in K$, $m \in M$, $\xi_i \in \mathfrak{k} \cap \mathfrak{q}_-$, $\eta_j \in \mathfrak{p} \cap \mathfrak{q}_-$, and

$$T_m : \mathfrak{k} \cap \mathfrak{q}_- \to \mathfrak{k} \cap \mathfrak{q}_- \text{ is the inverse of } p'_- \circ \mathrm{Ad}(m)\,|_{\mathfrak{k} \cap \mathfrak{q}_-}. \tag{3.11}$$

Lemma 2.24 says that T_m is well defined. Now compute

$$\begin{aligned}
&\omega_\varphi(k)(\xi_1, ..., \xi_p)(ml)(\eta_1, ..., \eta_q) \\
&\quad = \varphi(kml)(T_{ml}\xi_1, ..., T_{ml}\xi_p, \eta_1, ..., \eta_q) \\
&\quad = \psi(l)^{-1} \cdot \varphi(km)(\mathrm{Ad}(l)\, T_{ml}\xi_1, ..., \mathrm{Ad}(l)\, T_{ml}\xi_p, \mathrm{Ad}(l)\,\eta_1, ..., \mathrm{Ad}(l)\,\eta_q) \\
&\quad = \psi(l)^{-1} \cdot \varphi(km)(T_m\xi_1, ..., T_m\xi_p, \mathrm{Ad}(l)\,\eta_1, ..., \mathrm{Ad}(l)\,\eta_q) \\
&\quad = \psi(l)^{-1} \cdot \omega_\varphi(k)(\xi_1, ..., \xi_p)(m)(\mathrm{Ad}(l)\,\eta_1, ..., \mathrm{Ad}(l)\,\eta_q).
\end{aligned}$$

Thus $\omega_\varphi(k)(\xi_1,\ldots,\xi_p) \in A^q(M/L, \mathbf{V})$. Furthermore,

$$\begin{aligned}\omega_\varphi(kl)(\xi_1,\ldots,\xi_p)(m)(\eta_1,\ldots,\eta_q) &= \varphi(klm)(T_m\xi_1,\ldots,T_m\xi_p,\eta_1,\ldots,\eta_q)\\ &= \varphi(klm)(T_{lm}\operatorname{Ad}(l)\,\xi_1,\ldots,T_{lm}\operatorname{Ad}(l)\,\xi_p,\eta_1,\ldots,\eta_q)\\ &= \omega_\varphi(k)(\operatorname{Ad}(l)\,\xi_1,\ldots,\operatorname{Ad}(l)\,\xi_p)(lm)(\eta_1,\ldots,\eta_q).\end{aligned}$$

This says that $k \mapsto \omega_\varphi(k)$ has the transformation property under L such that $\omega_\varphi \in A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$.

Conversely, if $\omega \in A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$, then (3.10) suggests that we define $\varphi: G \to V \otimes \Lambda^{p+q}(\mathfrak{q}_-)^*$ by

$$\begin{aligned}\varphi(kmh)(T_m\xi_1,\ldots,T_m\xi_r,\eta_1,\ldots,\eta_{p+q-r}) &= 0, \quad \text{if } r \neq p,\\ &= \{\psi \otimes \Lambda^{p+q}(\operatorname{Ad}^*)\}(h^{-1}) \cdot \omega(k)(\xi_1,\ldots,\xi_p)(m)(\eta_1,\ldots,\eta_q),\\ &\qquad \text{if } r = p.\end{aligned} \tag{3.12}$$

Calculating as above, we see that φ is a well-defined element of $A^{p,q}$, and visibly $\omega = \omega_\varphi$. Thus $\varphi \mapsto \omega_\varphi$ maps $\mathscr{C}^{p,q}$ onto $A^p(K/L, \mathbf{A}^q(M/L), \mathbf{V}))$. This is continuous in the C^∞ topologies. The kernel is $\mathscr{C}^{p+1,q-1}$. Now $\varphi \mapsto \omega_\varphi$ induces a continuous one-to-one map of $E_0^{p,q}$ onto $A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$. It is a Fréchet space isomorphism by the open mapping theorem. Q.E.D.

Now identify $E_0^{p,q}$ with $A^p(K/L,\mathbf{A}^q(M/L, \mathbf{V}))$ as in Lemma 3.8. If $\omega \in E_0^{p,q}$, use (3.12) to represent it by $\varphi \in \mathscr{C}^{p,q}$. Then $d_0\omega \in E_0^{p,q+1}$ is represented by $\bar\partial\varphi \in \mathscr{C}^{p,q+1}$, so

$$(d_0\omega)(k)(\xi_1,\ldots,\xi_p)(m)(\eta_1,\ldots,\eta_{q+1}) = \bar\partial\varphi(km)(T_m\xi_1,\ldots,T_m\xi_p,\eta_1,\ldots,\eta_{q+1}). \tag{3.13}$$

In order to use this, we need a reasonably explicit formula for $\bar\partial\varphi$.

Recall the projections p', p'' of (2.13). If $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, then $p' \circ \operatorname{Ad}(\exp(\eta)): \mathfrak{k} \cap \mathfrak{q} \to \mathfrak{k} \cap \mathfrak{q}$ and $p'' \circ \operatorname{Ad}(\exp(\eta)): \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{p} \cap \mathfrak{h}$ are isomorphisms given by $\cosh(\operatorname{ad}(\eta))$. If $m \in M$, now

$$p' \circ \operatorname{Ad}(m): \mathfrak{k} \cap \mathfrak{q} \to \mathfrak{k} \cap \mathfrak{q} \qquad \text{and} \qquad p'' \circ \operatorname{Ad}(m): \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{p} \cap \mathfrak{h}$$

are invertible. Thus we can define

$$\begin{aligned} A_m&: \mathfrak{k} \cap \mathfrak{q} \to \mathfrak{k} \cap \mathfrak{q} && \text{is the inverse of } p' \circ \operatorname{Ad}(m^{-1})|_{\mathfrak{k}\cap\mathfrak{q}},\\ B_m&: \mathfrak{k} \cap \mathfrak{q} \to \mathfrak{p} \cap \mathfrak{h} && \text{is } -p'' \circ \operatorname{Ad}(m^{-1}) \circ A_m,\\ C_m&: \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{p} \cap \mathfrak{h} && \text{is the inverse of } p'' \circ \operatorname{Ad}(m)|_{\mathfrak{p}\cap\mathfrak{h}},\\ D_m&: \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{k} \cap \mathfrak{q} && \text{is } -p' \circ \operatorname{Ad}(m) \circ C_m.\end{aligned} \tag{3.14}$$

Notice, for $m \in M$ and $l_1, l_2 \in L$ that

$$A_{l_1 m l_2} = \mathrm{Ad}(l_1) \circ A_m \circ \mathrm{Ad}(l_2), \qquad B_{l_1 m l_2} = \mathrm{Ad}(l_2)^{-1} \circ B_m \circ \mathrm{Ad}(l_2),$$
$$C_{l_1 m l_2} = \mathrm{Ad}(l_2)^{-1} \circ C_m \circ \mathrm{Ad}(l_1)^{-1}, \qquad D_{l_1 m l_2} = \mathrm{Ad}(l_1) \circ D_m \circ \mathrm{Ad}(l_1)^{-1}. \tag{3.15}$$

View $m \mapsto D_m$ as a function $M \to \mathrm{Hom}(\mathfrak{p} \cap \mathfrak{h}, \mathfrak{k} \cap \mathfrak{q})$, and let $r(\eta)\, D_m$ denote its derivative on the right by $\eta \in \mathfrak{p} \cap \mathfrak{q}$. Then

$$\begin{aligned} A_m^{-1} \circ r(\eta)\, D_m \circ C_m^{-1} \\ &= p' \circ \mathrm{Ad}(m^{-1}) \circ p' \circ \mathrm{Ad}(m) \\ &\quad \circ \{-\mathrm{ad}(\eta) \circ p'' + C_m \circ p'' \circ \mathrm{Ad}(m) \circ \mathrm{ad}(\eta) \circ p''\} \\ &= p' \circ \mathrm{Ad}(m^{-1}) \circ (1 - p'') \\ &\quad \circ \mathrm{Ad}(m)\{-\mathrm{ad}(\eta) \circ p'' + C_m \circ p'' \circ \mathrm{Ad}(m) \circ \mathrm{ad}(\eta) \circ p''\} \\ &= -p' \circ \mathrm{ad}(\eta) \circ p'' + p' \circ \mathrm{Ad}(m^{-1}) \circ p'' \circ \mathrm{Ad}(m) \circ \mathrm{ad}(\eta) \circ p'' \\ &\quad + p' \circ C_m p'' \circ \mathrm{Ad}(m) \circ \mathrm{ad}(\eta) \circ p'' - p' \circ \mathrm{Ad}(m^{-1}) \circ p'' \\ &\quad \circ \mathrm{Ad}(m) \circ C_m p'' \circ \mathrm{Ad}(m) \circ \mathrm{ad}(\eta) \circ p''. \end{aligned}$$

But $p' \circ C_m = 0$ and $p'' \circ \mathrm{Ad}(m) \circ C_m \circ p'' = p''$ by their definitions, so the second and fourth terms cancel, and the third vanishes, leaving $-p' \circ \mathrm{Ad}(\eta) \circ p''$. Thus

$$\text{if } m \in M \text{ and } n \in \mathfrak{p} \cap \mathfrak{q}, \text{ then } r(\eta)\, D_m = -A_m \circ \mathrm{ad}(\eta) \circ C_m. \tag{3.16}$$

Now we compute the left and right action of various elements of $\mathfrak{g}$ at points $km \in G$, in the sense

$$\begin{aligned} &\text{at } km,\ l(\xi) \text{ is derivative along the curve } k \cdot \exp(t\xi) \cdot m, \\ &\text{at } km,\ r(\xi) \text{ is derivative along the curve } km \cdot \exp(t\xi). \end{aligned} \tag{3.17}$$

First, if $\xi \in \mathfrak{k} \cap \mathfrak{q}$, then $l(A_m \xi) = r(\mathrm{Ad}(m^{-1}) \circ A_m \xi) = r(p' \circ \mathrm{Ad}(m^{-1}) \circ A_m \xi) + r(p'' \circ \mathrm{Ad}(m^{-1}) \circ A_m \xi) = r(\xi) - r(B_m \xi)$, so

$$r(\xi) = l(A_m \xi) + r(B_m \xi) \qquad \text{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}. \tag{3.18}$$

Second, if $\zeta \in \mathfrak{p} \cap \mathfrak{h}$, then $r(C_m \zeta) = l(\mathrm{Ad}(m) \cdot C_m \zeta) = l(p' \circ \mathrm{Ad}(m) \circ C_m \zeta) + l(p'' \circ \mathrm{Ad}(m) \circ C_m \zeta) = -l(D_m \zeta) + l(\zeta)$, so

$$l(\zeta) = r(C_m \zeta) + l(D_m \zeta) \qquad \text{for} \quad \zeta \in \mathfrak{p} \cap \mathfrak{h}. \tag{3.19}$$

Consider a $\mathbf{V}$-valued $C^\infty(0, p+q)$-form on G/H, $\varphi: G \to V \otimes \Lambda^{p+q}(\mathfrak{q}_-)^*$, such that

$$\varphi(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*. \tag{3.20}$$

In other words, φ is a form constructed in (3.12). If $\xi_1,..., \xi_p \in \mathfrak{k} \cap \mathfrak{q}_-$ and $\eta_1,..., \eta_{q+1} \in \mathfrak{p} \cap \mathfrak{q}_-$ then, using (3.18),

$$\begin{aligned}
&\bar{\partial}\varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\quad = \sum_{i=1}^{p} (-1)^{i-1} r(\xi_i)\, \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\qquad + (-1)^p \sum_{j=1}^{q+1} (-1)^{j-1} r(\eta_j)\, \varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \hat{\eta}_j,..., \eta_{q+1})\\
&\quad = \sum_{i=1}^{p} (-1)^{i-1} l(A_m \xi_i)\, \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\qquad + \sum_{i=1}^{p} (-1)^{i-1} r(B_m \xi_i)\, \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\qquad + (-1)^p \sum_{j=1}^{q+1} (-1)^{j-1} r(\eta_j)\, \varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \hat{\eta}_j,..., \eta_{q+1}).
\end{aligned}$$

As $A_m \xi_i \in \mathfrak{k}$, $km \mapsto l(A_m \xi_i)\, \varphi(km)$ has values in $V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$, so each

$$l(A_m \xi_i)\, \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1}) = 0.$$

Since $B_m \xi_i \in \mathfrak{h}$,

$$\begin{aligned}
&r(B_m \xi_i)\, \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\quad = -d\psi(B_m \xi_i) \cdot \varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\qquad - \sum_{i' \neq i} \varphi(km)(\xi_1,..., [B_m \xi_i, \xi_{i'}],..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{q+1})\\
&\qquad - \sum_{j=1}^{q+1} \varphi(km)(\xi_1,..., \hat{\xi}_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., [B_m \xi_i, \eta_j],..., \eta_{q+1}).
\end{aligned}$$

The $d\psi(B_m \xi_i) \cdot \varphi(km)$ make sense because all values of

$$\varphi: G \times \Lambda^{p+q}(\mathfrak{q}_-) \to V$$

are C^∞ vectors for ψ. In effect, φ as just interpreted is a C^∞ function to V which satisfies a right invariance condition for H. Now the $d\psi(B_m \xi_i) \cdot \varphi(km)$

terms and the terms involving $[B_m\xi_i, \xi_{i'}] \in \mathfrak{p} \cap \mathfrak{q}$ vanish because $\varphi(km)$ has values in $V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$. The terms involving $[B_m\xi_i, \eta_j]$ are

$$\begin{aligned}&\varphi(km)(\xi_1,..., \hat{\xi}_i,..., \xi_p, \eta_1,..., \eta_{j-1}, [B_m\xi_i, \eta_j], \eta_{j+1},..., \eta_{q+1})\\&\quad = (-1)^{j-k+p-i+1}\varphi(km)(\xi_1,..., \xi_{i-1}, [\eta_j, B_m\xi_i], \xi_{i+1},..., \xi_p, \eta_1,..., \hat{\eta}_j,..., \eta_{q+1}).\end{aligned}$$

In summary, now

$$\begin{aligned}&\bar{\partial}\varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \eta_{q+1})\\&\quad = (-1)^p \sum_{i=1}^{p} \sum_{j=1}^{q+1} (-1)^{j-1}\varphi(km)(\xi_1,..., [\eta_j, B_m\xi_i],..., \xi_p, \eta_1,..., \hat{\eta}_j,..., \eta_{q+1})\\&\qquad + (-1)^p \sum_{j=1}^{q+1} (-1)^{j-1}\varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \hat{\eta}_j,..., \eta_{q+1}).\end{aligned} \tag{3.21}$$

In order to use (3.21) in (3.13) we must control the $[\eta_j, B_m T_m \xi_i]$. Recall the projections $p'_\pm$ of (2.23). Using (3.11) and (3.14), and writing $m = l_1 \cdot \exp(\eta) \cdot l_2$ with $l_i \in L$ and $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$,

$$T_m^{-1} = p'_- \circ \mathrm{Ad}(m)|_{\mathfrak{k}\cap\mathfrak{q}_-} = \mathrm{Ad}(l_1) \circ p'_- \circ \cosh(\mathrm{ad}(\eta)) \circ \mathrm{Ad}(l_2)|_{\mathfrak{k}\cap\mathfrak{q}_-}$$

and

$$\begin{aligned}p'_- \circ A_m|_{\mathfrak{k}\cap\mathfrak{q}_-} &= p'_- \circ \{p' \circ \mathrm{Ad}(m^{-1})|_{\mathfrak{k}\cap\mathfrak{q}}\}^{-1}|_{\mathfrak{k}\cap\mathfrak{q}_-}\\&= \mathrm{Ad}(l_1) \circ p'_- \circ \{\cosh(\mathrm{ad}(-\eta))|_{\mathfrak{k}\cap\mathfrak{q}}\}^{-1} \circ \mathrm{Ad}(l_2)|_{\mathfrak{k}\cap\mathfrak{q}_-}.\end{aligned}$$

They are equal because

$$\begin{aligned}&p'_- \circ \cosh(\mathrm{ad}(\eta))|_{\mathfrak{k}\cap\mathfrak{q}_-}\\&\quad = p'_- \exp(\mathrm{ad}(\eta))|_{\mathfrak{k}\cap\mathfrak{q}_-} = p'_- \circ \exp(\mathrm{ad}(-\eta))^{-1}|_{\mathfrak{k}\cap\mathfrak{q}_-}\\&\quad = p'_- \circ \{\exp(\mathrm{ad}(-\eta))|_{\mathfrak{k}\cap\mathfrak{q}_-}\}^{-1} = p'_- \circ \{\cosh(\mathrm{ad}(-\eta))|_{\mathfrak{k}\cap\mathfrak{q}_-}\}^{-1}.\end{aligned}$$

Now, with a glance at definition (2.27) of S_m,

$$\begin{aligned}&T_m : \mathfrak{k} \cap \mathfrak{q}_- \to \mathfrak{k} \cap \mathfrak{q}_- \quad &&\text{is the inverse of} \quad p'_- \circ A_m,\\&S_m : \mathfrak{k} \cap \mathfrak{q}_- \to \mathfrak{k} \cap \mathfrak{q}_+ \quad &&\text{is} \quad p'_+ \circ A_m \circ T_m.\end{aligned} \tag{3.22}$$

If $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$, then $A_m \circ T_m\xi = p'_+ \circ A_m \circ T_m\xi + p'_- \circ A_m \circ T_m\xi$, i.e.,

$$A_m \circ T_m\xi = \xi + S_m\xi \qquad \text{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}_-. \tag{3.23}$$

If $\eta \in \mathfrak{p} \cap \mathfrak{q}_-$, we compute

$$\begin{aligned} r(\eta)\, T_m &= -T_m \circ r(\eta)(p'_- \circ A_m) \circ T_m \\ &= -T_m \circ p'_- \circ A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ A_m \circ T_m \\ &= -T_m \circ p'_- \circ A_m \circ p'_- \circ \mathrm{ad}(\eta) \circ p'' \circ \mathrm{Ad}(m^{-1}) \circ A_m \circ T_m \\ &= p'_- \circ \mathrm{ad}(\eta) \circ B_m \circ T_m = \mathrm{ad}(\eta) \circ B_m \circ T_m \end{aligned}$$

and

$$\begin{aligned} r(\eta)\, S_m &= p'_+ \circ A_m \circ r(\eta)\, T_m + p'_+ \circ r(\eta) A_m \circ T_m \\ &= p'_+ \circ A_m \circ \mathrm{ad}(\eta) \circ B_m \circ T_m \\ &\quad + p'_+ \circ A_m \circ p' \circ \mathrm{ad}(\eta) \circ \mathrm{Ad}(m^{-1}) \circ A_m \circ T_m \\ &= 0. \end{aligned}$$

So we have

$$r(\eta)\, T_m = \mathrm{ad}(\eta) \circ B_m \circ T_m \quad \text{and} \quad r(\eta)\, S_m = 0 \qquad \text{for} \quad \eta \in \mathfrak{p} \cap \mathfrak{q}_-. \tag{3.24}$$

Now we can continue (3.21) as follows:

$$\begin{aligned} &\bar{\partial}\varphi(km)(T_m\xi_1, ..., T_m\xi_p, \eta_1, ..., \eta_{q+1}) \\ &= (-1) \sum_{i=1}^{p} \sum_{j=1}^{q+1} (-1)^{j-1} \varphi(km)(T_m\xi_1, ..., (r(\eta_j)\, T_m)\, \xi_i, ..., \xi_p, \eta_1, ..., \hat{\eta}_j, ..., \eta_{q+1}) \\ &\quad + (-1)^p \sum_{j=1}^{q+1} (-1)^{j-1} r(\eta_j)\, \varphi(km)(T_m\xi_1, ..., T_m\xi_p, \eta_1, ..., \hat{\eta}_j, ..., \eta_{q+1}) \\ &= (-1)^p \sum_{j=1}^{q+1} (-1)^{j-1} \omega_\varphi(k)(\xi_1, ..., \xi_p)(m \cdot \eta_j)(\eta_1, ..., \hat{\eta}_j, ..., \eta_{q+1}) \\ &= (-1)^p (\bar{\partial}_{M/L}\, \omega_\varphi(k)(\xi_1, ..., \xi_p))(m)(\eta_1, ..., \eta_{q+1}). \end{aligned}$$

In summary, we have proved

3.25. Proposition. *The spectral sequence differential* d_0 *on* $E_0^{p,q} = A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$ *is* $(-1)^p \bar{\partial}_{M/L}$.

Andreotti and Grauert showed [1, Theorem 1] that a short exact sequence of Fréchet bundles which is exact on the fibres over any point is exact on the global section level. In other words, if $0 \to \mathbf{K}_0 \to \mathbf{K}_1 \to \cdots$ is a complex of Fréchet vector bundles whose differentials have closed range on the fibres, then its cohomologies are the bundles whose fibres are the cohomology of the fibres in the complex. Lemmas 3.5 and 3.8 and Proposition 3.25 say that we can apply this to the complex $\{E_0^{p,*}, d_0\}$. Thus we have

$$E_1^{p,q} = A^p(K/L, \mathbf{H}^q(M/L, \mathbf{V})) \qquad \text{as Fréchet space.} \tag{3.26}$$

Using Lemma 3.5(iii), now

$$E_1^{p,q} = 0 \quad \text{for} \quad q > 0, \qquad \text{so} \qquad E_2^{p,0} = E_\infty^{p,0} = H^p(G/H, \mathbf{V}). \tag{3.27}$$

It remains to calculate $d_1 : E_1^{p,q} \to E_1^{p+1,q}$, and we need only do this for $q = 0$. But we write it out for $q \geqslant 0$ because we have some future variations in mind.

Let $c \in E_1^{p,q}$. Represent it by $\omega \in E_0^{p,q}$. Then $\omega = \omega_\varphi$, where $\varphi \in \mathscr{C}^{p,q}$ is given by (3.12) and satisfies (3.20). As

$$0 = d_0 \omega \in E_0^{p,q+1} = \mathscr{C}^{p,q+1}/\mathscr{C}^{p+1,q},$$

we have $\bar{\partial}\varphi \in \mathscr{C}^{p+1,q}$ and

$$\omega_{\bar{\partial}\varphi} \in E_0^{p+1,q} = A^{p+1}(K/L, \mathbf{A}^q(M/L, \mathbf{V})) \qquad \text{represents} \quad d_1 c. \tag{3.28}$$

Now let $\xi_1, \ldots, \xi_{p+1} \in \mathfrak{k} \cap \mathfrak{q}_-$ and $\eta_1, \ldots, \eta_q \in \mathfrak{p} \cap \mathfrak{q}_-$, and compute

$$\begin{aligned} &\bar{\partial}\varphi(km)(T_m\xi_1, \ldots, T_m\xi_{p+1}, \eta_1, \ldots, \eta_q) \\ &\quad = \sum_{i=1}^{p+1} (-1)^{i-1} r(T_m\xi_i)\, \varphi(km)(T_m\xi_1, \ldots, \widehat{T_m\xi_i}, \ldots, T_m\xi_{p+1}, \eta_1, \ldots, \eta_q) \\ &\qquad + (-1)^p \sum_{j=1}^{p} (-1)^{j-1} r(\eta_j)\, \varphi(km)(T_m\xi_1, \ldots, T_m\xi_{p+1}, \eta_1, \ldots, \hat{\eta}_j, \ldots, \eta_q). \end{aligned}$$

As $\eta_j \in \mathfrak{m}$, $km \mapsto r(\eta_j)\, \varphi(km)$ has values in $V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$, so each

$$r(\eta_j)\, \varphi(km)(T_m\xi_1, \ldots, T_m\xi_{p+1}, \eta_1, \ldots, \hat{\eta}_j, \ldots, \eta_q) = 0.$$

Also, $r(T_m\xi_i) = l(A_m T_m \xi_i) + r(B_m T_m \xi_i)$. Since $B_m T_m \xi_i \in \mathfrak{h}$, $r(B_m T_m \xi_i)$ acts on the arguments $T_m\xi_{i'}$, η_j of $\varphi(km)$ and the value $\varphi(km)(T_m\xi_1, \ldots, \widehat{T_m\xi_i}, \ldots, T_m\xi_{p+1}, \eta_1, \ldots, \eta_q) \in V$. The action on the arguments sends, in turn,

one of the p vectors $T_m\xi_1, \ldots, \widehat{T_m\xi_i}, \ldots, T_m\xi_{p+1}$ into $\mathfrak{p} \cap \mathfrak{q}$,

one of the q vectors $\eta_1, \ldots, \eta_q$ into $\mathfrak{k} \cap \mathfrak{q}$,

and these terms vanish because $\varphi(km) \in V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$.

Thus

$$r(T_m\xi_i)\,\varphi(km)(T_m\xi_1,\ldots,\widehat{T_m\xi_i},\ldots,T_m\xi_{p+1},\eta_1,\ldots,\eta_q)$$
$$= -d\psi(B_mT_m\xi_i)\cdot\varphi(km)(T_m\xi_1,\ldots,\widehat{T_m\xi_i},\ldots,T_m\xi_{p+1},\eta_1,\ldots,\eta_q).$$

Now we have, using $A_mT_m\xi_i = \xi_i + S_m\xi_i$,

$$\bar\partial\varphi(km)(T_m\xi_1,\ldots,T_m\xi_{p+1},\eta_1,\ldots,\eta_q)$$
$$= \sum_{i=1}^{p+1}(-1)^{i-l}l(A_mT_m\xi_i)\,\varphi(km)(T_m\xi_1,\ldots,\widehat{T_m\xi_i},\ldots,T_m\xi_{p+1},\eta_1,\ldots,\eta_q)$$
$$- \sum_{i=1}^{p+1}(-1)^{i-1}d\psi(B_mT_m\xi_i)\cdot\varphi(km)(T_m\xi_1,\ldots,\widehat{T_m\xi_i},\ldots,T_m\xi_{p+1},\eta_1,\ldots,\eta_q)$$
$$= \sum_{i=1}^{p+1}(-1)^{i-1}\omega_\varphi(k\cdot(\xi_i+S_m\xi_i))(\xi_1,\ldots,\hat\xi_i,\ldots,\xi_{p+1})(m)(\eta_1,\ldots,\eta_q)$$
$$- \sum_{i=1}^{p+1}(-1)^{i-1}d\psi(B_mT_m\xi_i)\cdot\omega_\varphi(k)(\xi_1,\ldots,\hat\xi_i,\ldots,\xi_{p+1})(m)(\eta_1,\ldots,\eta_q).$$

In other words,

3.29. PROPOSITION. *The spectral sequence differential d_1 on $E_1^{p,q} = A^p(K/L, \mathbf{H}^q(M/L,\mathbf{V}))$ is given as follows: Let $\omega\in A^p(K/L,\mathbf{A}^q(M/L,\mathbf{V}))$ represent $c\in E_1^{p,q}$. Define $\delta\omega\in A^{p+1}(K/L,\mathbf{A}^q(M/L,\mathbf{V}))$ by*

$$\delta\omega(k)(\xi_1,\ldots,\xi_{p+1})(m)$$
$$= \sum_{i=1}^{p+1}(-1)^{i-1}\omega(k\cdot\xi_i)(\xi_1,\ldots,\hat\xi_i,\ldots,\xi_{p+1})(m)$$
$$+ \sum_{i=1}^{p+1}(-1)^{i-1}\omega(k\cdot S_m\xi_i)(\xi_1,\ldots,\hat\xi_i,\ldots,\xi_{p+1})(m)$$
$$- \sum_{i=1}^{p+1}(-1)^{i-1}d\psi(B_mT_m\xi_i)\cdot\omega(k)(\xi_1,\ldots,\hat\xi_i,\ldots,\xi_{p+1})(m), \quad (3.30)$$

where $k\in K$, $m\in M$, and $\xi_i\in\mathfrak{k}\cap\mathfrak{q}_-$. Then $\delta\omega$ represents $d_1c\in E_1^{p+1,q}$.

Combine (3.26), (3.27), and Proposition 3.29. The result is

3.31. THEOREM. *The Dolbeault cohomology $H^p(G/H,\mathbf{V}) = H^p_\delta(K/L, \mathbf{H}^0(M/L,\mathbf{V}))$, where H^p_δ denotes p-cohomology for the operator*

$$\delta\colon A^p(K/L,\mathbf{H}^0(M/L,\mathbf{V}))\to A^{p+1}(K/L,\mathbf{H}^0(M/L,\mathbf{V}))$$

defined in (3.30).

Finally, we specify the operator δ in somewhat more familiar terms when $\pi: G/H \to K/L$ is holomorphic. That is based on

3.32. LEMMA. *Let $\pi: G/H \to K/L$ be holomorphic. Then $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ has a unique structure of holomorphic vector bundle in such a way that a local section $w: K \to H^0(M/L, \mathbf{V})$ is holomorphic if and only if*

$$w(k \cdot \xi)(m) = d\psi(B_m T_m \xi) \cdot w(k)(m) \qquad \textit{for} \quad \xi \in \mathfrak{k} \cap \mathfrak{q}_-. \tag{3.33}$$

Proof. The integrability condition for the asserted holomorphic vector bundle structure is that

$$\xi \mapsto \beta(\xi), \qquad [\beta(\xi) f](m) = d\psi(B_m T_m \xi) \cdot f(m)$$

be a linear representation of $\mathfrak{k} \cap \mathfrak{q}_-$ on $H^0(M/L, \mathbf{V})$ such that

$$\beta(\mathrm{Ad}(l)\xi) = \alpha(l)\, \beta(\xi)\, \alpha(l)^{-1} \qquad \text{for all} \quad l \in L,$$

where α is the natural action (restricted from M) of L on $H^0(M/L, \mathbf{V})$. See [38] for the details on this.

Let $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$ and $f \in H^0(M/L, \mathbf{V})$. We must check that $\beta(\xi) f \in H^0(M/L, \mathbf{V})$. Define $\tilde{f}: M \to V$ by

$$\tilde{f}(m) = d\psi(B_m T_m \xi) \cdot f(m).$$

From (3.14) and (3.22),

$$B_m T_m \xi = -p'' \circ \mathrm{Ad}(m^{-1}) \circ A_m \circ \{p'_- \circ A_m|_{\mathfrak{k} \cap \mathfrak{q}_-}\}^{-1} \xi$$

so

$$\tilde{f}(m) = -d\psi(p'' \circ \mathrm{Ad}(m^{-1})\xi) \cdot f(m).$$

If $l \in L$, now

$$\begin{aligned} \tilde{f}(ml) &= -d\psi(p'' \circ \mathrm{Ad}(l^{-1}) \circ \mathrm{Ad}(m^{-1})\xi) \circ f(ml) \\ &= -d\psi(\mathrm{Ad}(l^{-1})\{p'' \circ \mathrm{Ad}(m^{-1})\xi\}) \circ \psi(l)^{-1} f(m) \\ &= -\psi(l^{-1}) \circ d\psi(p'' \circ \mathrm{Ad}(m^{-1})) \circ \psi(l) \circ \psi(l)^{-1} f(m) \\ &= \psi(l)^{-1} \cdot \tilde{f}(m). \end{aligned}$$

Thus $\tilde{f}$ represents a section of $\mathbf{V} \to M/L$. If $\eta \in \mathfrak{p} \cap \mathfrak{q}_-$, then $f(m \cdot \eta) = 0$ because f represents a holomorphic section, so

$$\tilde{f}(m \cdot \eta) = d\psi(p'' \circ \mathrm{ad}(\eta) \circ \mathrm{Ad}(m^{-1})\xi) \circ f(m).$$

Since π is holomorphic, we have our roots ordered with $\mathfrak{p} \cap \mathfrak{q}_+ \subset \mathfrak{p}_+$ and $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, so $[\mathfrak{p}_0 \cap \mathfrak{q}_0, \mathfrak{k} \cap \mathfrak{q}_-] \subset \mathfrak{p}_+ \cap \mathfrak{h}$, and it follows that

$$\mathrm{Ad}(m^{-1})\xi \in (\mathfrak{k} \cap \mathfrak{q}_-) + (\mathfrak{p}_+ \cap \mathfrak{h}).$$

Now $\mathrm{ad}(\eta) \circ \mathrm{Ad}(m^{-1})\xi \in [\mathfrak{p} \cap \mathfrak{q}_-, \mathfrak{p}_+ \cap \mathfrak{h}] \subset \mathfrak{k} \cap \mathfrak{q}_-$, so $p'' \circ \mathrm{ad}(\eta) \circ \mathrm{Ad}(m^{-1})\xi = 0$, forcing $\tilde{f}(m \cdot \eta) = 0$. We have shown that $\tilde{f}$ represents a holomorphic section of $\mathbf{V} \to M/L$, so that $\beta(\xi) f \in H^0(M/L, \mathbf{V})$.

Next, we verify that the action β of $\mathfrak{k} \cap \mathfrak{q}_-$ on $H^0(M/L, \mathbf{V})$ is a representation. Let $\xi_1, \xi_2 \in \mathfrak{k} \cap \mathfrak{q}_-$. Then $[\xi_1, \xi_2] = 0$, so we must show that $[\beta(\xi_1), \beta(\xi_2)] = 0$. Compute in $\mathfrak{g}$ that

$$\begin{aligned}
[B_m T_m \xi_1, B_m T_m \xi_2] &= [p'' \circ \mathrm{Ad}(m^{-1})\, \xi_1, p'' \circ \mathrm{Ad}(m^{-1})\, \xi_2] \\
&\subset [p''\{(\mathfrak{k} \cap \mathfrak{p}_-) + (\mathfrak{p}_+ \cap \mathfrak{h})\}, p''\{(\mathfrak{k} \cap \mathfrak{q}_-) + (\mathfrak{p}_+ \cap \mathfrak{h})\}] \\
&= [\mathfrak{p}_+ \cap \mathfrak{h}, \mathfrak{p}_+ \cap \mathfrak{h}] \subset [\mathfrak{p}_+, \mathfrak{p}_+] = 0.
\end{aligned}$$

Now, if $f \in H^0(M/L, \mathbf{V})$, then

$$\begin{aligned}
[\beta(\xi_1), \beta(\xi_2)] f(m) &= [d\psi(B_m T_m \xi_1), d\psi(B_m T_m \xi_2)] f(m) \\
&= d\psi[B_m T_m \xi_1, B_m T_m \xi_2] f(m) = 0.
\end{aligned}$$

Thus β is a representation of $\mathfrak{k} \cap \mathfrak{q}_-$ on $H^0(M/L, \mathbf{V})$.

To see L-equivariance, let $l \in L$ and $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$, and compute

$$\begin{aligned}
\alpha(l)\, \beta(\xi)\, \alpha(l^{-1}) f &= \alpha(l)\, \beta(\xi)(m \mapsto f(lm)) \\
&= \alpha(l)(m \mapsto d\psi(B_m \circ T_m \xi) \circ f(lm)) \\
&= (m \mapsto d\psi(B_{l^{-1}m} \circ T_{l^{-1}m} \xi) \circ f(m)).
\end{aligned}$$

As $B_{l^{-1}m} \circ T_{l^{-1}m} = B_m \circ T_m \circ \mathrm{Ad}(l)$, this shows that $\alpha(l)\, \beta(l)\, \alpha(l^{-1}) = \beta(\mathrm{Ad}(l)\xi)$, as required. Q.E.D.

3.34. THEOREM. *Suppose that* $\pi: G/H \to K/L$ *is holomorphic and let* $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ *carry the holomorphic vector bundle structure of Lemma* 3.32. *Then the Dolbeault cohomology* $H^p(G/H, \mathbf{V}) = H^p(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$.

Proof. The operator δ of (3.30) is the $\bar{\partial}$ operator on $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ in the specified holomorphic structure. Now apply Theorem 3.31. Q.E.D.

3.35. *Remark.* We have implicitly been dealing with a "vanishing bidegree" decomposition for forms $\varphi \in A^m(G/H, \mathbf{V})$. The decomposition, which depends on our choice of θ, is

$$\varphi = \sum_{p+q=m} \varphi_{p,q} \qquad \text{where} \quad \varphi_{p,q}(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*. \tag{3.36}$$

In this decomposition, it is easy to check that $\bar{\partial}: A^m(G/H, \mathbf{V}) \to A^{m+1}(G/H, \mathbf{V})$ is the sum of three homogeneous terms,

$$\bar{\partial} = \bar{\partial}_{0,1} + \bar{\partial}_{1,0} + \bar{\partial}_{2,-1}. \tag{3.37}$$

Here $\bar{\partial}_{0,1}$ is given by the calculation just before Proposition 3.25; it specifies d_0. The next, $\bar{\partial}_{1,0}$, is given by the calculation just before Proposition 3.29; it specifies d_1. One can show that $\bar{\partial}_{2,-1}$ is given on $\varphi = \varphi_{p,q}$ by

$$\begin{aligned}
&\bar{\partial}\varphi(km)(\xi_1, \ldots, \xi_{p+2}, \eta_1, \ldots, \eta_{q-1}) \\
&\quad = \sum_{1 \leqslant i < j \leqslant p+2} (-1)^{i+j+p} \varphi(km)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+2}, \\
&\qquad\qquad [B_m \xi_i, \xi_j] - [B_m \xi_j, \xi_i], \eta_1, \ldots, \eta_{q-1}).
\end{aligned} \tag{3.38}$$

Since this would only affect the differential $d_2 = 0$, we do not need it now. But we remark that, by a glance at a local coordinate expression for φ in a holomorphic local trivialization,

$$\text{if } \pi: G/H \to K/L \text{ is holomorphic, then } \bar{\partial}_{2,-1} = 0. \tag{3.39}$$

4. The K-Spectrum of $H^p(G/H, \mathbf{V})$

In this section we will describe the natural representation of K (as a subgroup of G) on the Fréchet space $H^p(G/H, \mathbf{V})$, especially in the case where $\mathbf{V} \to G/H$ has finite-dimensional fibres. Later, we will need that description to study the corresponding square-integrable cohomology.

Recall (2.10), the θ-stable fundamental Cartan subgroup $T_G \subset H$ of G, (2.11), the positive $\mathfrak{t}_G$-root system Φ_G^+ on $\mathfrak{g}$, and (2.12), the common Cartan subgroup $T = T_G \cap K \subset L$ of K and M. If $\mathfrak{s}$ is an ad$(\mathfrak{t}_G)$-stable subspace of $\mathfrak{g}$, we write

$$\Phi_G(\mathfrak{s}) = \{\alpha \in \Phi_G : \mathfrak{g}_\alpha \subset \mathfrak{s}\} \qquad \text{and} \qquad \Phi_G(\mathfrak{s})^+ = \Phi_G(\mathfrak{s}) \cap \Phi_G^+.$$

Similarly, let Φ without the subscript denote the root system of $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{t} = \mathfrak{t}_G \cap \mathfrak{k}$ of $\mathfrak{t}_G$. From (2.11), $\Phi_G^+ = \{\alpha \in \Phi_G : \alpha(\xi) > 0\}$ for some $\xi \in \mathfrak{t}$, so $\Phi^+ = \{\alpha|_{\mathfrak{t}} : \alpha \in \Phi_G^+\}$ is a positive $\mathfrak{t}$-root system on $\mathfrak{g}$. If $\mathfrak{s}$ is an ad$(\mathfrak{t})$-stable subspace of $\mathfrak{g}$, we write

$$\Phi(\mathfrak{s}) = \{\beta \in \Phi : \mathfrak{g}_\beta \cap \mathfrak{s} \neq 0\} \qquad \text{and} \qquad \Phi(\mathfrak{s})^+ = \Phi(\mathfrak{s}) \cap \Phi^+.$$

In particular, we have positive $\mathfrak{t}$-root systems $\Phi(\mathfrak{k})^+$ for $\mathfrak{k}$, $\Phi(\mathfrak{m})^+$ for $\mathfrak{m}$, $\Phi(\mathfrak{l})^+ = \Phi(\mathfrak{k})^+ \cap \Phi(\mathfrak{m})^+$ for $\mathfrak{l}$.

If $\mathfrak{g}_{(i)}$ is a simple ideal of $\mathfrak{g}$ and $\mathfrak{g}_{(i)} \not\subset \mathfrak{h}$, then $\Phi_G(\mathfrak{g}_{(i)} \cap \mathfrak{h})$ is the subsystem of $\Phi_G(\mathfrak{g}_{(i)})$ generated by all but one of its $\Phi_G(\mathfrak{g})^+$-simple roots. If $\mathfrak{m}_{(i)}$ is a simple ideal of $\mathfrak{m}$ and $\mathfrak{m}_{(i)} \not\subset \mathfrak{l}$, then $\Phi(\mathfrak{m}_{(i)} \cap \mathfrak{l})$ is the subsystem of $\Phi(\mathfrak{m}_{(i)})$ generated by all but one of its $\Phi(\mathfrak{m})^+$-simple roots. If $\mathfrak{k}_{(i)}$ is a simple ideal of $\mathfrak{k}$ and $\mathfrak{k}_{(i)} \not\subset \mathfrak{l}$, then $\Phi(\mathfrak{k}_{(i)} \cap \mathfrak{l})$ is the subsystem of $\Phi(\mathfrak{k}_{(i)})$ generated by all but one of its $\Phi(\mathfrak{k})^+$-simple roots. Further, $\Phi_G(\mathfrak{q}_+) \subset \Phi_G^+$, $\Phi(\mathfrak{p} \cap \mathfrak{q}_+) \subset \Phi(\mathfrak{m})^+$, and $\Phi(\mathfrak{k} \cap \mathfrak{q}_+) \subset \Phi(\mathfrak{k})^+$.

Recall Schmid's formulation [30] for the L-module structure of $H^0(M/L, \mathbf{U}_\chi)$. Cascade up from the simple noncompact roots of $\mathfrak{m}$:

$$\begin{aligned}
&M_1, ..., M_c \text{ are the noncompact simple factors of } M;\\
&\tilde{\gamma}_i \text{ is the } \Phi(\mathfrak{m})^+\text{-simple root in } \Phi(\mathfrak{m}_i) \cap \Phi(\mathfrak{p} \cap \mathfrak{q}_+);\\
&\tilde{\gamma}_{i,1} = \tilde{\gamma}_i;\\
&\text{for } 1 < j \leqslant r_i = \operatorname{rank}_{\mathbf{R}} M_i,\ \tilde{\gamma}_{i,j} \text{ is the lowest root in}\\
&\quad \Phi(\mathfrak{m}_i) \cap \Phi(\mathfrak{p} \cap \mathfrak{q}_+) \text{ orthogonal to } \tilde{\gamma}_{i,1}, ..., \tilde{\gamma}_{i,j-1}.
\end{aligned} \tag{4.1}$$

Then L acts on the symmetric algebra $S(\mathfrak{p} \cap \mathfrak{q}_-)$ by the multiplicity-free sum of the irreducible representations of highest weights

$$-\tilde{n}\tilde{\gamma} = -\sum_{i=1}^{c} \sum_{j=1}^{r_i} n_{ij} \tilde{\gamma}_{i,j}, \text{ where the } n_{ij} \text{ are integers with } n_{i1} \geqslant n_{i2} \geqslant \cdots \geqslant n_{ir_i} \geqslant 0. \tag{4.2}$$

Write $U_{-\tilde{n}\tilde{\gamma}}$ for the irreducible L-module with highest weight $-\tilde{n}\tilde{\gamma}$ just described. Schmid's result implies

$$H^0(M/L, \mathbf{V})_L = \Sigma_{\tilde{n}}\, V_L \otimes U_{-\tilde{n}\tilde{\gamma}} \qquad \text{as } L\text{-module,} \tag{4.3}$$

where subscript L indicates the subspace consisting of L-finite vectors. Let $\mathbf{U}_{-\tilde{n}\tilde{\gamma}} \to K/L$ denote the homogeneous holomorphic vector bundle with fibre $U_{-\tilde{n}\tilde{\gamma}}$, and $\mathbf{V}_L \to K/L$ the homogeneous C^∞ vector bundle with fibre V_L. Then (4.3) gives us a C^∞ bundle decomposition

$$\mathbf{H}^0(M/L, \mathbf{V})_L = \sum_{\tilde{n}} \mathbf{V}_L \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}. \tag{4.4}$$

When $\dim V < \infty$, this can be made holomorphic.

4.5. THEOREM. *Suppose that ψ is unitary and finite dimensional. Give $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ the holomorphic vector bundle structure in which a* *local section* $w: K \to H^0(M/L, \mathbf{V})$ *is holomorphic if and only if* $w(k \cdot \xi) = 0$ *for all* $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$.

(1) *If* $\pi: G/H \to K/L$ *is holomorphic, then this coincides with the holomorphic vector bundle structure of Lemma* 3.32.

(2) *In any case,* $\mathbf{H}^0(M/L, \mathbf{V})_L$ *and the* $\mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}$ *are holomorphic sub-bundles,* (4.4) *gives a holomorphic bundle decomposition*

$$\mathbf{H}^0(M/L, \mathbf{V})_L = \sum_{\tilde{n}} \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}. \tag{4.6}$$

(3) *If* $\pi: G/H \to K/L$ *is holomorphic, then we have the K-module structure*

$$H^p(G/H, \mathbf{V})_K = \sum_{\tilde{n}} H^p(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}). \tag{4.7}$$

Proof. Since ψ is finite dimensional and unitary, it annihilates all noncompactly embedded factors of H, i.e., those with Lie algebra in $[\mathfrak{p} \cap \mathfrak{h}, \mathfrak{p} \cap \mathfrak{h}] + (\mathfrak{p} \cap \mathfrak{h})$. Now $d\psi(\mathfrak{p} \cap \mathfrak{h}) = 0$, so if $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$, then $d\psi(B_m T_m \xi) = 0$. Assertion (1) follows from (3.33).

The algebra $\mathfrak{k} \cap \mathfrak{q}_-$ acts trivially on local holomorphic sections of $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$. This of course preserves the sub-bundles in question, which thus are holomorphic, and (4.6) follows from (4.4). Assertion (2) is proved.

Assertion (3) now follows from Theorem 3.34. Q.E.D.

When V is unitary and finite dimensional and $\pi: G/H \to K/L$ is holomorphic, the Dolbeault cohomologies $H^p(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}})$ of (4.7) are accessible by means of the Bott–Borel–Weil theorem. Next, we do something similar when V remains finite dimensional but $\pi: G/H \to K/L$ is not required to be holomorphic. There, since all $d\psi(B_m T_m \xi) = 0$ for $\xi \in \mathfrak{p} \cap \mathfrak{h}$, we use the holomorphic bundle structure of Theorem 4.5 to express the operator δ of (3.30) as

$$\begin{aligned} \delta\omega(k)(\xi_1, ..., \xi_{p+1})(m) &= \bar{\partial}\omega(k)(\xi_1, ..., \xi_{p+1})(m) \\ &\quad + \sum_{i=1}^{p+1} (-1)^{i-1}\omega(k \cdot S_m \xi_i)(\xi_1, ..., \hat{\xi}_i, ..., \xi_{p+1})(m). \end{aligned} \tag{4.8}$$

Now filter $H^0(M/L, \mathbf{V})_L$ by order of vanishing at $1 \cdot L$,

$$H^0(M/L, \mathbf{V})^{(p)} = \sum_{|\tilde{n}| \geqslant p} V \otimes U_{-\tilde{n}\tilde{\gamma}} \qquad \text{as } L\text{-module}. \tag{4.9}$$

Here $|\tilde{n}| = \sum n_{ij}$ in the notation (4.2). In view of (4.6) we have associated holomorphic bundles

$$\mathbf{H}^0(M/L, \mathbf{V})^{(p)} = \sum_{|\tilde{n}| \geq p} \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}. \tag{4.10}$$

4.11. LEMMA. *Let* $'\mathscr{C}^{p,q} = A^{p+q}(K/L, \mathbf{H}^0(M/L, \mathbf{V})^{(p)})$. *Then* $\delta \cdot {'\mathscr{C}^{p,q}} \subset {'\mathscr{C}^{p,q+1}}$, *and if* $\omega \in {'\mathscr{C}^{p,q}}$ *then* $\delta\omega - \bar{\partial}\omega \in {'\mathscr{C}^{p+1,q}}$.

Proof. Let $\omega \in {'\mathscr{C}^{p,q}}$ and view it as a function, $\omega: K \to H^0(M/L, \mathbf{V})^{(p)} \otimes \Lambda^{p+q}(\mathfrak{k} \cap \mathfrak{q}_-)^*$. If $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$, define $r(S\xi)\omega$ by

$$[r(S\xi)\omega](k)(\xi_1,..., \xi_{p+q})(m) = \omega(k \cdot S_m\xi)(\xi_1,..., \xi_{p+q})(m). \tag{4.12}$$

As a function of m, (4.12) gives a section of $\mathbf{V} \to M/L$, and that section is holomorphic because (3.24) $r(\mathfrak{p} \cap \mathfrak{q}_-) S_m = 0$. Since $\omega \in {'\mathscr{C}^{p,q}}$ and $S_m = 0$ for $m \in L$, that holomorphic section is in $H^0(M/L, \mathbf{V})^{(p+1)}$. With a glance at (4.8), now $\delta\omega - \bar{\partial}\omega \in {'\mathscr{C}^{p+1,q}}$. But $\bar{\partial}\omega \in {'\mathscr{C}^{p,q+1}}$ because the bundles (4.10) are holomorphic. Thus, also, $\delta\omega \in {'\mathscr{C}^{p,q+1}}$. Q.E.D.

Lemma 4.11 ensures that $\{'\mathscr{C}^{p,q}, \delta\}$ is a complex. We have, for that complex, a decreasing filtration and resulting spectral sequence,

$$'\mathscr{F}^p = \sum_{q \geq 0} {'\mathscr{C}^{p,q}} \quad \text{and} \quad \{'E_r^{p,q}, 'd_r\} \quad \text{with} \quad 'E_0^{p,q} = {'\mathscr{C}^{p,q}}/{'\mathscr{C}^{p+1,q-1}}.$$

The second statement of Lemma 4.11 says that $'d_0: {'E_0^{p,q}} \to {'E_0^{p,q+1}}$, which by definition is $'d_0[\omega] = [\delta\omega]$, can be expressed $'d_0[\omega] = [\bar{\partial}\omega]$. Thus

$$'E_1^{p,q} = H^{p+q}\left(K/L, \mathbf{V} \otimes \sum_{|\tilde{n}| = p} \mathbf{U}_{-\tilde{n}\tilde{\gamma}}\right) \quad \text{as } K\text{-module}. \tag{4.13}$$

For any particular $\kappa \in \hat{K}$, the κ-isotopic subcomplex $\{('\mathscr{C}^{p,q})_\kappa, \bar{\partial}\}$ is finite dimensional, and $('\mathscr{C}^{p,q})_\kappa = 0$ for $p \gg 0$. Later we shall need a stronger result, Lemma 6.22, so we do not prove these assertions now. They allow us to manipulate the spectral sequence on the K-finite level as if the filtration were finite. For example,

4.14. LEMMA. *If* $\{'E_r, 'd_r\}$ *collapses (on the K-finite level) at* $'E_1$, *then* $H^p(G/H, \mathbf{V})_K = \sum_{\tilde{n}} H^p(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}})$ *as K-module.*

Proof. Using Theorem 3.31 and Lemma 4.11, we note that $'E_1 = {'E_\infty}$ would give us a sequence of K-module isomorphisms $H^r(G/V, \mathbf{V})_K \cong H^r_\delta(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)_K \cong \sum_{p+q=r} ('E_\infty^{p,q})_K \cong \sum_{p+q=r} ('E_1^{p,q})_K \cong H^r(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)_K \cong \sum_{\tilde{n}} H^r(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}})$ as asserted. Q.E.D.

In order to isolate a situation in which we have the disjointness that leads to the spectral sequence collapse, consider the *vanishing condition*

$$\text{if } \nu \in \hat{L} \text{ occurs on } H^0(M/L, \mathbf{V}), \text{ then } H^p(K/L, \mathbf{U}_\nu) = 0 \text{ for } p \neq s, \tag{4.15}$$

where we recall $s = \dim_{\mathbf{C}} K/L$.

4.16. PROPOSITION. *If ψ is unitary and* $\dim V < \infty$, *and* $\mathbf{V} \to G/H$ *satisfies the vanishing condition* (4.15), *then* $H^p(G/H, \mathbf{V})_K = 0$ *for* $p \neq s$, *and* $H^s(G/H, \mathbf{V})_K = \sum_{\tilde{n}} H^s(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}})$ *as K-module.*

Proof. In view of (4.14), the vanishing condition (4.15) says that $({}'E_1^{p,q})_K = 0$ for $p + q \neq s$. Thus every ${}'d_1 : {}'E_1^{p,q} \to {}'E_1^{p+1,q}$ is zero on the K-finite level. That is the hypothesis of Lemma 4.14. Q.E.D.

4.17. *Cautionary comment.* The Dolbeault cohomology groups

$$H^p(G/H, \mathbf{V}) = \operatorname{Ker}\{\bar{\partial}: A^p(G/H, \mathbf{V}) \to A^{p+1}(G/H, \mathbf{V})\}/\bar{\partial}A^{p-1}(G/H, \mathbf{V})$$

inherit a topology from the usual C^∞ topology on the $A^p(G/H, \mathbf{V})$. This inherited topology is Fréchet if and only if $\bar{\partial}$ has closed range. In any case, G acts continuously. The usual argument, convolution over K with an approximate identity which in turn is approximated by K-finite functions, shows that $H^p(G/H, \mathbf{V})_K$ is dense in $H^p(G/H, \mathbf{V})$. However, density in a non-Hausdorff space does not have the usual properties. To get around this difficulty, we define "topologically reduced" Dolbeault cohomologies

$$\bar{H}^p(G/H, \mathbf{V}) = \operatorname{Ker}\{\bar{\partial}: A^p(G/H, \mathbf{V}) \to A^{p+1}(G/H, \mathbf{V})\}/\operatorname{closure}\{\bar{\partial}A^{p-1}(G/H, \mathbf{V})\}.$$

They are Fréchet spaces by definition. For each $\kappa \in \hat{K}$ we have an exact sequence

$$0 \to (\text{Image } \bar{\partial})_\kappa \to (\text{kernel } \bar{\partial})_\kappa \to H^*(H/H, \mathbf{V})_\kappa \to 0$$

in which the last term is finite dimensional by Proposition 4.16. Thus $(\text{Image } \bar{\partial})_\kappa$ is closed, by the open mapping theorem. This shows

$$H^p(G/H, \mathbf{V})_K = \bar{H}^p(G/H, \mathbf{V})_K.$$

In other words, $H^p(G/H, \mathbf{V})$ and $\bar{H}^p(G/H, \mathbf{V})$ have the same underlying Harish-Chandra module. This distinction is only important in the case where G acts on $H^p(G/H, \mathbf{V})$ with singular infinitesimal character; in Section 5 we will prove that $\bar{\partial}$ has closed range in the case of nonsingular infinitesimal character.

Suppose now that $\mathbf{V} \to G/H$ is a Hermitian line bundle, i.e., that ψ is a unitary character on H. Identify $\chi = \psi|_L$ with its weight. Then the L-types on

$H^0(M/L, \mathbf{V})$ are precisely those with highest weights $\chi - \tilde{n}\tilde{\gamma}$. If $\nu \in \mathfrak{t}^*$ is K-dominant integral we denote

$$\begin{aligned} \tau_\nu&: \quad \text{irreducible representation of } K \text{ with highest weight } \nu, \\ W_\nu&: \quad \text{representation space of } \tau_\nu. \end{aligned} \tag{4.18}$$

The Bott–Borel–Weil theorem says that the vanishing condition (4.15) is equivalent to the condition that

$$\begin{aligned} &\text{if} \quad \langle \chi - \tilde{n}\tilde{\gamma} + \rho_K, \alpha \rangle > 0 \quad \text{for some} \quad \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+), \\ &\text{then} \quad \langle \chi - \tilde{n}\tilde{\gamma} + \rho_K, \alpha \rangle = 0 \quad \text{for some} \quad \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+) \end{aligned} \tag{4.19}$$

for all $\tilde{n}$.

Consider the Weyl group element

$$w_0 \in W_k \qquad \text{with} \quad w_0 \Phi(\mathfrak{l})^+ \subset \Phi(\mathfrak{k})^+ \quad \text{and} \quad w_0 \Phi(\mathfrak{k} \cap \mathfrak{q}_+) \subset -\Phi(\mathfrak{k})^+. \tag{4.20}$$

In other words, w_0 interchanges the positive systems $\Phi(\mathfrak{l})^+ \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_\pm)$. Then, given (4.19), the Bott–Borel–Weil theorem says

$$\begin{aligned} H^s(K/L, \mathbf{U}_{\chi - \tilde{n}\tilde{\gamma}}) & \\ &= 0, \qquad \text{if} \quad \langle \chi - \tilde{n}\tilde{\gamma} + \rho_K, \alpha \rangle = 0 \quad \text{for some} \quad \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+), \\ &= W_{w_0(\chi - \tilde{n}\tilde{\gamma} + \rho_K) - \rho_K}, \qquad \text{otherwise}. \end{aligned} \tag{4.21}$$

We modify (4.1) by cascading down from the maximal noncompact roots of $\mathfrak{m}$:

$$\begin{aligned} &M_1, \ldots, M_c \text{ are the noncompact simple factors of } M; \\ &\gamma_i \text{ is the } \Phi(\mathfrak{m})^+\text{-maximal root in } \Phi(\mathfrak{m}_i) \cap \Phi(\mathfrak{p} \cap \mathfrak{q}_+); \\ &\gamma_{i,1} = \gamma_i; \\ &\text{for } 1 < j \leqslant r_i = \operatorname{rank}_{\mathbf{R}} M_i,\ \gamma_{i,j} \text{ is the highest root in} \\ &\quad \Phi(\mathfrak{m}_i) \cap \Phi(\mathfrak{p} \cap \mathfrak{q}_+) \text{ orthogonal to } \gamma_{i,1}, \ldots, \gamma_{i,j-1}. \end{aligned} \tag{4.22}$$

Denote the Weyl group element

$$w_1 \in W_L \qquad \text{such that} \quad w_1 \Phi(\mathfrak{l})^+ = -(\mathfrak{l})^+. \tag{4.23}$$

Then $w_1 \Phi(\mathfrak{p} \cap \mathfrak{q}_+) = \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$, and so, making proper choices where there are several highest roots in $\Phi(\mathfrak{m}_i) \cap \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ orthogonal to $\gamma_{i,1}, \ldots, \gamma_{i,j-1}$, we may suppose

$$w_1 \tilde{\gamma}_{i,j} = \gamma_{i,j} \qquad \text{for} \quad 1 \leqslant i \leqslant c \quad \text{and} \quad 1 \leqslant j \leqslant r_i. \tag{4.24}$$

Note that $w_1(\chi)=\chi$ because $\langle \chi, \Phi(\mathfrak{l})\rangle = 0$, and $w_1 w_0 \in W_k$ sends $\Phi(\mathfrak{k})^+$ to $-\Phi(\mathfrak{k})^+$. Now

$$W^*_{w_0(\chi-\tilde{n}\tilde{\gamma}+\rho_K)-\rho_K} = W_{-w_1w_0\{w_0(\chi-\tilde{n}\tilde{\gamma}+\rho_K)-\rho_K\}}$$

and

$$\begin{aligned}-w_1 w_0\{w_0(\chi-\tilde{n}\tilde{\gamma}+\rho_K)-\rho_K\} &= -w_1(\chi-\tilde{n}\tilde{\gamma}+p_K)-\rho_K \\ &= -\chi+\tilde{n}\gamma - w_1(\rho_K)-\rho_K = -\chi+\tilde{n}\gamma-2\rho_{K/L},\end{aligned}$$

where

$$\tilde{n}\gamma = \sum_{i=1}^{c}\sum_{j=1}^{r_i} n_{ij}\gamma_{i,j} \qquad \text{and} \qquad \rho_{K/L} = \tfrac{1}{2}\textstyle\sum_{\Phi(\mathfrak{l}\cap\mathfrak{q}_+)}\alpha. \tag{4.25}$$

Apply w_1 to (4.19). Since w_1 preserves $\Phi(\mathfrak{k}\cap\mathfrak{q}_+)$, the vanishing condition (4.15) is equivalent to

$$\begin{aligned}&\text{if} \quad \langle \chi-\tilde{n}\gamma+2\rho_{K/L}-\rho_K, \alpha\rangle > 0 \qquad \text{for some} \quad \alpha\in\Phi(\mathfrak{k}\cap\mathfrak{q}_+),\\ &\text{then} \quad \langle \chi-\tilde{n}\gamma+2\rho_{K/L}-\rho_K, \alpha\rangle = 0 \qquad \text{for some} \quad \alpha\in\Phi(\mathfrak{k}\cap\mathfrak{q}_+).\end{aligned} \tag{4.26}$$

And, as we have just seen, if the vanishing condition holds, then

$$\begin{aligned}H^s(K/L, \mathbf{U}_{\chi-\tilde{n}\tilde{\gamma}}) &= 0, && \text{if} \quad \langle \chi-\tilde{n}\gamma+2\rho_{K/L}-\rho_K, \alpha\rangle = 0,\\ & && \text{some } \alpha\in\Phi(\mathfrak{k}\cap\mathfrak{q}_+),\\ &= W^*_{-\chi+\tilde{n}\gamma-2\rho_{K/L}}, && \text{otherwise.}\end{aligned} \tag{4.27}$$

Finally, we combine this with Proposition 4.17 to obtain

4.28. THEOREM. *Suppose that* $\mathbf{V}\to G/H$ *is a Hermitian line bundle and let* $\chi=\psi|_L$. *Then* $\mathbf{V}\to G/H$ *satisfies the vanishing condition* (4.15) *if and only if* (4.26) *holds for every* $\tilde{n}$. *In that case,* $H^p(G/H,\mathbf{V})_K = 0$ *for* $p\neq s$, *and* $H^s(G/H,\mathbf{V})_K$ *is the sum over* $\tilde{n}$ *of the* K*-modules*

$$\begin{aligned}&0, \qquad \text{if} \quad \langle \chi-\tilde{n}\gamma+2\rho_{K/L}-\rho_K, \alpha\rangle = 0 \ \textit{for some} \ \alpha\in\Phi(\mathfrak{k}\cap\mathfrak{q}_+),\\ &W^*_{-\chi+\tilde{n}\gamma-2\rho_{K/L}}, \qquad \textit{otherwise.}\end{aligned}$$

In particular, the vanishing condition implies that $H^s(G/H,\mathbf{V})$ *is* K*-multiplicity-free.*

Remark. Comment 4.17 applies here.

We now relax the condition that V be finite dimensional, and instead assume that ψ has some formal resemblance to a holomorphic discrete series representation of H. Remember, here V is a Fréchet H-module, not necessarily unitary. The framework for studying this case is

4.29. LEMMA. *Let $\pi: G/H \to K/L$ be holomorphic, so G/K is Hermitian symmetric with holomorphic tangent space represented by $\mathfrak{p}_+ = \mathrm{ad}(\mathscr{U}(\mathfrak{k}))(\mathfrak{p} \cap \mathfrak{q}_+)$. Then H/L is Hermitian symmetric with holomorphic tangent space represented by $\mathfrak{p}_+ \cap \mathfrak{h}$, and if $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$ and $m \in M$, then $B_m T_m \xi \in \mathfrak{p}_+ \cap \mathfrak{h}$.*

Proof. The statement about G/K is part of Proposition 2.21 and its proof. Now $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$ implies $[\mathfrak{l}, \mathfrak{p}_+ \cap \mathfrak{h}] \subset (\mathfrak{p}_+ \cap \mathfrak{h})$, so H/L is Hermitian symmetric in such a way that $\mathfrak{p}_+ \cap \mathfrak{h}$ represents its holomorphic tangent space. For the last statement we may assume $m^{-1} = \exp(\eta)$, where $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, and decompose $\eta = \eta_+ + \eta_-$, where $\eta_+ \in \mathfrak{p} \cap \mathfrak{q}_+ = \mathfrak{p}_+ \cap \mathfrak{q}_+$ and $\eta_- = \bar{\eta}_+ \in \mathfrak{p}_- \cap \mathfrak{q}_-$. Then

$$\mathrm{ad}(\eta)\xi = [\eta_+, \xi] + [\eta_-, \xi] = [\eta_+, \xi] \in \mathfrak{p}_+ \cap \mathfrak{h}$$

and

$$\mathrm{ad}(\eta)^2 \xi \in [\mathfrak{p}_+ \cap \mathfrak{q}_+, \mathfrak{p}_+ \cap \mathfrak{h}] + [\mathfrak{p}_- \cap \mathfrak{q}_-, \mathfrak{p}_+ \cap \mathfrak{h}] \subset (\mathfrak{k} \cap \mathfrak{q}_-).$$

Now $\mathrm{Ad}(m^{-1})\xi = \sinh(\mathrm{ad}(\eta))\xi + \cosh(\mathrm{ad}(\eta))\xi$ with $\sinh(\mathrm{ad}(\eta))\xi \in \mathfrak{p}_+ \cap \mathfrak{h}$ and $\cosh(\mathrm{ad}(\eta))\xi \in \mathfrak{k} \cap \mathfrak{q}_-$. So $B_m T_m \xi = -p'' \circ \mathrm{Ad}(m^{-1})\xi = -\sinh(\mathrm{ad}(\eta))\xi \in \mathfrak{p}_+ \cap \mathfrak{h}$. Q.E.D.

Now let $\pi: G/H \to K/L$ be holomorphic and, in the notation of Lemma 4.29, suppose that ψ has a highest L-type relative to $\Phi(\mathfrak{p}_+ \cap \mathfrak{h})$, i.e., that

> $\psi|_L$ has an irreducible summand of highest weight χ in such a way that every irreducible summand of $\psi|_L$ has highest weight of the form $\chi - \sum b_i \beta_i$, $b_i \geqslant 0$, $\beta_i \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$. (4.30)

Then V has an increasing L-invariant filtration

$$V = V^0 \subset V^1 \subset \cdots, \qquad \bigcup V^p = V_L,$$

defined by

$$V^p \text{ is the closed span of the } L\text{-types } \chi - \sum b_i \beta_i \text{ with } \sum b_i \leqslant p. \quad (4.31)$$

That in turn gives us

$$\mathbf{V}^p \to M/L: \quad \text{holomorphic sub-bundle of } \mathbf{V}|_{H/L} \text{ with fibre } V^p. \quad (4.32)$$

Note that $d\psi(\mathfrak{p}_+ \cap \mathfrak{h}) \cdot V^p \subset V^{p-1}$, so Lemma 4.29 ensures

$$\text{if } \xi \in \mathfrak{k} \cap \mathfrak{q}_- \text{ and } m \in M, \text{ then } d\psi(B_m T_m \xi) \cdot V^p \subset V^{p-1}. \tag{4.33}$$

Now a glance back at (3.30) and Lemma 3.32 shows that

$$\mathbf{H}^0(M/L, \mathbf{V}^p) \to K/L \text{ is a holomorphic sub-bundle of } \mathbf{H}^0(M/L, \mathbf{V}) \tag{4.34}$$

and that the

$$''\mathscr{C}^{p,q} = A^{p+q}(K/L, \mathbf{H}^0(M/L, \mathbf{V}^p)) \text{ satisfy } \bar{\partial}''\mathscr{C}^{p,q} \subset {}''\mathscr{C}^{p,q+1}. \tag{4.35}$$

Thus we have an increasing filtration and resulting spectral sequence

$$''\mathscr{F}^p = \sum_{q>0} {}''\mathscr{C}^{p,q} \quad \text{and} \quad \{''E_r^{p,q}, {}''d_r\} \text{ with } {}''E_0^{p,q} = {}''\mathscr{C}^{p,q}/''\mathscr{C}^{p-1,q+1}. \tag{4.36}$$

If $[\omega] \in {}''E_0^{p,q}$ is represented by $\omega \in {}''\mathscr{C}^{p,q}$, then $''d_0[\omega] = [\bar{\partial}\omega] \in {}''E_0^{p,q+1}$ is specified by

$$\begin{aligned}
&\bar{\partial}\omega(k)(\xi_1, \ldots, \xi_{p+q+1})(m) \\
&\quad = \sum_{i=1}^{p+q+1} (-1)^{i-1} \omega(k \cdot \xi_i)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+q+1})(m) \\
&\qquad - \sum_{i=1}^{p+q+1} (-1)^{i-1} d\psi(B_m T_m \xi_i) \cdot \omega(k)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+q+1})(m).
\end{aligned}$$

Here, in view of (4.33), the $d\psi$ terms take values in V^{p-1} as functions of m, so those terms are in $''\mathscr{C}^{p-1,q+1}$. Thus $''d_0[\omega] = [\bar{\partial}_1 \omega]$, where $\bar{\partial}_1 \omega(k)(\xi_1, \ldots, \xi_{p+q+1})(m) = \sum_{i=1}^{p+q+1} (-1)^{i-1} \omega(k \cdot \xi_i)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+q+1})(m)$. But this $\bar{\partial}_1$ is the Dolbeault operator on K/L for the bundle

$$\mathbf{H}^0(M/L, \mathbf{V}^p)/\mathbf{H}^0(M/L, \mathbf{V}^{p+1}) = \mathbf{H}^0(M/L, \mathbf{V}^p/\mathbf{V}^{p+1}).$$

In other words,

$$''E_1^{p,q} = H^{p+q}(K/L, \mathbf{H}^0(M/L, \mathbf{V}^p/\mathbf{V}^{p+1})). \tag{4.37}$$

The spectral sequence $\{''E_r^{p,q}, d_r\}$ eventually collapses because $''d_r : {}''E_r^{p,q} \to {}''E_r^{p-r,q+1+r}$ must be trivial when one of $''\mathscr{C}^{p,q}$ and $''\mathscr{C}^{p-r,q+1+r}$ is zero, e.g., when $r > s$. Thus the characteristic

$$\begin{aligned}
\sum_{r=0}^{s} (-1)^r H^r(K/L, \mathbf{H}^0(M/L, \mathbf{V}))_K &= \sum_{r=0}^{s} (-1)^r \sum_{p+q+r} (''E_\infty^{p,q})_K \\
&= \sum_{r=0}^{s} (-1)^r \sum_{p+q+r} (''E_1^{p,q})_K
\end{aligned}$$

as virtual K-modules. In view of (4.37), and with a glance back at (4.4), we summarize as follows:

4.38. THEOREM. *Let $\pi: G/H \to K/L$ be holomorphic, give H/L the Hermitian symmetric space structure of Lemma* 4.29, *and suppose that ψ is a holomorphic representation of H in the sense* (4.30) *that it has a highest K-type. Then, as virtual K-modules,*

$$\sum_{r=0}^{s} (-1)^r H^r(G/H, \mathbf{V})_K = \sum_{r=0}^{s} (-1)^r \sum_{p,\tilde{n}} H^r(K/L, (\mathbf{V}^p/\mathbf{V}^{p-1}) \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}})_K. \tag{4.39}$$

4.40. *Remark.* In Theorem 4.38, the right-hand side is computable from (i) the explicit L-spectrum of ψ, which specifies the finite dimensional L-modules V^p/V^{p-1}, (ii) the explicit decomposition of the $(V^p/V^{p-1}) \otimes U_{-\tilde{n}\tilde{\gamma}}$ into L-irreducibles, and (iii) the Bott–Borel–Weil theorem.

The differential $''d_1: {}''E_1^{p,q} \to {}'E_1^{p-1,q+1}$ is given by the $d\psi$-terms in the formula for $\bar{\partial}$. They cause the cancellation in passing from $''E_1$ to $''E_2$ which means that, in contrast to the finite-dimensional case, the K-spectrum of $H^p(G/H, \mathbf{V})$ is more sparse than it would be were $H^0(M/L, \mathbf{V})_L \to K/L$ a holomorphic sum of finite-dimensional bundles. We will have more to stay on this in Section 12.

5. INFINITESIMAL AND DISTRIBUTION CHARACTERS

We now study the characters of the representation π_V of G on $H^s(G/H, \mathbf{V})$ when $\dim V < \infty$. Our main result, Theorem 5.23, gives the infinitesimal character and explicit alternating sum formulae for the distribution- and K-characters of G on the $H^p(G/H, \mathbf{V})$ in the case where $\pi: G/H \to K/L$ is holomorphic. After proving it, we briefly indicate how to drop the assumption that π be holomorphic. The distribution character formula is not really explicit in that more general case.

The reader whose knowledge of semisimple representation theory leaves something to be desired should just study the statement of Corollary 5.27 below and then go on to Section 6.

For the moment, now, our working hypothesis is that

$$\pi: G/H \to K/L \text{ is holomorphic and}$$
$$\psi \text{ is unitary and finite dimensional.} \tag{5.1}$$

Then H/L is a bounded symmetric domain where $\mathfrak{p}_+ \cap \mathfrak{h}$ is the holomorphic tangent space. Also, $\chi = \psi|_L$ is irreducible. If $\nu \in \hat{L}$, we have denoted its representation space by U_v and have written $\mathbf{U}_v \to K/L$ and $\mathbf{B}_v \to M/L$ for the associated bundles over K/L and M/L. Write

$$\mathbf{F}_v \to H/L: \quad \text{homogeneous holomorphic bundle associated to } U_v. \qquad (5.2)$$

We will also need the length function l on the Weyl groups $W_{\mathfrak{g}}$ and $W_{\mathfrak{h}}$: $l(w)$ is the number of positive roots α such that $w(\alpha)$ is negative. If $v \in W_{\mathfrak{h}}$, then $v(\rho - \rho_H) = \rho - \rho_H$, so $v(\rho_H) - \rho_H = v(\rho) - \rho$, and $l(v)$ is the same for $W_{\mathfrak{g}}$ and for $W_{\mathfrak{h}}$. Set

$$W_{\mathfrak{h},p} = \{v \in W_{\mathfrak{h}}: l(v) = p \text{ and } v(\rho) - \rho \text{ is } \Phi(\mathfrak{l})^+\text{-dominant}\}. \qquad (5.3)$$

Note $W_{\mathfrak{h},0} = \{1\}$. We now prove

5.4. Proposition. *There is an exact sequence of H-modules*

$$0 \to V \to H^0(H/L, \mathbf{F}_\chi) \to \cdots \to \bigoplus_{v \in W_{\mathfrak{h},p}} H^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho}) \to \cdots. \qquad (5.5)$$

Remarks. Such a resolution exists for any finite-dimensional irreducible H-module V, unitary or not, and the following proof can be modified slightly to cover that more general case. Also, although we will not use this fact, (5.5) is dual to Lepowsky's generalized Bernstein–Gelfand–Gelfand resolution [20].

Proof. Since H/L is a Stein manifold, its de Rham cohomology can be computed from the holomorphic de Rham complex $0 \to H^0(H/L, \Omega^0) \to H^0(H/L, \Omega^1) \to \cdots$. Since H/L is contractible, the de Rham complex is acyclic, so we have an exact sequence

$$0 \to \mathbf{C} \to H^0(H/L, \Omega^0) \to H^0(H/L, \Omega^1) \to \cdots. \qquad (5.6)$$

The homomorphisms in this sequence commute with the action of H. Also, Ω^p is the sheaf of germs of holomorphic sections of the homogeneous vector bundle $\Lambda^p(\mathfrak{p}_+ \cap \mathfrak{h})^* \to H/L$ associated to the representation of L on $\Lambda^p(\mathfrak{p}_+ \cap \mathfrak{h})^* \cong \Lambda^p(\mathfrak{p}_- \cap \mathfrak{h})$. Kostant proved [19, Corollary 8.2] that the L-module $\Lambda^p(\mathfrak{p}_- \cap \mathfrak{h})$ is direct sum over $W_{\mathfrak{h},p}$ of the $U_{v(\rho_H)-\rho_H} = U_{v(\rho)-\rho}$. Thus the exact sequence (5.6) is an exact sequence

$$0 \to \mathbf{C} \to H^0(H/L, \mathbf{F}_0) \to \cdots \to \bigoplus_{v \in W_{\mathfrak{h},p}} H^0(H/L, \mathbf{F}_{v(\rho)-\rho}) \to \cdots \qquad (5.7)$$

of H-modules.

Since V is a finite-dimensional H-module, we may view it as the fibre of a holomorphically trivial homogeneous vector bundle $\mathbf{F}_\chi \to H/L$. Thus

$$V \otimes H^0(H/L, \mathbf{F}_{v(\rho)-\rho}) = H^0(H/L, \mathbf{F}_\chi \otimes \mathbf{F}_{v(\rho)-\rho}) \qquad (5.8)$$

as H-module. The group $L = L_1 \cdot L_2$, quotient of a direct product by a finite subgroup $L_1 \cap L_2$, where L_1 is the intersection of L with the product of the noncompact simple factors of H and L_2 is the product of the center with the compact simple factors. Here L_1 acts trivially on V—this is the point that needs some care if we replace V by an arbitrary irreducible finite-dimensional H-module—and L_2 acts trivially on $\mathfrak{p}_- \cap \mathfrak{h}$. Thus $V \otimes U_{v(\rho)-\rho} = U_{v(\chi+\rho)-\rho}$, so $\mathbf{F}_\chi \otimes \mathbf{F}_{v(\rho)-\rho} = \mathbf{F}_{v(\chi+\rho)-\rho}$. Substituting this into (5.8), we obtain our assertion (5.5) by tensoring V with the exact sequence (5.7). Q.E.D.

Each of the Fréchet H-modules in (5.5) determines a homogeneous holomorphic Fréchet bundle over G/H, and these form an exact sequence

$$0 \to \mathbf{V} \to \mathbf{H}^0(H/L, \mathbf{F}_\chi) \to \cdots \to \bigoplus_{v \in W_{\mathfrak{h},p}} \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho}) \to \cdots. \quad (5.9)$$

That will give us a useful spectral sequence converging to $H^*(G/V, \mathbf{V})$. To state the result, denote

$$W_v^q = H^q(K/L, \mathbf{U}_{v(\chi+\rho)-\rho}) \qquad \text{for} \quad v \in W_{\mathfrak{h},p}. \quad (5.10)$$

Notice that $(v(\chi+\rho)-\rho)+\rho_K$ is $\Phi(\mathfrak{k})$-singular if and only if $v(\chi+\rho)$ is, because $(\rho-\rho_K, \alpha)=0$ for all $\alpha \in \Phi(\mathfrak{k})$. If $v(\chi+\rho)$ is $\Phi(\mathfrak{k})$-nonsingular, choose $w \in W_k$ such that $wv(\chi+\rho)$ is $\Phi(\mathfrak{k})^+$-dominant. Now $w(v(\chi+\rho)-\rho+\rho_K)-\rho_K = wv(\chi+\rho)-\rho$, so the Bott–Borel–Weil theorem identifies the K-module W_v^q as

$$\begin{aligned} W_v^q &= 0 && \text{if} \quad q \neq l(w), \\ &= W_{wv(\chi+\rho)-\rho} && \text{if} \quad q = l(w). \end{aligned} \quad (5.11)$$

Denote

$$\mathbf{W}_v^q \to G/K: \quad \text{homogeneous holomorphic bundle for } W_v^q. \quad (5.12)$$

Now we can state

5.13. Proposition. *There is a G-equivariant spectral sequence converging to* $H^*(G/H, \mathbf{V})$, *for which* $E_1^{p,q} = \bigoplus_{v \in W_{\mathfrak{h},p}} H^0(G/K, \mathbf{W}_v^q)$.

Proof. The resolution (5.9) leads to a spectral sequence, converging to $H^*(G/H, \mathbf{V})$, for which $E_1^{p,q} = \bigoplus_{v \in W_{\mathfrak{h},p}} H^q(G/H, \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho}))$. That is standard from homological algebra. In effect, writing (5.9) as $0 \to \mathbf{V} \to \mathbf{V}_0 \to \mathbf{V}_1 \to \cdots$ and $A^q(\mathbf{V}_p)$ for $A^q(G/H, \mathbf{V}_p)$, the fact that $\bar{\partial}$ commutes with the bundle maps gives us a double complex

$$\begin{array}{ccccccc}
A^0(\mathbf{V}_0) & \to & A^0(\mathbf{V}_1) & \to \cdots \to & A^0(\mathbf{V}_p) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
A^1(\mathbf{V}_0) & \to & A^1(\mathbf{V}_1) & \to \cdots \to & A^1(\mathbf{V}_p) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

The pth row is exact when augmented by $A^p(\mathbf{V})$, so the horizontal filtration gives a spectral sequence $'E$ with $'E_1^{p,q} = H^q(p\text{th row}) = A^p(\mathbf{V})$ for $q = 0$, $= 0$ for $q > 0$. The sequence collapses so H^* (double complex) $= H^*(\mathbf{V})$. The spectral sequence E associated to the vertical filtration (the usual one for a double complex) has $E_1^{p,q} = H^q(p\text{th column}) = H^q(\mathbf{V}_p)$. It also converges to H^* (double complex), which we saw to be $H^*(\mathbf{V})$.

We now need to compare $H^q(G/H, \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho}))$ with $H^0(G/K, \mathbf{W}_v^q)$. Apply the discussion of Section 3 to the holomorphic fibration $G/L \to G/H$, fibre H/L. There the Leray sequence collapses because H/L is Stein, so

$$H^q(G/H, \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho})) \cong H^q(G/L, \mathbf{Y}_{v(\chi+\rho)-\rho}),$$

where $\mathbf{Y}_v \to G/L$ is the homogeneous holomorphic bundle over G/L with the same fibre U_v as $\mathbf{F}_v \to H/L$. Similarly, since the base of $G/L \to G/K$, fibre K/L, is Stein, the associated spectral sequence collapses and

$$H^q(G/L, \mathbf{Y}_{v(\chi+\rho)-\rho}) \cong H^0(G/K, \mathbf{H}^q(K/L, \mathbf{F}_{v(\chi+\rho)-\rho})).$$

Combining these two isomorphisms with (5.10) and (5.12), we have $H^q(G/H, \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho})) \cong H^0(G/K, \mathbf{W}_v^q)$. The proposition follows.

Q.E.D.

Let λ be any W_K-regular $\Phi(\mathfrak{k})^+$-dominant weight. Then $\lambda - \rho$ still is $\Phi(\mathfrak{k})^+$-dominant because $\rho - \rho_K \perp \Phi(\mathfrak{k})$, so there is an irreducible K-module $W_{\lambda-\rho}$ of highest weight $\lambda - \rho$. It gives a homogeneous holomorphic vector bundle $\mathbf{W}_{\lambda-\rho} \to G/K$, and the Fréchet G-module $H^0(G/K, \mathbf{W}_{\lambda-\rho})$ belongs to the "continuation of the holomorphic discrete series" studied by Harish-Chandra. Its subspace $H^0(G/K, \mathbf{W}_{\lambda-\rho})_K$ of K-finite vectors has the following properties:

it has a composition series of finite length; (5.14)

the center of $\mathscr{U}(\mathfrak{g})$ acts on it by the character of Harish-Chandra parameter λ, i.e., G acts on $H^0(G/K, \mathbf{W}_{\lambda-\rho})$ with infinitesimal character χ_λ, in Harish-Chandra's notation [11]; (5.15)

$H^0(G/K, \mathbf{W}_{\lambda-\rho})_K \cong W_{\lambda-\rho} \otimes S(\mathfrak{p}_-)$ as K-module; (5.16)

$H^0(G/K, \mathbf{W}_{\lambda-\rho})_K$ has finite T-multiplicities and $\lambda - \rho$ is a highest weight for it. (5.17)

In particular we have, well defined,

$$\Theta(\lambda): \quad \text{distribution character of } G \text{ on } H^0(G/K, \mathbf{W}_{\lambda-\rho}). \tag{5.18}$$

Hecht [15] worked out an explicit global formula for the characters $\Theta(\lambda)$.

We extend the definition (5.18) of $\Theta(\lambda)$ to all elements of the weight lattice Λ by the symmetry condition

$$\Theta(w\lambda) = \varepsilon(w)\,\Theta(\lambda) \qquad \text{for} \quad \lambda \in \Lambda, \quad w \in W_K, \quad \varepsilon(w) = \operatorname{sign}(w). \tag{5.19}$$

In other words, if λ is W_K-singular, then $\Theta(\lambda) = 0$, and if λ is W_K-regular, then $\Theta(\lambda) = \varepsilon(w)\,\Theta(w\lambda)$, where $w \in W_K$ such that $w(\lambda)$ is $\Phi(\mathfrak{k})^+$-dominant and where $\Theta(w\lambda)$ is defined by (5.18).

By (5.17), $H^0(G/K, \mathbf{W}_{\lambda-\rho})$ has finite K-multiplicities, so there is a well-defined

$$\Theta_K(\lambda): \quad \text{formal character of } K \text{ on } H^0(G/K, \mathbf{W}_{\lambda-\rho}). \tag{5.20}$$

In view of (5.16) it is given on the regular set K' by $\operatorname{Ad}(K)$-invariance and by this formula on $T \cap K'$:

$$\Theta_K(\lambda) = \sum_{n_1,\ldots,n_t=0}^{\infty} \frac{\sum_{w \in W_K} \varepsilon(w)\, e^{w(\lambda - \rho + \rho_K - n_1\beta_1 - \cdots - n_t\beta_t)}}{\prod_{\alpha \in \Phi(\mathfrak{k})^+} (e^{\alpha/2} - e^{-\alpha/2})}, \tag{5.21}$$

where $\Phi(\mathfrak{p}_+) = \{\beta_1, \ldots, \beta_t\}$. Since W_K permutes the β_i and fixes $\rho - \rho_K$, (5.21) remains valid when we extend the definition (5.20) of the distributions $\Theta_K(\lambda)$ on K by

$$\Theta_K(w\lambda) = \varepsilon(w)\,\Theta_K(\lambda) \qquad \text{for} \quad \lambda \in \Lambda \quad \text{and} \quad w \in W_K. \tag{5.22}$$

We can now state and prove

5.23. THEOREM. *Let $\pi: G/H \to K/L$ be holomorphic,* $\dim V < \infty$, *and $\chi = \psi|_L$. Then for every integer $p \geqslant 0$, $H^p(G/H, \mathbf{V})_K$ is a Harish-Chandra module of finite length which is T-finite with weights bounded from above and which has infinitesimal character of Harish-Chandra parameter $\chi + \rho$. Write $\sum_v$ for summation over $\bigcup_{p \geqslant 0} W_{\mathfrak{h},p} = \{v \in W_{\mathfrak{h}}: v(\rho) - \rho$ is $\Phi(\mathfrak{l})^+$-dominant$\}$. Then the distribution characters $\Theta(H^p(G/H, \mathbf{V}))$ exist and satisfy*

$$\sum (-1)^p \Theta(H^p(G/H, \mathbf{V})) = \sum\nolimits_v \varepsilon(v)\,\Theta(v(\chi + \rho)). \tag{5.24}$$

The formal K-characters $\Theta_K(H^p(G/H, \mathbf{V}))$ *exist and satisfy*

$$\sum (-1)^p \Theta_K(H^p(G/H, \mathbf{V})) = \sum_v \varepsilon(v)\, \Theta_K(v(\chi + \rho)), \tag{5.25}$$

$$= \sum_v \varepsilon(v) \sum_{n_1, \ldots, n_t = 0}^{\infty} \frac{\sum_{w \in W_K} \varepsilon(w)\, e^{w(v(\chi+\rho) - \rho + \rho_K - n_1\beta_1 - \cdots - n_t\beta_t)}}{\prod_{\alpha \in \Phi(\mathfrak{k})^+} (e^{\alpha/2} - e^{-\alpha/2})}, \tag{5.26}$$

where $\Phi(\mathfrak{p}_+) = \{\beta_1, \ldots, \beta_t\}$.

Proof. The spectral sequence of Proposition 5.13 has only finitely many nonzero $E_1^{p,q}$, each of which is a finite sum of G-modules $H^0(G/K, \mathbf{W}_{\lambda-\rho})$ for various choices of λ. These each have finite composition series (5.14), each is T-finite with weights bounded from above (5.17), and each has infinitesimal character $\chi + \rho$ by (5.11) and (5.15). It follows that $H^p(G/H, \mathbf{V})_L$ has finite composition series, is T-finite with weights bounded from above, and has infinitesimal character $\chi + \rho$. Now, in the spectral sequence we take Euler characteristics as follows:

$$\sum_{p \geq 0} (-1)^p \Theta(H^p(G/H, \mathbf{V})) = \sum_{p,q \geq 0} \Theta(E_1^{p,q})$$

$$= \sum_{v,q} (-1)^{q + l(v)} \Theta(H^0(G/K, \mathbf{W}_v^q)).$$

By (5.11), (5.18), and (5.19),

$$(-1)^q \Theta(H^0(G/K, \mathbf{W}_v^q)) = \varepsilon(w)\, \Theta(wv(\chi + \rho)) = \Theta(v(\chi + \rho)).$$

Thus

$$\sum_{p \geq 0} (-1)^p \Theta(H^p(G/H, \mathbf{V})) = \sum_v \varepsilon(v)\, \Theta(v(\chi + \rho)),$$

which is (5.24). Since $\Theta(\lambda)$ and $\Theta_K(\lambda)$ satisfy the same W_K-symmetry conditions, (5.25) follows, and then (5.21) gives (5.26). Q.E.D.

We remark that one can derive (5.26) directly from (4.7), and it is of course equivalent to the case $\dim V < \infty$ of (4.39), but we omit the computation. If one does proceed from (4.7), he first obtains the K-character and infinitesimal character, but has a hard time showing that $H^p(G/H, \mathbf{V})_L$ is a $(\mathfrak{g}_0, K)$-module of finite length whose distribution character is determined by the K-character. The resolutions (5.5) and (5.9) efficiently avoid this problem. In fact, proceeding from (4.7), it is only feasible to get the generic but special case

5.27. Corollary. *Let* $\pi\colon G/H \to K/L$ *be holomorphic,* $\dim V < \infty$, *and*

$\chi = \psi|_L$. *Suppose that we have the vanishing condition* (*recall Comment* 4.17 *here*)

$$H^p(G/H, \mathbf{V})_K = 0 \qquad \text{for} \quad p \neq s. \tag{5.28}$$

Then $H^s(G/H, \mathbf{V})_K$ *is a* $(\mathfrak{g}_0, K)$*-module of finite length, T-finite with weights bounded from above, with infinitesimal character of Harish-Chandra parameter* $\chi + \rho$, *with distribution character*

$$\Theta(H^s(G/H, \mathbf{V})) = (-1)^s \frac{1}{|W_L|} \sum_{v \in W_{\mathfrak{h}}} \varepsilon(v)\, \Theta(v(\chi + \rho)), \tag{5.29}$$

and with K-character $\Theta_K(H^s(G/H,)\mathbf{V})_K)$ *given by*

$$\frac{(-1)^s}{|W_L|} \sum_{v \in W} \varepsilon(v) \sum_{n_1, \ldots, n_t = 0}^{\infty} \frac{\sum_{w \in W_K} \varepsilon(w)\, e^{w(v(\chi+\rho) - \rho + \rho_K - n_1\beta_1 - \cdots - n_t\beta_t)}}{\prod_{\alpha \in \Phi(\mathfrak{k})^+} (e^{\alpha/2} - e^{-\alpha/2})}, \tag{5.30}$$

where $\{\beta_1, \ldots, \beta_t\} = \Phi(\mathfrak{p}_+)$.

Consider the generic negative case where $\chi + \rho$ is "very nonsingular" in the sense that $(\chi + \rho, \alpha) \ll 0$ for $\alpha \in \Phi(\mathfrak{q}_+)$. More precisely, since $v(\chi) = \chi$ for all $v \in W_{\mathfrak{h}}$, suppose that

$$(v(\chi + \rho), \alpha) = ((\chi + \rho) + (v\rho - \rho), \alpha) < 0 \quad \text{for all} \quad v \in W_{\mathfrak{h}} \quad \text{with}$$
$$v\rho - \rho \ L\text{-dominant and all } \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+). \tag{5.31}$$

Then $W_v^q = 0$ unless $q = s$, and $W_v^s = W_{w_0 v(\chi+\rho) - \rho}$, where w_0 is given by (4.20). In particular, the spectral sequence of Proposition 5.13 collapses at E_1. Thus,

5.32. Corollary. *If* (5.31) *holds, then the vanishing condition* (5.28) *holds, so we have the conclusions of Corollary* 5.27. *Also, there is an exact sequence*

$$0 \to H^s(G/H, \mathbf{V}) \to H^0(G/K, \mathbf{W}_{w_0(\chi+\rho)-\rho}) \to$$
$$\cdots \to \bigoplus_{v \in W_{\mathfrak{h},p}} H^0(G/K, \mathbf{W}_{w_0 v(\chi+\rho)-\rho}) \to \cdots \tag{5.33}$$

resolving the representation of G on $H^s(G/H, \mathbf{V})$ *by "full holomorphic" representations.*

Now we can prove the closed range result mentioned at the end of Comment 4.17.

5.34. Proposition. *If* $(\chi + \rho, \alpha) < 0$ *for all* $\alpha \in \Phi(\mathfrak{q}_+)$, *then each* $\bar{\partial}: A^p(G/H, \mathbf{V}) \to A^{p+1}(G/H, \mathbf{V})$ *has closed range, so each* $H^p(G/H, \mathbf{V})$ *is a Fréchet space and* $H^p(G/H, \mathbf{V}) = 0$ *for* $p \neq s$.

Proof. In the proof of Proposition 5.13, each

$$E_1^{p,q} = H^q(G/H, \mathbf{V}_p) = H^q\left(G/H, \bigoplus_{v \in W_{\mathfrak{h},p}} \mathbf{H}^0(H/L, \mathbf{F}_{v(\chi+\rho)-\rho})\right)$$

has a natural C^∞ topology, which makes the spectral sequence differential d_1 continuous. The Andreotti–Grauert result [1, Theorem 1], quoted just after Proposition 3.25, and the open mapping theorem show that the isomorphisms $E_1^{p,q} \cong \bigoplus_{v \in W_{\mathfrak{h},p}} H^0(G/K, \mathbf{W}_v^q)$ of Proposition 5.13 are topological isomorphisms.

Suppose that $\chi + \rho$ is "very nonsingular" in the sense that (5.31) holds. Now, the connecting maps of (5.33), including the injection $H^s(G/H, \mathbf{V}) \to H^0(G/K, \mathbf{W}_{w_0(\chi+\rho)-\rho})$, are continuous. All the $H^0(G/K, \mathbf{W}_{w_0 v(\chi+\rho)-\rho})$ visibly are Fréchet, so now $H^s(G/H, \mathbf{V})$ is Fréchet. Since $H^p(G/H, \mathbf{V}) = 0$ for $p \neq s$ by Theorem 3.34 and our negativity hypothesis, the closed range statement follows.

In general, choose a weight $\mu \perp \Phi(\mathfrak{h})$ such that $\langle \mu, \alpha \rangle \gg 0$ for all $\alpha \in \Phi(\mathfrak{q}_+)$, and define $\chi' = \chi - \mu$. Let V' be the irreducible H-module of highest weight μ. According to the "very nonsingular" case, $H^s(G/H, \mathbf{V}' \otimes \mathbf{Y}) \cong H^s(G/H, \mathbf{V}') \otimes Y$ is a Fréchet G-module. Note that $\mathbf{V}$ is a quotient of $\mathbf{V}' \otimes \mathbf{Y}$ and that all other composition factors of $V' \otimes \mathbf{Y}$ are modeled on irreducible H-modules of highest weights ν such that $\nu + \rho \notin W_{\mathfrak{g}}(\chi + \rho)$. We collect them all together as $\mathbf{V}''$ to form an exact sequence

$$0 \to \mathbf{V}'' \to \mathbf{V}' \otimes \mathbf{Y} \to \mathbf{V} \to 0.$$

If $\mathbf{Z}$ is any composition factor, we get a spectral sequence, like that of Proposition 5.3, for $H^*(G/H, \mathbf{Z})$. In particular, since every vector there is a differentiable vector, the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra acts on $H^*(G/H, \mathbf{Z})$ by a character. We conclude that

$$H^*(G/H, \mathbf{V}' \otimes \mathbf{Y}) \text{ is } \mathcal{Z}(\mathfrak{g})\text{-finite}$$

and

$$H^*(G/H, \mathbf{V}) \text{ and } H^*(G/H, \mathbf{V}'') \text{ have no } \mathcal{Z}(\mathfrak{g})\text{-character in common,}$$

so

$$\mathrm{Hom}_{\mathfrak{g}}(H^*(G/H, \mathbf{V}), H^*(G/H, \mathbf{V}'')) = 0.$$

In particular, the connecting homomorphisms are trivial in the long exact cohomology sequence for $0 \to \mathbf{V}'' \to \mathbf{V}' \otimes \mathbf{Y} \to \mathbf{V} \to 0$, so we have short exact sequences

$$0 \to H^p(G/H, \mathbf{V}'') \to H^p(G/H, \mathbf{V}' \otimes \mathbf{Y}) \to H^p(G/H, \mathbf{V}) \to 0.$$

We cannot apply the open mapping theorem, since we have not yet shown that $H^p(G/H, \mathbf{V})$ is Hausdorff. But arguing again with $\mathcal{Z}(\mathfrak{g})$-characters, we see that the sequence splits topologically, concluding that $H^p(G/H, \mathbf{V})$ inherits a Fréchet space structure from $H^p(G/H, \mathbf{V}' \otimes \mathbf{Y})$. Q.E.D.

In the preceding discussion we assumed that the fibration $\pi: G/H \to K/L$ is holomorphic, in particular that G/K is Hermitian symmetric. In fact, when properly rephrased, Theorem 5.23 remains valid without this assumption: $H^p(G/H, \mathbf{V})_K$ is no longer a highest weight module, but (5.24)–(5.26) still hold. We now indicate very briefly just how the statement and proof must be modified for the much more general case where we assume only that rank $K =$ rank G.

Initially, fix a positive root system $\Phi^+ = \Phi(\mathfrak{g})^+$ and give G/T the invariant complex structure such that the holomorphic tangent space is the sum of the positive roots spaces. Let $\mathscr{C}$ denote the *negative* Weyl chamber. If $\lambda \in \Lambda$ is "very negative," i.e., $(\lambda, \alpha) \ll 0$ for all $\alpha \in \Phi(\mathfrak{g})^+$, then the techniques of Schmid ([29, 31]) show

if $p \neq \dim_{\mathbf{C}} K/T$, then $H^p(H/T, \mathbf{C}_\lambda) = 0$;

if $p = \dim_{\mathbf{C}} K/T$, then $H^p(G/T, \mathbf{C}_\lambda)$ is infinitesimally equivalent to the discrete series representation with character $\Theta_{\lambda+\rho}$. (5.35)

Here $\mathbf{C}_\lambda \to K/T$ denotes the line bundle associated by the character $e^\lambda \in \hat{T}$, and $\Theta_{\lambda+\rho}$ is as defined by Harish-Chandra [14].

Let $\Theta(\mathscr{C}, \lambda)$ denote the family of invariant eigendistributions on G defined in [33] and characterized there by the two properties

$$\Theta(\mathscr{C}, \lambda) = \Theta_\lambda \text{ if } \lambda \in \Lambda \text{ is regular and lies in } \mathscr{C}, \tag{5.36a}$$

$$\Theta(\mathscr{C}, \lambda) \text{ depends coherently on the parameter } \lambda \in \Lambda. \tag{5.36b}$$

According to (5.35), if λ is very negative, then

$$\sum_{p \geqslant 0} (-1)^p \Theta(H^p(G/T, \mathbf{C}_{\lambda-\rho})) = (-1)^{\dim_{\mathbf{C}} K/T} \Theta(\mathscr{C}, \lambda). \tag{5.37}$$

A tensoring argument and some homological algebra show that

the $H^p(G/T, \mathbf{C}_{\lambda-\rho})$ are Fréchet G-modules, and their spaces of K-finite vectors are Harish-Chandra modules of finite length, for any $\lambda \in \Lambda$ (5.38)

and

the identity (5.37) holds for any $\lambda \in \Lambda$. (5.39)

Let $\Omega^p \to G/T$ denote the sheaf of germs of holomorphic p-forms. Then $0 \to \mathbf{C} \to \Omega^0 \to \Omega^1 \to \cdots$ is exact. However, neither the constant sheaf $\mathbf{C}$ nor the holomorphic sheaves Ω^p need be acyclic, so the analog of (5.6) does not carry over directly. But there is an analog if one takes Euler characteristic:

$$|W_K| \cdot 1 = \sum_{p,q} (-1)^{p+q} \Theta(H^q(G/T, \Omega^p)). \tag{5.40}$$

Here $|W_K|$ enters as the Euler characteristic of $\mathbf{C} \to G/T$. Use (5.37), valid for any $\lambda \in \Lambda$, to rewrite (5.40) as

$$|W_K| \cdot 1 = (-1)^{\dim_{\mathbf{C}} K/T} \sum_{w \in W_{\mathfrak{g}}} \varepsilon(w)\, \Theta(\mathscr{C}, w\rho). \tag{5.41}$$

Let $\mathrm{ch}(Y_\mu)$ denote the character of the irreducible finite-dimensional G-module Y_μ of highest weight μ. Under the process of coherent continuation, (5.41) implies

$$(-1)^{\dim_{\mathbf{C}} K/T} |W_K| \cdot \mathrm{ch}(Y_\mu) = \sum_{w \in W_{\mathfrak{h}}} \varepsilon(w)\, \Theta(\mathscr{C}, w(\mu + \rho)). \tag{5.42}$$

We return to the geometric situation of Section 3. Suppose $\operatorname{rank} K = \operatorname{rank} G$, so that the invariant eigendistributions $\Theta(\mathscr{C}, \lambda)$ are defined. Choose $\Phi(\mathfrak{g})^+$ so that $G/T \to G/H$, fibre H/T, is holomorphic. Let V_χ denote a finite-dimensional, not necessarily unitary, irreducible H-module of highest weight χ. Then $\operatorname{rank} L = \operatorname{rank} H$ and we have the analog

$$(-1)^{\dim_{\mathbf{C}} L/T} |W_L| \cdot \mathrm{ch}_H(V_\chi) = \sum_{u \in W_{\mathfrak{h}}} \varepsilon(u)\, \Theta_H(\mathscr{C}_H, u(\chi + \rho_H)) \tag{5.43}$$

of (5.42). View (5.43) as an identity between virtual Fréchet H-modules. These H-modules determine homogeneous holomorphic Fréchet vector bundles over G/H. They satisfy the identity

$$|W_L|\, \mathbf{V}_\chi = \sum_{p \geqslant 0} \sum_{u \in W_{\mathfrak{h}}} (-1)^p \varepsilon(u)\, \mathbf{H}^p(H/T, \mathbf{C}_{u(\chi+\rho)-\rho}) \tag{5.44}$$

in the sense of virtual homogeneous holomorphic bundles. For essentially formal reasons one can equate Euler characteristics of the characters of the cohomology groups. That yields

$$\begin{aligned} &|W_L| \sum_{p \geqslant 0} (-1)^p \Theta(H^p(G/H, \mathbf{V}_\chi)) \\ &\quad = \sum_{p,q} \sum_{u \in W_{\mathfrak{h}}} (-1)^{p+q} \varepsilon(u)\, \Theta(H^q(G/H, \mathbf{H}^p(H/T, \mathbf{C}_{u(\chi+\rho)-\rho}))). \end{aligned}$$

Using the Leray sequence for $G/T \to G/H$, fibre H/T, this becomes

$$\sum_{q \geqslant 0} \sum_{u \in W_{\mathfrak{h}}} (-1)^q \varepsilon(u)\, \Theta(H^q(G/T, \mathbf{C}_{u(\chi+\rho)-\rho})).$$

According to (5.37) and (5.39), this is equal to

$$(-1)^{\dim_{\mathbf{C}} K/T} \sum_{u \in W_{\mathfrak{h}}} \varepsilon(u)\, \Theta(\mathscr{C}, u(\chi+\rho)).$$

Thus we have

$$|W_L| \sum_{p \geqslant 0} (-1)^p \Theta(H^p(G/H, \mathbf{V}_\chi)) = (-1)^{\dim_{\mathbf{C}} K/T} \sum_{u \in W_{\mathfrak{h}}} \varepsilon(u)\, \Theta(\mathscr{C}, u(\chi+\rho)). \tag{5.45}$$

In deriving (5.45) we implicitly asserted that the $H^p(G/H, \mathbf{V}_\chi)_K$ are Harish-Chandra modules of finite length, so that the $\Theta(H^p(G/H, \mathbf{V}_\chi))$ are defined. In the case where $\pi\colon G/H \to K/L$ is holomorphic, the $\Theta(\lambda)$ defined in (5.18) and (5.19) are equal to $(-1)^{\dim_{\mathbf{C}} K/T} \Theta(\mathscr{C}, \lambda)$. This follows, for example, from the Leray sequence for the fibration $G/T \to K/T$. Thus, (5.44) is equivalent, when $\pi\colon G/H \to K/L$ is holomorphic, to the main assertion of Theorem 5.23, the alternating sum formula (5.24) for the distribution characters $\Theta(H^p(G/H, \mathbf{V}_\chi))$.

Caution. If $\Phi(\mathfrak{g})^+$ is not chosen so that $G/T \to G/K$ is holomorphic, then (5.22) holds only for those $u \in W_K$ that are products of reflections in compact simple roots; see [16].

The invariant eigendistributions are virtual characters, so we can define

$\Theta_K(\mathscr{C}, \lambda)$: formal K-character of the virtual Harish-Chandra module with distribution character $\Theta(\mathscr{C}, \lambda)$. (5.46)

According to [16],

$$\Theta_K(\mathscr{C}, \lambda) = (-1)^{\dim_{\mathbf{C}} K/T} \sum_{n_1, \ldots, n_t = 0}^{\infty} \frac{\sum_{w \in W_K} \varepsilon(w)\, e^{w(\lambda - \rho + \rho_K - n_1\beta_1 - \cdots - n_t\beta_t)}}{\prod_{\alpha \in \Phi(\mathfrak{k})^+} (e^{\alpha/2} - e^{\alpha/2})}. \tag{5.47}$$

Compare with this with (5.21). An immediate consequence of (5.45) is

$$|W_L| \sum_{p \geqslant 0} (-1)^p \Theta_K(H^p(G/H, \mathbf{V}_\chi)) = (-1)^{\dim_{\mathbf{C}} K/T} \sum_{u \in W_{\mathfrak{h}}} \varepsilon(u)\, \Theta_K(\mathscr{C}, u(\chi+\rho)). \tag{5.48}$$

In the case where $\pi: G/H \to K/L$ is holomorphic, (5.47) and (5.48) are equivalent to the K-character assertions of Theorem 5.23, which are the alternating sum formulae (5.25) and (5.26).

We now have shown how Theorem 5.23 extends to the case where $\pi: G/H \to K/L$ is not necesarily holomorphic and $\mathbf{V} \to G/H$ is not necessarily Hermitian, when we replace the extended holomorphic characters $\Theta(\lambda)$ by the coherent family $\Theta(\mathscr{C}, \lambda)$ and reformulate (5.24) as (5.45), and (5.25) and (5.26) as (5.48) and (5.47).

6. Harmonic Forms on G/H

Throughout this section we assume that H operates unitarily on V. The complex semisimple symmetric space G/H carries a G-invariant indefinite Kähler metric induced by the Killing form of $\mathfrak{g}_0$. This metric together with the Hermitian metric on $\mathbf{V} \to G/H$ specifies the conjugate linear Kodaira–Hodge orthocomplementation operator

$$\#: E^{p,q}(G/H, \mathbf{V}) \to E^{n-p,n-q}(G/H, \mathbf{V}^*), \tag{6.1}$$

where $E^{a,b}(G/H, \mathbf{V})$ denotes the space of C^∞ $\mathbf{V}$-valued (a, b)-forms on G/H, $n = \dim_{\mathbf{C}} G/H$, and $\mathbf{V}^* \to G/H$ is the dual bundle. As in the positive definite case,

$$\bar{\partial}: E^{p,q}(G/H, \mathbf{V}) \to E^{p,q+1}(G/H, \mathbf{V})$$

has formal adjoint

$$\bar{\partial}^* = -\#\,\bar{\partial}\#: E^{p,q+1}(G/H, \mathbf{V}) \to E^{p,q}(G/H, \mathbf{V}) \tag{6.2}$$

with respect to the global G-invariant, generally indefinite, Hermitian inner product

$$\langle \varphi, \varphi' \rangle = \int_{G/H} \varphi(x)\, \bar{\wedge}\, \#\varphi'(x)\, d(xH), \tag{6.3}$$

where $\bar{\wedge}$ denotes exterior product followed by contraction of V against V^*. Since $\langle\ ,\ \rangle$ need not be definite, the Kodaira–Hodge–Beltrami–Laplace operator

$$\square = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial}: E^{p,q}(G/H, \mathbf{V}) \to E^{p,q}(G/H, \mathbf{V}) \tag{6.4}$$

need not be elliptic, so we will not use it to define "harmonic" forms. Instead, we define that

$$\varphi \in E^{p,q}(G/H, \mathbf{V}) \text{ is } \textit{harmonic} \text{ if } \bar{\partial}\varphi = 0 \text{ and } \bar{\partial}^*\varphi = 0. \tag{6.5}$$

This implies $\square\varphi = 0$, but $\square\varphi = 0$ does not imply that φ is harmonic.

In this section we study the possibility of representing a Dolbeault cohomology class $c \in H^p(G/H, \mathbf{V})$ by a harmonic form φ. This will be done in such a way that, in Section 7, it will be easy to see whether φ is square integrable over G/H.

Let $\varphi \in E^{0,p+q}(G/H, \mathbf{V}) = A^{p+q}(G/H, \mathbf{V})$ such that, viewing $\varphi: G \to V \otimes \Lambda^{p+q}(\mathfrak{q}_-)^*$,

$$\varphi(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*. \tag{6.6}$$

Choose bases

$$\begin{aligned} &\{X_1,..., X_s\} \text{ of } \mathfrak{k} \cap \mathfrak{q}_- \text{ such that } B(X_i, \bar{X}_j) = -\delta_{ij}, \\ &\{Y_1,..., Y_t\} \text{ of } \mathfrak{p} \cap \mathfrak{q}_- \text{ such that } B(Y_i, \bar{Y}_j) = +\delta_{ij}, \end{aligned} \tag{6.7}$$

where $B(\cdot, \cdot)$ is the Killing form. Write e and i for exterior and interior product,

$$e(w)W = w \wedge W \qquad \text{and} \qquad i(w)(w \wedge W) = \|w\|^2 W.$$

Then, on any $E^{a,b}(G/H, \mathbf{V})$,

$$\bar{\partial} = -\sum_{i=1}^{s} r(X_i) \otimes e(\bar{X}_i) + \sum_{j=1}^{t} r(Y_j) \otimes e(\bar{Y}_j),$$

where $\bar{\mathfrak{q}}_- = \mathfrak{q}_+$ is identified to $(\mathfrak{q}_-)^*$ by the Killing form. That gives us

$$\bar{\partial}^* = -\sum_{i=1}^{s} r(\bar{X}_i) \otimes i(X_i) - \sum_{j=1}^{t} r(\bar{Y}_j) \otimes i(Y_j). \tag{6.8}$$

Now, if $\xi_i \in \mathfrak{k} \cap \mathfrak{q}_-$ and $\eta_j \in \mathfrak{p} \cap \mathfrak{q}_-$,

$$\begin{aligned} &\bar{\partial}^* \varphi(km)(\xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) \\ &\quad = -\sum_{i=1}^{s} r(\bar{X}_i)\, \varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) \\ &\qquad - \sum_{j=1}^{t} r(\bar{Y}_j)\, \varphi(km)(Y_j, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q). \end{aligned}$$

As $\bar{Y}_j \in \mathfrak{m}$ and $Y_j \in \mathfrak{p} \cap \mathfrak{q}_-$, (6, 6) forces

$$r(\bar{Y}_j)\, \varphi(km)(Y_j, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) = 0.$$

Now, using (3.18),

$$\begin{aligned} &\bar{\partial}^* \varphi(km)(\xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) \\ &\quad = -\sum_{1 \leqslant i \leqslant s} \{l(A_m \bar{X}_i) + r(B_m \bar{X}_i)\}\, \varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) \end{aligned}$$

Since $B_m\bar{X}_i \in \mathfrak{h}$,

$$-r(B_m\bar{X}_i)\,\varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q)$$
$$= \varphi(km)([B_m\bar{X}_i, X_i], \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q)$$
$$+ \sum_{a=1}^{p-1} \varphi(km)(X_i, \xi_1,..., [B_m\bar{X}_i, \xi_a],..., \xi_{p-1}, \eta_1,..., \eta_q)$$
$$+ \sum_{b=1}^{q} \varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., [B_m\bar{X}_i, \eta_b],..., \eta_q)$$
$$+ d\psi(B_m\bar{X}_i)\cdot\varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q).$$

As $[B_m\bar{X}_i, X_i]$ and the $[B_m\bar{X}_i, \xi_a]$ are in $\mathfrak{p}\cap\mathfrak{q}_-$, and each $[B_m\bar{X}_i, \eta_b] \in \mathfrak{k}\cap\mathfrak{q}_-$, the first three terms vanish by (6.6). That leaves

$$\bar{\partial}^*\varphi(km)(\xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q)$$
$$= -\sum_{1\leqslant i\leqslant s} l(A_m\bar{X}_i)\cdot\varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q)$$
$$+ \sum_{1\leqslant i\leqslant s} d\psi(B_m\bar{X}_i)\cdot\varphi(km)(X_i, \xi_1,..., \xi_{p-1}, \eta_1,..., \eta_q) \qquad (6.9)$$

Also,

$$\bar{\partial}^*\varphi(km)(\xi_1,..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$= -\sum_{i=1}^{s} r(\bar{X}_i)\,\varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$- \sum_{j=1}^{t} r(\bar{Y}_j)\,\varphi(km)(Y_j, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1}).$$

We note

$$l(A_m\bar{X}_i)\,\varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1}) = 0$$

by (6.6) because $A_m\bar{X}_i \in \mathfrak{k}$, so (3.18) gives us

$$-r(\bar{X}_i)\,\varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$= -r(B_m\bar{X}_i)\,\varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$= \varphi(km)([B_m\bar{X}_i, X_i], \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$+ \sum_{a=1}^{p} \varphi(km)(X_i, \xi_1,..., [B_m\bar{X}_i, \xi_a],..., \xi_p, \eta_1,..., \eta_{q-1})$$
$$+ \sum_{b=1}^{q-1} \varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., [B_m\bar{X}_i, \eta_b],..., \eta_{q-1})$$
$$+ d\psi(B_m\bar{X}_i)\cdot\varphi(km)(X_i, \xi_1,..., \xi_p, \eta_1,..., \eta_{q-1}).$$

As X_i, $[B_m\bar{X}_i, \eta_b] \in \mathfrak{k} \cap \mathfrak{q}_-$, the last two terms vanish by (6.6). Permuting some vectors, that leaves

$$\begin{aligned}
&\bar{\partial}^*\varphi(km)(\xi_1,\ldots,\xi_p,\eta_1,\ldots,\eta_{q-1})\\
&\quad= (-1)^{p-1}\sum_{i=1}^{s}\varphi(km)(\xi_1,\ldots,\xi_p,[B_m\bar{X}_i, X_i],\eta_1,\ldots,\eta_{q-1})\\
&\qquad -\sum_{i=1}^{s}\sum_{a=1}^{p}(-1)^{p-a}\varphi(km)(X_i,\xi_1,\ldots,\hat{\xi}_a,\ldots,\xi_p;[B_m\bar{X}_i,\xi_a],\eta_1,\ldots,\eta_{q-1})\\
&\qquad +(-1)^p\sum_{j=1}^{t} r(\bar{Y}_j)\,\varphi(km)(\xi_1,\ldots,\xi_p,Y_j,\eta_1,\ldots,\eta_{q-1}). \qquad (6.10)
\end{aligned}$$

Arguing as for (6.9) and (6.10), one sees

$$\begin{aligned}
&\bar{\partial}^*\varphi(km)(\xi_1,\ldots,\xi_{p-2},\eta_1,\ldots,\eta_{q+1})\\
&= (-1)^p\sum_{i=1}^{s}\sum_{j=1}^{q+1}(-1)^{j-1}\varphi(km)(X_i,[B_m\bar{X}_i,\eta_b],\xi_1,\ldots,\xi_{p-2},\eta_1,\ldots,\hat{\eta}_b,\ldots,\eta_{q+1}). \qquad (6.11)
\end{aligned}$$

Remark 3.35 shows that $\bar{\partial}^*$ is completely determined by (6.9)–(6.11) and that the term (6.11) vanishes if $\pi\colon G/H \to K/L$ is holomorphic.

6.12. Lemma. *Suppose that $\pi\colon G/H \to K/L$ is holomorphic. Let $\varphi \in A^s(G/H, \mathbf{V})$, $s = \dim_{\mathbb{C}} K/L$, such that $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Then $\bar{\partial}^*\varphi = 0$ if and only if $l(A_m\bar{\xi})\,\varphi(km) = d\psi(B_m\bar{\xi}) \cdot \varphi(km)$ for $k \in K$, $m \in M$, $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$.*

Proof. The terms (6.10) and (6.11) drop out and (6.9) reduces to

$$\begin{aligned}
\bar{\partial}^*\varphi(km)(X_1,\ldots,\hat{X}_i,\ldots,X_s) &= (-1)^i l(A_m\bar{X}_i)\,\varphi(km)(X_1,\ldots,X_s)\\
&\quad + (-1)^{i-1} d\psi(B_m\bar{X}_i)\cdot\varphi(km)(X_1,\ldots,X_s).
\end{aligned}$$

That vanishes for each i if and only if every $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$ gives $l(A_m\bar{\xi})\,\varphi(km) - d\psi(B_m\bar{\xi})\cdot\varphi(km) = 0$. Q.E.D.

We are going to compare the $\bar{\partial}^*$-closed condition of Lemma 6.12 with the $\bar{\partial}^*_{K/L}$-closed condition for a form $\omega \in A^s(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$. At first glance we need two things in order to define $\bar{\partial}^*_{K/L}$: a holomorphic vector bundle structure on $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ and a Hermitian structure on that bundle. We will assume $\pi\colon G/H \to K/L$ holomorphic and use the holomorphic vector bundle structure of Lemma 3.32 for $\mathbf{H}^0(M/L, \mathbf{V})$ and its subbundle $\mathbf{H}^0(M/L, \mathbf{V})_L$ with typical fibre the space $H^0(M/L, V)_L$ of L-finite holomorphic sections of $V|_{M/L} \to M/L$. Use (4.3) and the natural $\mathfrak{h}$-invariant inner product on V_L, and choose an L-invariant inner product on each $U_{-\tilde{n}\tilde{\gamma}}$, to define a conjugate linear isomorphism of $H^0(M/L, \mathbf{V})_L$ with its L-finite dual space

$$H^0(M/L, \mathbf{V})_L^* = \sum_{\tilde{n}} V_L^* \otimes U_{-\tilde{n}\tilde{\gamma}}^*,$$

and thus, as in (4.4), of $\mathbf{H}^0(M/L, \mathbf{V})_L^*$ with its "L-finite dual bundle"

$$\mathbf{H}^0(M/L, \mathbf{V})_L^* = \sum_{\tilde{n}} V_L^* \otimes U_{-\tilde{n}\tilde{\gamma}}^*. \tag{6.13}$$

We then have Hodge–Kodeira orthocomplementation operators

$$\#: E^{p,q}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L) \to E^{s-p,s-q}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L^*),$$

which depend, of course, on the choice of inner product on the $U_{-\tilde{n}\tilde{\gamma}}$. Now $\bar{\partial}: E^{p,q}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L) \to E^{p,q+1}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)$ has formal adjoint $\bar{\partial}_{K/L}^* = -\#\bar{\partial}\#$. That is our

$$\bar{\partial}_{K/L}^*: E^{p,q}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L) \to E^{p,q-1}(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L). \tag{6.14}$$

As we shall see presently, $\bar{\partial}_{K/L}^*$ does not depend on any arbitrary choices. Glancing back to (3.33), we see that

$$\bar{\partial} = -\sum_{i=1}^{s} r_K(X_i) \otimes e(\bar{X}_i) + \sum_{i=1}^{s} d\psi(BTX_i) \otimes e(\bar{X}_i),$$

where $r_K(\xi) f(k) = f(k \cdot \xi)$, the X_i are given by (6.7), $\bar{X}_i \in \mathfrak{k} \cap \mathfrak{q}_+$ is identified to an element of $(\mathfrak{k} \cap \mathfrak{q}_-)^*$, and $d\psi(BTX)$ acts on $H^0(M/L, \mathbf{V})_L$ by

$$[d\psi(BTX)f](m) = d\psi(B_m T_m X) \cdot f(m).$$

That gives us

$$\bar{\partial}_{K/L}^* = -\sum_{i=1}^{s} r_K(\bar{X}_i) \otimes i(X_i) + \sum_{i=1}^{s} d\psi(\overline{BTX_i}) \otimes i(X_i), \tag{6.15}$$

independently of the choices that were made to define $\#$. If $\omega \in A^p(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)$, now

$$\begin{aligned}
&\bar{\partial}_{K/L}^* \omega(k)(\xi_1, \ldots, \xi_{p-1})(m) \\
&\quad = -\sum_{i=1}^{s} \omega(k \cdot \bar{X}_i)(X_i, \xi_1, \ldots, \xi_{p-1})(m) \\
&\qquad + \sum_{i=1}^{s} d\psi(\overline{B_m T_m X_i}) \cdot \omega(k)(X_i, \xi_1, \ldots, \xi_{p-1})(m).
\end{aligned} \tag{6.16}$$

If $p = s$, then (6.16) reduces to

$$\bar{\partial}^*_{K/L}\,\omega(k)(X_1,..., \hat{X}_i,..., X_s)(m) = (-1)^i\omega(k \cdot X_i)(X_1,..., X_s)(m)$$
$$+ (-1)^{i-1}d\psi(B_m T_m X_i) \cdot \omega(k)(X_1,..., X_s)(m).$$

But $\overline{B_m T_m X_i} = B_m \overline{T_m X_i}$ from the definition (3.14), and $T_m X_i = A_m^{-1} X_i$ because $\pi: G/H \to K/L$ is holomorphic. Thus

6.17. LEMMA. *Let $\omega \in A^s(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)$. Then $\bar{\partial}^*_{K/L}\,\omega = 0$ if and only if $\omega(k \cdot \bar{\xi})(\cdot)(m) = d\psi(B_m \overline{A_m^{-1}\xi}) \cdot \omega(k)(\cdot)(m)$ for $k \in K$, $m \in M$, $\xi \in \mathfrak{k} \cap \mathfrak{q}_-$.*

Since $A_m : \mathfrak{k} \cap \underline{\mathfrak{q}_-} \to \mathfrak{k} \cap \mathfrak{q}_-$ isomorphically when $\pi: G/H \to K/L$ is holomorphic, and $\overline{A\xi} = A\bar{\xi}$, Lemmas 6.12 and 6.17 combine to yield

6.18. PROPOSITION. *Suppose that $\pi: G/H \to K/L$ is holomorphic. Let $\omega \in A^s(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)$, where $s = \dim_{\mathbb{C}} K/L$, and let $\varphi \in A^s(G/H, \mathbf{V})$ be the corresponding (3.12) form with $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Then $\bar{\partial}^*\varphi = 0$ if and only if $\bar{\partial}^*_{K/L}\,\omega = 0$.*

Let $c \in H^s(G/H, \mathbf{V})_K$, where $\pi: G/H \to K/L$ is holomorphic. If the corresponding Dolbeault class $c' \in H^s(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L)$ has a harmonic representative $\omega \in A^s(K/L, \mathbf{H}^s(M/L, \mathbf{V})_L)$, then the form $\varphi \in A^s(G/H, \mathbf{V})$ of Proposition 6.18 will be a harmonic representative for c. Since $\mathbf{H}^0(M/L, \mathbf{V})_L \to K/L$ has infinite-dimensional fibres, we must work a bit to show the existence of ω.

If $\nu \in \hat{L}$, we denote the ν-isotypic component of $H^0(M/L, \mathbf{V})$ by $H^0(M/L, \mathbf{V})_\nu$. Similarly, $V_{\nu'}$ is the ν'-isotypic component of V. Every $\dim V_{\nu'} < \infty$ because ψ is an irreducible unitary representation of H. If ν occurs in $V_{\nu'} \otimes U_{-\tilde{n}\tilde{\gamma}}$, then $\nu = \nu' - \tilde{n}\tilde{\gamma} + \lambda$, where λ is a sum of roots of L and $-\tilde{n}\tilde{\gamma} + \lambda$ is a weight of L on $U_{-\tilde{n}\tilde{\gamma}}$. Let ζ be the central element of $\mathfrak{h}$ with $\mathrm{ad}(\zeta) = \pm i$ on $\mathfrak{q}_\pm$. Then $\nu(\zeta) = '(\zeta) - \tilde{n}\tilde{\gamma}(\zeta) = \nu'(\zeta) - |\tilde{n}|\, i$. There are only finitely many possibilities for $\nu'(\zeta)$ because

$$V_L = d\psi(\mathscr{U}(\mathfrak{h}))\, V_{\nu''} = \sum_{n=0}^{\infty} d\psi(\mathfrak{p} \cap \mathfrak{h})^n V_{\nu''}$$

for any L-isotypic component $V_{\nu''}$ of V and because any sum of roots from $\mathfrak{p} \cap \mathfrak{h}$ vanishes on ζ. Given ν, now, there are only finitely many possibilities for $|\tilde{n}|$, thus for $\tilde{n}$, thus for $-\tilde{n}\tilde{\gamma} + \lambda$. Now, any $\nu \in \hat{L}$ can occur on only a finite number of the $V_{\nu'} \otimes U_{-\tilde{n}\tilde{\gamma}}$. As $H^0(M/L, \mathbf{V})_L = \sum_{\nu' \in \hat{L}} \sum_{\tilde{n}} V_{\nu'} \otimes U_{-\tilde{n}\tilde{\gamma}}$, we conclude

$$\text{if } \nu \in \hat{L}, \text{ then } \dim H^0(M/L, \mathbf{V})_\nu < \infty. \tag{6.19}$$

Let $\mathbf{H}^0(M/L, \mathbf{V})_\nu \to K/L$ denote the sub-bundle of $\mathbf{H}^0(M/L, \mathbf{V}) \to K/L$ with typical fibre $H^0(M/L, \mathbf{V})_\nu$. Then (6.19) says that if is a finite-dimensional bundle.

Let $\square_{K/L} = \bar\partial\,\bar\partial^*_{K/L} + \partial^*_{K/L}\,\bar\partial$, the Kodaira–Hodge–Beltrami–Laplace operator for $\mathbf{H}^0(M/L, \mathbf{V})_L \to K/L$. If $\kappa \in \hat{K}$, denote

$$W^p_\kappa\text{: the } \kappa\text{-isotypic component of } A^p(K/L, \mathbf{H}^0(M/L, \mathbf{V})_L). \quad (6.20)$$

By equivariance,

$$\bar\partial W^p_\kappa \subset W^{p+1}_\kappa, \qquad \bar\partial^*_{K/L} W^{p+1}_\kappa \subset W^p_\kappa, \qquad \square_{K/L} W^p_\kappa \subset W^p_\kappa. \quad (6.21)$$

6.22. Lemma. *If $\kappa \in \hat{K}$, then there is a finite subset $F(\kappa) \subset \hat{L}$ such that $W^p_\kappa \subset A^p(K/L, \sum_{\nu \in F(\kappa)} \mathbf{H}^0(K/L, \mathbf{V})_\nu)$ for $0 \leqslant p \leqslant s$.*

Proof. Fix p. Let B be the (finite) set of L-types that occur in $\kappa|_L$. Frobenius reciprocity for (K, L) says that κ occurs in $A^p(K/L, H^0(M/L, \mathbf{V})_{\nu'})$ if and only if some $\nu \in B$ occurs in $H^0(M/L, \mathbf{V})_{\nu'} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Let

$$F^p(\kappa) = \{\nu' \in \hat{L}\text{: some } \nu \in B \text{ occurs in } U_{\nu'} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*\}.$$

The Lemma is equivalent to the statement that each $F^p(\kappa)$ is finite; we then take $F(\kappa) = \bigcup_{p=0}^s F^p(\kappa)$.

Enumerate $\Phi(\mathfrak{k} \cap \mathfrak{q}_+) = \{\alpha_1, \ldots, \alpha_s\}$. Every L-type that occurs in $U_{\nu'} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$ has form

$$\nu = \nu' + \alpha_{i_1} + \cdots + \alpha_{i_p}, \qquad 1 \leqslant i_1 < \cdots < i_p \leqslant s.$$

If $\nu \in B$ occurs on $U_{\nu'} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$, now ν' must be of the form $\nu - (\alpha_{i_1} + \cdots + \alpha_{i_p})$. There are only finitely many possibilities for each $\nu \in B$.

Q.E.D.

Let $c' \in H^p(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$ be K-finite. Then it is a finite sum of classes $c'_\kappa \in H^p(K/L, \mathbf{H}^0(M/L, \mathbf{V}))_\kappa$, $\kappa \in \hat{K}$. According to (6.21), c'_κ is calculable in the complex $\{W^p_\kappa, \bar\partial\}$. Lemma 6.22, together with (6.19), say that $c'_\kappa \in H^p(K/L, \mathbf{U})$, where $\mathbf{U} \to K/L$ is Hermitian, holomorphic, and finite dimensional. Using (6.21) again, standard Hodge theory provides a harmonic form $\omega_\kappa \in W^p_\kappa$ whose Dolbeault class is c'_κ. Now $\omega = \sum \omega_\kappa$ is a harmonic representative for c'. We have proved

6.23. Proposition. *Every K-finite class in $H^p(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$ has a unique K-finite harmonic representative.*

Let $c \in H^s(G/H, \mathbf{V})_K$ correspond to $c' \in H^s(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$. Then c' is K-finite and so has a unique K-finite harmonic representative ω. Let

$\varphi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^*$ correspond to ω as in (3.12). As ω takes values in $\mathbf{H}^0(M/L, \mathbf{V})$ and $\bar{\partial}\omega = 0$, the calculation just before Proposition 3.29 says $\bar{\partial}\varphi = 0$. Proposition 6.18 thus says that φ is harmonic. With a glance back to Theorem 3.34, we now have

6.24. THEOREM. *Suppose that $\pi: G/H \to K/L$ is holomorphic. Then every K-finite class in $H^s(G/H, \mathbf{V})$ has a unique K-finite harmonic representative $\varphi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^*$ such that $\varphi(KM) \subset V_L \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$.*

7. SQUARE INTEGRABILITY

In addition to its G-invariant indefinite Kähler metric, our complex semisimple symmetric space G/H carries a distinguished K-invariant positive definite Hermitian metric that is not G-invariant. Let B denote the Killing form of $\mathfrak{g}$ as before. The metrics are specified on the complexified tangent space $\mathfrak{q}$ at $1 \cdot H$ by

$$\begin{aligned} &G\text{-invariant indefinite Kähler metric:} && (\xi, \eta) = B(\xi, \bar{\eta}), \\ &\text{positive definite Hermitian metric:} && ((\xi, \eta)) = -(\xi, \theta\bar{\eta}). \end{aligned} \tag{7.1}$$

They are given at an arbitrary point of G/H, say kmH, with $k \in K$ and $m \in M$, through translation by km of the metric at $1 \cdot H$, as follows: If $k \in K$ and $m \in M$, then every $\xi \in \mathfrak{q} = T_{1 \cdot H}(G/H)_{\mathbb{C}}$ defines a tangent vector $\xi_{km} = dt_k \cdot dt_m(\xi)$ at kmH, where $t_x: gH \mapsto (xg)H$ is translation by $x \in G$. Since $((\ ,\ ))$ is $\mathrm{Ad}(L)$-invariant, ξ_{km} has well-defined positive definite square norm given by $\|\xi_{km}\|^2 = -B(\xi, \theta\bar{\xi})$. The action of G on G/H distorts the norm by

$$\|dt_x(\xi_{km})\|^2 = -B(\mathrm{Ad}(h)\xi, \theta \cdot \mathrm{Ad}(h)\bar{\xi}) \quad \text{for } h \in H \text{ with } xkm \in KMh. \tag{7.2}$$

Our first task is to bound this distortion.

7.3. LEMMA. *If $x \in G$, then dt_x is uniformly bounded on the tangent spaces of G/H, relative to the positive definite metric $((\ ,\ ))$, with bound continuous in x.*

Proof. Let $k \in K$ and $m \in M$, and factor $xkm = k'm'h$, where $k' \in K$, $m' \in M$ and $h = h(x, k, m) \in H$ is well defined modulo L. In view of (7.2), we must show that $\mathrm{Ad}(h)$: $\mathfrak{q} \mapsto \mathfrak{q}$ has norm bounded by some continuous function of x. For that, we may absorb k, k' into x and assume $h \in \exp(\mathfrak{p}_0 \cap \mathfrak{h}_0)$. Thus, if the Lemma fails then we have

(i) a bounded sequence $\{x_n\} \subset G$,

(ii) sequences $\{m_n\}$, $\{m'_n\} \subset M$,

(iii) a sequence $\{\eta_n\} \subset \mathfrak{p}_0 \cap \mathfrak{h}_0$ with $\|\eta_n\| = 1$, and

(iv) a sequence $\{t_n\} \to \infty$ of real numbers,

such that $x_n m_n = m'_n \cdot \exp(t_n \eta_n)$. Passing to a subsequence we may further suppose $\{\eta_n\} \to \eta \in \mathfrak{p}_0 \cap \mathfrak{h}_0$.

If $\theta\tau$ is an inner automorphism of G, then denote $G' = G$. Otherwise, let G' be the semidirect product $G \cdot \mathbf{Z}_2$ given by $\theta\tau$. In either case, let M' denote the G'-normalizer of M, so $\theta\tau \in M'$ and G' is generated by M' and $\exp(\mathfrak{p}_0 \cap \mathfrak{h}_0)$.

We need a nontrivial finite-dimensional G'-module U whose space of M'-fixed vectors has $\dim U^{M'} = 1$. Let $\{\xi_1, \ldots, \xi_r\}$ be a basis of $\mathfrak{m}_0$ and $u = (\xi_1 \wedge \cdots \wedge \xi_r)^2 \in S^2 \Lambda^r(\mathfrak{g})$. Every $m' \in M'$ acts on $\mathfrak{m}_0$ with determinant ± 1, so $m'(u) = u$. Let U be the G'-subspace of $S^2 \Lambda^r(\mathfrak{g})$ generated by u. Then u spans $U^{M'}$, and we have a unique M'-invariant splitting $U = U' \oplus U''$ with $U' = U^{M'}$.

Recall $\eta = \lim(\eta_n)$ and decompose $u = \sum_{i=0}^k u_i$, where $\eta u_i = \lambda_i u_i$, λ_i distinct and real, and $\lambda_0 = 0$. Asymptotically as $n \to \infty$,

$$(m'_n)^{-1} x_n u = (m'_n)^{-1} x_n m_n u = \exp(t_n \eta_n) u$$
$$\sim \exp(t_n \eta) \cdot u = \sum_{i=0}^{k} e^{t_n \lambda_i} u_i.$$

As $\{x_n u\}$ is bounded and the $(m'_n)^{-1}$ fix U'-components, now $\sum_{i=0}^k e^{t_n \lambda_i} u_i$ has bounded U'-component. Thus $u_i \in U''$ whenever $\lambda_i > 0$. As $(\theta\tau)\eta = -\eta$, we also have $u_i \in U''$ whenever $\lambda_i < 0$. That leaves us with

$$u = u'_0 + \sum_{i=0}^{k} u''_i \qquad \text{where} \quad u_0 = u'_0 + u''_0, \quad u_i = u''_i \quad \text{for} \quad i > 0,$$

$u'_0 \in U'$ and $u''_i \in U''$. It follows that $u = u_0 \in U'$. Now u is fixed under the subgroup of G' generated by M' and $\{\exp(t\eta)\}$. We may in fact assume G simple for the proof of Lemma 7.3. Then M' and $\{\exp(t\eta)\}$ generate G', so u is G'-fixed, and U as constructed is just the span of u. In the construction of U that forces $\mathfrak{m}$ to be an ideal in $\mathfrak{g}$, which is a contradiction. Q.E.D.

As in (6.1), we have Kodaira–Hodge orthocomplementation operators

$$\#: E^{p,q}(G/H, \mathbf{V}) \to E^{n-p,n-q}(G/H, \mathbf{V}^*) \tag{7.4}$$

relative to the positive definite metric. That gives a global K-invariant positive definite inner product on $\mathbf{V}$-valued (p, q)-forms,

$$\langle\langle \varphi, \varphi' \rangle\rangle = \int_{G/H} \varphi(x) \bar{\wedge} \# \varphi'(x) \, d(xH). \tag{7.5}$$

That in turn gives rise to Hilbert spaces

$$L_2^{p,q}(G/H, \mathbf{V}): \quad \text{completion of the space of } C_0^\infty \text{ forms in } E^{p,q}(G/H, \mathbf{V}) \text{ relative to inner product } \langle\!\langle\, , \rangle\!\rangle. \tag{7.6}$$

Of course the inner product on $L_2^{p,q}(G/H, \mathbf{V})$ depends on the choice of K, i.e., on the choice of Cartan involution θ of G that stabilizes H. But in fact $L_2^{p,q}(G/H, \mathbf{V})$ itself is independent of this choice because, from Lemma 7.3,

$$\text{if } g \in G, \text{ then the natural action of } g \text{ on } E^{p,q}(G/H, \mathbf{V}) \text{ defines a bounded operator on } L_2^{p,q}(G/H, \mathbf{V}). \tag{7.7}$$

Since the definite noninvariant metric was obtained from the indefinite invariant metric by "reversing signs," every form $\varphi \in L_2^{p,q}(G/H, \mathbf{V})$ is square integrable with respect to the indefinite Hermitian inner product defined in (6.3). Thus, the Hilbert space $L_2^{p,q}(G/H, \mathbf{V})$ carries the continuous G-invariant indefinite Hermitian form $\langle\, , \rangle$.

In this section we consider the harmonic forms φ of Theorem 6.4 and study the conditions under which $\varphi \in L_2^{0,s}(G/H, \mathbf{V})$. Later, in Section 8, we will analyse $\langle\, , \rangle$ on the space of all harmonic (relative to the invariant metric) forms on $L_2^{0,s}(G/H, \mathbf{V})$.

7.8. LEMMA. *If Haar measures are properly normalized, then*

$$\int_G f(x)\, dx = \int_{K/L} \int_{M/L} \int_H f(kmh) \det(A_m)^{-1} dh\, dm\, dk \qquad \textit{for} \quad f \in C_c(G)$$

and

$$\int_{G/H} f(xH)\, d(xH) = \int_{K/L} \int_{M/L} f(kmH) \det(A_m)^{-1}\, dm\, dk \qquad \textit{for} \quad f \in C_c(G/H).$$

Proof. The differential of $\pi: G/H \to K/L$ on the horizontal space at kmH is given by $A_m: \mathfrak{k} \cap \mathfrak{q} \to \mathfrak{k} \cap \mathfrak{q}$. Q.E.D.

Let $\omega \in A^p(K/L, A^q(M/L, \mathbf{V}))$, and let $\varphi \in A^{p+q}(G/H, \mathbf{V})$ correspond to ω by (3.12). Thus $\varphi: G \to V \otimes \Lambda^{p+q}(\mathfrak{q}_+)^*$ with $\varphi(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^q(\mathfrak{p} \cap \mathfrak{q}_-)^*$. By Lemma 7.8, φ has global square norm

$$\|\varphi\|_{G/H}^2 = \int_{K/L} \int_{M/L} \|\varphi(km)\|^2 \det(A_m)^{-1}\, dm\, dk \tag{7.9}$$

with respect to the positive definite metric. Formula (7.9) also specifies the invariant Hermitian form, as it gives $(-1)^p \langle \varphi, \varphi \rangle$.

We recast (7.9) in terms of the form $\omega \in A^p(K/L, \mathbf{A}^q(M/L, \mathbf{V}))$, related by

$$\varphi(km)(\xi_1,...,)\xi_p, \eta_1,..., \eta_q) = \omega(k)(A_m\xi_1,..., A_m\xi_p)(m)(\eta_1,..., \eta_q). \quad (7.10)$$

For this, denote

$$\Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*: \quad \text{trivial (product) bundle } (M/L) \times \Lambda^q(\mathfrak{k} \cap \mathfrak{q}_-)^* \to M/L \quad (7.11)$$

with the nontrivial Hermitian metric

$(w_\lambda(m), w_\mu(m)) = \sum_I \lambda(\Lambda^p(A_m)\,\xi_I)\,\overline{\mu(\Lambda^p(A_m)\,\xi_I)}\det(A_m)^{-1}$ where $\{\xi_I\}$ is an orthonormal basis of $\Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)$ and w_λ, w_μ are the "constant" sections of $\Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$ for $\lambda, \mu \in \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$. (7.12)

We have "fibre restrictions" of φ given by

$$\varphi_k: \quad \omega(k) \text{ viewed as an element of } A^q(M/L, \mathbf{V} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*). \quad (7.13)$$

Compute

$$\begin{aligned}\|\varphi(km)\|^2 \det(A_m)^{-1} &= \sum_{I,J} \|\varphi(km)(\xi_I, \eta_J)\|^2 \det(A_m)^{-1} \\ &= \sum_{I,J} \|\omega(k)(\Lambda^p(A_m)\,\xi_I)(m)(\eta_J)\|^2 \det(A_m)^{-1} \\ &= \sum_J \|\varphi_k(m)(\eta_J)\|^2 = \|\varphi_k(m)\|^2.\end{aligned}$$

Now (7.9) becomes

$$\|\varphi\|^2_{G/H} = \int_{K/L} \|\varphi_k\|^2_{M/L}\, dk. \quad (7.14)$$

Suppose that $\pi: G/H \to K/L$ is holomorphic. Then A_m preserves each $\mathfrak{k} \cap \mathfrak{q}_\pm$, and $\det(A_m) = |\det(A_m|_{\mathfrak{k} \cap \mathfrak{q}_-})|^2$. Since $A_m \to 0$ as $m \to \infty$ inside $\exp(\mathfrak{p}_0 \cap \mathfrak{q}_0)$

$$\|\omega(k)(\Lambda^p(A_m)\,\xi_I)(m)\|^2 \det(A_m)^{-1} \leqslant \|\omega(k)(\xi_I)(m)\|^2$$

with equality in case $p = s$. In other words,

7.15. Lemma. *If $\pi: G/H \to K/L$ is holomorphic, then $\|\varphi\|^2_{G/H} \leqslant \|\omega\|^2_{K/L}$ with equality in case $p = s$.*

In $\mathbf{V} \to G/H$, $\mathfrak{q}_-$ acts trivially on the typical fibre V, so $\mathfrak{p} \cap \mathfrak{q}_-$ acts trivially on the typical fibre of $\mathbf{V}|_{M/L} \to M/L$. Thus the decomposition of V

into L-isotypic components V_ν leads to orthogonal Hermitian holomorphic bundle decompositions

$$\mathbf{V}|_{M/L} = \sum_{\nu \in \hat{L}} \mathbf{V}_\nu \qquad \text{and} \qquad \mathbf{V}_\nu = \mathbf{B}_\nu \otimes \mathbf{C}^{n(\nu)}, \tag{7.16}$$

where $\mathbf{V}_\nu \to M/L$ has typical fibre V_ν, $\mathbf{B}_\nu \to M/L$ is the Hermitian homogeneous holomorphic bundle associated to the irreducible representation ν of L, and $n(\nu)$ is the multiplicity of ν in $\psi|_L$. Denote

$$\mathscr{H}_2(M/L, \cdot): \quad \text{Hilbert space of } L_2 \text{ holomorphic sections.} \tag{7.17}$$

Then (7.16) gives us

$$\mathscr{H}_2(M/L, \mathbf{V}) = \sum_{\nu \in \hat{L}} \mathscr{H}_2(M/L, \mathbf{B}_\nu) \otimes \mathbf{C}^{n(\nu)}. \tag{7.18}$$

Recall (4.22) the set $\{\gamma_1, ..., \gamma_c\}$ of maximal roots of the noncompact simple factors of M. The Harish-Chandra existence criterion for the holomorphic discrete series says

$\mathscr{H}_2(M/L, \mathbf{B}_\nu) \neq 0$ if, and only if, $(\nu + \rho_M, \gamma_i) < 0$ for $1 \leqslant i \leqslant c$; and in that case $\mathscr{H}_2(M/L, \mathbf{B}_\nu)$ contains every L-finite holomorphic section of $\mathbf{B}_\nu \to M/L$. (7.19)

We are going to combine this with Lemma 7.15 and prove

7.20. Proposition. *Let* $\pi: G/H \to K/L$ *be holomorphic. Let* $\varphi \in A^s(G/H, \mathbf{V})$ *correspond to a form* $\omega \in A^s(K/L, \mathbf{A}^0(M/L, \mathbf{V}))$, *i.e.*, $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. *Define*

$$S(V) = \{\nu \in \hat{L} : V_\nu \neq 0 \text{ and } (\nu + \rho_M, \gamma_i) < 0 \text{ for } 1 \leqslant i \leqslant c\}. \tag{7.21}$$

If φ *is* $\bar{\partial}$*-closed and K-finite, then*

$$\|\varphi\|^2_{G/H} < \infty \text{ if, and only if, } \varphi(KM) \subset \Big(\sum_{\nu \in S(V)} V_\nu\Big) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*. \tag{7.22}$$

Proof. Since $\bar{\partial}\varphi = 0$, $\omega \in A^s(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$, so $\varphi_k \in H^0(M/L, \mathbf{V} \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*) = H^0(M/L, \mathbf{V}) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Here, with π holomorphic and $p = s$, (7.12) reduces to the product metric, $(w_\lambda(m), w_\mu(m)) = (\lambda, \mu)$, so $\|\varphi_k\|^2_{M/L} < \infty$ if and only if $\varphi_k \in \mathscr{H}_2(M/L, \mathbf{V}) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$.

If φ is K-finite and L_2, then every $\|\varphi_k\|^2_{M/L} < \infty$ with $\|\varphi_k\|_{M/L}$ continuous in k. To see this, choose a basis $\{\varphi_1, ..., \varphi_d\} \subset L_2^{0,s}(G/H, \mathbf{V})$ of the span of the K-translates of φ. We have sets $S_l \subset K$ of measure zero such that $\|(\varphi_l)_k\|^2_{M/L} < \infty$ for $k \notin S_l$. Set $S = \bigcup S_l$. If $\varphi' \in \text{span}\{\varphi_l\}$, then $\|\varphi'_k\|^2_{M/L} < \infty$

for $k \notin S$. As span$\{\varphi_i\}$ is K-invariant, this remains true with S replaced by any kS, and thus also with S replaced by $\bigcap_{k \in K} kS$, which is empty. So $\|\varphi_k\|^2_{M/L} < \infty$ for all k, and $\varphi_k = ((k^{-1})^* \varphi)_1$ gives continuity of the norm.

Now suppose φ to be K-finite and L_2. Then every $\varphi_k \in \mathscr{H}_2(M/L, \mathbf{V}) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Further, if $\Xi \in \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)$, then $\varphi_k(\Xi)$ is L-finite, so $\varphi_k \in \mathscr{H}_2(M/L, \sum_F \mathbf{V}_\nu) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$ for some finite set $F = F_k \subset \hat{L}$. Using (7.19), $F \subset S(V)$. This proves half of (7.22).

Conversely, suppose that φ is K-finite with $\varphi(KM) \subset (\sum_{\nu \in S(V)} V_\nu) \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Then every $\varphi_k(\Xi)$ is an L-finite holomorphic section, hence square integrable by (7.19), with L_2-norm continuous in k as above. Now (7.14) ensures $\|\varphi\|^2_{G/H} < \infty$. That is the other half of (7.22). Q.E.D.

We now combine Proposition 7.20 with Theorem 6.24 to obtain the main result of Sections 6 and 7.

7.23. THEOREM. *Let $\pi: G/H \to K/L$ be holomorphic and, as before, let $\{\gamma_i\}$ be the maximal roots of the noncompact simple factors of M. Suppose that, for every L-type ν with $V_\nu \neq 0$,*

$$(\nu + \rho_M, \gamma_i) < 0 \qquad \textit{for each } i. \tag{7.24}$$

Then every K-finite class in $H^s(G/H, \mathbf{V})$ is represented uniquely by a form

$$\varphi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^* \qquad \textit{such that} \quad \varphi(KM) \subset V_L \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$$

which is harmonic relative to the invariant metric and L_2 relative to the positive definite metric.

If $\nu = \chi - \sum b_j \beta_j$ with $b_j \geqslant 0$ and $\beta_j \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$, then $(\beta_j, \gamma_i) \geqslant 0$ forces $(\nu + \rho_M, \gamma_i) \leqslant (\chi + \rho_M, \gamma_i)$. Thus we have

7.25. COROLLARY. *Let $\pi: G/H \to K/L$ be holomorphic. Suppose that ψ has a highest L-type χ in the sense of* (4.30), *as is automatic, for example, in case ψ is finite dimensional. If*

$$(\chi + \rho_M, \gamma_i) < 0 \qquad \textit{for} \quad i = 1, ..., c, \tag{7.26}$$

then every K-finite class in $H^s(G/H, \mathbf{V})$ has a unique L_2 harmonic representative φ with $\varphi(KM) \subset V_L \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$.

In fact, Corollary 7.25 is very close to the most general case to which Theorem 7.23 applies, because of

7.27. LEMMA. *Let H' be the identity component of the kernel of the action of G on G/H. Then there are θ-stable local direct product decompositions*

$$H = H' \cdot H'' \quad \textit{and} \quad G = H' \cdot G'' \qquad \textit{with} \quad G/H = G''/H''$$

and corresponding tensor product decompositions

$$V = V' \otimes V'' \quad \text{and} \quad \psi = \psi' \otimes \psi'' \qquad \text{with} \quad \psi' \in \hat{H}' \quad \text{and} \quad \psi'' \in \hat{H}''.$$

Set $L'' = L \cap H''$. Then ψ satisfies L_2 condition (7.24) *if and only if ψ'' satisfies it, and in that case ψ'' has a highest L''-type in the sense of* (4.30).

Proof. All assertions are clear except the last one, that (7.24) implies (4.30) for ψ''. For that, we may suppose $G = G''$ and $\psi = \psi''$. In other words, we may suppose that $\mathfrak{h}$ contains no nonzero ideal of G.

Fix an L-type $V_\nu \neq 0$. Then $V_L = \sum_{n=0}^{\infty} d\psi(\mathfrak{p} \cap \mathfrak{h})^n \cdot V_\nu$ by the irreducibility of ψ. Suppose that (4.30) fails for ψ. Then $d\psi(\mathfrak{p}_+ \cap \mathfrak{h})^n \cdot V_\nu \neq 0$ for every integer $n \geqslant 0$, for otherwise we choose $V_\chi \neq 0$ in the last nonzero $d\psi(\mathfrak{p}_+ \cap \mathfrak{h})^n \cdot V_\nu$, and $d\psi(\mathfrak{p}_+ \cap \mathfrak{h}) \cdot V_\chi = 0$ forcing (4.30). Now we have a sequence $\{\nu_n\} \subset \hat{L}$ with $0 \neq U_{\nu_n} \subset d\psi(\mathfrak{p}_+ \cap \mathfrak{h})^n \cdot V_\nu$. Given (7.24), at least one $\delta \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$ has $(\delta, \gamma) \leqslant 0$ for all $\gamma \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+) = \Phi(\mathfrak{p}_+ \cap \mathfrak{q}_+)$ (see (2.21)). Since $\delta + \gamma$ is not a root, now

$$\Sigma = \{\delta \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h}) \colon \delta \text{ strongly orthogonal to } \Phi(\mathfrak{p} \cap \mathfrak{q}_+)\}$$

is nonempty. Thus

$$\mathfrak{r}\colon \quad \text{subalgebra of } \mathfrak{h} \text{ generated by all } \mathfrak{g}_\delta + \mathfrak{g}_{-\delta}, \delta \in \Sigma,$$

is nonzero. As $\mathfrak{r}$ is semisimple, ad($\mathfrak{l}$)-stable and θ-invariant, it is an ideal in $\mathfrak{h}$ by the theory of orthogonal involutive Lie algebras. But $\mathfrak{r}$ centralizes $\mathfrak{p} \cap \mathfrak{q}$, which together with $\mathfrak{h}$ generates $\mathfrak{g}$, so $\mathfrak{r}$ is an ideal in $\mathfrak{g}$. This contradiction completes the proof of the Lemma. Q.E.D.

In view of Theorem 7.23 we define

$$\mathscr{H}_2^s(G/H, \mathbf{V}) = \{\varphi \in L_2^{0,s}(G/H, \mathbf{V}) \colon \bar{\partial}\varphi = 0 \text{ and } \bar{\partial}^*\varphi = 0 \text{ as distributions}\}. \tag{7.27}$$

This is our L_2 harmonic space. It is a closed subspace of $L_2^{0,s}$ $(G/H, \mathbf{V})$, its C^∞ elements are harmonic in the usual sense (6.5), and its K-finite elements are C^∞. Theorem 7.23 says that, assuming (7.24), every K-finite class in $H^s(G/H, \mathbf{V})$ has a unique representative $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})_{\bar{K}}$ such that $\varphi(KM) \subset V_L \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. In particular, the natural map of $\bar{\partial}$-closed C^∞ forms to their Dolbeault classes maps $\mathscr{H}_2^s(G/H, \mathbf{V})_K$ onto $H^s(G/H, \mathbf{V})_K$. One can compute Dolbeault cohomology from distribution forms as well as from C_∞ forms; see [34]. Thus every $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$ has a well-defined Dolbeault class $[\varphi] \in H^s(G/H, \mathbf{V})$, and the natural map

$$\mathscr{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V}) \qquad \text{by} \quad \varphi \mapsto [\varphi] \tag{7.28}$$

has image that contains $H^s(G/H, \mathbf{V})_K$ if (7.24) holds.

8. Construction of the Unitary Representation: Nonsingular Case

In this section we prove that, under certain conditions, the global G-invariant indefinite Hermitian form $\langle\, ,\, \rangle$ is semidefinite on $\mathscr{H}_2^s(G/H, \mathbf{V})$ and that its kernel there coincides with the kernel of the nature map $\varphi \mapsto [\varphi] \in H^s(G/H, \mathbf{V})$ to Dolbeault cohomology. The technical device amounts to tracing square integrability through the spectral sequence of Section 3 to show that $[\varphi] = 0$ implies $\varphi = \lim\{\bar{\partial}\eta_i\}$ for some square-integrable η_i. The conditions amount to the requirement that the resulting unitary representation of G have nonsingular infinitesimal character. In Section 9 we will relax these conditions, obtaining slightly less elegant unitary representations with singular infinitesimal character.

We start with some preliminiaries on bounded domains, which later will be the fibres of $\pi: G/H \to K/L$.

Let $D \subset \mathbf{C}^n$ be a bounded domain, holomorphically convex, such that $tD \subset D$ for $|t| \leqslant 1$ with tD relatively compact in D for $|t| < 1$. Let $\mathbf{W} \to D$ be a Hermitian vector bundle with a particular holomorphic trivialization. We use the natural (from $D \subset \mathbf{C}^n$) holomorphic trivialization on the holomorphic tangent bundle $\mathbf{T}^{1,0}(D) \to D$ and associated bundles. Now suppose that D carries a Hermitian metric with the following property:

> Compare the metric on $\mathbf{V} \otimes \Lambda^q \mathbf{T}^{0,1}(D)^* \otimes \Lambda^n \mathbf{T}^{1,0}(D)$ at $z \in D$ and tz using the isomorphism of fibres given by the specified trivializations. Then the metric at tz bounds the metric at z, uniformly for $z \in D$ and $|t| \leqslant 1$. (8.1)

In particular, for $t = 0$, this implies that

> the metric on $\mathbf{V} \otimes \Lambda^q \mathbf{T}^{0,1}(D)^* \otimes \Lambda^n \mathbf{T}^{1,0}(D)$ is uniformly bounded by any metric that is constant with respect to the specified trivializations. (8.2)

If $\varphi \in A^q(D, \mathbf{V})$, say $\varphi = \sum f_{i,J} v_i \otimes d\bar{z}^J$, where $\{v_i\}$ is a complete orthonormal set in the typical fibre V of $\mathbf{V} \to D$ and the multi-index J has size $|J| = q$, then denote

$$\varphi^t(z) = \sum \bar{t}^q f_{i,J}(tz)\, v_i \otimes d\bar{z}^J \qquad \text{for} \quad |t| \leqslant 1. \tag{8.3}$$

8.4. Lemma. *If φ is square integrable, then so is φ^t. Moreover, the map $\varphi \mapsto \varphi^t$ is bounded in L_2-norm, uniformly in t for $\varepsilon \leqslant t \leqslant 1$ whenever $0 < \varepsilon \leqslant 1$.*

Proof. Let $a_{iI,jJ}(z) = r(z)\langle v_i \otimes d\bar{z}^I, v_j \otimes d\bar{z}^J\rangle_z$, where $r(z)$ is the ratio of the Hermitian volume element on D to the Euclidean volume element $dx \wedge dy$ on the ambient $\mathbf{C}^n$. By assumption (8.1),

$$(a_{iJ,jJ}(z)) \leqslant C(a_{iI,jJ}(tz)), \qquad \text{some} \quad C > 0,$$

as positive definite Hermitian matrices, for all $z \in D$ and $|t| \leqslant 1$. Here $r(z)$ accounts for the $\Lambda^n \mathbf{T}^{1,0}(D)$ in (8.1). Thus,

$$\begin{aligned}
\|\varphi^t\|_D^2 &= \int_D \sum \bar{t}^q f_{iI}(tz)\, t^q \overline{f_{jJ}(tz)}\, a_{iJ,jJ}(z)\, dx \wedge dy \\
&\leqslant C|t|^{2q} \int_D \sum f_{iI}(tz) \overline{f_{jJ}(tz)}\, a_{iI,jJ}(tz)\, dx \wedge dy \\
&= C\,|t|^{2q-2n} \int_{tD} \sum f_{iI}(z) \overline{f_{jJ}(z)}\, a_{iI,jJ}(z)\, dx \wedge dy \\
&\leqslant C\,|t|^{2q-2n} \int_D \sum f_{iI}(z) \overline{f_{jJ}(z)}\, a_{iI,jJ}(z)\, dx \wedge dy \\
&= C\,|t|^{2q-2n}\, \|\varphi\|_b^2.
\end{aligned}$$

The lemma is immediate from this. Q.E.D.

8.5. LEMMA. *If φ is square integrable, then φ^t tends to φ in L_2 norm as $t \to 1$ with $|t| < 1$.*

Proof. Let $\varepsilon > 0$ and choose a C_c^∞ form φ' with $\|\varphi - \varphi'\| < \varepsilon$. By uniform continuity, $\varphi'^t \to \varphi'$ uniformly and thus in L_2 norm. Lemma 8.4 ensures $\|\varphi^t - \varphi'^t\| \leqslant (\text{const})\, \varepsilon$. The assertion follows. Q.E.D.

8.6. *Remark.* Let L be a group of automorphisms of $\mathbf{V} \to D$ that acts linearly on D and preserves all the bundle trivializations. Then $\varphi \mapsto \varphi^t$ commutes with the action of L on $\mathbf{V}$-valued forms.

This ends the preliminaries. We now return to the setting of Sections 2–7 and assume now that $\pi\colon G/H \to K/L$ is holomorphic. The fibre M/L is identified, through the Harish-Chandra embedding, as a bounded symmetric domain $D \subset \mathfrak{p} \cap \mathfrak{q}_+$, and there the universal factor of automorphy construction gives a distinguished L-invariant holomorphic trivialization of $\mathbf{V}|_{M/L} \to M/L$.

We use the fibre restrictions φ_k defined in (7.13). Since $\mathbf{V}|_{M/L}$ is trivialized and $\mathbf{A}^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \to M/L$ is defined (7.11) as trivial, we write $\varphi_{kM/L}$ for φ_k viewed as a section $kM/L \to V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$ of the trivialized bundle $\mathbf{V} \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^* \to kM/L$ over the fibre $kM/L = \pi^{-1}(kL)$.

The isotropy subgroup L acts linearly on $M/L = D \subset \mathfrak{p} \cap \mathfrak{q}_+$ and preserves the trivialization of the tangent bundle. Thus the operations

$\varphi_{kM/L} \mapsto \varphi^t_{kM/L}$ on the fibres of $\pi: G/H \to K/L$ combine to give a linear map $\varphi \mapsto \varphi^t$, which we may interpret as a map of $E_0^{p,q} = \mathscr{C}^{p,q}/\mathscr{C}^{p+1,q-1}$ to itself. Split $\bar\partial = \bar\partial_{0,1} + \bar\partial_{1,0}$ as described in Remark 3.35. Then $\bar\partial_{0,1}(\varphi^t) = (\bar\partial_{0,1}\varphi)^t$ because any coefficient f of $\varphi_{kM/L}$ (in the sense of the $f_{I,J}$ of (8.3)) satisfies $(\partial/\partial\bar z^i)(f(tz)) = \bar t(\partial f/\partial \bar z^i)(tz)$. Then $\bar\partial_{1,0}(\varphi^t) = (\bar\partial_{1,0}\varphi)^t$ because $\bar\partial_{1,0}$ is the $\bar\partial$ operator on the base and does not act on the coefficients of φ on the fibres. Thus

$$\bar\partial(\varphi^t) = (\bar\partial\varphi)^t \qquad \text{for} \quad \varphi \in E_0^{p,q}. \tag{8.7}$$

In order to globalize Lemmas 8.4 and 8.5, we must know when the condition (8.1) holds on the fibres. Denote $\Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^* = \sum_{p=0}^s \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$, product bundle with metric (7.12) and typical fibre $\Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^* = \sum_{p=0}^s \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$.

8.8. LEMMA. *For every weight ν of L on V, every weight μ of L on $\Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^*$, and every $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$, suppose*

$$(\nu + \mu + 2\rho_{M/L}, \beta) \leqslant 0. \tag{8.9}$$

Then (8.1) *holds uniformly on the fibres of $\pi: G/H \to K/L$, for the bundle $\mathbf{V} \otimes \Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^*$ in place of* $\mathbf{V}$.

Proof. By L-invariance of the metrics it suffices to check (8.1) when $z = mL$ lies in the maximal totally geodesic polydisc $D' = (\mathfrak{a}_+ \cap D) \subset \mathfrak{a}_+$ of $D \subset \mathfrak{p} \cap \mathfrak{q}_+$. Here $\mathfrak{a}_+$ is the sum of the root spaces for, say, the maximal strongly orthogonal set $\{\gamma_{i,j}\}$ of (4.22).

Now $\mathbf{V}|_{D'}$ is the orthogonal sum of homogeneous holomorphic line sub-bundles $\mathbf{L}_\nu \to D'$ as ν ranges over the L-weights of V.

We are going to show that $\Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^*|_{D'}$ is the orthogonal direct sum of homogeneous holomorphic line sub-bundles $\mathbf{L}_\mu$ as μ ranges over the weights on $\Lambda(\mathfrak{k} \cap \mathfrak{q}_-)^*$, i.e., the weights on $\Lambda(\mathfrak{k} \cap \mathfrak{q}_+)$.

Choose root vectors $E_{ij} \in \mathfrak{m}_{\gamma_{i,j}}$ with $B(E_{ij}, \overline{E_{ij}}) = 1$. The $E_{ij} \in \mathfrak{p} \cap \mathfrak{q}_+$ and the $Y_{ij} = E_{ij} + \overline{E_{ij}}$ span $\mathfrak{a}_0$. Set $H_{ij} = [E_{ij}, \overline{E_{ij}}] \in \mathfrak{t}$. Let $m = \exp \sum_{i,j} t_{ij} Y_{ij}$. The standard identity

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & \tanh(t) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\cosh(t) & 0 \\ 0 & \cosh(t) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tanh(t) & 1 \end{pmatrix}$$

gives $m \in \exp(\mathfrak{p} \cap \mathfrak{q}_+) \cdot \tilde m \cdot \exp(\mathfrak{p} \cap \mathfrak{q}_-)$, where

$$\tilde m^{-1} = \exp\left(\sum_{i,j} \log\cosh(t_{ij})\, H_{ij}\right)$$

$$= \prod_{i,j} \exp(\log\cosh(t_{ij})\, H_{ij}).$$

If $\alpha_1,\ldots,\alpha_r \in \Phi(\mathfrak{k}\cap\mathfrak{q}_+)$ are distinct and $0\neq\xi_l\in\mathfrak{k}_{\alpha_l}$, then the Hermitian homogeneous holomorphic line bundle over D' with fibre $(\xi_1\wedge\cdots\wedge\xi_r)\mathbf{C}\subset\Lambda^r(\mathfrak{k}\cap\mathfrak{q}_+)$ has "constant section" given at $m\in\exp(\mathfrak{a}_0)$ by

$$\begin{aligned}\mathrm{Ad}(\tilde{m}^{-1})&(\xi_1\wedge\cdots\wedge\xi_r)\\ &=\left\{\prod_{i,j,k}\cosh(t_{ij})^{2\langle\alpha_k,\gamma_{i,j}\rangle/\langle\gamma_{i,j},\gamma_{i,j}\rangle}\right\}\xi_1\wedge\cdots\wedge\xi_r\\ &=\left\{\prod_k\cosh(t_k)\right\}\xi_1\wedge\cdots\wedge\xi_r,\end{aligned}$$

where

$$\begin{aligned}&\text{if } \alpha_k\perp \text{every } \gamma_{i,j}, && \text{then } t_k=0;\\ &\text{if } \alpha_k\not\perp\gamma_{i,j}, \text{ so } \alpha_k-\tfrac{1}{2}\gamma_{i,j}\perp \text{every } \gamma_{u,v}, && \text{then } t_k=t_{i,j}.\end{aligned}$$

In particular, that constant section has length

$$\|\mathrm{Ad}(\tilde{m}^{-1})\,\xi_1\wedge\cdots\wedge\xi_r\|=\left\{\prod_{k=1}^{r}\cosh(t_k)\right\}\|\xi_1\wedge\cdots\wedge\xi_r\|. \tag{8.10}$$

On the other hand, $A_m^{-1}=\cosh(\mathrm{ad}\sum t_{ij}Y_{ij})|_{\mathfrak{k}\cap\mathfrak{q}}$ multiplies ξ_l by $\cosh(t_l)$. If $\Phi(\mathfrak{k}\cap\mathfrak{q}_+)=\{\alpha_1,\ldots,\alpha_s\}$, $\xi_l\in\mathfrak{k}_{\alpha_l}$, and $\bar{\xi}_l^*$ is the element of $(\mathfrak{k}\cap\mathfrak{q}_-)^*$ dual to $\bar{\xi}_l$ by the Killing form, then (7.12) says that the constant section of $\Lambda^{s-r}(\mathfrak{k}\cap\mathfrak{q}_-)^*$ for $\bar{\xi}_{r+1}^*\wedge\cdots\wedge\bar{\xi}_s^*=\lambda$ has square norm

$$\|w_\lambda\|^2=\left\{\prod_{k=1}^{r}\cosh(t_k)\right\}^2\|\xi_1\wedge\cdots\wedge\xi_r\|^2. \tag{8.11}$$

Comparing (8.10) and (8.11) we have the asserted decomposition of $\Lambda(\mathfrak{k}\cap\mathfrak{q}_-)^*|_{D'}$.

The bundle $\Lambda^q\mathbf{T}^{0,1}(D)^*\otimes\Lambda^{\dim D}\mathbf{T}^{1,0}(D)$ restricts on D' to an orthogonal direct sum of line bundles $\mathbf{L}_E\otimes\mathbf{L}_{2\rho_{M/L}}$, where E runs over the sums of q distinct elements of $\Phi(\mathfrak{p}\cap\mathfrak{q}_-)$. The restriction is isometric but *not holomorphic*. Note that $\langle E,\beta\rangle\leqslant 0$ for every $\beta\in\Phi(\mathfrak{p}\cap\mathfrak{q}_+)$. Now our hypothesis (8.9) says that $\mathbf{V}\otimes\Lambda(\mathfrak{k}\cap\mathfrak{q}_-)^*\otimes\Lambda^q\mathbf{T}^{0,1}(D)^*\otimes\Lambda^{\dim D}\mathbf{T}^{1,0}(D)$ restricts on D' to an orthogonal direct sum of line bundles $\mathbf{L}_\delta$ with $\langle\delta,\beta\rangle\leqslant 0$ for all $\beta\in\Phi(\mathfrak{p}\cap\mathfrak{q}_+)$. On each disc factor of D', any such $\mathbf{L}_\delta$ is a nonnegative power of the holomorphic cotangent bundle, whose metric $1-|z|^2\downarrow 0$ as $|z|\to 1$. That gives (8.1) for $\mathbf{V}\otimes\Lambda(\mathfrak{k}\cap\mathfrak{q}_-)^*$. Q.E.D.

If $\varphi\in A^r(G/H,\mathbf{W})$, we decompose $\varphi=\sum_{i=0}^r\varphi_i$, where φ_i corresponds by (3.12) to an element of $A^i(K/L,\mathbf{A}^{r-i}(M/L,\mathbf{V}))$, i.e., where $\varphi_i(KM)\subset V\otimes\Lambda^i(\mathfrak{k}\cap\mathfrak{q}_-)^*\otimes\Lambda^{r-i}(\mathfrak{p}\cap\mathfrak{q}_-)^*$. Then we define $\varphi^t=\sum_{i=0}^r\varphi_i^t$. Applying (8.7) to each φ_i we have $\bar{\partial}(\varphi^t)=(\bar{\partial}\varphi)^t$ for $|t|\leqslant 1$.

8.12. LEMMA. *Assume* (8.9). *If* $\varphi \in A^r(G/H, \mathbf{V})$ *is* L_2, *then so is* φ^t *for* $|t| \leqslant 1$, *and* $\{\varphi^t\} \to \varphi$ *in* L_2*-norm. If* φ *is* L_2 *and* $\bar{\partial}$*-exact, say* $\varphi = \bar{\partial}\eta$, *then* η^t *is* L_2 *for* $|t| < 1$ *and* $\{\bar{\partial}\eta^t\} \to \varphi$ *in* L_2*-norm.*

Proof. If η is any C^∞ $\mathbf{V}$-valued form on G/H, we assert that η^t is L_2 for $|t| < 1$. First, the coefficients of η^t are bounded because the coefficients of η are bounded on $|z| <$ (const) < 1. Second, (8.2) holds uniformly on the fibres of $\pi: G/H \to K/L$ because of Lemma 8.8, so smooth forms with bounded coefficients are L_2. That gives the L_2 assertions for φ^t and η^t in the statement of the lemma. Now $\{\varphi^t\} \to \varphi$ in L_2 as in Lemma 8.5, and $\{\bar{\partial}\eta^t\} = \{(\bar{\partial}\eta)^t\} = \{\varphi^t\} \to \varphi$ in L_2 as asserted. Q.E.D.

To complete the results of this section we will put hypothesis (8.9) into a more familiar form. If $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ with $(\alpha, \beta) \neq 0$, then $-s_\beta(\alpha) \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$ and $(-s_\beta(\alpha), \beta) = (\alpha, \beta)$. Thus we can replace μ, which is a sum of distinct elements of $\Phi(\mathfrak{k} \cap \mathfrak{q}_+)$, by $-s_\beta(\mu)$, in (8.9). Since $(\alpha, \beta) \geqslant 0$ here, we need only consider the case $\mu = 2\rho_{K/L}$, and our argument shows $(\rho_{K/L}, \beta) = (-s_\beta(\rho_{K/L}), \beta) = (\rho_{H/L}, \beta)$. Thus (8.9) is equivalent to the requirement that

$$(\nu + 2\rho_{M/L} + \rho_{K/L} + \rho_{H/L}, \beta) \leqslant 0 \tag{8.13}$$

for every weight ν of L on V and every $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$.

The ρ's in (8.13), and the set of weights ν there, all are W_L-invariant. Since every $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ is a convex linear combination of W_L-translates of $\mathfrak{m}$-simple roots in $\Phi(\mathfrak{p} \cap \mathfrak{q}_+)$, it suffices to check (8.13) when β is $\mathfrak{m}$-simple. In that case (ν, β) is maximized, on the weights ν of an L-isotypic component of V, at the lowest weight of that isotypic component. With β $\mathfrak{m}$-simple, $2(\rho_M, \beta)/(\beta, \beta) = 1$, so $\frac{1}{2}(\beta, \beta) = (\rho_M, \beta) = (\rho_L, \beta) + (\rho_{M/L}, \beta)$. Putting all this into (8.13), we see that (8.9) is equivalent to the inequality

$$(-\nu^* - \rho_L + \rho_{M/L} + \rho_{K/L} + \rho_{H/L}, \beta) \leqslant -\tfrac{1}{2}(\beta, \beta) \tag{8.14}$$

for every highest weight ν of L on V (corresponding to lowest weight $-\nu^*$) and every noncompact simple root β of $\mathfrak{m}$. The opposition (maximal length) element $w_1 \in W_L$ of (4.23) carries these simple roots β to the maximal roots γ_i of (4.22), fixes $\rho_{M/L}$, $\rho_{K/L}$, and $\rho_{H/L}$, sends ρ_L to its negative, and transforms $-\nu^*$ to ν. Note that $\rho_L + \rho_{M/L} + \rho_{K/L} + \rho_{H/L}$ is $\rho = \rho_G$. Thus we have reformulated (8.14) so that Lemmas 8.8 and 8.12 combine to yield

8.15. THEOREM. *Suppose that* $\pi: G/H \to K/L$ *is holomorphic. Let* $\{\gamma_1, ..., \gamma_c\}$ *be the maximal roots of the noncompact simple ideals of* $\mathfrak{m}$. *If* $\nu \in \hat{L}$ *with* $V_\nu \neq 0$, *suppose that*

$$2(\nu + \rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1 \qquad \text{for} \quad 1 \leqslant i \leqslant c. \tag{8.16}$$

If $\varphi \in A^r(G/H, \mathbf{V})$ *is* L_2 *and* $\bar\partial$*-exact, then* $\varphi = \lim\{\bar\partial\eta_i\}$ *in* L_2*-norm, for some* L_2*-forms* $\eta_i \in A^{r-1}(G/H, \mathbf{V})$.

8.17. *Remark.* We shall see later that if G (or even just M) is a linear group, or if V is finite dimensional, then (8.16) can be written: $2(\nu+\rho, \gamma_i)/(\gamma_i, \gamma_i) < 0$ for $1 \leqslant i \leqslant c$. If $\dim V < \infty$, this will turn out to be the condition that the unitary representation we eventually obtain has nonsingular infinitesimal character.

Recall (7.25) the space $\mathscr{H}_2^s(G/H, \mathbf{V})$ of all C^∞ $\mathbf{V}$-valued (O, s)-forms on G/H that are square integrable for the positive definite metric and harmonic for the G-invariant metric.

8.18. THEOREM. *Let* $\pi: G/H \to K/L$ *be holomorphic and suppose that every* $2(\nu+\rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1$, *where* $\{\gamma_i\}$ *are the maximal roots of the noncompact simple factors of* M *and* ν *runs over the classes in* $\hat{L}$ *with* $V_\nu \neq 0$. *Then the natural* G*-invariant Hermitian form* $\langle,\rangle_{G/H}$ *on* $L_2^{0,s}(G/H, \mathbf{V})$ *and the natural map* $\mathscr{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V})$ *to Dolbeault cohomology, are related as follows*:

(1) $(-1)^s \langle,\rangle_{G/H}$ *is positive semidefinite on* $\mathscr{H}_2^s(G/H, \mathbf{V})$.

(2) *The null space of* $\langle,\rangle_{G/H}$ *on* $\mathscr{H}_2^s(G/H, \mathbf{V})$ *is the kernel of the map* $\varphi \mapsto [\varphi] \in H^s(G/H, \mathbf{V})$ *to Dolbeault cohomology.*

(3) *The space* $\mathscr{H}_2^s(G/H, \mathbf{V})_K$ *of* K*-finite elements maps onto* $H^s(G/H, \mathbf{V})_K$.

Proof. Let $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$ be in the kernel of map (7.28) to Dolbeault cohomology. Choose an approximate identity $\{f_n\} \subset C^\infty(G)$. Then $\{f_n * \varphi\} \to \varphi$ in $L_2^{0,s}(G/H, \mathbf{V})$ and each $f_n * \varphi$ is C^∞. As $[\varphi] = 0$, there is a distribution-coefficient $\mathbf{V}$-valued $(0, s-1)$-form η on G/H with $\varphi = \bar\partial\eta$ in the sense of distributions. Now $f_n * \eta$ is C^∞ and $\bar\partial(f_n * \eta) = f_n * (\bar\partial\eta) = f_n * \varphi$. Theorem 8.15 applies to the $f_n * \varphi$, providing sequences $\{\eta_{n,i}\}_{i=1,2,\ldots}$ of L_2 forms in $A^{s-1}(G/H, \mathbf{V})$ such that $\lim_{i\to\infty} \bar\partial\eta_{n,i} = f_n * \varphi$ in $L_2^{0,s}(G/H, \mathbf{V})$. Passing to a suitable diagonal sequence φ_{n,i_n}, we find $\varphi = \lim_{n\to\infty} \bar\partial\eta_{n,i_n}$ in $L_2^{0,s}(G/H, \mathbf{V})$. Let $\varphi' \in \mathscr{H}_2^s(G/H, \mathbf{V})$. Since φ' is L_2 and $\bar\partial^*$-closed in the invariant metric,

$$\langle\varphi, \varphi'\rangle_{G/H} = \lim\langle\bar\partial\eta_{n,i_n}, \varphi'\rangle_{G/H} = \lim\langle\eta_{n,i_n}, \bar\partial^*\varphi'\rangle = 0.$$

Thus φ is in the null space of $\langle,\rangle_{G/H}$ on $\mathscr{H}_2^s(G/H,\mathbf{V})$.

If $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$, viewed as $\varphi: G \to V \otimes \Lambda^s(\mathfrak{q}_-)^*$, has $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$, then $(-1)^s \langle\varphi, \varphi\rangle_{G/H} = \|\varphi\|_{G/H}^2 \geqslant 0$ and vanishes just when $\varphi = 0$. We have (7.24) because each $(\rho_M, \gamma_i) \leqslant (\rho, \gamma_i)$; for if $\beta \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h}) \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$, then $\beta + \gamma_i$ is not a root. Now Theorem 7.23 says that every K-

finite class in $H^s(G/H, \mathbf{V})$ is of the form $[\varphi]$, where $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$ is K-finite with $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. So, on the K-finite level,

$$\mathscr{H}_2^s(G/H, \mathbf{V})_K \to H^s(G/H, \mathbf{V})_K$$

is surjective from a $(-1)^s \langle , \rangle_{G/H}$ positive definite subspace and has kernel in the null space of $(-1)^s \langle , \rangle_{G/H}$. It follows that $(-1)^s \langle , \rangle_{G/H}$ is positive semidefinite on $\mathscr{H}_2^s(G/H, \mathbf{V})_K$ and that its null space there is exactly the kernel of the map to $H^s(G/H, \mathbf{V})_K$. Since $\mathscr{H}_2^s(G/H, \mathbf{V})$ is closed in $L_2^{0,s}(G/H, \mathbf{V})$ and $\langle , \rangle_{G/H}$ is jointly continuous on $L_2^{0,s}(G/H, \mathbf{V})$, now $(-1)^s \langle , \rangle_{G/H}$ is positive semidefinite on $\mathscr{H}_2^s(G/H, \mathbf{V})$, its null space there is an $L_2^{0,s}(G/H, \mathbf{V})$-closed subspace, and that null space is exactly the kernel of map (7.26) from $\mathscr{H}_2^s(G/H, \mathbf{V})$ to $H^s(G/H, \mathbf{V})$. Q.E.D.

In view of Theorem 8.18, we define *reduced* square-integrable harmonic spaces by

$$\bar{\mathscr{H}}_2^s(G/H, \mathbf{V}) = \mathscr{H}_2^s(G/H, \mathbf{V})/(\text{null space of } \langle , \rangle_{G/H}) \tag{8.19}$$

with Hermitian form $\langle , \rangle$ induced by $(-1)^s \langle , \rangle_{G/H}$.

Here $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ carries a Hilbert space structure as quotient of a closed subspace of $L_2^{0,s}(G/H, \mathbf{V})$ by a smaller closed subspace and $\langle , \rangle$ is a jointly continuous positive definite Hermitian form on $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$.

8.20. LEMMA. *Under the hypotheses of Theorem* 8.17, *the pre-Hilbert space structure on $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ defined by $\langle , \rangle$ coincides with the Hilbert space structure obtained by realizing $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ as a quotient of closed subspaces of $L_2^{0,s}(G/H, \mathbf{V})$.*

Proof. Every class in $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ is represented by a unique form $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$ such that $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. This is clear for K-finite classes through comparison with Dolbeault cohomology and follows by continuity for all classes.

Let $\bar{\varphi} \in \bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$; represent it by $\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V})$ such that $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$ and also represent it by $\varphi' \in \mathscr{H}_2^s(G/H, \mathbf{V})$ which is $L_2^{0,s}$-orthogonal to the kernel of $\mathscr{H}_2^s \to \bar{\mathscr{H}}_2^s$. To prove Lemma 8.20, we must prove $\|\varphi'\|_{G/H}^2 = (-1)^s \langle \varphi, \varphi \rangle_{G/H}$. First, since φ' is the $L_2^{0,s}$-shortest element of $\mathscr{H}_2^s$ representing $\bar{\varphi}$,

$$\|\varphi'\|_{G/H}^2 \leqslant \|\varphi\|_{G/H}^2 = (-1)^s \langle \varphi, \varphi \rangle_{G/H}.$$

Second, since the kernel of $\mathscr{H}_2^s \to \bar{\mathscr{H}}_2^s$ is the null space of $\langle , \rangle_{G/H}$ on $\mathscr{H}_2^s$,

$$(-1)^s \langle \varphi, \varphi \rangle_{G/H} = (-1)^s \langle \varphi', \varphi' \rangle_{G/H} \leqslant \|\varphi'\|_{G/H}^2.$$

The lemma follows. Q.E.D.

8.21. THEOREM. *Let $\pi: G/H \to K/L$ be holomorphic and suppose that every $2(\nu+\rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1$, where $\{\gamma_i\}$ are the maximal roots of the noncompact simple factors of M and ν runs over the classes in $\hat{L}$ with $V_\nu \neq 0$. Then the natural action of G on $\mathbf{V}$-valued differential forms on G/H preserves $\mathscr{H}_2^s(G/H, \mathbf{V})$ and passes to the quotient $\mathscr{H}_2^s(G/H, \mathbf{V})$/(null space of $\langle , \rangle_{G/H}$) as*

$$\pi_V: \quad \text{unitary representation of } G \text{ on } \bar{\mathscr{H}}_2^s(G/H, \mathbf{V}). \tag{8.22}$$

The natural map from $\mathscr{H}_2^s(G/H, \mathbf{V})$ to Dolbeault cohomology induces a $(\mathfrak{g}_0, K)$-module isomorphism of $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})_K$ onto $H^s(G/H, \mathbf{V})_K$, and thus π_V unitarizes the action of G on Dolbeault cohomology $H^s(G/H, \mathbf{V})$.

8.23. *Remark.* If the highest weights ν of L on V all are integral in $\mathfrak{m}$, then the hypothesis $2(\nu+\rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1$ of Theorems 8.18 and 8.21 coincides with the nonsingularity condition $(\nu+\rho, \gamma_i) < 0$. Of course the ν are integral if M is linear, in particular if G is linear. They also are integral if $\dim V < \infty$, for then the highest weight χ of L on V is orthogonal to $\Phi(\mathfrak{p} \cap \mathfrak{h})$, hence $\mathfrak{h}$-integral, and χ is $\mathfrak{k}$-integral because it lifts to a well-defined character e^χ of the torus $T \subset K$. Thus χ is $\mathfrak{g}$-integral (every simple $\mathfrak{g}$-root is a root of either $\mathfrak{h}$ or $\mathfrak{k}$), and in particular $\mathfrak{m}$-integral.

8.24. *Remark.* If V has a highest L-type χ in the sense of (4.30), then the hypothesis of Theorems 8.18 and 8.21 that

$$2(\nu+\rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1 \qquad \text{all } i, \quad \text{all } \nu \in \hat{L} \text{ with } V_\nu \neq 0$$

reduces to

$$2(\chi+\rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1 \qquad \text{for all } i$$

as in the passage from Theorem 7.23 to Corollary 7.25. According to Lemma 7.27, existence of the highest L-type χ is not a restrictive condition here.

Proof of Theorem 8.21. All the statements follow directly from Theorem 8.18 and Lemma 8.20, except weak continuity of $G \times \bar{\mathscr{H}}_2^s(G/H, \mathbf{V}) \to \bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$. That weak continuity follows from joint continuity of the Hermitian form $\langle , \rangle_{G/H}$ on $L_2^{0,s}(G/H, \mathbf{V})$, which is part of the statement of Lemma 7.3. Q.E.D.

9. CONSTRUCTION OF THE UNITARY REPRESENTATION: GENERAL CASE

In Section 8 we examined the map $\mathscr{H}_2^s(G/H, \mathbf{V}) \to H^s(G/H, \mathbf{V})$ and saw, under certain circumstances, that its kernel coincides with the kernel of the

G-invariant form $\langle , \rangle_{G/H}$ and that $(-1)^s \langle , \rangle_{G/H}$ is positive semidefinite. That led to a Hilbert space structure on the quotient $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ and a unitary representation π_V of G on $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ that unitarizes the action of G on the Dolbeault cohomology $H^s(G/H, \mathbf{V})$. The key was the study of the space of L_2 "special" harmonic forms,

$$\mathscr{S} = \{\varphi \in \mathscr{H}_2^s(G/H, \mathbf{V}) \colon \varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\}, \tag{9.1}$$

on which $(-1)^s \langle , \rangle_{G/H}$ is positive definite and whose K-finite subset $\mathscr{S}_K$ maps isomorphically to $H^s(G/H, \mathbf{V})_K$. Here we show under somewhat weaker conditions that $\mathscr{S}$ still corresponds to a G-invariant Hilbert space structure on an appropriate quotient $\bar{\mathscr{S}}(G/H, \mathbf{V})$ of $\mathscr{U}(\mathfrak{g})\,\mathscr{S}$, resulting in a unitary representation of G that unitarizes its action on $H^s(G/H, \mathbf{V})$.

The space $\mathscr{U}(\mathfrak{g})\,\mathscr{S}_K$ is the smallest $\mathfrak{g}$-invariant subspace of $\mathscr{H}_2^s(G/H, \mathbf{V})$ that contains $\mathscr{S}_K$, and $\mathscr{S}_K$ injects to the quotient $\mathscr{U}(\mathfrak{g})\,\mathscr{S}_K/\{\mathscr{U}(\mathfrak{g})\,\mathscr{S}_K\}^\perp$ (relative to $\langle\, , \rangle_{G/H}$) because $(-1)^s \langle\, , \rangle_{G/H}$ is positive definite on $\mathscr{S}$. We need to know that the image is invariant.

9.2. THEOREM. *Let $\pi \colon G/H \to K/L$ be holomorphic and suppose that every $2(\nu + \rho_M, \gamma_i)/(\gamma_i, \gamma_i) < -\frac{1}{2}$, where $\{\gamma_i\}$ are the maximal roots of the noncompact simple factors of M and ν runs over the classes in $\hat{L}$ with $V_\nu \neq 0$. Then the image of $\mathscr{S}_K$ in $\mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}_K/\{\mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}_K\}^\perp$ is $\mathfrak{g}$-invariant.*

The reader may wish to view this section as a disgression and proceed directly to the discussion of unitary representations in Sections 10 and 11. But first he should note that the hypothesis $2(\nu + \rho_M, \gamma_i)/(\gamma_i, \gamma_i) < -\frac{1}{2}$ of Theorem 9.2 is very close to the condition $2(\nu + \rho_M, \gamma_i)/(\gamma_i, \gamma_i) < 0$ for the existence of L_2 harmonic representatives in Theorem 7.23. See Remark 9.18 for the fact that they coincide if M is linear or if $\dim V < \infty$.

We start by estimating the growth of forms $\varphi \in \mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}$. Haar measures on $G = KAH$ and $M = LAL$ are given through certain functions δ_G, δ_M: $A \to \mathbf{R}^+$ by

$$dg = \delta_G(a)\, dk \cdot da \cdot dm, \qquad dm = \delta_M(a)\, dl \cdot da \cdot dl,$$
$$\delta_G(a) \cdot \det A_a = \delta_M(a), \tag{9.3}$$

where we use Lemma 7.8. A Cayley transform c carries $\mathfrak{a}^*$ to the span of the maximal set $\{\gamma_{ij}\}$ of strongly orthogonal roots of (4.22). Its inverse carries the positive root system $\Phi(\mathfrak{m})^+$ for the compact Cartan $\mathfrak{t}$ to a positive root system for the maximally split Cartan

$$\{x \in \mathfrak{t} \colon \text{each } \gamma_{ij}(x) = 0\} + \mathfrak{a},$$

which in turn specifies a positive restricted root system $\Delta_\mathfrak{a}^+$ on $\mathfrak{m}$. Let $\gamma_{i,j}^C \in \Delta_\mathfrak{a}^+$ denote the restricted root sent by c to $\gamma_{i,j}$.

9.4. LEMMA. *Assume the conditions of Theorem* 9.1. *If* $\varphi \in \mathscr{U}(\mathfrak{g})\mathscr{S}_K$, *then there exist constants* C, $\varepsilon > 0$ *such that*

$$\|\varphi(ka)\| \leqslant C\delta_G(a)^{-1/2} \exp\left(-(\tfrac{1}{4}+\varepsilon)\sum_{i,j} \gamma_{i,j}^c\right)(a)$$

uniformly for $k \in K$ *and* $a \in A^+$.

Proof. We first must show that every $\varphi \in \mathscr{U}(\mathfrak{g})\mathscr{S}_K$ can be expressed as a finite sum on KM of the form

$$\varphi(km) = (\det A_m|_{\mathfrak{k}\cap\mathfrak{q}_-}) \sum b_j(k)\, c_j(m) f_j(m)\, \theta_j, \tag{9.5}$$

where the b_j are coefficients of finite-dimensional unitary representation of K, the c_j are polynomials in the matrix coefficients of C_m and D_m (defined in (3.14)), the f_j are L-finite holomorphic sections of bundles $\mathbf{V}_\nu \to M/L$, where $\nu \in \hat{L}$ with $V_\nu \neq 0$, and the $\theta_j \in \Lambda^s(\mathfrak{q}_-)^*$.

Suppose that $\varphi \in \mathscr{S}_K$. Then $\varphi(km)(\xi_1,..., \xi_s) = \omega_\varphi(k)(A_m\xi_1,..., A_m\xi_s)(m) = (\det A_m|_{\mathfrak{k}\cap\mathfrak{q}_-})\, \omega_\varphi(k)(\xi_1,..., \xi_s)(m)$, $\omega_\varphi(k)(\xi_1,..., \xi_s)$ is an L-finite holomorphic section of $\mathbf{V} \to M/L$, and ω_φ is K-finite. So, here, $\varphi(km)$ has expression (9.5) with $c_j \equiv 1$.

As $\mathscr{U}(\mathfrak{g})\mathscr{S}_K = \bigcup_{n=0}^\infty \mathfrak{p}^n \cdot \mathscr{S}_K$, it now suffices to show that if $\varphi \in \mathscr{U}(\mathfrak{g})\mathscr{S}_K$ has form (9.5), and if $\zeta \in \mathfrak{p}$, then $\zeta \cdot \varphi$ has form (9.5). By (3.17), $(\zeta \cdot \varphi)(km) = \varphi(-\zeta \cdot km) = -\varphi(k \cdot \mathrm{Ad}(k^{-1})\,\zeta \cdot m) = -l(\mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)$. Of course, $-\mathrm{Ad}(k^{-1})\,\zeta$ is a linear combination of elements of a basis of $\mathfrak{p}$ where the coefficients are matrix coefficients of finite-dimensional unitary representations of K. Now to prove (9.5) we need only show: if $\varphi \in \mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}_K$ has form (9.5), and if either $\zeta \in \mathfrak{p} \cap \mathfrak{h}$ or $\zeta \in \mathfrak{p} \cap \mathfrak{q}$, then $l(\zeta)\,\varphi$ has form (9.5).

Let $\zeta \in \mathfrak{p} \cap \mathfrak{h}$. By (3.19), $l(\zeta)\,\varphi(km) = r(C_m\zeta)\,\varphi(km) + l(D_m\zeta)\,\varphi(km)$. Here $D_m\colon \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{k} \cap \mathfrak{q}$, so $l(D_m\zeta)\,\varphi(km) = \varphi((k \cdot D_m\zeta)\, m)$, which inherits form (9.5) from φ. Also $C_m\colon \mathfrak{p} \cap \mathfrak{h} \to \mathfrak{p} \cap \mathfrak{h}$, so if $\eta_1,..., \eta_s \in \mathfrak{q}_-$, then

$$\begin{aligned} r(C_m\zeta)\,\varphi(km)(\eta_1,..., \eta_s) &\\ &= \varphi(km \cdot C_m\zeta)(\eta_1,..., \eta_s) \\ &= -d\psi(C_m\zeta) \cdot \varphi(km)(\eta_1,..., \eta_s) \\ &\quad - \sum_{i=1}^{s} (-1)^{i-1}\, \varphi(km)([C_m\zeta, \eta_i], \eta_1,..., \hat{\eta}_i,..., \eta_s). \end{aligned}$$

These terms all inherit form (9.5) from φ.

Let $\zeta \in \mathfrak{p} \cap \mathfrak{q}$, so $l(\zeta)$ acts by differentiating the M-variable on the left,

$l(\zeta)\,\varphi(km) = \varphi(k \cdot \zeta m)$. We look at its action on the $(\det A_m|_{\mathfrak{k}\cap\mathfrak{q}_-})$ term of (9.5). Making repeated use of the definitions (3.14),

$$\begin{aligned}
l(\zeta)\,A_m &= l(\zeta) \cdot \{p' \circ \mathrm{Ad}(m^{-1})\}^{-1} \\
&= -A_m \circ l(\zeta)\{p' \circ \mathrm{Ad}(m^{-1})\} \circ A_m \\
&= A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ \mathrm{ad}(\zeta) \circ A_m \\
&= A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ p'' \circ \mathrm{ad}(\zeta) \circ A_m \\
&= A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ p'' \circ \mathrm{Ad}(m) \circ p'' \circ C_m \circ \mathrm{ad}(\zeta) \circ A_m \\
&= A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ p'' \circ \mathrm{Ad}(m) \circ C_m \circ \mathrm{ad}(\zeta) \circ A_m \\
&= A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ (1 - p') \circ \mathrm{Ad}(m) \circ C_m \circ \mathrm{ad}(\zeta) \circ A_m \\
&= -A_m \circ p' \circ \mathrm{Ad}(m^{-1}) \circ p' \circ \mathrm{Ad}(m) \circ C_m \circ \mathrm{ad}(\zeta) \circ A_m \\
&= -p' \circ \mathrm{Ad}(m) \circ C_m \circ \mathrm{ad}(\zeta) \circ A_m \\
&= D_m \circ \mathrm{ad}(\zeta) \circ A_m
\end{aligned}$$

so

$$l(\zeta)(\det A_m|_{\mathfrak{k}\cap\mathfrak{q}_-}) = (\mathrm{trace}\, D_m \circ \mathrm{ad}(\zeta)|_{\mathfrak{k}\cap\mathfrak{q}_-}) \det(A_m|_{\mathfrak{k}\cap\mathfrak{q}_-}).$$

Thus, the part of $l(\zeta)\,\varphi$ that involves differentiating the $\det(A_m|_{\mathfrak{k}\cap\mathfrak{q}_-})$ term in expression (9.5) for φ, retains form (9.5). Evidently the same holds for the part of $l(\zeta)\,\varphi$ that involves derivatives of the f_j. As for the $c_j(\zeta \cdot m)$, we obtain the same conclusion by noting that

$$\begin{aligned}
l(\zeta)\,C_m &= -C_m \circ p'' \circ \mathrm{ad}(\zeta) \circ \mathrm{Ad}(m) \circ C_m \\
&= -C_m \circ \mathrm{ad}(\zeta) \circ p' \circ \mathrm{Ad}(m) \circ C_m \\
&= C_m \circ \mathrm{ad}(\zeta) \circ D_m
\end{aligned}$$

and

$$\begin{aligned}
l(\zeta)\,D_m &= -p' \circ \mathrm{ad}(\zeta) \circ \mathrm{Ad}(m) \circ C_m - p' \circ \mathrm{Ad}(m) \circ l(\zeta)\,C_m \\
&= -\mathrm{ad}(\zeta) \circ p'' \circ \mathrm{Ad}(m) \circ C_m - p' \circ \mathrm{Ad}(m) \circ C_m \circ \mathrm{ad}(\zeta) \circ D_m \\
&= -\mathrm{ad}(\zeta) + D_m \circ \mathrm{ad}(\zeta) \circ D_m,
\end{aligned}$$

so the derivatives of the C_m, D_m coefficients do not occur. This completes the proof that every $\varphi \in \mathscr{U}(\mathfrak{g})\,\mathscr{S}_K$ has form (9.5).

The following estimate is implicit in Harish-Chandra's analysis [11–13] of the coefficients of the holomorphic discrete series. See his formula [12, Lemma 14] for his function ψ and his estimates [13, Sect. 9] on the ψ_Λ. A very slightly weaker estimate, valid for the entire discrete series, comes out of Miličić's work [21] on characters and asymptotics.

Let $f\colon M \to V_\nu$ be an L-finite holomorphic section of $\mathbf{V}_\nu \to M/L$. Let $\varepsilon > 0$ be such that $(\nu + \rho_M, \gamma_i) < -(\frac{1}{4} + \varepsilon)(\gamma_i, \gamma_i)$ for all i. The estimate: there is a number $C > 0$ such that

$$\|f(lal')\| = C\delta_M(a)^{-1/2} \exp\left(-(\tfrac{1}{4} + \varepsilon)\sum_{i,j} \gamma^c_{i,j}\right)(a) \tag{9.6}$$

for all l, $l' \in L$ and all $a \in A^+$.

If $m = \exp(\eta)$, $\eta \in \mathfrak{p}_0 \cap \mathfrak{q}_0$, we earlier noted that $A_m = \cosh(\mathrm{ad}(\eta))|_{\mathfrak{k}\cap\mathfrak{q}}$ because $\mathrm{ad}(\eta)$ exchanges $\mathfrak{p} \cap \mathfrak{h}$ and $\mathfrak{k} \cap \mathfrak{q}$. Similarly, $C_m = \{\cosh(\mathrm{ad}(\eta))|_{\mathfrak{p}\cap\mathfrak{h}}\}^{-1}$ and $D_m = -\sinh(\mathrm{ad}(\eta)) \circ \{\cosh(\mathrm{ad}(\eta))|_{\mathfrak{p}\cap\mathfrak{h}}\}^{-1} = -\tanh(\mathrm{ad}(\eta))|_{\mathfrak{p}\cap\mathfrak{h}}$. In view of (3.15), now C_m and D_m are bounded uniformly in $m \in M$, so the functions $c_j(m)$ are uniformly bounded. In proving the Lemma we may thus ignore the c_j in expression (9.5) for φ. That having been seen possible, (9.5) and (9.6) combine using (9.3) to give the estimate of Lemma 9.4. Q.E.D.

The next step in the proof of Theorem 9.1 is

9.7. Lemma. *Let* $\varphi \in A^p(G/H, \mathbf{V})$ *with* $\varphi(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$. *Let* $\zeta \in \mathfrak{p}$ *and define* $\beta = \beta_\zeta : KM \to V \otimes \Lambda^{p-1}(\mathfrak{k} \cap \mathfrak{q}_-)^*$ *by*

$$\omega_\beta(k)(\xi_1, \ldots, \xi_{p-1})(m) = (-1)^p\, \omega_\varphi(k)(\xi_1, \ldots, \xi_{p-1}, p'_- \circ D_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m), \tag{9.8}$$

where $\xi_i \in \mathfrak{k} \cap \mathfrak{q}_-$ *and* $\sigma\colon \mathfrak{p} \to \mathfrak{p} \cap \mathfrak{h}$ *is projection with kernel* $\mathfrak{p} \cap \mathfrak{q}$. *Then* β *is well defined as an element of* $A^{p-1}(G/H, \mathbf{V})$, *and if* $\bar\partial\varphi = 0$, *then* $(\zeta \cdot \varphi + \bar\partial\beta)\colon KM \to V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$.

Proof. To see that β is well defined as an element of $A^{p-1}(G/H, \mathbf{V})$, i.e., that ω_β is well defined as an element of $A^{p-1}(K/L, \mathbf{A}^0(M/L, \mathbf{V}))$, we compute using (3.15)

$$\begin{aligned}\omega_\beta(k)(\xi_1, \ldots, \xi_{p-1})(ml) &= (-1)^p\, \omega_\varphi(k)(\xi_1, \ldots, \xi_{p-1}, p' \circ D_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(ml)\\ &= \psi(l)^{-1} \cdot \omega_\beta(k)(\xi_1, \ldots, \xi_{p-1})(m)\end{aligned}$$

and

$$\begin{aligned}&\omega_\beta(kl)(\xi_1, \ldots, \xi_{p-1})(m)\\ &= (-1)^p\, \omega_\varphi(kl)(\xi_1, \ldots, \xi_{p-1}, p'_- \circ D_m \circ \sigma \circ \mathrm{Ad}(l^{-1}) \circ \mathrm{Ad}(k^{-1})\,\zeta)(m)\\ &= (-1)^p\, \omega_\varphi(k)(\mathrm{Ad}(l)\,\xi_1, \ldots, \mathrm{Ad}(l)\,\xi_{p-1},\\ &\qquad \mathrm{Ad}(l) \circ p'_- \circ D_m \circ \sigma \circ \mathrm{Ad}(l^{-1}) \circ \mathrm{Ad}(k^{-1})\,\zeta)(m)\\ &= (-1)^p\, \omega_\varphi(k)(\mathrm{Ad}(l)\,\xi_1, \ldots, \mathrm{Ad}(l)\,\xi_{p-1}, p'_- \circ D_{lm} \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m)\\ &= \omega_\beta(k)(\mathrm{Ad}(l)\,\xi_1, \ldots, \mathrm{Ad}(l)\,\xi_{p-1})(m).\end{aligned}$$

Now suppose $\bar{\partial}\varphi = 0$, and decompose $\zeta \cdot \varphi = \varphi' + \varphi''$, where $\varphi'(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$ and $\varphi''(KM) \subset V \otimes \Lambda^{p-1}(\mathfrak{k} \cap \mathfrak{q}_-)^* \otimes \Lambda^1(\mathfrak{p} \cap \mathfrak{q}_-)^*$. Set $\omega' = \omega_{\varphi'} \in A^p(K/L, \mathbf{A}^0(M/L, \mathbf{V}))$ and $\omega'' = \omega_{\varphi''} \in A^{p-1}(K/L, \mathbf{A}^1(M/L, \mathbf{V}))$. Note, using (3.19),

$$\begin{aligned}
&\omega''(k)(\xi_1,..., \xi_{p-1})(m)(\eta)\\
&\quad = (\zeta \cdot \varphi)(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\quad = -l(\mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\quad = -l(\sigma \cdot \mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\qquad -l((1-\sigma) \cdot \mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\quad = -l(D_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\qquad -r(C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\qquad -l((1-\sigma) \circ \mathrm{Ad}(k^{-1})\,\zeta)\,\varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta).
\end{aligned}$$

Here the first and third terms vanish because $\varphi(KM) \subset V \otimes \Lambda^p(\mathfrak{k} \cap \mathfrak{q}_-)^*$. That leaves

$$\begin{aligned}
&\omega''(k)(\xi_1,..., \xi_{p-1})(m)(\eta)\\
&\quad = d\psi(C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta) \circ \varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, \eta)\\
&\qquad + \sum_{i=1}^{p-1} (-1)^{i-1}\,\varphi(km)([C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta, T_m\xi_i],\\
&\qquad\qquad T_m\xi_1,..., \widehat{T_m\xi_i},..., T_m\xi_{p-1}, \eta)\\
&\qquad + \varphi(km)(T_m\xi_1,..., T_m\xi_{p-1}, [C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta, \eta]).
\end{aligned}$$

Here the first and second terms vanish as before. Using (3.16),

$$\begin{aligned}
&\omega''(k)(\xi_1,..., \xi_{p-1})(m)(\eta)\\
&\quad = \omega_\varphi(k)(\xi_1,..., \xi_{p-1}, A_m[C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta, \eta])(m)\\
&\quad = \omega_\varphi(k)(\xi_1,..., \xi_{p-1}, p'_- \circ A_m \circ \mathrm{ad}(\eta) \circ C_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m)\\
&\quad = -\omega_\varphi(k)(\xi_1,..., \xi_{p-1}, p'_- \circ (r(\eta)\,D_m) \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m).
\end{aligned}$$

Since $\bar{\partial}\varphi = 0$, $-\omega_\varphi(k)(\xi_1,..., \xi_{p-1}, \xi_p)(m \circ \eta) = 0$ for each m; given m we take $\xi_p = p'_- \circ D_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta$. Now,

$$\begin{aligned}
&\omega''(k)(\xi_1,..., \xi_{p-1})(m)(\eta)\\
&\quad = -\omega_\varphi(k)(\xi_1,..., \xi_{p-1}, p'_- \circ (r(\eta)\,D_m) \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m)\\
&\qquad -\omega_\varphi(k)(\xi_1,..., \xi_{p-1}, p'_- \circ D_m \circ \sigma \circ \mathrm{Ad}(k^{-1})\,\zeta)(m \cdot \eta)\\
&\quad = (-1)^{p-1}\,\omega_\beta(k)(\xi_1,..., \xi_{p-1})(m \cdot \eta).
\end{aligned}$$

We write that as

$$\omega''(k)(\xi_1,\dots,\xi_{p-1}) = (-1)^{p-1}\,\bar\partial_{M/L}(\omega_\beta(k)(\xi_1,\dots,\xi_{p-1})).$$

In view of Proposition 3.25, or more precisely, the calculation just before its statement, this is equivalent to

$$\varphi''(km)(\xi_1,\dots,\xi_{p-1},\eta) = -\bar\partial\beta(km)(\xi_1,\dots,\xi_{p-1},\eta).$$

Thus $(\zeta\circ\varphi + \bar\partial\beta) = \varphi' + \bar\partial_{1,0}(\beta)$, which proves the lemma. Q.E.D.

9.9. LEMMA. *Let $\varphi \in \mathscr{S}_K$ and $\zeta \in \mathfrak{p}$. Define $\beta = \beta_\zeta$ by* (9.8). *Under the conditions of Theorem* 9.1, *$\bar\partial\beta$ is square integrable and $\langle \bar\partial\beta, \mathscr{U}(\mathfrak{g})\,\mathscr{S}_K\rangle_{G/H} = 0$.*

Proof. We have that $\zeta\cdot\varphi$ is square integrable because φ is a C^∞ vector for G on $L_2^{0,s}(G/H;\mathbb{V})$; or one can use Lemma 9.4. Now set $\tilde\varphi = \zeta\cdot\varphi + \bar\partial\beta$. A glance at (9.8) shows that β inherits $K-$finiteness from φ, so $\tilde\varphi$ is K-finite. Also, $\bar\partial\tilde\varphi = \bar\partial(\zeta\cdot\varphi) + \bar\partial(\bar\partial\beta) = \bar\partial(\zeta\cdot\varphi) = \zeta\cdot\bar\partial\varphi = 0$. Thus, using Lemma 9.7, $\tilde\varphi \in A^s(G/H,\mathbb{V})$ is K-finite and $\bar\partial$-closed, and maps KM into $V\otimes\Lambda^s(\mathfrak{k}\cap\mathfrak{q}_-)^*$. Proposition 7.20 says that $\tilde\varphi$ is square integrable, and square integrability follows for $\bar\partial\beta = \tilde\varphi - \zeta\cdot\varphi$.

One can write (9.8) as $\beta(km)(\xi_1,\dots,\xi_{p-1}) = (-1)^p\,\varphi(km)(\xi_1,\dots,\xi_{p-1}, p'\circ \mathrm{Ad}(m^{-1})\circ p'_-\circ D_m\circ\sigma\circ\mathrm{Ad}(k^{-1})\,\zeta)$, which we use here with $p = s$. Note that

$$\begin{aligned}
&p'\circ\mathrm{Ad}(m^{-1})\circ p'_-\circ D_m\\
&\quad= p'_-\circ\mathrm{Ad}(m^{-1})\circ D_m \qquad \text{(because } \pi \text{ is holomorphic)}\\
&\quad= -p'_-\circ\mathrm{Ad}(m^{-1})\circ p'\circ\mathrm{Ad}(m)\circ C_m\\
&\quad= -p'_-\circ\mathrm{Ad}(m^{-1})\circ(1-p'')\circ\mathrm{Ad}(m)\circ C_m\\
&\quad= p'_-\circ\mathrm{Ad}(m^{-1})\circ p''\circ\mathrm{Ad}(m)\circ C_m \qquad (\text{because } p'\circ C_m = 0)\\
&\quad= p'_-\circ\mathrm{Ad}(m^{-1}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\beta(km)(\xi_1,\dots,\xi_{s-1})\\
&\quad= (-1)^s\,\varphi(km)(\xi_1,\dots,\xi_{s-1}, p'_-\circ\mathrm{Ad}(m^{-1})\circ\sigma\circ\mathrm{Ad}(k^{-1})\,\zeta). \qquad (9.10)
\end{aligned}$$

Every $\alpha \in \Phi(\mathfrak{p}\cap\mathfrak{h})$ has projection to the span of the $\gamma_{i,j}$ which is 0 or is $\pm\frac{1}{2}\gamma_{i,j}$ for some unique (i,j). If $a \in A^+$, that gives $\|\mathrm{Ad}(a^{-1})|_{\mathfrak{p}\cap\mathfrak{h}}\| \leqslant \exp(\frac{1}{2}\sum_{i,j}\gamma^c_{i,j})(a)$. Use this while applying Lemma 9.4 to φ in (9.10),

$$\|\beta(ka)\| \leqslant C'\delta_G(a)^{-1/2}\exp\left((\tfrac{1}{4}-\varepsilon)\sum_{i,j}\gamma^c_{i,j}\right)(a) \qquad (9.11)$$

uniformly for $k \in K$ and $a \in A^+$.

Now let $\eta \in \mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}_K$. Applying Lemma 9.4 to η and (9.11) to β, $\|(\beta \bar{\wedge} \# \eta)(ka)\| \leqslant C'' \delta_G(a)^{-1} \exp(-2\varepsilon \sum_{i,j} \gamma_{i,j}^c)(a)$, which is integrable over G/H. Choose a sequence $\{f_n\}$ of compactly supported C^∞ functions on G/H such that $0 \leqslant f_n(x) \leqslant 1$ for all $x \in G/H$, if $F \subset G/H$ is compact, then $f_n \equiv 1$ on F for n sufficiently large, and the df_n are uniformly bounded. For example, if $h: \mathbf{R} \to \mathbf{R}$ is C^∞, $h(t) = 0$ for $t \leqslant 0$, $h(t) = 1$ for $t \geqslant 1$, and $h'(t) \geqslant 0$ for all t, one could use $f_n(x) = h((1/n)(\text{distance } x \text{ to } x_0)^2)$. Compute

$$
\begin{aligned}
&\langle \bar{\partial}\beta, \eta \rangle_{G/H} \\
&= \int_{G/H} \bar{\partial}\beta \,\bar{\wedge}\, \# \eta = \lim_{n \to \infty} \int_{G/H} f_n \bar{\partial}\beta \,\bar{\wedge}\, \# \eta \\
&= \lim_{n \to \infty} \int_{G/H} \{\bar{\partial}(f_n \beta) \,\bar{\wedge}\, \# \eta - \bar{\partial} f_n \wedge \beta \wedge \# \eta\} \\
&= -\lim_{n \to \infty} \int_{G/H} \bar{\partial} f_n \wedge \beta \,\bar{\wedge}\, \# \eta \quad (\text{because } f_n\beta \text{ is compactly} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{supported and } \bar{\partial}^* \eta = 0) \\
&= 0 \quad (\text{because } \bar{\partial} f_n \text{ is uniformly bounded on } G/H, \\
&\qquad \text{and on any compact set } \bar{\partial} f_n \equiv 0 \text{ for } n \gg 0).
\end{aligned}
$$

Lemma 9.9 is proved. Q.E.D.

Proof of Theorem 9.2. Let $\varphi \in \mathscr{S}_K$ and $\zeta \in \mathfrak{p}$; we must show that $\zeta \cdot \varphi \in \mathscr{S}_K + \{\mathscr{U}(\mathfrak{g}) \cdot \mathscr{S}_K\}^\perp$. Let $\beta = \beta_\zeta \in A^{s-1}(G/H, \mathbf{V})$ as in (9.8), so $\beta \in \{\mathscr{U}(\mathfrak{g}) \mathscr{S}_K\}^\perp$ by Lemma 9.9, and $\tilde{\varphi} = \zeta \cdot \varphi + \bar{\partial}\beta$ is square integrable, K-finite, and $\bar{\partial}$-closed, and maps KM to $V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. We cannot conclude that $\tilde{\varphi}$ is in $\mathscr{S}_K$ because it need not be harmonic.

Theorem 7.23 provides an element of $\mathscr{S}_K$ that belongs to the same Dolbeault class as $\tilde{\varphi}$, say $\tilde{\varphi} + \bar{\partial}\beta'$. Necessarily, $\bar{\partial}\beta'$ is K-finite and $\bar{\partial}\beta'(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. From the latter, the components of β' of fibre degree >0 are $\bar{\partial}$-closed, so we may assume (i) $\beta'(KM) \subset V \otimes \Lambda^{s-1}(\mathfrak{k} \cap \mathfrak{q}_-)^*$ and (ii) $\bar{\partial}_{0,1}\beta' = 0$ using (3.37), i.e., $\omega_{\beta'} \in A^{s-1}(K/L, \mathrm{H}^0(M/L, \mathbf{V}))$. If $\xi_1, \ldots, \xi_{s-1} \in \mathfrak{k} \cap \mathfrak{q}_-$, now $\omega_{\beta'}(k)(\xi_1, \ldots, \xi_{s-1})$ is an L-finite holomorphic section of $\mathbf{V} \to M/L$, hence satisfies (9.6), and $\omega_{\beta'}$ is K-finite. Thus for $a \in A^+$,

$$\beta'(ka)(\xi_1, \ldots, \xi_{s-1})(a) = \omega_{\beta'}(k)(A_a \xi_1, \ldots, A_a \xi_{s-1})(a)$$

has norm bounded by some

$$C\delta_M(a)^{-1/2} \exp\Big(-(\tfrac{1}{4} + \varepsilon) \sum_{i,j} \gamma_{i,j}^c\Big)(a) \|\Lambda^{s-1}(A_a|_{\mathfrak{k} \cap \mathfrak{q}_-})\|.$$

Every $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ projects, on the span of the $\gamma_{i,j}$, to 0 or to some $\frac{1}{2}\gamma_{i,j}$, so

$$|\det(A_a|_{\mathfrak{k}\cap\mathfrak{q}_-})|^{-1} \, \|\Lambda^{s-1}(A_a|_{\mathfrak{k}\cap\mathfrak{q}_-})\| \leqslant \exp\left(\tfrac{1}{2}\sum_{i,j} \gamma_{i,j}^c\right)(a).$$

Combining the last two inequalities with (9.3) we get

$$\|\beta'(ka)\| \leqslant C'\delta_G(a)^{-1/2} \exp\left((\tfrac{1}{4} - \varepsilon)\sum_{i,j} \gamma_{i,j}^c\right)(a).$$

Just as in the last paragraph of the proof of Lemma 9.9, it follows that $\langle \bar\partial\beta', \mathscr{U}(\mathfrak{g})\,\mathscr{S}_K\rangle_{G/H} = 0$. We now have $\zeta \circ \varphi = (\tilde\varphi + \bar\partial\beta') - \bar\partial(\beta + \beta')$ with $\tilde\varphi + \bar\partial\beta' \in \mathscr{S}_K$ and $\bar\partial(\beta + \beta') \in \{\mathscr{U}(\mathfrak{g})\,\mathscr{S}_K\}^\perp \cap \{\mathscr{U}(\mathfrak{g})\,\mathscr{S}_K\}$. Q.E.D.

Now, in analogy to the situation of Section 8, we define

$$\mathscr{S}(G/H, \mathbf{V}) = \text{smallest closed } G\text{-invariant subspace of } \mathscr{H}_2^s(G/H, \mathbf{V}) \text{ containing } \mathscr{S}_K \tag{9.12}$$

and the corresponding "reduced" special L_2 harmonic space

$$\bar{\mathscr{S}}(G/H, \mathbf{V}) = \mathscr{S}(G/H, \mathbf{V})/(\text{nullspace of } \langle\, ,\rangle_{G/H}) \text{ with Hermitian form induced by } \langle\, ,\rangle_{G/H}. \tag{9.13}$$

In the quotient topology from $L_2^{0,s}(G/H, \mathbf{V})$, which is its topology induced by $\langle\, ,\rangle_{G/H}$, $\bar{\mathscr{S}}(G/H, \mathbf{V})$ is a closed G-invariant positive definite subspace of $\bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$.

9.14. THEOREM. *Let $\pi\colon G/H \to K/L$ be holomorphic and let $\{\gamma_1, ..., \gamma_c\}$ be the maximal roots of the noncompact simple factors of M. Suppose that all*

$$2(\nu + \rho_M, \gamma_i)/(\gamma_i, \gamma_i) < -\tfrac{1}{2} \qquad \textit{for} \quad 1 \leqslant i \leqslant c, \tag{9.15}$$

where ν runs over the classes in $\hat{L}$ with $V_\nu \neq 0$. Then $(-1)^s\langle\, ,\rangle_{G/H}$ is positive semidefinite on $\mathscr{S}(G/H, \mathbf{V})$, its null space there is the kernel of the map $\varphi \mapsto [\varphi] \in H^s(G/H, \mathbf{V})$ to Dolbeault cohomology, and $\mathscr{S}(G/H, \mathbf{V})_K$ maps onto $H^s(G/H, \mathbf{V})_K$. The quotient $\bar{\mathscr{S}}(G/H, \mathbf{V})$ is a Hilbert space and the action of G on $\mathbf{V}$-valued forms defines

$$\pi_V\text{: unitary representation of } G \text{ on } \bar{\mathscr{S}}(G/H, \mathbf{V}) \tag{9.16}$$

which unitarizes the action of G on $H^s(G/H, \mathbf{V})$. If the stronger (than (9.15)) condition

$$2(\nu + \rho, \gamma_i)/(\gamma_i, \gamma_i) \leqslant -1 \qquad \textit{for} \quad 1 \leqslant i \leqslant c \tag{9.17}$$

holds, then $\bar{\mathscr{S}}(G/H, \mathbf{V}) = \bar{\mathscr{H}}_2^s(G/H, \mathbf{V})$ *and* π_V *coincides with the representation of Theorem* 8.21.

Proof. Since $\mathscr{S}_K$ maps onto $H^s(G/H, \mathbf{V})_K$ isomorphically by Theorem 7.23, our assertions follow by the arguments of Theorem 8.18, Lemma 8.20, and Theorem 8.21, except that we use Theorem 9.2 in place of Theorem 8.15. Q.E.D.

9.18. *Remark.* If the highest weights ν of L on V all are integral in $\mathfrak{m}$, then hypothesis (9.15) of Theorem 8.15 coincides with the L_2 condition $(\nu + \rho_M, \gamma_i) < 0$ of Theorem 7.23. As in Remark 8.23, if M is linear, in particular if G is linear, or if dim $V < \infty$, then it is automatic that the ν are $\mathfrak{m}$-integral.

9.19. *Remark.* If V has a highest L-type χ in the sense of (4.30), then we may replace (9.15) by the condition that $2(\chi + \rho_M, \gamma_i)/(\gamma_i, \gamma_i) < -\frac{1}{2}$ for $1 \leqslant i \leqslant c$.

10. Antidominance Conditions on the Representations

We now start to analyse and identify the unitary representation π_V of Sections 8 and 9. In order to do that we first examine the interplay between certain negativity conditions on the representation of L on V. In this section we carry out the examination for the four conditions listed below, in case dim $V < \infty$. A few of the results also hold for dim $V = \infty$ and we discuss those at the end of the section. Then in Section 11 we will look at irreducibility and characters for the unitary representations π_V.

Assume, now, that V is finite dimensional. Thus $\chi = \psi|_L$ is irreducible and every weight of χ is orthogonal to $\Phi(\mathfrak{p} \cap \mathfrak{h})$. As usual, we also write χ for the highest weight. Also assume π: $G/H \to K/L$ holomorphic.

The condition of Theorem 7.23, which ensures the existence of special square-integrable harmonic representatives for K-finite Dolbeault classes $c \in H^s(G/H, \mathbf{V})$ is, for dim $V < \infty$, the

$$L_2 \text{ condition:} \quad (\chi + \rho_M, \gamma) < 0 \qquad \text{for all} \quad \gamma \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+). \tag{10.1}$$

A condition which will give us a reasonable starting point for the analysis of the K-spectrum of π_V is the

highest K-type condition:

$$(\chi + \rho_K, \alpha) < 0 \qquad \text{for all} \quad \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+). \tag{10.2}$$

It says that $H^s(K/L, \mathbf{H}^0(M/L, \mathbf{V})_\chi)$ is nonzero, occurs as a K-type in $H^s(G/H, \mathbf{V})$ with multiplicity 1, and is the highest K-type there.

The results of Section 5 show that π_V has infinitesimal character of Harish-Chandra parameter $\chi + \rho$. The

$$\text{nonsingularity condition:} \quad (\chi + \rho, \alpha) < 0 \qquad \text{for all} \quad \alpha \in \Phi(\mathfrak{q}_+) \tag{10.3}$$

ensures that $\chi + \rho$ is interior to the Weyl chamber that contains a generic negative χ, translated by ρ.

Finally, as we saw in Theorem 4.5, the condition that $H^p(G/H, \mathbf{V}) = 0$ for $p \neq s$ is equivalent to the

vanishing condition:

$$\begin{aligned} &\text{if } \nu \in \hat{L} \text{ occurs in } V \otimes S(\mathfrak{p} \cap \mathfrak{q}_-) \\ &\text{and if } (\nu + \rho_K, \alpha) > 0 \text{ for some } \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+) \\ &\text{then } (\nu + \rho_K, \alpha') = 0 \text{ for some } \alpha' \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+). \end{aligned} \tag{10.4}$$

The various conditions are related as follows:

10.5. THEOREM. *Let π: $G/H \to K/L$ be holomorphic, let* $\dim V < \infty$, *and set $\chi = \psi|_L$. Then we have the implications*

L_2 condition (10.1)

nonsingularity (10.3) $\Rightarrow$ *L_2 condition* (10.1); *nonsingularity* (10.3) $\Rightarrow$ *highest K-type* (10.2); *L_2 condition* (10.1) $\Rightarrow$ *vanishing* (10.4); *highest K-type* (10.2) $\Rightarrow$ *vanishing* (10.4); *L_2 condition* (10.1) $\to$ *highest K-type* (10.2) (*see below*).

If G/H has no irreducible factor of the form

$$\begin{aligned} &SO(2n-2, 2)/U(n-1, 1), && n \geqslant 5, \\ &SU(k+l, m)/S(U(k) \times U(l, m)), && 0 < m < l, \end{aligned}$$

then the L_2 condition implies the highest K-type condition.

We break the proof into a series of lemmas, the first three of which are the easy implications and some of which are of independent interest.

10.6. LEMMA. *Nonsingularity implies L_2.*

Proof. Let $\gamma \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$. If $\beta \in \Phi^+ \backslash \Phi(\mathfrak{m})^+$, then either $\beta \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$ or $\beta \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$. In either case, $\beta + \gamma$ is not a root, so $(\beta, \gamma) \geqslant 0$. Now $(\rho_M, \gamma) \leqslant (\rho, \gamma)$ and (10.3) implies (10.1). Q.E.D.

10.7. LEMMA. *Nonsingularity implies highest K-type.*

Proof. If $\alpha \in \Phi(\mathfrak{k})$, then $(\rho_K, \alpha) = (\rho, \alpha)$ because $[\mathfrak{k}, \mathfrak{k}]$ acts with trace 0 on $\mathfrak{p}_+$. Q.E.D.

10.8. LEMMA. *Highest K-type implies vanishing.*

Proof. Any L-type occurring on $V \otimes S(\mathfrak{p} \cap \mathfrak{q}_-)$ has form $\nu = \chi - \sum n_i \beta_i$ with $\beta_i \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$. If $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$, then $\alpha + \beta_i$ is not a root, so $(\alpha, \beta_i) \geqslant 0$ and $(\nu + \rho_K, \alpha) \leqslant (\chi + \rho_K, \alpha) < 0$. Q.E.D.

We start the proof that L_2 implies vanishing. Without loss of generality we may, and do, assume that

$$G \text{ is simple.} \tag{10.9}$$

Then there is exactly one simple root $\alpha_0 \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and exactly one simple root $\beta_0 \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$. Each has multiplicity one in the maximal root. Here $\Phi(\mathfrak{q}_+)$, resp. $\Phi(\mathfrak{p}_+)$, consists of all roots whose expressions as a linear combination of simple roots carries α_0, resp. β_0, with coefficient 1. In particular, the highest root $\gamma_1 \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ connects to any $\gamma \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ by a chain of roots whose successive differences belong to $\Phi(\mathfrak{l})^+$. Thus $\mathfrak{p} \cap \mathfrak{q}_+$ is L-irreducible, i.e.,

$$M/L \text{ is an irreducible Hermitian symmetric space.} \tag{10.10}$$

Only A_n, D_n, and E_6 have more than one simple root of multiplicity 1 in the highest root, and they have only one root length. Thus we may renormalize the inner product on $\mathfrak{t}^*$ and assume

$$\|\alpha\|^2 = 2 \qquad \text{for all} \quad \alpha \in \Phi. \tag{10.11}$$

In view of (10.10) we denote maximal strongly orthogonal subsets of $\Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ by

$$\begin{aligned} \gamma_1, \dots, \gamma_r&: \quad \text{cascade down from the maximal root } \gamma_1, \\ \tilde{\gamma}_1, \dots, \tilde{\gamma}_r&: \quad \text{cascade up from the } \mathfrak{m}\text{-simple root } \tilde{\gamma}_1. \end{aligned} \tag{10.12}$$

We noted (4.24) that the opposition element $w_1 \in W_L$ sends γ_i to $\tilde{\gamma}_i$ for $1 \leqslant i \leqslant r$. Similarly,

$$\alpha_1 = w_1(\alpha_0) \qquad \text{and} \qquad \beta_1 = w_1(\beta_0) \tag{10.13}$$

are the respective highest roots in $\Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and $\Phi(\mathfrak{p}_+ \cap \mathfrak{h})$. Also note that the orthogonal projections

$$p: \mathfrak{t}^* \to \operatorname{span}\{\gamma_i\} \quad \text{and} \quad \tilde{p}: \mathfrak{t}^* \to \operatorname{span}\{\tilde{\gamma}_i\} \tag{10.14}$$

satisfy $w_1 \circ p = \tilde{p} \circ w_1$.

The following lemma is analogous to the Restricted Root Theorem for Hermitian symmetric spaces.

10.15. LEMMA. *Every root* $\alpha \in \Phi^+$ *satisfies one of the conditions*

(i) $\tilde{p}\alpha = 0$, *in which case* $\alpha \in \Phi^+(\mathfrak{l}) \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_+) \cup \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$;

(ii) $\tilde{p}\alpha = -\frac{1}{2}\tilde{\gamma}_i$ *for some i, in which case* $\alpha \in \Phi^+(\mathfrak{l})$;

(iii) $\tilde{p}\alpha = \frac{1}{2}\tilde{\gamma}_i$ *for some i, in which case* $\alpha \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+) \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_+) \cup \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$;

(iv) $\tilde{p}\alpha = \frac{1}{2}(\tilde{\gamma}_i - \tilde{\gamma}_j)$ *with* $i > j$, *in which case* $\alpha \in \Phi^+(\mathfrak{l})$;

(v) $\tilde{p}\alpha = \frac{1}{2}(\tilde{\gamma}_i + \tilde{\gamma}_j)$ *with* $i \neq j$, *in which case* $\alpha \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$;

(vi) $\tilde{p}\alpha = \tilde{\gamma}_i$ *for some i, in which case* $\alpha = \tilde{\gamma}_i \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$.

Proof. Since all roots have length 2, $(\alpha, \tilde{\gamma}_i)$ is 0 or ± 1 or ± 2, and $(\alpha, \tilde{\gamma}_i) = \pm 2$ if and only if $\alpha = \tilde{\gamma}_i$.

By maximality of $\{\tilde{\gamma}_1, ..., \tilde{\gamma}_r\}$, $\tilde{p}\alpha = 0$ implies $\alpha \notin \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$.

Now $\mathrm{ad}(\mathfrak{p} \cap \mathfrak{q}_-)^2$ vanishes on $\mathfrak{k} + \mathfrak{h} + (\mathfrak{p} \cap \mathfrak{q}_-)$ and $\mathrm{ad}(\mathfrak{p} \cap \mathfrak{q}_+)$ vanishes on $\mathfrak{p} \cap \mathfrak{q}_+$, so $(\alpha, \tilde{\gamma}_l) > 0$ for at most two of the $\tilde{\gamma}_l$, and that can only happen if $\alpha \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$. If $(\alpha, \tilde{\gamma}_i) < 0$, then $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{h})$.

To complete the proof we need only verify that if $\alpha \in \Phi(\mathfrak{l})$ with $\tilde{p}\alpha = \frac{1}{2}\tilde{\gamma}_i$ or $\frac{1}{2}(\tilde{\gamma}_i - \tilde{\gamma}_j)$, where $i < j$, then $-\alpha \in \Phi^+$. To do that, note $\tilde{\gamma}_i - \alpha \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ and $(\tilde{\gamma}_i - \alpha) \perp \tilde{\gamma}_l$ for $l < i$, so $\tilde{\gamma}_i - \alpha$ is higher than $\tilde{\gamma}_i$ in our root order.

Q.E.D.

From the Lemma, if A, A' are sums of roots in $\Phi^+(\mathfrak{l})$ with $\tilde{p}A = -\tilde{p}A'$, then $\tilde{p}A = \tilde{p}A' = 0$ because A, A' each has inner product $\leqslant 0$ with $\sum_i (r + 1 - i)\, \tilde{\gamma}_i$, and that inner product vanishes only when $\tilde{p}A = \tilde{p}A' = 0$. Apply w_1. Then $pA = -pA'$ just when $pA = pA' = 0$.

10.16. LEMMA. *We have* $\tilde{p}\alpha_0 = \tilde{p}\beta_0 = \frac{1}{2}\tilde{\gamma}_1$ *and* $p\alpha_1 = p\beta_1 = \frac{1}{2}\gamma_1$.

Proof. Here $\tilde{\gamma}_1$ is the lowest root in $\mathfrak{p} \cap \mathfrak{q}_+$, so necessarily $\tilde{\gamma}_1 = \alpha_0 + \beta_0 + \sum \delta_i$, where δ_i are the simple roots between α_0 and β_0 in the Dynkin diagram. In particular,

$$\tilde{\gamma}_1 - \alpha_0 \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h}) \qquad \text{and} \qquad \tilde{\gamma}_1 - \beta_0 \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+).$$

Thus $(\alpha_0, \tilde{\gamma}_1) = 1 = (\beta_0, \tilde{\gamma}_1)$, so $\tilde{p}\alpha_0 = \frac{1}{2}\tilde{\gamma}_1 = \tilde{p}\beta_0$. Apply w_1 to get the second assertion.

Q.E.D.

10.17. LEMMA. *Suppose that there is a root* $\delta \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ *such that* $(\rho_K, \delta) \leqslant (\rho_M, \gamma_1)$, $\tilde{p}\delta = \tilde{p}\alpha_1$, *and, in case H has a compact simple factor, also* $p\delta = p\alpha_1$. *Then* L_2 *implies vanishing.*

Proof. Let $\nu \in \hat{L}$ occur in $V \otimes S(\mathfrak{p} \cap \mathfrak{q}_-)$ and let $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ with

$(\nu+\rho_K, \alpha_1)>0$. Since ν is L-dominant, we then have $(\nu+\rho_K, \alpha_1)>0$. Express $\nu=\tau-\tilde{n\gamma}$, where τ is a weight of ν and $\tilde{n\gamma}=n_1\tilde{\gamma}_1+\cdots+n_r\tilde{\gamma}_r$, $n_1\geqslant\cdots\geqslant n_r\geqslant 0$, so that $-\tilde{n\gamma}$ is a highest weight of L on $S(\mathfrak{p}\cap\mathfrak{q}_-)$.

Let α denote an arbitrary root between δ and α_1, i.e., any root in $\Phi(\mathfrak{k}\cap\mathfrak{q}_+)$ such that $\alpha-\delta$ is a sum of (0 or more) roots from $\Phi(\mathfrak{l})^+$. As α_1 is maximal in $\Phi(\mathfrak{k}\cap\mathfrak{q}_+)$, $\alpha_1-\alpha$ also is such a sum. The remark just after Lemma 10.15, with $A=\alpha-\delta$ and $A'=\alpha_1-\alpha$, shows $\tilde{p}\delta=\tilde{p}\alpha=\tilde{p}\alpha_1$. Thus $(\tilde{n\gamma},\alpha)=(\tilde{n\gamma},\alpha_1)$.

If H has a compact simple factor, then $p\delta=p\alpha=p\alpha_1=\frac{1}{2}\gamma_1$ as just above, using Lemma 10.16. Then $\gamma_1-\alpha, \gamma_1-\alpha_1\in\Phi(\mathfrak{p}_+\cap\mathfrak{h})$. So $(\tau,\gamma_1-\alpha)=0=(\tau,\gamma_1-\alpha_1)$. This uses $\dim V<\infty$ in a crucial manner. Also, $\chi-\tau$ is a sum from $\Phi(\mathfrak{l})^+$ and $(\chi-\tau,\gamma_1)\geqslant 0$. We conclude

$$(\tau,\alpha)=(\tau,\alpha_1)=(\tau,\gamma_1)\leqslant(\chi,\gamma_1). \tag{10.18}$$

The statement (10.18) remains true, with $(\tau,\gamma_1)=(\chi,\gamma_1)$, if H has no compact simple factor. For then $\dim V=1$ and $\tau=\chi$, which necessarily is orthogonal to every simple root except α_0. In any case, (10.18) combines with $(\tilde{n\gamma},\alpha)=(\tilde{n\gamma},\alpha_1)$ to give

$$(\nu,\delta)=(\nu,\alpha_1)=(\nu,\alpha) \qquad \text{for all possibilities of } \alpha \tag{10.19}$$

because δ is one of the possibilities for α. Also, use (10.18) with $\alpha=\delta$, and our assumptions on δ, for

$$\begin{aligned}(\chi+\rho_M,\gamma_1)&\geqslant(\tau,\delta)+(\rho_M,\gamma_1)\geqslant(\tau,\delta)+(\rho_K,\delta)\\ &=(\nu,\delta)+(\rho_K,\delta).\end{aligned}$$

Glancing back to (10.1), now

$$L_2 \qquad \text{implies} \qquad (\nu+\rho_K,\delta)<0. \tag{10.20}$$

As α ranges from δ to α_1, $(\nu+\rho_K,\alpha)$ ranges from negative to positive. By (10.19), (ν,α) is constant as α varies in that range, and by (10.11), (ρ_K,α) varies over all integers from (ρ_K,δ) to (ρ_K,α_1). Thus $(\nu+\rho_K,\alpha)=0$ for some α between δ and α_1. Q.E.D.

One can prove that the hypothesis of Lemma 10.17 is valid, and thus that L_2 implies vanishing, without classification theory ((10.11) can be done without classification). We give a proof using classification because it is shorter and it yields the additional information of Theorem 10.5.

The height $h(\alpha)$ of a root $\alpha\in\Phi$ is the sum of its coefficients when expressed as a linear combination of simple roots. By (10.11), $h(\alpha)=(\rho,\alpha)$. The analogous height function h_K on $\Phi(\mathfrak{k})$ satisfies

$$(\rho_K,\alpha)=h_K(\alpha)=h(\alpha)=(\rho,\alpha) \qquad \text{for} \quad \alpha\in\Phi(\mathfrak{k})$$

because $\Phi(\mathfrak{k})^+$-simple roots are Φ^+-simple. The $\Phi(\mathfrak{m})^+$-simple roots are Φ^+-simple except for $\tilde{\gamma}_1$, so there we have

$$(\rho_M, \alpha) = h_M(\alpha) = h(\alpha) - h(\tilde{\gamma}_1) + 1 \qquad \text{for} \quad \alpha \in \Phi(\mathfrak{m}).$$

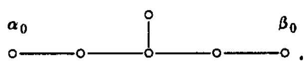
Since $\tilde{\gamma}_1 = \alpha_0 + \beta_0 +$ (sum of all simple roots between α_0 and β_0),

$$h(\tilde{\gamma}_1) = 2 + (\text{number of simple roots between } \alpha_0 \text{ and } \beta_0).$$

Finally, $h(\gamma_1)$ and $h(\alpha_1)$ are visible from the Dynkin diagram of Φ^+ and its subdiagrams obtained by erasing β_0 and those simple roots that α_0 separates from β_0, for γ_1 and α_1 are the highest roots of the resulting systems.

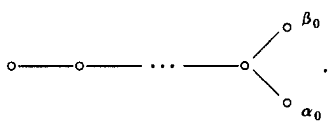
We run through the inequivalent connected Dynkin diagrams with two distinct simple roots α_0, β_0 of multiplicity 1 in the highest root.

10.21. *Case* E_6.



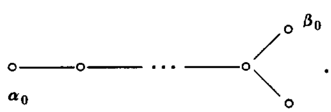
Here $h(\gamma_1) = 11$, $h(\tilde{\gamma}_1) = 5$, and (system D_5) $h(\alpha_1) = 7$. Thus $(\rho_M, \gamma_1) = h_M(\gamma_1) = h(\gamma_1) - h(\tilde{\gamma}_1) + 1 = 7 = h(\alpha_1) = (\rho_K, \alpha_1)$, so the hypothesis of Lemma 10.17 is satisfied with $\delta = \alpha_1$.

10.22. *Case* $D_n, n \geqslant 4$:



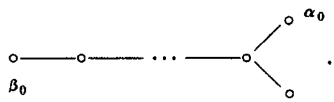
Here $h(\gamma_1) = 2n - 3$, $h(\tilde{\gamma}_1) = 3$, and (system A_{n-1}) $h(\alpha_1) = n - 1$. Thus $(\rho_M, \gamma_1) = h_M(\gamma_1) = h(\gamma_1) - h(\tilde{\gamma}_1) + 1 = 2n - 5 \geqslant n - 1 = (\rho_K, \alpha_1)$, so the hypothesis of Lemma 10.17 is satisfied with $\delta = \alpha_1$.

10.23. *Case* $D_n, n \geqslant 5$:



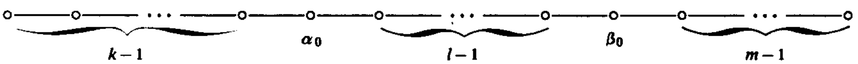
Here $h(\gamma_1) = 2n - 3$, $h(\tilde{\gamma}_1) = n - 1$, and $h(\alpha_1) = n - 1$, so $(\rho_M, \gamma_1) = n - 1 = (\rho_K, \alpha_1)$. Thus the hypothesis of Lemma 10.17 is satisfied with $\delta = \alpha_1$.

10.24. *Case* $D_n n \geqslant 5$:



Label the simple roots $x_1 - x_2 = \beta_0$, $x_2 - x_3, \ldots, x_{n-1} - x_n, x_{n-1} + x_n = \alpha_0$. Then $\tilde{\gamma}_1 = x_1 + x_n$ with $h(\tilde{\gamma}_1) = n - 1$, and we define $\delta = x_{n-2} + x_{n-1}$ which belongs to $\Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and has $h(\delta) = 3$. Then $(\rho_K, \delta) = (\rho, \delta) = 3 < n - 1 = h(\gamma_1) - h(\tilde{\gamma}_1) + 1 = h_M(\gamma_1) = (\rho_M, \gamma_1)$. Here $r = 1$ and $\delta \perp \tilde{\gamma}_1$ so $\tilde{p}\delta = 0$, and $\alpha_0 \perp \gamma_1$ implies $\alpha_1 \perp \tilde{\gamma}_1$ so $\tilde{p}\alpha_1 = 0$. Since H has no noncompact simple factor here, δ satisfies the hypothesis of Lemma 10.17.

10.25. *Case* A_n, $n \geqslant 2$:



Here k, l, $m \geqslant 1$ are integers with $n = k + l + m - 1$. Label the simple roots $x_1 - x_2, \ldots, x_k - x_{k+1} = \alpha_0, \ldots, x_{k+l} - x_{k+l+1} = \beta_0, \ldots, x_{k+l+m-1} - x_{k+l+m}$. Then $\gamma_1 = x_1 - x_{k+l+m}$, $\gamma_2 = x_2 - x_{k+l+m-1}, \ldots, \gamma_r = x_r - x_{k+l+m+1-r}$, where $r = \min(k, m)$. Similarly, $\tilde{\gamma}_1 = x_k - x_{k+l+1}$, $\tilde{\gamma}_2 = x_{k-1} - x_{k+l+2}, \ldots, \tilde{\gamma}_r = x_{k+1-r} - x_{k+l+r}$. Also, $\alpha_1 = x_1 - x_{k+l}$. If $m \geqslant l$, then

$$\begin{aligned}(\rho_M, \gamma_1) &= (k + l + m - 1) - (l + 1) + 1 = k + m - 1 \\ &\geqslant k + l - 1 = (\rho, \alpha_1) = (\rho_K, \alpha_1)\end{aligned}$$

so the hypothesis of Lemma 10.17 is satisfied with $\delta = \alpha_1$. Now suppose $m < l$ and let $\delta = x_1 - x_{k+1} \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$. Then $\alpha_1 - \delta = x_{k+1} - x_{k+l}$ is orthogonal to all γ_i and all $\tilde{\gamma}_i$ so $\tilde{p}\delta = \tilde{p}\alpha_1$ and $p\delta = p\alpha_1$, and

$$(\rho_M, \gamma_1) = k + m - 1 \geqslant k = h(\delta) = (\rho_K, \delta),$$

so δ satisfies the hypothesis of Lemma 10.17.

Proof of Theorem 10.5. Lemmas 10.6–10.8 give the easy implications. For the fact that L_2 implies vanishing, combine Lemma 10.17 with the case by case verification of its hypotheses. In that verification, we were able to take $\delta = \alpha_1$ except in the cases

$$\begin{aligned}&G/H = SO(2n-2, 2)/U(n-1, 1) &&\text{with} \quad n \geqslant 5, \\ &G/H = SU(k+l, m)/S(U(k) \times U(l, m)) &&\text{with} \quad 0 < m < l.\end{aligned}$$

If $\delta = \alpha_1$ satisfies the hypothesis of Lemma 10.17, then

$$\begin{aligned}(\chi + \rho_M, \gamma_1) &\geqslant (\chi, \gamma_1) + (\rho_K, \alpha_1) \\ &\geqslant (\chi, \alpha_1) + (\rho_K, \alpha_1) \qquad \text{using (10.18)},\end{aligned}$$

which is $\geqslant (\chi + \rho_K, \alpha)$ for all $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$. In that case, the L_2 condition implies the highest K-type condition. Q.E.D.

The highest K-type condition implies, of course, that $H^s(G/H, \mathbf{V}) \neq 0$. It can happen that the L_2 condition holds but the highest K-type condition fails, and we look at that now.

Let $\pi: G/H \to K/L$ be holomorphic with G simple and $\dim V < \infty$. Suppose that (10.1) holds but (10.2) fails. Then $(\chi + \rho_K, \alpha_1) \geqslant 0 > (\chi + \rho_M, \gamma_1)$. There are two cases.

Case 1: $\tilde{p}\alpha_1 = \frac{1}{2}\tilde{\gamma}_r$. If n is a large integer, then the L-type $\chi - n\tilde{\gamma} = \chi - n(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_r)$ in $V \otimes S(\mathfrak{p} \cap \mathfrak{q}_-)$ satisfies $(\chi - n\tilde{\gamma} + \rho_K, \alpha_1) = (\chi + \rho_K, \alpha_1) - n < 0$, so $H^s(K/L, \mathbf{V} \otimes \mathbf{U}_{-n\tilde{\gamma}}) \neq 0$ and thus $H^s(G/H, \mathbf{V}) \neq 0$.

Case 2: $\tilde{p}\alpha_1 = 0$. If $\mathbf{V} \to G/H$ is a line bundle, then every L-type $\nu = \chi - \tilde{n}\tilde{\gamma}$ on $V \otimes S(\mathfrak{p} \cap \mathfrak{q}_-)$ satisfies $(\nu + \rho_K, \alpha_1) = (\chi - \rho_K, \alpha_1) \geqslant 0$, so $H^s(K/L, \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}) = 0$, and thus $H^s(G/H, \mathbf{V}) = 0$. Since L_2 implies vanishing, here in fact $H^p(G/H, \mathbf{V}) = 0$ for all p. The situation where $1 < \dim V < \infty$ occurs only for $SU(k+l, m)/S(U(k) \times U(l, m))$ and will be treated in Sections 11 and 13.

Case 1 occurs, for example, with $G/H = SU(k+l, m)/S(U(k) \times U(l, m))$, $k \leqslant m < l$. Case 2 occurs for $G/H = SO(2n-2, 2)/U(n-1, 1)$ with $n \geqslant 5$ and for $G/H = SU(k+l, m)/S(U(k) \times U(l, m))$ with $m < \min(k, l)$.

If V is infinite dimensional then, according to Lemma 7.27, we may as well assume that it has a highest L-type χ in the sense of (4.30). That done,

(i) (10.1) is the correct formulation of the L_2 condition,

(ii) (10.2) is the correct formulation of the highest K-type condition,

and

(iii) (10.3) is the correct formulation of the nonsingularity condition.

The vanishing condition (10.4) needs revision to take into account the cancellation mentioned in the last paragraph of Section 4.

Nonsingularity implies L_2 by the argument of Lemma 10.6. Nonsingularity also implies highest K-type by the argument of Lemma 10.7. In effect, for these one need only consider the highest L-type χ of ψ.

As mentioned above, the vanishing condition (10.4) is too restrictive for infinite-dimensional V. The argument of Lemma 10.8, that highest K-type implies vanishing in the sense of (10.4), fails for $\dim V = \infty$ because we must replace χ by $\chi - \sum n_k \beta_k$, $n_k \geqslant 0$ and $\beta_k \in \Phi(\mathfrak{p}_+ \cap \mathfrak{h})$, and $(\beta_k, \alpha) \geqslant 0$ there. The argument of Lemma 10.17, that L_2 implies vanishing in the sense of (10.4), fails for $\dim V = \infty$ because $\dim V < \infty$ is necessary for the proof of (10.18) there. Finally, the argument that L_2 usually implies highest K-type, also fails for $\dim V = \infty$ because it makes use of (10.18).

11. Irreducibility and Characters of the Unitary Representations

In this section we prove that the representation π_V of Sections 8 and 9 are irreducible. Specifically, we will show

11.1. Theorem. *Let $\pi: G/H \to K/L$ be holomorphic. Suppose that $\dim V < \infty$, $\chi = \psi|_L$, and V satisfies the L_2 condition. Then $H^s(G/H, \mathbf{V})_K$ is an irreducible Harish-Chandra module for $(\mathfrak{g}_0, K)$. In particular, the unitary representations π_V of Theorems 8.20 and 9.14 are irreducible with infinitesimal character of Harish-Chandra parameter $\chi + \rho$ and with distribution and K-characters (5.29) and (5.30).*

The character statement follows from Corollary 5.27 because L_2 implies the vanishing condition. The approach to irreducibility is based on

11.2. Lemma. *Let X be a T-finite $(\mathfrak{g}_0, K)$-module with weights bounded from above, and suppose that X has infinitesimal character of Harish-Chandra parameter λ. Let S denote the set of highest weights of K-irreducible constituents of X. Suppose that there is just one element $\mu \in S$ such that $\|\mu + \rho\| = \|\lambda\|$, and suppose that the corresponding K-type has multiplicity 1 in X. Then X is irreducible.*

Proof. Suppose that X reduces. Then we have two composition factors X_1, X_2. They are irreducible and T-finite with weights bounded from above, so they have highest weights μ_1, μ_2, and thus they are quotients of the corresponding Verma modules. Since X_i inherits the infinitesimal character of Harish-Chandra parameter λ from X, now $\mu_i + \rho = w_i(\lambda)$ for some w_1, $w_2 \in W_{\mathfrak{g}}$; in particular, $\|\mu_i + \rho\| = \|\lambda\|$, contradicting our hypothesis that μ be unique and of multiplicity 1. Q.E.D.

Theorem 4.5 tells us that $H^s(G/H, \mathbf{V})_K$ is the sum of the $H^s(K/L, \mathbf{U}_\chi \otimes \mathbf{U}_{-\tilde{n}\gamma})$, where $\chi = \psi|_L$ is identified with its highest weight. As in (4.20) let $w_0 \in W_K$ be the element that interchanges $\Phi(\mathfrak{l})^+ \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_-)$ and $\Phi(\mathfrak{l})^+ \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$. The highest weights of K on $H^s(K/L, \mathbf{U}_\chi \otimes \mathbf{U}_{-\tilde{n}\gamma})$ are the elements of

$$S(\chi, \tilde{n}) = \{w_0(\nu + \rho_K) - \rho_K : \nu \in \hat{L} \text{ occurs in } U_\chi \otimes U_{-\tilde{n}\gamma} \\ \text{and } (\nu + \rho_K, \alpha) < 0 \text{ for all } \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)\}. \tag{11.3}$$

Here note that $w_0(\nu + \rho_K) - \rho_K = w_0(\nu + \rho) - \rho$ because $\rho - \rho_K$ is orthogonal to every compact root.

According to Theorem 5.23, $H^s(G/H, \mathbf{V})_K$ is a T-finite $(\mathfrak{g}_0, K)$-module with all weights bounded from above and with infinitesimal character χ_λ, $\lambda = \chi + \rho$. Thus Lemma 11.2 immediately specializes to

11.4. LEMMA. *Let S be the union of the sets $S(\chi, \tilde{n})$ of* (11.3). *Suppose that there is just one element $\mu \in S$ such that $\|\mu + \rho\| = \|\chi + \rho\|$, and that the corresponding K-type has multiplicity* 1 *in $H^s(G/H, \mathbf{V})_K$. Then $H^s(G/H, \mathbf{V})_K$ is $(\mathfrak{g}_0, K)$-irreducible.*

Now the case of nonsingular infinitesimal character is easy.

11.5. LEMMA. *Under the nonsingularity condition, $H^s(G/H, \mathbf{V})_K$ is $(\mathfrak{g}_0, K)$-irreducible.*

Proof. Let $\nu \in \hat{L}$ occur in $U_\chi \otimes U_{-\tilde{n}\gamma}$. Then ν is of the form $\chi - B$, where B is a sum of $|\tilde{n}|$ roots in $\Phi(\mathfrak{p} \cap \mathfrak{q}_+)$. By nonsingularity, $(\chi + \rho, B) \leqslant 0$ with equality only when $B = 0$, i.e., when $|\tilde{n}| = 0$. The K-type $\mu \in S(\chi, \tilde{n})$ corresponding to ν is $\mu = w_0(\nu + \rho) - \rho = w_0(\chi + \rho - B) - \rho$, so $\|\mu + \rho\|^2 = \|\chi + \rho - B\|^2 = \|\chi + \rho\|^2 - 2(\chi + \rho, B) + \|B\|^2 \geqslant \|\chi + \rho\|^2$ with equality only when $|\tilde{n}| = 0$. Of course, $|\tilde{n}| = 0$ gives $\nu = \chi$, which cannot occur in $U_\chi \otimes U_{-\tilde{n}\gamma}$ for $|\tilde{n}| \neq 0$ by its value on the central element of $\mathfrak{h}$. Lemma 11.3 applies. Q.E.D.

When nonsingularity fails, we must be more precise about the $S(\chi, \tilde{n})$. Let χ^* denote the highest weight of U_χ^*. Parthasarathy, Ranga-Rao, and Varadarajan proved [24, Corollary 2, p. 394] that $-\chi^* - \tilde{n}\gamma$ is a weight of every $\nu \in \hat{L}$ that occurs in $U_\chi \otimes U_{-\tilde{n}\gamma}$. Let us denote

$$\nu_0(\tilde{n}) = w(-\chi^* - \tilde{n}\gamma) \quad \text{where} \quad w \in W_L \quad \text{with} \quad \nu_0(\tilde{n}) \quad L\text{-dominant.} \tag{11.6}$$

Then, as an immediate consequence, $\|\nu_0(\tilde{n})\|$ minimizes $\|\nu\|$ and $\|\nu_0(\tilde{n}) + \rho_L\|$ minimizes $\|\nu + \rho_L\|$ as ν ranges over the classes in $\hat{L}$ that occur in $U_\chi \otimes U_{-\tilde{n}\gamma}$, and if ν does occur, then $\nu - \nu_0(\tilde{n})$ is a sum of positive roots of $\mathfrak{l}$. From the last, $(\nu - \nu_0(\tilde{n}), \rho - \rho_L) = 0$, so $\nu \neq \nu_0$ gives

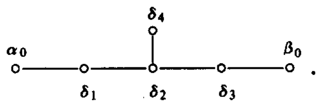
$$\begin{aligned}\|\nu + \rho\|^2 &= \|\nu + \rho_L\|^2 + 2(\nu + \rho_L, \rho - \rho_L) + \|\rho - \rho_L\|^2\\ &\geqslant \|\nu_0(\tilde{n}) + \rho_L\|^2 + 2(\nu_0(\tilde{n}) + \rho_L, \rho - \rho_L) + \|\rho - \rho_L\|^2\\ &> \|\nu_0(\tilde{n}) + \rho_L\|^2.\end{aligned}$$

Also, if $H^s(K/L, \mathbf{U}_{\nu_0(\tilde{n})}) = 0$, so $(\nu_0(\tilde{n}) + \rho_K, \alpha) \geqslant 0$ for some $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and in particular for the highest element $\alpha = \alpha_1$ there, we use $(\delta, \alpha_1) \geqslant 0$ for all $\delta \in \Phi(\mathfrak{l})^+$ to conclude $(\nu + \rho_K, \alpha_1) \geqslant 0$, so $H^s(K/L, \mathbf{U}_\nu) = 0$. Now we can sharpen Lemma 11.4 as follows:

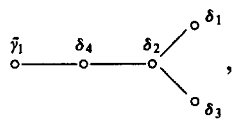
11.7. LEMMA. *If there is only one choice of $\tilde{n}$ such that $\|\nu_0(\tilde{n}) + \rho\|^2 \leqslant \|\chi + \rho\|^2$ and $(\nu_0(\tilde{n}) + \rho, \alpha_1) < 0$, then $H^s(G/H, \mathbf{V})_K$ is $(\mathfrak{g}_0, K)$-irreducible. If there is no such choice, then $H^s(G/H, \mathbf{V})_K = 0$.*

Now we are forced to run through the classification. As in Section 10, we assume G simple, we write α_0 for the simple root in $\Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ and β_0 for the simple root in $\Phi(\mathfrak{p}_+ \cap \mathfrak{h})$, and we denote $\alpha_1 = w_1(\alpha_0)$ and $\beta_1 = w_1(\beta_0)$, where $w_1 \in W_L$ is the opposition element. Roots are normalized by $\|\alpha\|^2 = 2$.

11.8. *Case* E_6:



Here, the simple roots of $\mathfrak{m}$ are

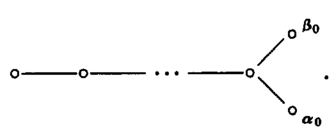


where $\tilde{\gamma}_1 = \alpha_0 + \delta_1 + \delta_2 + \delta_2 + \beta_0$. Also, $\tilde{\gamma}_2 = \tilde{\gamma}_1 + 2\delta_4 + 2\delta_2 + \delta_1 + \delta_3 = \gamma_1$ and χ is orthogonal to δ_1, δ_2, δ_3, δ_4, and β_0, so dim $V = 1$. Since our roots are normalized to $(\text{length})^2 = 2$, $(\rho_M, \gamma_1) = 7$, so our L_2 condition says $(\chi, \alpha_0) = (\chi, \gamma_1) < -7$, so

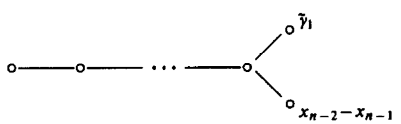
$$(\chi + \rho, \tilde{\gamma}_1) = (\chi, \tilde{\gamma}_1) + (\rho, \tilde{\gamma}_1) < -7 + 5 < 0,$$
$$(\chi + \rho, \tilde{\gamma}_2) = (\chi + \rho, \tilde{\gamma}_1) = -2 < 0.$$

Of course dim $V = 1$ gives $v_0(\tilde{n}) = \chi - \tilde{n}\tilde{\gamma}$, so now $\|v_0(\tilde{n}) + \rho\|^2 = \|\chi - \tilde{n}\tilde{\gamma} + \rho\|^2 \geqslant \|\chi + \rho\|^2$ with equality just when $\tilde{n} = (0, 0)$. Since L_2 implies highest K-type here, we also have $(\chi + \rho, \alpha_1) < 0$. Thus the conditions of Lemma 11.7 are satisfied.

11.9. *Case* D_n, $n \geqslant 4$:



We use the standard orthonormal basis $\{x_i\}$ of $\mathfrak{l}^*$ in which the simple roots are $x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n = \beta_0$, $x_{n-1} + x_n = \alpha_0$. Then the simple roots of $\mathfrak{m}$ are



and $\tilde{\gamma}_t = x_{n-2t} + x_{n-2t+1}$ for $1 \leqslant t \leqslant [n/2]$. Now χ has form

$-(k/2)(x_1 + \cdots + x_n)$, where k is an integer. Here $k \geqslant n$ because the L_2 condition implies the highest K-type condition $0 > (\chi + \rho_K, \alpha_1) = (\chi + \rho, \alpha_1) = (\chi + \rho, x_1 + x_n) = -k + n - 1$. Now

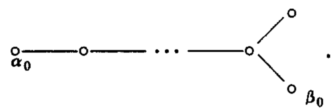
$$(\chi + \rho, \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_t) = \left(\chi + \rho, \sum_{n-2t}^{n-1} x_i\right) = -kt + \sum_{n-2t}^{n-1} (n - i)$$

$$= -kt + \sum_{1}^{2t} i = t(2t + 1 - k) \leqslant t$$

$$< 2t = \|\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_t\|^2.$$

Here $\dim V = 1$, so $\nu_0(\tilde{n}) = \chi - \tilde{n}\tilde{\gamma}$; so now $\|\nu_0(\tilde{n}) + \rho\|^2 = \|\chi - \tilde{n}\tilde{\gamma} + \rho\|^2 \geqslant \|\chi + \rho\|^2$ with equality just when $\tilde{n} = (0,..., 0)$. We have verified the conditions of Lemma 11.7.

11.10. *Case* $D_n, n \geqslant 5$:



Here $\alpha_0 = x_1 - x_2$ and $\beta_0 = x_{n-1} + x_n$, and $\chi = -kx_1$ for some integer k; M has real rank $r = 1$ and $\tilde{\gamma}_1 = x_1 + x_n$. Also, $\alpha_1 = x_1 - x_n$, $\gamma_1 = x_1 + x_2$, and the L_2 condition implies the highest K-type condition $0 > (\chi + \rho_K, \alpha_1) = (\chi + \rho, \alpha_1) = -k + n - 1$. Thus

$$(\chi + \rho, \tilde{\gamma}_1) = -k + n - 1 < 0.$$

Again, $\|\nu_0(\tilde{n}) + \rho\|^2 = \|\chi - \tilde{n}\tilde{\gamma} + \rho\|^2 \geqslant \|\chi + \rho\|^2$ with equality just when $\tilde{n} = (0)$. The conditions of Lemma 11.7 are verified.

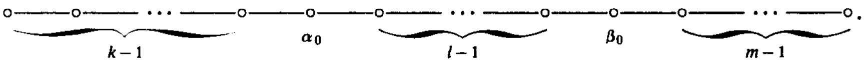
11.11. *Case* $D_n, n \geqslant 5$:

Here $\alpha_0 = x_{n-1} + x_n$, $\beta_0 = x_1 - x_2$, $\alpha_1 = x_2 + x_3$, and $\chi = -(k/2)(x_1 + \cdots + x_n)$ for some integer k; M is the same as in Case 11.10. The L_2 condition $0 > (\chi + \rho_M, \gamma_1) = -k + n - 1$ says $k \geqslant n$, and the highest K-type condition $0 > (\chi + \rho_K, \alpha_1) = (\chi + \rho, \alpha_1) = -k + 2n - 5$ says $k \geqslant 2n - 4$. Compute

$$(\chi + \rho, \tilde{\gamma}_1) = -k + n - 1 < 0$$

to see $\|\nu_0(\tilde{n}) + \rho\|^2 = \|\chi - \tilde{n}\tilde{\gamma} + \rho\|^2 \geqslant \|\chi + \rho\|^2$ with equality just when $\tilde{n} = (0)$. The conditions of Lemma 11.7 are verified, but $H^s(G/H, \mathbf{V})_K = 0$ in the range $n \leqslant k \leqslant 2n - 5$.

11.12. *Case* A_n, $n \geqslant 2$:



Here k, l, $m \geqslant 1$ are integers with $n = k + l + m - 1$, and we use the standard labelling $x_1 - x_2, \ldots, x_k - x_{k+1} = \alpha_0, \ldots, x_{k+l} - x_{k+l+1} = \beta_0, \ldots, x_{k+l+m-1} - x_{k+l+m}$ on the simple roots. By adding an appropriate multiple of $x_1 + \cdots + x_{n+1}$ to χ we may suppose

$$\chi = -(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k), \quad a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k \text{ integers.} \tag{11.13}$$

The L_2 condition says $0 > (\chi + \rho_M, \gamma_1) = -a_1 + k + m - 1$, i.e., $a_1 \geqslant k + m$. Also, $-\chi^* = -\sum_{i=1}^{k} a_{k+1-i} x_i$. Since $\tilde{\gamma}_t = x_{k+1-t} - x_{k+l+t}$, $1 \leqslant t \leqslant r = \min(k, m)$, that gives

$$v_0(\tilde{n}) = w\left(-\sum_{i=1}^{k} a_{k+1-i} x_i - \sum_{i=1}^{r} n_i (x_{k+1-i} - x_{k+l+i})\right), \tag{11.14}$$

where w, as a permutation of $\{x_1, \ldots, x_n\}$, only moves $\{x_1, \ldots, x_k\}$. Among all such permutations, w is the one that makes $-\chi^* - \tilde{n}\tilde{\gamma}$ dominant, i.e., that maximizes $\|w(-\chi^* - \tilde{n}\tilde{\gamma}) + \rho\|$. Notice that, since α_1 is L-dominant, this choice of w also maximizes $(w(-\chi^* - \tilde{n}\tilde{\gamma}) + \rho, \alpha_1)$, which thus is negative for the correct w if and only if it is negative for all permutations of $\{x_1, \ldots, x_k\}$ in place of w. Those permutations act on $\alpha_1 = x_1 - x_k + l$ sending it to any $x_i - x_{k+l}$, $1 \leqslant i \leqslant k$. Thus, using (11.14), we have

$$\begin{aligned} (v_0(\tilde{n}) + \rho, \alpha_1) < 0 &\Leftrightarrow -(a_i + n_i) < -(k + l - 1), && \text{for} \quad 1 \leqslant i \leqslant r, \\ &\Leftrightarrow -a_i < -(k + l - 1), && \text{for} \quad r + 1 \leqslant i \leqslant k. \end{aligned} \tag{11.15}$$

Now we have

$$H^s(G/H, \mathbf{V})_K \neq 0 \quad \Leftrightarrow \quad a_i \geqslant k + l \text{ for } r + 1 \leqslant i \leqslant k \tag{11.16}$$

and in that case the highest K-type is given by $v_0(\tilde{n})$, where $n_i = \max(k + l - a_i, 0)$ for $1 \leqslant i \leqslant r$. In computing $\|v_0(\tilde{n}) + \rho\|^2 - \|\chi + \rho\|^2$ we may add a multiple of $x_1 + \cdots + x_n$ to ρ, thus replacing ρ by $-\sum_{i=1}^{n} i x_i$. Now

$$\begin{aligned} &\|v_0(\tilde{n}) + \rho\|^2 - \|\chi + \rho\|^2 \\ &\quad = \|-\chi^* - \tilde{n}\tilde{\gamma} + w^{-1}\rho\|^2 - \|\chi + \rho\|^2 \end{aligned}$$

$$\geqslant \left\| -\sum_{i=1}^{k} a_{k+1-i} x_i - \sum_{i=1}^{r} n_i (x_{k+1-i} - x_{k+l+i}) - \sum_{i=1}^{k} (k+1-i)\, x_i - \sum_{i=k+1}^{n} i x_i \right\|^2$$

$$- \left\| -\sum_{i=1}^{k} (a_i + i)\, x_i - \sum_{i=k+1}^{n} i x_i \right\|^2$$

$$= \sum_{i=1}^{r} (a_i + n_i + i)^2 + \sum_{i=r+1}^{k} (a_i + i)^2 + \sum_{i=1}^{r} (k+l+i-n_i)^2 - \sum_{i=1}^{k} (a_i + i)^2 - \sum_{i=1}^{r} (k+l+i)^2$$

$$= 2 \sum_{i=1}^{r} n_i (n_i + a_i - k - l).$$

Under the condition $(\nu_0(\tilde{n}) + \rho, \alpha_1) < 0$, which is $n_i \geqslant \max(k + l - a_i, 0)$ for $1 \leqslant i \leqslant r$, each $n_i(n_i + a_i - k - l) \geqslant 0$ with equality just when $n_i = \max(k + l - a_i, 0)$. Thus, $\| \nu_0(\tilde{n}) + \rho \|^2 \geqslant \| \chi + \rho \|^2$ with equality just when each $n_i = \max(k + l - a_i, 0)$. This verifies the conditions of Lemma 11.7.

The proof of irreducibility is complete. In each case we have seen just when $H^s(G/H, \mathbf{V}) \neq 0$ and we have found its highest K-type. Q.E.D.

12. Infinite-Dimensional Fibre

In this section we investigate the results of our construction in the case of an infinite-dimensional fibre V. The square-integrability criterion (7.24) essentially restricts our choice of V to a highest weight module, as noted in Lemma 7.27. At one extreme, we may let V be a holomorphic discrete series representation of H, at the other V may be one of the singular representations attached to an indefinite Kähler-symmetric space of H. The former produces a holomorphic discrete series representation of G, the latter a singular unitary representation, which is not quite as highly singular as those corresponding to finite-dimensional bundles over G/H. We shall also iterate our construction to obtain representations between these two extremes.

To be more specific, we consider a subgroup $H_1 \subset G$ with compactly embedded center, which is the centralizer of its own center. Replacing H_1 by one of its conjugates, if necessary, we make it θ-stable. The intersection

$$L_1 = H_1 \cap K \tag{12.1}$$

is then a maximal compactly embedded subgroup of H_1. We assume that the quotients G/H_1, G/K, and G/L_1 carry invariant complex structures which make the two fibrations

$$\begin{array}{ccc} & G/L_1 & \\ \swarrow & & \searrow \\ G/H_1 & & G/K \end{array} \tag{12.2}$$

holomorphic. In addition, we require the existence of a sequence of θ-stable intermediate subgroups

$$H_1 \subset H_2 \subset \cdots \subset H_{N-1} \subset H_N = G \tag{12.3}$$

such that G/H_i fibres holomorphically over G/H_{i+1}, for $1 \leqslant i \leqslant N-1$, with fibres H_{i+1}/H_i that are indefinite-Kähler symmetric in the sense of Section 2. These assumptions are hereditary, as we shall see shortly. If $N=2$, they merely reproduce the situation described in Section 2; for higher N, they will allow us to apply our construction inductively.

The homogeneous complex structure of G/H_1 corresponds to an Ad H_1-invariant splitting

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{q}_{1,+} \oplus \mathfrak{q}_{1,-}, \qquad \mathfrak{q}_{1,-} = \bar{\mathfrak{q}}_{1,+}, \tag{12.4}$$

such that $\mathfrak{q}_{1,+}$represents the holomorphic tangent space at $1 \cdot H_1$. Since H_1 is θ-stable and coincides with the centralizer in G of its own center, which is compactly embedded in G, L_1 contains a fundamental Cartan subgroup T of G; T is compactly embedded since G/K has a Hermitian symmetric structure. As usual, we denote the root system of $(\mathfrak{g}, \mathfrak{t})$ by Φ. The hypotheses concerning the fibrations (12.2) can be rephrased as follows: there exists a system of positive roots $\Phi^+ \subset \Phi$ so that

$$\Phi(\mathfrak{q}_{1,+}) \cup \Phi(\mathfrak{p}_+) \subset \Phi^+ \tag{12.5}$$

and

$$\Phi(\mathfrak{h}_1) \text{ is the sub-root system of } \Phi \text{ generated by a subset of the set of simple roots.} \tag{12.6}$$

In analogy to (12.4), $\mathfrak{g}$ splits Ad H_i-invariantly as

$$\mathfrak{g} = \mathfrak{h}_i \oplus \mathfrak{q}_{i,+} \oplus \mathfrak{q}_{i,-}, \qquad \mathfrak{q}_{i,-} = \bar{\mathfrak{q}}_{i,+}. \tag{12.7}$$

The fibrations $G/H_i \to G/H_{i+1}$ are holomorphic if and only if $\mathfrak{q}_{i,+} \supset \mathfrak{q}_{i+1,+}$, $1 \leqslant i \leqslant N-1$, or in other words, if and only if

$$\Phi(\mathfrak{q}_{1,+}) \supset \Phi(\mathfrak{q}_{2,+}) \supset \cdots \supset \Phi(\mathfrak{q}_{N-1,+}). \tag{12.8}$$

The H_i all contain the compactly embedded Cartan subgroup T, and thus have compactly embedded centers. Since $\mathfrak{h}_{i+1} \cap \mathfrak{q}_{i,+}$ represents the holomorphic tangent space of the fibre H_{i+1}/H_i, the final hypothesis on these quotients is equivalent to

$$\begin{aligned}&\text{every simple root for } \Phi(\mathfrak{h}_i) \cap \Phi^+ \text{ is simple also for } \Phi(\mathfrak{h}_{i+1}) \cap \\ &\Phi^+, \text{ and the sum of the coefficients of the remaining simple roots} \\ &\text{in any } \alpha \in \Phi(\mathfrak{h}_{i+1}) \text{ does not exceed one.}\end{aligned} \tag{12.9}$$

In particular, any two of the remaining simple roots belong to different simple factors of $\mathfrak{h}_{i+1}$.

If we replace G by H_{N-1} and $\mathfrak{q}_i$ by $\mathfrak{h}_{N-1} \cap \mathfrak{q}_i$, $1 \leqslant i \leqslant N-2$, the conditions (12.5)–(12.9) reproduce themselves: if the pair (G, H_1) satisfies the assumptions stated at the beginning of this section, then so does the pair (H_{N-1}, H_1).

We shall study the cohomology of a homogeneous holomorphic vector bundle $\mathbf{V}_1 \to G/H_1$, modeled on a finite-dimensional irreducible unitary H_1-module V_1 whose highest weight χ_1 satisfies the nonsingularity condition analogous to (10.3),

$$(\chi_1 + \rho, \alpha) < 0 \qquad \text{for all} \quad \alpha \in \Phi(\mathfrak{q}_{1,+}). \tag{12.10}$$

This condition can be relaxed slightly, but we shall not attempt to pin down the weakest possible hypotheses, as we did in Sections 9 and 10. The quotient K/L_1 can be identified with the K-orbit of the identity coset in G/H_1. It is a compact subvariety; we denote its complex dimension by s_1. Our next result describes the cohomology groups which will be unitarized later in this section.

12.11. THEOREM. *Assume that V_1 satisfies the negativity condition* (12.10). *Then $H^p(G/H_1, \mathbf{V}_1) = 0$ for $p \neq s_1$, whereas $H^{s_1}(G/H_1, \mathbf{V}_1)$ is a nonzero irreducible Fréchet G-module*[3] *with finite T-multiplicities, whose global character equals $(-1)^{s_1} \sum_v \varepsilon(v)\, \Theta(v(\chi_1 + \rho))$, cf.* (5.18) *and* (5.19). *In this summation v ranges over all those elements of $W(\mathfrak{h}_1, \mathfrak{t})$ which make $v(\chi_1 + \rho)$ dominant with respect to $\Phi^+(\mathfrak{l}_1)$. Let w_1 be the element of W_K which maps every $\alpha \in \Phi^+(\mathfrak{l}_1)$ to a positive root and reverses the sign of every $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_{1,+})$. All K-types occurring in $H^{s_1}(G/H_1, \mathbf{V}_1)$ have highest weights of the form $w_1(\chi_1 + \rho - B) - \rho$, with B equal to a (possibly empty) sum of roots in $\Phi(\mathfrak{p} \cap \mathfrak{q}_{1,+})$. The weight $w_1(\chi_1 + \rho) - \rho$ occurs exactly once and is the highest weight in $H^{s_1}(G/H_1, \mathbf{V}_1)$.*

Proof. We first use induction on the length N of the chain (12.3) to show

$$H^p(G/H_1, \mathbf{V}_1) = 0 \qquad \text{for} \quad p > s_1. \tag{12.12}$$

[3] Fréchet with respect to the natural C^∞ topology; cf. Section 5.

In order to relate our present situation to that of Sections 2–5, we set $H = H_{N-1}$, $\mathfrak{q} = \mathfrak{q}_{N-1}$. The fibration $G/H \to K/L$ is then holomorphic because of (12.5) and (12.8). As the weight of the one-dimensional H-module $\Lambda^d \mathfrak{q}_+$, $d = \dim \mathfrak{q}_+$, $2(\rho - \rho_H)$ is perpendicular to $\Phi(\mathfrak{h})$, so

$$(\chi_1 + \rho_H, \alpha) = (\chi_1 + \rho, \alpha) < 0 \qquad \text{for} \quad \alpha \in \Phi(\mathfrak{h} \cap \mathfrak{q}_{1,+}); \tag{12.13}$$

in other words, the nonsingularity condition (12.10) descends to H/H_1. By induction on N we may assume that $H^*(H/H_1, \mathbf{V}_1)$ satisfies the assertions of the theorem, with H taking the place of G. In particular,

$$H^p(H/H_1, \mathbf{V}_1) = 0 \qquad \text{for} \quad p \neq \tilde{s}, \tag{12.14}$$

with $\tilde{s} = \dim_{\mathbf{C}} L/L_1$. The holomorphic fibration

$$H/H_1 \to G/H_1 \to G/H \tag{12.15}$$

determines a Leray spectral sequence, abutting to $H^*(G/H_1, \mathbf{V}_1)$. In view of (12.14), it degenerates at the E_2-term. We let V denote the Fréchet H-module $H^{\tilde{s}}(H/H_1, \mathbf{V}_1)$; then

$$H^p(G/H_1, \mathbf{V}_1) = H^{p-\tilde{s}}(G/H, \mathbf{V}).$$

We now appeal to Theorem 3.34 to conclude

$$H^p(G/H_1, \mathbf{V}_1) = H^{p-\tilde{s}}(K/L, \mathbf{H}^0(M/L, \mathbf{V})). \tag{12.16}$$

This implies (12.12), since $s_1 = s + \tilde{s}$.

We now prove $H^p(G/H, \mathbf{V}_1) = 0$ for $p < s_1$ as follows: Both statement and proof of Proposition 5.4 carry over to the present situation if we let H_1, L_1, and χ_1 play the roles of, respectively, H, L, and χ. Whenever $\chi_1 + \rho$ is "very nonsingular" in the sense that $(\chi_1 + \rho, \alpha) \ll 0$ for $\alpha \in \Phi(\mathfrak{q}_{1,+})$, the resulting spectral sequence collapses at E_1 and gives the vanishing of cohomology below dimension s_1, as in Corollary 5.32. The final conclusion remains valid under the simple nonsingularity hypothesis (12.10); this follows from a tensoring argument entirely analogous to that in the proof of Proposition 5.34.

Once the cohomology groups $H^p(G/H_1, \mathbf{V}_1)$ are known to vanish for all $p \neq s_1$, we can proceed as in Proposition 5.34 to deduce the Fréchet property of $H^{s_1}(G/H_1, \mathbf{V}_1)$ and as in Theorem 5.23 to prove the character formula.

We recall the spectral sequence (4.36) that we used in Theorem 4.38. Every K-type of the G-module (12.24) must occur already at the E_1-level. In other words, in the notation of the filtrations (4.31) and (4.32), the space of K-finite vectors injects, noncanonically, into

$$\bigoplus_{l=0}^{\infty} H^{p-\tilde{s}}(K/L, \mathbf{H}^0(M/L, \mathbf{V}^l/\mathbf{V}^{l+1})_L). \tag{12.17}$$

There can be cancellation, but only of K-types which show up for more than one value of p. According to the Borel–Weil–Bott theorem, the irreducible summands of the K-module (12.7) have highest weights of the form

$$w(\mu+\rho_K)-\rho_K, \qquad w\in W_K, \quad l(w)=p-\tilde{s}, \tag{12.18}$$

with μ equal to the highest weight of an L-type occurring in $\oplus_{l=0}^{\infty} H^0(M/L, \mathbf{V}^l/\mathbf{V}^{l+1})$. Any such μ can be expressed as

$$\mu=\nu-B_1, \tag{2.19}$$

where B_1 stands for a sum of roots in $\Phi(\mathfrak{p}\cap\mathfrak{q}_{2,+})$ and ν has the highest weight of an L-type in V. By induction on N, there exists B_2, a sum of roots in $\Phi(\mathfrak{h}\cap\mathfrak{p}\cap\mathfrak{q}_{1,+})$, such that

$$\nu=v_1(\chi_1+\rho_H-B_2)-\rho_H; \tag{12.20}$$

$v_1\in W_L$ reverses the sign of every $\alpha\in\Phi(\mathfrak{k}\cap\mathfrak{q}_+)$ and preserves the sign of every $\alpha\in\Phi^+(\mathfrak{l})$. Since L normalizes $\mathfrak{q}_+$, $v_1(\rho-\rho_H)=\rho-\rho_H$; similarly, $w(\rho-\rho_K)=\rho-\rho_K$. Combining (12.18)–(12.20), we now find

$$w(\mu+\rho_K)-\rho_K=wv_1(\chi_1+\rho-B)-\rho, \tag{12.21}$$

with $B=B_2+v_1^{-1}B_1$, which is a sum of roots in $\Phi(\mathfrak{p}\cap\mathfrak{q}_+)$. If p is to equal $s_1=s+\tilde{s}$, $l(w)$ must equal s, which happens precisely when $wv_1=w_1$, as defined in the statement of the theorem: w maps the $\Phi^+(\mathfrak{l})$-dominant weight $\mu+\rho_K$ to another $\Phi^+(\mathfrak{l})$-dominant weight (in fact, to one that is $\Phi^+(\mathfrak{k})$-dominant), and hence does not reverse the sign of any $\alpha\in\Phi^+(\mathfrak{l})$. The weight (12.21) coincides with $w_1(\chi_1+\rho)-\rho$ only if $B=0$ and $wv=w_1$. For this weight, there can be no cancellation. It is higher than every other among the weights (12.21), and consequently higher than all other weights which occur in $H^{s_1}(G/H_1,\mathbf{V}_1)$.

Because of the character formula, the center $\mathscr{Z}(\mathfrak{g})$ of $\mathscr{U}(\mathfrak{g})$ acts on $H^{s_1}(G/H_1,\mathbf{V}_1)$ according to the infinitesimal character of Harish-Chandra parameter $\chi_1+\rho$. The irreducibility now follows readily from the criterion (10.2) and the properties of the K-structure that were just established: any nonzero sum B of roots in $\Phi(\mathfrak{p}\cap\mathfrak{q}_{1,+})$ has a strictly negative inner product with $\chi_1+\rho$ (cf. (12.10)), so

$$\begin{aligned}\|w_1(\chi_1+\rho-B)\|^2&=\|\chi_1+\rho-B\|^2\\&=\|\chi_1+\rho\|^2-2(\chi_1+\rho,B)+\|B\|^2>\|\chi_1+\rho\|^2.\end{aligned}$$

This completes the proof of the theorem. Q.E.D.

Our objective is to unitarize the sheaf cohomology group $H^{s_1}(G/H_1,\mathbf{V}_1)$

by identifying it with a quotient of a space of harmonic forms. We continue to write H instead of H_{N-1}, L instead of L_{N-1}, etc. As was pointed out in the proof of Theorem 12.11, the nonsingularity assumption (12.10) descends to H/H_1. Thus we may assume inductivity that

there exists a unitary H-module V which is infinitesimally equivalent to $H^{\tilde{s}}(H/H_1, \mathbf{V}_1)$ (12.22)

($\tilde{s} = \dim_{\mathbf{C}} L/L_1$); in other words, the spaces of L-finite vectors in these two H-modules are isomorphic over $\mathscr{U}(\mathfrak{h})$.

12.23. Lemma. *The G-modules $H^p(G/H_1, \mathbf{V}_1)$ and $H^{p-\tilde{s}}(G/H, \mathbf{V})$ are infinitesimally equivalent.*

As was pointed out in Section 4, a nonzero topological G-module which fails to be Hausdorff may well be infinitesimally equivalent to zero. Although it seems unlikely, the $\bar{\partial}$-operator in the Dolbeault complex for $\mathbf{V}$ might not have closed range. In this sense, the lemma makes an assertion only about the space of K-finite vectors in $H^*(G/H, \mathbf{V})$.

Proof of Lemma. In order to simplify the notation we set $V_F = H^{\tilde{s}}(H/H_1, \mathbf{V}_1)$. Be careful here: V_F is a Fréchet module and V is a Hilbert space, in contrast to the notation used in the proof of Theorem 12.11, where V denoted our present Fréchet space V_F. According to the proof of Theorem 12.11, the Leray spectral sequence of the fibration (12.15) provides an isomorphism

$$H^p(G/H_1, \mathbf{V}_1) \cong H^{p-\tilde{s}}(G/H, \mathbf{V}_F).$$

We now use the spectral sequence (4.36) to compare the cohomology groups of $\mathbf{V}$ and $\mathbf{V}_F$. In both cases, that spectral sequence calculates the K-finite part of the cohomology. The successive quotients $\mathbf{V}^l/\mathbf{V}^{l+1}$ and the differentials d_r depend only on the Harish-Chandra module of L-finite vectors in V, which coincides with the Harish-Chandra module attached to V_F (cf. (12.22)). We conclude: $H^{p-\tilde{s}}(G/H, \mathbf{V})$ is infinitesimally equivalent to $H^{p-\tilde{s}}(G/H, \mathbf{V}_F)$, and hence also to $H^p(G/H_1, \mathbf{V}_1)$. Q.E.D.

Let us check that the nonsingularity assumption (12.10) implies the hypotheses of Theorem 8.21, at least if G is linear. The highest weight ν of any irreducible L-constitutent V can be expressed as $\nu = v(\chi_1 + \rho_H - B) - \rho_H$, with $v \in W_L$ and B equal to a sum of roots in $\Phi(\mathfrak{p}_+ \cap \mathfrak{q}_1 \cap \mathfrak{h})$; this follows from Theorem 12.11 applied to the quotient H/H_1. Since L normalizes $\mathfrak{q}_+$, $v(\rho_H) - \rho_H$ equals $v\rho - \rho$, hence

$$(\nu + \rho, \gamma_l) = (v(\chi_1 + \rho - B), \gamma_l) = (\chi_1 + \rho - B, v^{-1}\gamma_l)$$

(cf. (12.20)). As a root in $\Phi(\mathfrak{p}_+ \cap \mathfrak{q}_+)$, $v^{-1}\gamma_i$ has a nonnegative inner product with every $\beta \in \Phi(\mathfrak{p}_+)$. Coupled with (12.10), this implies

$$(\nu + \rho, \gamma_i) \leqslant (\chi_1 + \rho, \gamma_i) < 0. \tag{12.24}$$

If G is linear or if

$$2(\chi_1 + \rho, \gamma)/(\gamma, \gamma) \leqslant -1 \qquad \text{for} \quad \gamma \in \Phi(\mathfrak{q}_{1,+}), \tag{12.25}$$

(12.24) can be sharpened to $(\chi_1 + \rho, \gamma_i) \leqslant -\frac{1}{2}(\gamma_i, \gamma_i)$. In either case we may apply Theorem 8.21. We also recall Lemma 12.23 and note that $s_1 = s + \tilde{s}$. This proves

12.26. THEOREM. *If* $\mathbf{V}_1$ *satisfies the nonsingularity assumption* (12.10) *and* G *is linear, or if* $\mathbf{V}_1$ *satisfies the more restrictive condition* (12.25), *the invariant Hermitian form* $(-1)^s\langle\,,\,\rangle_{G/H}$ *induces a Hilbert space structure on the reduced* L_2 *harmonic space*

$$\mathscr{H}_2^s(G/H, \mathbf{V})/(\textit{radical of } \langle\,,\,\rangle_{G/H}).$$

The resulting unitary G-module is infinitesimally equivalent to $H^{s_1}(G/H_1, \mathbf{V}_1)$.

Some comments are in order. If the isotropy group H_1 intersects every noncompact simple factor of G in a noncompact subgroup, (12.25) follows directly from (12.10), whether or not G is linear, as in Remark 8.23. On the other hand, if G does have a simple factor G' whose intersection with G is compact, the chain (12.3) can be altered so that

(i) $G' \cap H$ is compact, and

(ii) $G'' \cap H = G''$, for every simple factor G'' of G, other than G'.

In this case $G/H \simeq G'/G' \cap H$ has a positive definite invariant metric and (12.26) becomes a consequence of standard results about the holomorphic discrete series, provided only that $\mathbf{V}_1$ satisfies the nonsingularity hypothesis (12.10). In other words, if we choose the chain (12.3) judiciously, our geometric construction unitarizes the sheaf cohomology of $\mathbf{V}_1$ even for a nonlinear group G as soon as (12.10) holds.

It is natural to ask whether Theorem 8.21 extends to nonsymmetric quotients of the type discussed in the beginning of this section. We have partial results: the quotients G/H_1 carry distinguished positive definite noninvariant metrics which can be used to define spaces $\mathscr{H}_2^p(G/H_1, \mathbf{V}_1)$ of L_2 harmonic $\mathbf{V}_1$-valued $(0, p)$-forms on G/H_1, again every K-finite Dolbeault cohomology class has a representative in $\mathscr{H}_2^p(G/H_1, \mathbf{V}_1)$, and again the kernel of the map to Dolbeault cohomology contains the radical of $\langle\,,\,\rangle_{G/H_1}$;

so a Hilbert space quotient $\bar{\mathscr{H}}_2^p(G/H_1, \mathbf{V}_1)$ maps isomorphically, on the K-finite level, onto Dolbeault cohomology. What is missing is the statement that the radical of the invariant Hermitian form on $\mathscr{H}_2^p(G/H_1, \mathbf{V}_1)$ contains the kernel of the natural map to Dolbeault cohomology. A direct attack on this question leads into some subtle analytic problems.

13. Example: $U(k+l, m+n)/U(k) \times U(l, m) \times U(n)$

We studied the representations π_V for finite-dimensional $\mathbf{V} \to U(k+l, m)/U(k) \times U(l, m)$ in Sections 10.25 and 11.12. Here we put that information together a bit more systematically, especially as regards highest K-type, and show how the case $k = 1 = \dim V$ produces the ladder representations of the indefinite unitary groups. Finally, we specialize the iterative procedure of Section 12 and see that we come close to the realization of all the unitarizable highest weight modules of indefinite unitary groups.

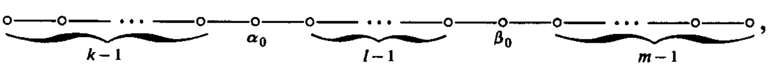
We start with the case $n = 0$, which is

$$G/H = U(k+l, m)/U(k) \times U(l, m) \qquad \text{with} \quad k, l, m \geqslant 1. \tag{13.1}$$

The Lie algebra $\mathfrak{g} = \mathfrak{u}(k+l, m)_{\mathbf{C}} = \mathfrak{gl}(k+l+m; \mathbf{C})$ and various subspaces are given in matrix block form by

$$\begin{matrix} \begin{pmatrix} \mathfrak{k} \cap \mathfrak{h} & \mathfrak{k} \cap \mathfrak{q}_+ & \mathfrak{p} \cap \mathfrak{q}_+ \\ \mathfrak{k} \cap \mathfrak{q}_- & \mathfrak{k} \cap \mathfrak{h} & \mathfrak{p} \cap \mathfrak{h} \\ \underbrace{\mathfrak{p} \cap \mathfrak{q}_-}_{k} & \underbrace{\mathfrak{p} \cap \mathfrak{h}}_{l} & \underbrace{\mathfrak{k} \cap \mathfrak{h}}_{m} \end{pmatrix} & \begin{matrix} \}k \\ \}l \\ \}m \end{matrix} \end{matrix} \tag{13.2}$$

Here $\pi: G/H \to K/L$ is holomorphic, the compact Cartan subgroup T has complexified Lie algebra $\mathfrak{t}$ consisting of the diagonal matrices, and we use the simple root system $\{x_i - x_{i+1}\}$, where $x_i: \mathfrak{t} \to \mathbf{C}$ is the ith diagonal entry. Thus we have Dynkin diagram



(13.3)

where $\alpha_0 = x_k - x_{k+1}$ is the simple root in $\Phi(\mathfrak{q}_+)$ and β_0 is the simple root in $\Phi(\mathfrak{p}_+)$.

Consider the case dim $V < \infty$, let $\chi = \psi|_L$, and ad an appropriate multiple of $x_1 + \cdots + x_{k+l+m}$ to obtain the normalization

$$\chi = -(a_1 x_1 + \cdots + a_k x_k), \qquad a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k \text{ integers.} \tag{13.4}$$

The L_2 condition $(\chi + \rho_M, \gamma_1) < 0$ is

$$L_2 \text{ condition:} \quad a_1 \geqslant k + m. \tag{13.5}$$

From the results of Section 10 we know that it implies the vanishing condition $H^p(G/H, \mathbf{V}) = 0$ for $p \neq s$. Let $r = \operatorname{rank}_{\mathbf{R}} M = \min(k, m)$. Then, as noted in (11.16),

$$H^s(G/H, \mathbf{V}) \neq 0 \Leftrightarrow a_i \geqslant k + l \qquad \text{for} \quad r + 1 \leqslant i \leqslant k. \tag{13.6}$$

Given (13.6), the highest K-type of $H^s(G/H, \mathbf{V})$ is $H^s(K/L, \mathbf{U}_{\nu_0})$, where

$$\nu_0 = x(-\chi^* - \tilde{n}\tilde{\gamma}), \qquad n_i = \max(k + l - a_i, 0) \quad \text{for} \quad 1 \leqslant i \leqslant r,$$

where $\chi^* = \sum_{i=1}^k a_{k+1-i} x_i$ is the dual of χ, where $\tilde{\gamma}_t = x_{k+1-t} - x_{k+l+t}$ for $1 \leqslant t \leqslant r$, and where $w \in W_L$ is the element of that Weyl group such that $w(-\chi^* - \tilde{n}\tilde{\gamma})$ is $\Phi(\mathfrak{l})^+$-dominant. Now we simplify

$$-\chi^* - \tilde{n}\tilde{\gamma} = -\sum_{i=1}^{r} (a_i + n_i)\, x_{k+1-i} - \sum_{i=r+1}^{k} a_i x_{k+1-i} + \sum_{i=1}^{r} n_i x_{k+l+i}$$

by noting that $a_i + n_i = \max(a_i, k + l)$, so

$$\begin{aligned} -\chi^* - \tilde{n}\tilde{\gamma} = &-\sum_{i=1}^{r} \max(a_i, k+l)\, x_{k+1-i} \\ &- \sum_{i=r+1}^{k} a_i x_{k+1-i} + \sum_{i=1}^{r} \max(k + l - a_i, 0)\, x_{k+l+i}. \end{aligned}$$

Of course $w \in W_L$ simply permutes within the sets $\{x_1, \ldots, x_k\}$, $\{x_{k+1}, \ldots, x_{k+l}\}$, and $\{x_{k+l+1}, \ldots, x_{k+l+m}\}$, and $\Phi(\mathfrak{l})^+$-dominance means that the coefficients of the x_i are nonincreasing within each of these three sets. As $a_1 \leqslant \cdots \leqslant a_k$ by (13.4), and $a_i \geqslant k + l$ for $r < i \leqslant k$ by (13.6),

$$-\max(a_1, k + l) \geqslant \cdots \geqslant -\max(a_r, k + l) \geqslant -a_{r+1} \geqslant \cdots \geqslant -a_k$$

and

$$\max(k + l - a_1, 0) \geqslant \cdots \geqslant \max(k + l - a_r, 0).$$

Thus, an element of W_L that carries $-\chi^* - \tilde{n}\tilde{\gamma}$ to ν_0 is the one given by

$$x_i \mapsto x_{k+1-i} \text{ for } 1 \leqslant i \leqslant k \quad \text{and} \quad x_j \mapsto x_j \text{ for } j > k.$$

In summary now,

under the nontriviality condition (13.6), $H^s(G/H, \mathbf{V})$ has highest K-type $H^s(K/L, \mathbf{U}_{\nu_0})$, where (13.7)

$$\nu_0 = -\sum_{i=1}^{r} \max(a_i, k+l)\, x_i - \sum_{r+1}^{k} a_i x_i + \sum_{i=1}^{r} \max(k+l-a_i, 0)\, x_{k+l+i}.$$

In particular, the highest K-type condition is $a_1 \geqslant k+l$, so

$$L_2 \text{ plus highest } K\text{-type:} \quad a_1 \geqslant k + \max(l, m). \tag{13.8}$$

Assuming those conditions, (13.7) reduces to $\nu_0 = \chi$. Without the highest K-type condition, (13.7) just says $\nu_0 = \chi - \sum_{1<i<r} \max(k+l-a_i)(x_i - x_{k+l+i})$.

The K-type $H^s(K/L, \mathbf{U}_{\nu_0}) = \mathbf{W}_{w_0(\nu_0+\rho_K)-\rho_K}$, where $w_0 \in W_K$ is the Weyl group element that interchanges $\Phi(\mathfrak{l})^+ \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$ with $\Phi(\mathfrak{l})^+ \cup \Phi(\mathfrak{k} \cap \mathfrak{q}_-)$. As permutation on $\{1,..., k+l+m\}$, w_0 is given by $\{1,..., k+l\} \mapsto \{k+1,..., k+l; 1,..., k\}$, and so

$$w_0(\rho_K) - \rho_K = -k \sum_{i=1}^{l} x_i + l \sum_{j=1}^{k} x_{l+j}.$$

Thus (13.7) becomes

under the nontriviality condition (13.6), $H^s(G/H, \mathbf{V})$ has highest K-type

$$\begin{aligned} \mu_\chi = &- \sum_{i=1}^{l} kx_i - \sum_{i=1}^{r} (\max(a_i, k+l) - l)\, x_{l+i} \\ &- \sum_{i=r+1}^{k} (a_i - l)\, x_{l+i} + \sum_{i=1}^{r} \max(k+l-a_i, 0)\, x_{k+l+i}. \end{aligned} \tag{13.9}$$

If the highest K-type condition is satisfied, then this reduces to $\mu_\chi = -\sum_{i=1}^{l} kx_i - \sum_{i=1}^{k} (a_i - l)\, x_{l+i}$. In general, the dual $H^s(G/H, \mathbf{V})^*$ has lowest K-type

$$\begin{aligned} \mu_\chi^* = &\sum_{i=1}^{k-r} (a_{k+1-i} - l)\, x_i + \sum_{i=k-r+1}^{k} (\max(a_{k+1-i}, k+l) - l)\, x_i \\ &+ \sum_{i=1}^{l} kx_{k+i} - \sum_{i=1}^{r} \max(k+l-a_i, 0)\, x_{k+l+m+1-i}. \end{aligned} \tag{13.10}$$

Here, if all $a_i \geqslant k+l$, then $\mu_\chi^* = \sum_{i=1}^{k} (a_{k+1-i} - l)\, x_i + \sum_{i=1}^{l} kx_{k+i}$.

We specialize these results to the case $k=1$, which is

$$G/H = U(1+l, m)/U(1) \times U(l, m) \qquad \text{with} \quad l, m \geqslant 1. \tag{13.11}$$

There, adding an appropriate multiple of $x_1 + \cdots + x_{n+1}$, we have the normalization and L_2 condition

$$\chi = -a x_1, \qquad a \text{ integer}, \quad a > m, \tag{13.12}$$

and the nontriviality condition (13.6) is vacuous because $r = k$ $(=1)$. Thus (13.9) $H^s(G/H, \mathbf{V})$ has highest K-type

$$\mu_\chi = -(x_1 + \cdots + x_l) - (\max(a, l+1) - l)\, x_{l+1} + \max(l+1-a, 0)\, x_{l+2}. \tag{13.13}$$

If $a > l$, this reduces to $\mu_\chi = -(x_1 + \cdots + x_{l-1}) - (a-l-1)\, x_{l+1}$. Let us denote

$$\nu_d : d\text{-th ladder representation of } U(1+l, m) \tag{13.14}$$

in the notation of Sternberg and Wolf [37]. Then [37, Proposition 5.6] ν_d^* has highest K-type μ_χ for $d = a-l-1 \geqslant 0$, and so [37, Theorem 5.8] π_V is unitarily equivalent to ν_d^* for $d = a-l-1 \geqslant 0$. Alternatively, by (13.10), $H^s(G/H, \mathbf{V})^*$ has lowest K-type

$$\mu_\chi^* = (\max(a, l+1) - l)\, x_1 + (x_2 + \cdots + x_{l+1}) - \max(l+1-a, 0)\, x_{m+l+1}. \tag{13.15}$$

If $a > l$, this reduces to $\mu_\chi^* = (a-l-1)\, x_1 + (x_1 + \cdots + x_{l+1})$, so, again, [37, Proposition 5.6 and Theorem 5.8] show π_V^* unitarily equivalent to ν_d for $d = a-l-1 \geqslant 0$.

We have just seen that the setup (13.11), (13.12) produces the ladder representations ν_d for $d = a-l-1$ with $a > \max(l, m)$, i.e., for $d \geqslant \max(l, m) - l$. Similarly, the setup

$$G/H = U(l, m+1)/U(l, m) \times U(1) \qquad \text{with} \quad l, m \geqslant 1,$$

$$\chi = a x_{l+m+1}, \qquad a \text{ integral}, \quad a > l,$$

gives ν_d for $d = m+1-a$ with $a > \max(l, m)$, i.e., for $d \leqslant m - \max(l, m)$. Combining this with the result of (13.11) and (13.12), we conclude that

> the ladder representations $\nu_d \in U(p, q)\hat{}$, $d > \max(p-1, q) - p$, are the π_V^* for line bundles $\mathbf{V} \to U(p, q)/U(1) \times U(p-1, q)$ that satisfy the L_2 and highest K-type conditions, (13.16)

and that

> the ladder representations $\nu_d \in U(p,q)\hat{}$, $d < q - \max(p, q-1)$, are the π_V^* for line bundles $\mathbf{V} \to U(p,q)/U(p,q-1) \times U(1)$ that satisfy the L_2 and highest K-type conditions. (13.17)

Specifically, the ranges of d in (13.16) and (13.17) are

$$\begin{aligned} &\text{if } p < q: \quad d \leqslant 0 &&\text{and} \quad d \geqslant q-p+1, \\ &\text{if } p = q: \quad d \leqslant -1 &&\text{and} \quad d \geqslant 1, \\ &\text{if } p > q: \quad d \leqslant q-p-1 &&\text{and} \quad d \geqslant 0, \end{aligned} \tag{13.18}$$

and ladder representations not in those ranges always have singular infinitesimal character.

We now turn to the somewhat more general case

$$G/H_1 = U(k+l, m+n)/U(k) \times U(l,m) \times U(n) \quad \text{with} \quad l, m, k+n \geqslant 1. \tag{13.19}$$

Then $\mathfrak{g}$ and various subspaces are given in matrix block form by

$$\begin{pmatrix} \mathfrak{k}\cap\mathfrak{h}_1 & \mathfrak{k}\cap\mathfrak{q}_{1,+} & \mathfrak{p}\cap\mathfrak{q}_{1,+} & \mathfrak{p}\cap\mathfrak{q}_{1,+} \\ \mathfrak{k}\cap\mathfrak{q}_{1,-} & \mathfrak{k}\cap\mathfrak{h}_1 & \mathfrak{p}\cap\mathfrak{h}_1 & \mathfrak{p}\cap\mathfrak{q}_{1,+} \\ \mathfrak{p}\cap\mathfrak{q}_{1,-} & \mathfrak{p}\cap\mathfrak{h}_1 & \mathfrak{k}\cap\mathfrak{h}_1 & \mathfrak{k}\cap\mathfrak{q}_{1,+} \\ \underbrace{\mathfrak{p}\cap\mathfrak{q}_{1,-}}_{k} & \underbrace{\mathfrak{p}\cap\mathfrak{q}_{1,-}}_{l} & \underbrace{\mathfrak{k}\cap\mathfrak{q}_{1,-}}_{m} & \underbrace{\mathfrak{k}\cap\mathfrak{h}_1}_{n} \end{pmatrix} \begin{matrix} \}k \\ \}l \\ \}m \\ \}n \end{matrix} \tag{13.20}$$

We consider a finite-dimensional $\mathbf{V}_1 \to G/H_1$, let χ_1 be the action of $L_1 = K \cap H_1$ on V_1, and add an appropriate multiple of $x_1 + \cdots + x_{k+l+m+n}$ to χ_1 in order to obtain the normalization

$$\chi_1 = -(a_1 x_1 + \cdots + a_k x_k) + (b_1 x_{k+l+m+1} + \cdots + b_n x_{k+l+m+n})$$
$$\text{where} \quad a_1 \leqslant \cdots \leqslant a_k \text{ and } b_1 \geqslant \cdots \geqslant b_n \text{ are integers.} \tag{13.21}$$

The element $w_1 \in W_K$ of Theorem 12.11, which keeps $\Phi(\mathfrak{l}_1)^+$ in $\Phi(\mathfrak{k})^+$ but sends $\Phi(\mathfrak{k} \cap \mathfrak{q}_{1,+})$ into $-\Phi(\mathfrak{k})^+$, is

$$\begin{aligned} w_1: \quad & x_i \to x_{l+i}, && \text{for} \quad 1 \leqslant i \leqslant k, \\ & \to x_{i-k}, && \text{for} \quad k < i \leqslant k+l; \\ w_1: x_{k+l+j} & \to x_{k+l+n+j}, && \text{for} \quad 1 \leqslant j \leqslant m, \\ & \to x_{k+l+j-m}, && \text{for} \quad m < j \leqslant m+n \end{aligned} \tag{13.22}$$

so

$$\begin{aligned} w_1(\rho_K) - \rho_K = -k \sum_{i=1}^{l} x_i + l \sum_{i=1}^{k} x_{l+i} \\ - m \sum_{i=1}^{m} x_{k+l+i} + n \sum_{i=1}^{m} x_{k+l+n+i} \end{aligned} \tag{13.23}$$

and thus $\mu = w_1(\chi_1 + \rho) - \rho = w_1(\chi) + w_1(\rho_K) - \rho_K$ is

$$\begin{aligned} \mu = -k \sum_{i=1}^{l} x_i + \sum_{i=1}^{k} (l - a_i)\, x_{l+i} \\ - \sum_{i=1}^{n} (m - b_i)\, x_{k+l+i} + n \sum_{i=1}^{m} x_{k+l+n+i}. \end{aligned} \tag{13.24}$$

It will be convenient to note that we can shift this by $k(x_1 + \cdots + x_{k+l+m+n})$, so $\mu' = \mu + k(x_1 + \cdots + x_{k+l+m+n})$ is

$$\begin{aligned} \mu' = \sum_{i=1}^{k} (k + l - a_i)\, x_{l+i} + \sum_{i=1}^{n} (b_i - m - n)\, x_{k+l+i} \\ + (k + n) \sum_{j=1}^{m+n} x_{k+l+j}. \end{aligned} \tag{13.25}$$

With a glance at (13.20) and (13.21) we note that

$$\begin{aligned} (\chi_1 + \rho_K, \alpha) < 0 \quad \text{for all} \quad \alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_{1,+}) \Leftrightarrow \\ a_1 \geqslant k + l \quad \text{and} \quad b_n \geqslant m + n, \end{aligned} \tag{13.26}$$

and

$$\begin{aligned} (\chi + \rho, \alpha) < 0 \quad \text{for all} \quad \alpha \in \Phi(\mathfrak{q}_{1,+}) \Leftrightarrow \\ a_1 \geqslant k + l + m \quad \text{and} \quad b_n \geqslant l + m + n. \end{aligned} \tag{13.27}$$

According to Theorem 12.26, now

> if $a_1 \geqslant k + l + m$ and $b_n \geqslant l + m + n$, then the representation π_{V_1} of $G = U(k + l, m + n)$ on $H^{s_1}(U(k + l, m + n)/U(k) \times U(l, m) \times U(n), \mathbf{V}_1)$ is infinitesimally equivalent to an irreducible unitary representation of G on a reduced L_2 harmonic space (13.28)

and, with a glance at (13.25),

$$\pi_{V_1} \otimes \det^k \text{ has highest weight } \mu' \text{ of (13.25).} \tag{13.29}$$

The unitarizable highest weight modules of indefinite unitary groups $U(p, q)$ were classified by Jakobsen [18]. They are the ones with highest weight of the form

$$\nu = -\sum_{i=1}^{k} u_i x_{p-k+i} + \sum_{i=1}^{n} v_i x_{p+i} + t \sum_{j=1}^{q} x_{p+j},$$

where $0 \leqslant u_1 \leqslant \cdots \leqslant u_k$ and $v_1 \geqslant \cdots \geqslant v_n \geqslant 0$ are integers, t is an integer, $k \leqslant p$, $n \leqslant q$, and $t \geqslant k+n$ (13.30)

plus a multiple of $x_1 + \cdots + x_{p+q}$.

The representation of highest weight ν belongs to the holomorphic discrete series if and only if $(\nu+\rho, \gamma) < 0$, where γ is the maximal root. Since $\gamma = x_1 - x_{p+q}$ and $(\rho, \gamma) = p+q-1$, we formulate that

holomorphic descrete series condition:

$$\delta_{k,p} u_1 + \delta_{n,q} v_n \geqslant p+q-t. \tag{13.31}$$

Since the holomorphic discrete series representations come out of the classical case of our construction, we limit our attention now to comparison of the representations π_{ν_1} with those of highest weight ν of (13.30) such that

not holomorphic discrete series:

$$t < p+q \text{ and } \delta_{k,p} u_1 + \delta_{n,q} v_n < p+q-t. \tag{13.32}$$

The relation between the weights ν of (13.30) and μ' of (13.25) is

$$k+l=p, \qquad m+n=q, \qquad a_i = u_i + k + l,$$
$$b_i = v_i + m + n, \qquad t = k+n. \tag{13.33}$$

Given ν, not corresponding to a holomorphic discrete series representation, we first consider the case $t > k+n$, reducing it to the case $t = k+n$. Since $t < p+q$, either $k < p$ or $n < q$ or both. If $k < p$, we can increase k to $k+1$ by substituting

$$-(0x_{p-k} + u_1 x_{p-k+1} + \cdots + u_k x_p)$$
$$\text{for} \qquad -(u_1 x_{p-k+1} + \cdots + u_k x_p).$$

Then the new u_1 is 0, and for $2 \leqslant i \leqslant k+1$ the new u_i is u_{i-1}. Similarly, if $n < q$, we can increase n to $n+1$ by substituting

$$(v_1 x_{p+1} + \cdots + v_n x_{p+n} + 0x_{p+n+1})$$
$$\text{for} \qquad (v_1 x_{p+1} + \cdots v_n x_{p+n}).$$

The v_i are unchanged for $1 \leqslant i \leqslant n$, and the new v_{n+1} is 0. In this way, we come to the case $t = k + n$ by increasing k, n, or both. The cost: if we increase k we go to a situation with $u_1 = 0$, and if we increase n we go to a situation with $v_n = 0$.

Now suppose $t = k + n$. Then (13.33) specifies G/H_1 and $\chi_1 \in \hat{L}_1$, yielding

$$\mathbf{V}_1 \to G/H_1 = U(k+l, m+n)/U(k) \times U(l, m) \times U(n)$$

such that (13.26) is satisfied with $\mu' = \nu$. The condition (13.27), used in Section 12 to construct an irreducible unitary representation $\pi_{\nu, 1}$, is $u_1 \geqslant m$ (unless $k = 0$) and $v_n \geqslant l$ (unless $n = 0$). This matches (13.18), which in effect is the case $k + n = 1$. We will deal with the corresponding "gaps" in a future publication.

APPENDIX A: HISTORICAL NOTE

During the 1960s, the metaplectic (= oscillator) representation μ_m of the 2-sheeted covering group of

$$Sp(m; \mathbf{R}): \quad \text{automorphism group of } (\mathbf{R}^{2m}, \omega), \text{ where}$$

$$\omega(x, y) = \sum_{i=1}^{m} (x_i y_{m+i} - x_{m+i} y_i)$$

come into use in the study of boson operators, Bose–Einstein creation and annihilation operators, certain facets of the wave equation, and some aspects of supersymmetry. See Bargmann's survey article [2]. For more current developments, see mention below of some recent work of Flato and Fronsdal.

For various reasons one expects to get μ_m by geometric quantization of the symplectic homogeneous space $X = (\mathbf{R}^{2m} \setminus \{0\}, \omega)$ of $Sp(m; \mathbf{R})$. That does not work directly because there is no invariant polarization. During the early 1970s Blattner, Kostant, and Sternberg extended the Kostant–Souriau quantization procedure to handle some cases of moving polarization. In the case of $Sp(m; \mathbf{R})$ acting on X, this "half form method" produces the metaplectic representation μ_m. See Guillemin and Sternberg [9, Chap. 5] for details.

The Euclidean space $\mathbf{R}^{2m}$ supports the unitary structure $(\mathbf{C}^m, h)$, where h is a positive definite Hermitian form with imaginary part ω. In 1974, Simms [35; also see 36, pp. 106–110] used half-forms and a certain polarization $\mathscr{F}_m$ on X to quantize the function $h(x, x)$, obtaining the spectrum and multiplicities appropriate to the harmonic oscillator. Rawnsley noted the suitability of $\mathscr{F}_m$ for quantizing the action of $U(m)$ on X; the polarization $\mathscr{F}_m$ is real in the direction of the Hamiltonian vector field for $h(x, x)$, which generates the center of $U(m)$, and is totally complex on the quotient $P^{m-1}(\mathbf{C})$ of any $U(m)$-orbit by that vector field. Rawnsley [25] proved that $H^p(X, S(\mathscr{F}_m)) = 0$ for $p \neq 1$ and that $H^1(X, S(\mathscr{F}_m))$ is the sum over the set of integral (Bohr–Sommerfeld condition) $U(m)$-orbits of the zero-cohomology of the pushed-down bundles on the quotients $P^{m-1}(\mathbf{C})$. Those pushed-down bundles are the various nonnegative powers of the hyperplane bundle, so $H^1(X, S(\mathscr{F}_m))$ as $U(m)$-module gives the $U(m)$-spectrum of μ_m.

Further, $\mathbf{R}^{2m}$ supports indefinite-unitary structures $(\mathbf{C}^m, h) = \mathbf{C}^{p,q}$, where $p + q = m$, the Hermitian form h has signature $(p + {}'s, q - {}'s)$, and ω is the imaginary part of h. This realizes the indefinite unitary group $U(p, q)$ inside $Sp(m; \mathbf{R})$. In 1975, Sternberg and Wolf [37] worked out the $U(p, q)$-spectrum of μ_m and studied the structure of the resulting "ladder representations" of $U(p, q)$. They tried to match the corresponding subspaces of Satake's L_2 cohomology version [27, 28] of the Bargmann–Fock realization of μ_m, with cohomologies of the pushed-down bundles over quotients of the integral $U(p, q)$-orbits on X. Those quotients, open subsets of $P^{m-1}(\mathbf{C})$, have homogeneous space structure

$$U(p, q)/U(1) \times U(p-1, q) \qquad \text{and} \qquad U(p, q)/U(p, q-1) \times U(1).$$

At that point it becamse clear that an intrinsic higher L_2 cohomology theory was need for Hermitian holomorphic line bundles over indefinite Kähler manifolds.

In the meantime, Atiyah had observed that the transformation laws in Penrose's twistor theory were such that various twistor fields could be viewed as elements of Dolbeault cohomologies $H^1(D, \mathbf{H}^m)$, where $\mathbf{H} \to P^3(\mathbf{C})$ is the hyperplane bundle and D is one of its two open $U(2, 2)$-orbits. Thus

$$D = U(2, 2)/U(1) \times U(1, 2) \qquad \text{or} \qquad D = U(2, 2)/U(2, 1) \times U(1).$$

There was considerable interest in the possibility of reformulating the Penrose inner product in a manner natural to this Dolbeault cohomology setting.

In 1977, Rawnsley and Wolf looked at the $U(p, q)$-invariant polarization on X that is analogous to Simms' $\mathscr{F}_m$. They did not see how to associate L_2 cohomology to the pushed-down bundles. (This was the point at which Sternberg and Wolf had been stopped.) They did see how Penrose's inner product fit into the picture, and they made tentative arrangements to work together in 1978/79 on the program that eventually resulted in the present paper.

During the following academic year, 1977/78, Blattner and Wolf discussed the use of invariant indefinite global Hermitan inner products for a higher L_2 cohomology theory that would specialize to produce the ladder representations of the $U(p, q)$. Sternberg was involved in one of those discussions, and a calculation of Rawnsley [26] played a key role. Blattner succeeded in associating square-integrable cohomology spaces H_2^i to X for the case $(p, q) = (1, 1)$. The result was the one stated in the next paragraph, but the method did not extend past the $U(1, 1)$ case.

Subsequently, in the fall of 1978, Blattner and Rawnsley associated square-integrable cohomology groups H_2^i to X with its standard $U(p, q)$-invariant indefinite Hermitian metric h. They used the polarization $\mathscr{F}$ spanned by the $\partial/\partial \bar{z}_j$ for $1 \leqslant j \leqslant p$ and the $\partial/\partial z_j$ for $p < j \leqslant m = p + q$. They proved [3] that the intrinsic global Hermitian form on H_2^i is identically zero for $i \neq q$ and is semidefinite on H_2^q, and that H_2^q/(nullspace of the global Hermitian form) has the same $U(p, q)$-spectrum as μ_m. Here H_2^i consists of all $(0, i)$ forms φ in a certain L_2 space with $D_{\mathscr{F}}\varphi = 0$ and $D_{\mathscr{F}}^* \varphi = 0$, where $D_{\mathscr{F}}$ is the $\mathscr{F}$-analog of the $\bar{\partial}$-operator and $D_{\mathscr{F}}^*$ is its adjoint relative to h. This gives a direct but still extrinsic geometric realization of the sum of the ladder representations of $U(p, q)$ on an L_2 cohomology space.

Rawnsley and Wolf then looked into the question of carrying out a similar program to obtain intrinsic realizations of the individual ladder representations of $U(p, q)$, quantizing with the invariant pseudo-Kähler metric on $U(p, q)/U(1) \times U(p-1, q)$ and $U(p, q)/U(p, q-1) \times U(1)$, for homogeneous Hermitian holomorphic line bundles $\mathbf{L}$. They worked this out explicitly for $G/H = U(1, 2)/U(1, 1) \times U(1)$. There, they defined $H_2^i(G/H, \mathbf{L})$ to be the space of $\mathbf{L}$-valued $(0, i)$-forms on G/H, finite under the maximal compact subgroup $K = U(1) \times U(2)$ of G, and L_2 relative to the usual positive definite Kähler metric on the $P^2(\mathbf{C})$ containing G/H, which are annihilated by $\bar{\partial}$ and its adjoint $\bar{\partial}^*$ relative to the G-invariant indefinite metric. They showed (i) $H_2^i(G/H, \mathbf{L}) = 0$ for $i \neq 1$, (ii) the intrinsic global Hermitian form is semidefinite on $H_2^1(G/H, \mathbf{L})$, (iii) the kernel of that global form is the kernel of the natural map from $H_2^1(G/H, \mathbf{L})$ to Dolbeault cohomology, and (iv) the action of $U(1, 2)$ on the Hilbert space completion of $H_2^1(G/H, \mathbf{L})$/(kernel) is the expected ladder representation. That calculation was completed in September 1979, and in the following months Rawnsley and Wolf made some progress in carrying out a similar computation for $U(1, n)/U(1, n-1) \times U(1)$. These results were not published formally because they were subsumed by the present paper.

In the meantime, Zuckerman [43; or see 39] developed an algebraic analog of Dolbeault cohomology for elliptic co-adjoint orbits of real reductive groups. His construction, which is functorial, gives a Harish-Chandra module that can be located within Langlands' classification scheme, that can be described by K-type, and whose global character can be specified. The first question here is that of unitarizability, and of course the basic idea behind the construction calls out for a intrinsic higher L_2 cohomology theory for holomorphic line bundles over these indefinite Kähler manifolds.

During the summer of 1979, Schmid considered the possibility of unitarizing some of Zuckerman's representations using an indefinite Hermitian metric. He examined the case of the ladder representations of $U(1, 2)$ and the action of $U(1, 2)$ on the space of all L_2 bundle-valued $(0, 1)$-forms on $U(1, 2)/U(1, 1) \times U(1)$ that are harmonic relative to the invariant indefinite metric. He saw that those harmonic spaces lack a certain finiteness property enjoyed by the ladder representations and concluded that they could not be used to realize the ladder representations. When he learned of the Rawnsley–Wolf computation in late 1979, he and Wolf discussed the apparent disparity and clarified the role of the kernel of the global Hermitian inner product on harmonic forms. Schmid then carried out a computation parallel to that of Rawnsley and Wolf, purely in terms of the structure of $U(1, 2)$ as real reductive Lie group.

Schmid and Wolf then looked at the case of a negative line bundle $\mathbf{L} \to G/H$, where G/H is indefinite Hermitian symmetric, G/K is Hermitian symmetric, and a consistency condition holds between the complex structures. They looked in the dimension s of the maximal compact subvariety $K/(K \cap H)$ of G/H, used the KAH decomposition of G to explicitly solve the equations for certain ("special") sorts of $\bar{\partial}$-closed $\bar{\partial}^*$-closed $\mathbf{L}$-valued $(0, s)$-forms on G/H, and calculated which of the K-finite forms are square integrable in a suitable sense. Schmid and Wolf showed that the natural map from the space $\mathscr{H}_2^s(G/H, \mathbf{L})$ of square-integrable harmonic $\mathbf{L}$-valued $(0, s)$-forms on G/H to Dolbeault cohomology is an isomorphism from the space of K-finite special forms onto the K-finite subspace of Dolbeault cohomology, and that its kernel is equal to the kernel of the global Hermitian inner product on $\mathscr{H}_2^s(G/H, \mathbf{L})$. Thus G acts on $\mathscr{H}_2^s(G/H, \mathbf{L})/(\text{kernel})$ by an irreducible unitary representation specified as to character and K-spectrum. This unitarizes some of the most singular of Zuckerman's representations and does it in a natural way. In the case $G = U(p, q)$ it produces ladder representations intrinsically and shows, through the mechanism of the Kashdan–Kostant–Sternberg moment map, that $\mu_m|_{U(p,q)}$ is given by restriction of X to the symplectic quotient of its $U(p, q)$-Bohr–Sommerfeld set. Thus the original Sternberg–Wolf problem was settled. This work was completed in summer 1980.

At the end of the summer of 1980, Schmid and Wolf saw that their results could be simplified significantly by using the fibration $\pi: G/H \to K/L$ to eliminate many explicit calculations. Wolf took advantage of that simplification, when he wrote the first draft of this paper in early 1981, to replace the line bundle $\mathbf{L} \to G/H$ by a possibly infinite-dimensional bundle $\mathbf{V} \to G/H$. The present version was completed by Schmid and Wolf in summer 1981.

There are two important parallel developments which we understood only after this paper was first drafted. The more important of these developments is an aspect of quantum electrodynamics. The Gupta–Bleuler method [4, 10] is a quantization in an indefinite Hilbert space that Dirac encountered [44, especially the last few sentences] in 1936 and more formally introduced in the early 1940s [45, 46]. The study of gauge invariance there was further developed by Mack, Salam, and Todorov in the late 1960s [53, 54]. Up to and including that work, the invariant subspace corresponding to nonobservables (such as longitudinal photons) and the quotient corresponding one–one to observables (such as transverse photons) were not associated to the action of a semisimple group. The group representations were first recognized by Fronsdal [51] in 1975; he identified the Dirac–Gupta–Bleuler space as an indecomposable module for the Lie algebra of $SO(3, 2)$. Fronsdal and Fang described the gravitational analog [47] in 1978, followed by [52] and [48]. Indefinite metric $SO(3, 2)$-quantization in electrodynamics was further developed by Fronsdal and Flato in their work [49, 50] on Dirac singletons. If one views $SO(3, 2)$ as (quotient of the double cover of) the real symplectic group $Sp(2, \mathbf{R})$, those singletons are the two irreducible constituents of the metaplectic representation μ_2, and they appear in the context of this paper on $Sp(2; \mathbf{R})/U(1, 1)$. Also see Segal [59], Segal *et al.* [60], and Paneitz and Segal [58].

The other parallel development has some results in common with this paper. In 1978, Parthasarathy [23] gave a sufficient condition for a highest weight module, whose infinitesimal character has certain nondegeneracy and integrality, to be unitarizable. Already in 1975, Wallach [40] had obtained all unitarizable highest weight modules with one-dimensional highest K-type by continuing the K-central character of that K-type past the holomorphic discrete series range. He showed that unitarizability is true precisely for a certain finite set of equispaced discrete values of the character together with a ray of values which, of course, includes the holomorphic discrete series case. In February and March of 1981, Enright, Howe, and Wallach announced an extension of Wallach's result which eliminates the condition that the highest K-type be one-dimensional. They relied on [40], on Enright's completion functor [6], on Parthasarathy's condition [23], on various results of Jantzen, on Howe's theory of dual reductive pairs, and, at least initially, on some information from this paper. See [7].

Also in 1981, Jakobsen [55, 56] independently described the highest weight unitary representations in a somewhat different but presumably equivalent form. He determined the "last possible place of unitarity" and argued that all other unitary highest weight modules are obtained from those extreme ones by tensoring with one of the representations determined by Wallach [40]. He used the Bernstein–Gelfand–Gelfand resolution, Kashiwara and Vergne [57], and the fact of the Kashiwara–Vergne conjecture.

We also mention the paper of Garland and Zuckerman [8], in which the unitarizability criterion that enters the proof of Enright *et al.* is established by purely algebraic means.

The spaces of K-finite vectors in the unitary representations constructed here are highest weight modules. In this sense our representations appear on the list of all unitarizable highest weight modules. Our goal, however, is quite different. We are after a geometric construction that will unitarize all formally unitarizable functorial representations attached to integral elliptic coadjoint orbits.

APPENDIX B. EXPLICIT EXPRESSIONS FOR SPECIAL HARMONIC FORMS

As noted in Appendix A, we originally constructed the unitary representations π_ν from explicit formulae for the special harmonic forms

$$\varphi \in A^s(G/H, \mathbf{V}) \quad \text{with} \quad \bar{\partial}\varphi = 0, \quad \bar{\partial}^*\varphi = 0, \quad \varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$$

in the case where $\pi: G/H \to K/L$ is holomorphic and dim $V = 1$. Those explicit formulae were obtained by solving the differential equations. It is, of course, much easier to proceed directly from Propositions 6.18 and 6.23: for each L-type $U_\nu \subset H^0(M/L, \mathbf{V})$ write down the harmonic forms $\omega \in A^s(K/L, \mathbf{U}_\nu)$ and then the corresponding forms $\varphi \in A^s(G/H, \mathbf{V})$. This works well because K/L is a Hermitian symmetric space of compact type, but one has to take U_ν explicitly because ν could occur with multiplicity > 1 when dim $V > 1$.

First, recall the explicit form of the Bott–Borel–Weil theorem in degree s on K/L. Fix $\nu \in \hat{L}$, so we have its representation space U_ν and the homogeneous holomorphic bundle $\mathbf{U}_\nu \to K/L$. Let $\omega: K \to U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$ be a $\mathbf{U}_\nu$-valued $(0, s)$-form. Then $\bar{\partial}\omega = 0$ by degree, and $\bar{\partial}^*\omega = 0$ if and only if the right action of $\mathfrak{k} \cap \mathfrak{q}_+$ annihilates ω. Compute

$$\begin{aligned}
&\{L_2(K) \otimes U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\}^{r(\mathfrak{k} \cap \mathfrak{q}_+)} \\
&\quad = \sum_{\kappa \in \hat{K}} W_\kappa \otimes \{W_\kappa^* \otimes U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\}^{\mathfrak{k} \cap \mathfrak{q}_+} \\
&\quad = \sum_{\kappa \in \hat{K}} W_\kappa \otimes \{[W_\kappa^*]_L \otimes U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\},
\end{aligned}$$

where $[W_\kappa^*]_L$ denotes the highest L-component of W_K^*. Write w_K for the opposition element of W_K; then $W_\kappa^* = W_{-w_K(\kappa)}$, so $[W_\kappa^*]_L = U_{-w_K(\kappa)}$. Now the space of $\mathbf{U}_\nu$-valued harmonic $(0, s)$-forms on K/L is

$$\begin{aligned}
&\{L_2(K) \otimes U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\}^{r(L) \cup r(\mathfrak{k} \cap \mathfrak{q}_+)} \\
&\quad = \sum_{\kappa \in K} W_\kappa \otimes \{U_{-w_K(\kappa)} \otimes U_\nu \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*\}^L.
\end{aligned}$$

Since $\Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^* = U_{2\rho_{K/L}}$, the term in braces vanishes unless $U_{-w_K(\kappa)}$ is the L-module dual to $U_{\nu+2\rho_{K/L}}$, i.e., unless $-w_K(\kappa) = -w_L(\nu + 2\rho_{K/L})$, i.e., unless $\kappa = w_0(\nu+\rho_K) - \rho_K$, where w_0 is given by (4.20). That requires $w_0(\nu+\rho_K)-\rho_K$ to be $\Phi(\mathfrak{k})^+$-dominant, or in other words, $(\nu+\rho_K, \alpha) < 0$ for all $\alpha \in \Phi(\mathfrak{k} \cap \mathfrak{q}_+)$, which we now assume. So

$$\begin{aligned}\{L_2(K) \otimes U_\nu \otimes \Lambda^s(\mathfrak{k}\cap\mathfrak{q}_-)^*\}^{r(L)\cup r(\mathfrak{k}\cap\mathfrak{q}_+)} \\ = W_{w_0(\nu+\rho_K)-\rho_K} \otimes \{[W^*_{w_0(\nu+\rho_K)-\rho_K}]_L \otimes U_\nu \otimes \Lambda^s(\mathfrak{k}\cap\mathfrak{q}_-)^*\}^L.\end{aligned}$$

That, of course, shows $H^s(K/L, \mathbf{U}_\nu) = W_{w_0(\nu+\rho_K)-\rho_K}$ as K-module, but also it gives a formula for the harmonic forms, as follows: Let $\{w_i^*\}$ be a basis of $[W^*_{w_0(\nu+\rho)-\rho}]_L$, let $0 \neq \omega_0 \in \Lambda^s(\mathfrak{k}\cap\mathfrak{q}_-)^*$, and choose a basis $\{u_i\}$ of U_ν such that $\{u_i \otimes \omega_0\}$ is dual to $\{w_i^*\}$. If $w \in W_{w_0(\nu+\rho)-\rho}$, then the corresponding harmonic form $f_w: K \to U_\nu \otimes \Lambda^s(\mathfrak{k}\cap\mathfrak{q}_-)^*$ is given by

$$f_w(k) = \sum_i (k^{-1}\cdot w, w_i^*)\, u_i \otimes \omega_0. \tag{B.1}$$

Note that the proof of (B.1) is valid more generally for compact homogeneous Kähler manifolds.

Second, we suppose that $\mathbf{V} \to G/H$ is a line bundle, so $H^0(M/L, \mathbf{V})_L = \sum U_{\chi - \tilde{n}\tilde{\gamma}}$ as in (4.3) with $\chi = \psi|_L$, and we recall the explicit form of the holomorphic section $h: M \to V$ of $\mathbf{V} \to M/L$ that is a lowest weight vector of L on $U_{\chi-\tilde{n}\tilde{\gamma}}$.

The Harish-Chandra embedding $\zeta: M/L \to \mathfrak{p}\cap\mathfrak{q}_+$ is given by $m \in \exp(\zeta(mL))\cdot L_{\mathbf{C}} \cdot \exp(\mathfrak{p}\cap\mathfrak{q}_-)$. Choose $E_{ij} \in \mathfrak{m}_{\gamma_{i,j}}$ and $E_k \in \mathfrak{m}_{\beta_k}$, where $\Phi(\mathfrak{p}\cap\mathfrak{q}_+) = \{\gamma_{ij}\} \cup \{\beta_k\}$, such that $B(E_{ij}, \bar{E}_{ij}) = 1$. Choose $v_0 \neq 0$ in V. Then

$$h'(mL) = \left\{\prod z_{ij}^{n_{ij}}\right\} v_0, \qquad \text{where}$$

$$\zeta(mL) = \sum z_{ij}E_{ij} + \sum z_k E_k \tag{B.2}$$

is a holomorphic function from M/L to V, and

$$h(m) = \left\{\prod z_{ij}^{n_{ij}}\right\} \chi(\tilde{m}^{-1})\, v_0,$$

$$m \in \exp(\mathfrak{p}\cap\mathfrak{q}_+)\cdot \tilde{m} \cdot \exp(\mathfrak{p}\cap\mathfrak{q}_-) \quad \text{with} \quad \tilde{m} \in L_{\mathbf{C}}, \tag{B.3}$$

is the corresponding holomorphic section of $\mathbf{V} \to M/L$. By direct calculation, if $\xi \in \mathfrak{t}$, then $\exp(t\xi)$ sends h to

$$[\exp(t\xi)\, h](m) = h(-\exp(t\xi)\, m) = \chi(\exp(t\xi))\, e^{-t\tilde{n}\gamma(\xi)} h(m),$$

so h is a weight vector of weight $\chi - \tilde{n}\gamma$ for T on $H^0(M/L, \mathbf{V})$. Now let us check that it is a lowest weight vector for L. Fix $\alpha \in \Phi(\mathfrak{l})^+$. If there is no root $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ such that $\beta - \alpha = \gamma_{i,j}$ for some (i,j), then $[E_{-\alpha}, \zeta(mL)]$ is a linear combination of the E_k, and it follows that $E_{-\alpha}$ annihilates h'. Since $(\chi, \Phi(\mathfrak{h})) = 0$, $E_{-\alpha}$ also annihilates v_0 and thus kills h. Now suppose that $\beta \in \Phi(\mathfrak{p} \cap \mathfrak{q}_+)$ such that $\beta - \alpha = \gamma_{i,j}$. Apply Lemma 10.15 to the ith noncompact simple factor of $\mathfrak{m}$ and then use (4.24):

$$p\alpha = \tfrac{1}{2}(\gamma_{i,l} - \gamma_{i,j}) \qquad \text{and} \quad p\beta = \tfrac{1}{2}(\gamma_{i,l} + \gamma_{i,j}) \quad \text{for some } l < j.$$

In particular, α specifies one such (i,j) and

$$[E_{-\alpha}, \zeta(mL)] \equiv \left\{ \sum_{\beta_k - \alpha = \gamma_{i,j}} a_k z_k \right\} E_{i,j} \qquad \text{modulo span}\,\{E_k\},$$

where $[E_{-\alpha}, E_{\beta_k}] = a_k E_{ij}$. Since $l < j$, so $n_{il} \geqslant n_{ij}$, it follows that $E_{-\alpha}$ annihilates h' and thus kills h. We have proved that

> the holomorphic section $h: M \to V$ given by (B.3) is a lowest weight vector, weight $\chi - \tilde{n}\gamma$, for L on $H^0(M/L, \mathbf{V})$. (B.4)

Note that in the proof of (B.4) we only used the assumption that $\mathbf{V} \to G/H$ is a line bundle, in order to facilitate the decomposition of $V \otimes U_{-\tilde{n}\tilde{\gamma}}$ into L-irreducible constituents. An analogous result follows whenever that decomposition is made explicit.

Third, we will combine (B.1) and (B.3) to write down the harmonic forms ω representing classes in $H^s(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$, such that ω takes values in the L-constituent $\mathbf{U}_{\chi - \tilde{n}\tilde{\gamma}} = \mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}$ of $\mathbf{H}^0(M/L, \mathbf{V})$. The $\omega(k)(\xi_1, ..., \xi_s)(m)$ will be explicit for $m \in A$. Suppose that $w_0(\chi - \tilde{n}\tilde{\gamma} + \rho) - \rho$ is $\Phi(\mathfrak{k})^+$-dominant, so that $\mathbf{V} \otimes \mathbf{U}_{-\tilde{n}\tilde{\gamma}}$ contributes to $H^s(K/L, \mathbf{H}^0(M/L, \mathbf{V}))$. The subspace $U_{\chi - \tilde{n}\tilde{\gamma}} = V \otimes U_{-\tilde{n}\tilde{\gamma}}$ of $H^0(M/L, \mathbf{V})$ has a basis $\{h_i\}$ such that $h_1 = h$ as in (B.3) and the other h_i are of the form $dv(E_{\alpha_1}) \cdot \cdots \cdot dv(E_{\alpha_p}) \cdot h$, where $v = \chi - \tilde{n}\tilde{\gamma}$, $\alpha_a \in \Phi(\mathfrak{l})^+$, $0 \neq E_{\alpha_a} \in \mathfrak{l}_{\alpha_a}$, and $p > 0$. If $\zeta(mL) = \sum z_{ij} E_{ij} + \sum z_k E_k$ as in (B.2), then $h_i'(mL) = \chi(\tilde{m})\, h_i(m)$ is a linear combination of monomials

$$\left\{ \prod_{i,j} z_{i,j}^{b_{ij}} \right\} \left\{ \prod_k z_k^{b_k} \right\} \qquad \text{with} \quad \sum b_{ij}\gamma_{ij} + \sum b_k \beta_k = \sum n_{ij}\gamma_{ij} - \sum \alpha_a$$

because it has weight $-\tilde{n}\gamma + \sum \alpha_a$. As the number p of α_a is at least one, some $b_k > 0$, and so $h'(mL) = 0$ for all m in

$$A = \exp(\mathfrak{a}_0) \qquad \text{where} \quad \mathfrak{a}_0 \text{ is then span of the} Y_{ij} = E_{ij} + \overline{E_{ij}}. \qquad \text{(B.5)}$$

In other words, we have exhibited

$$\{h_i\}: \quad \text{basis of } U_{\chi-\tilde{n}\tilde{\gamma}} \text{ with } h_1 = h \text{ and } h_i|_A = 0 \text{ for } i > 1. \tag{B.6}$$

Fix $0 \neq \omega_0 \in \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$ and define

$$\begin{aligned} \{w_i^*\}: \quad & \text{basis of } [W^*_{w_0(\chi-\tilde{n}\tilde{\gamma})-\rho}]_L = [W_{-\chi+\tilde{n}\tilde{\gamma}-2\rho_{K/L}}]_L \\ & \text{that is dual to the basis } \{h_i \otimes \omega_0\} \text{ of } U_{\chi-\tilde{n}\tilde{\gamma}} \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^* \end{aligned} \tag{B.7}$$

Here recall that W_ν is the irreducible K-module of highest weight ν and $[W_\nu]_L \cong U_\nu$ is its highest L-component. Now, according to (B.1), the isomorphism $W_{w_0(\chi-\tilde{n}\tilde{\gamma}+\rho)-\rho} \cong H^s(K/L, \mathbf{U}_{\chi-\tilde{n}\tilde{\gamma}})$ is made explicit by $w \leftrightarrow \omega_w$, where ω_w is the harmonic form defined by

$$\omega_w(k)(\xi_1, \ldots, \xi_s)(m) = \sum_i (k^{-1}w, w_i^*)\, h_i(m), \tag{B.8}$$

where $\{\xi_1, \ldots, \xi_s\}$ is a basis of $\mathfrak{k} \cap \mathfrak{q}_-$ such that $\omega_0(\xi_1, \ldots, \xi_s) = 1$. Now let us specialize this to the case

$$m = \exp\left(\sum t_{ij} Y_{ij}\right) \in A. \tag{B.9}$$

Then, as noted in the proof of Lemma 8.8,

$$\zeta(mL) = \sum_{i,j} \tanh(t_{ij})\, E_{ij}$$

and

$$\tilde{m}^{-1} = \prod_{i,j} \exp(\log\cosh(t_{ij})\, H_{ij}), \qquad \text{where} \quad H_{ij} = [E_{ij}, \overline{E_{ij}}].$$

In view of (B.7) we need only consider the term $i = 1$ of (B.8), where $h_1 = h$ and w_1^* (which we now call w^*) is the highest weight vector of $[W^*_{w_0(\chi-\tilde{n}\tilde{\gamma}+\rho)-\rho}]_L$. For that term, our evaluations of $\tilde{m}^{-1}$ and $\zeta(mL)$ give

$$\omega_w(k)(\xi_1, \ldots, \xi_s)(m) = (k^{-1}w, w^*) \left\{ \prod_{i,j} \cosh(t_{ij})^{\chi(H_{ij})} \tanh(t_{ij})^{n_{ij}} \right\} v_0. \tag{B.10}$$

Here note $\chi(H_{ij}) = 2(\chi, \gamma_{i,j})/(\gamma_{i,j}, \gamma_{i,j})$.

Fourth, we carry (B.10) over to a statement on harmonic forms $\varphi \in A^s(G/H, \mathbf{V})$ such that $\varphi(KM) \subset V \otimes \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. The relation, of course, is $\varphi(km)(\xi_1, \ldots, \xi_s) = \omega(k)(A_m \xi_1, \ldots, A_m \xi_s)(m) = (\det A_m|_{\mathfrak{k}\cap\mathfrak{q}_-})\omega(k)(\xi_1, \ldots, \xi_s)(m)$. As in the proof of Lemma 8.8 we enumerate $\Phi(\mathfrak{k} \cap \mathfrak{q}_+) = \{\alpha_1, \ldots, \alpha_s\}$, we suppose $\xi_a \in \mathfrak{k}_{-\alpha_a}$, and we denote

$$\begin{aligned} &\text{if } \alpha_a \perp \text{every } \gamma_{i,j}, && \text{then } t_a = 0 \\ &\text{if } \alpha_a \not\perp \gamma_{i,j}, \text{ so } \alpha_a - \tfrac{1}{2}\gamma_{i,j} \perp \text{every } \gamma_{u,v}, && \text{then } t_a = t_{ij}. \end{aligned}$$

If m is given by (B.9), then $A_m^{-1} = \cosh(\operatorname{ad} \sum t_{ij} Y_{ij})$ sends ξ_a to $\cosh(t_a)\, \xi_a$, so

$$\det(A_m|_{\mathfrak{k}\cap\mathfrak{q}_-}) = \prod_{1 \leqslant a \leqslant s} \cosh(t_a)^{-1} = \prod_{i,j,a} \cosh(t_{ij})^{-\alpha_a(H_{ij})}$$

$$= \prod_{i,j} \cosh(t_{ij})^{-2\rho_{K/L}(H_{ij})}.$$

Combine this with (B.10). The result is

$$\varphi_w(km) = (k^{-1}w, w^*) \left\{ \prod_{i,j} \cosh(t_{ij})^{(\chi - 2\rho_{K/L})(H_{ij})} \tanh(t_{ij})^{n_{ij}} \right\} v_0 \otimes \omega_0, \quad \text{(B.11)}$$

where $m \in A$ as in (B.9) and $(\chi - 2\rho_{K/L})(H_{ij}) = 2(\chi - 2\rho_{K/L}, \gamma_{i,j})/(\gamma_{i,j}, \gamma_{i,j})$.

Fifth, we incorporate the right action of H in (B.11). Since

$$\varphi_w(kmh) = \{\chi \otimes \Lambda^s(\mathrm{Ad}^*)\}(h)^{-1} \cdot \varphi_w(km),$$

this is just a matter of replacing v_0 by $\chi(h)^{-1} v_0$ and ω_0 by $\Lambda^s \mathrm{Ad}^*(h)^{-1} \omega_0$. Now, since $G = KAH$ as a cosequence of (2.6), we can summarize as follows:

B.12. THEOREM. *Suppose that $\pi: G/H \to K/L$ is holomorphic and $\mathbf{V} \to G/H$ is a line bundle. Let $\chi = \psi|_L$ and fix nonzero $v_0 \in V$ and $\omega_0 \in \Lambda^s(\mathfrak{k} \cap \mathfrak{q}_-)^*$. Then $H^s(G/H, \mathbf{V})_K$ is the algebraic direct sum of those K-types $W_{w_0(\chi - \tilde{n}\tilde{\gamma} + \rho) - \rho}$ for which $w_0(\chi - \tilde{n}\tilde{\gamma} + \rho) - \rho$ is $\Phi(\mathfrak{k})^+$-dominant, and in such a K-type the special harmonic form corresponding to a vector w is*

$$\varphi_w(kah) = (k^{-1}w, w^*) \left\{ \prod_{i,j} \cosh(t_{ij})^{(\chi - 2\rho_{K/L})(H_{ij})} \tanh(t_{ij})^{n_{ij}} \right\}$$
$$\times \chi(h)^{-1} v_0 \otimes \Lambda^s \mathrm{Ad}^*(h)^{-1} \omega_0 \qquad \text{(B.13)}$$

using $G = KAH$ with $k \in K, h \in H$, and $a = \exp \sum t_{ij} Y_{ij}$ as in (B.9), and where w^* is a highest weight vector for the L-module $[W^*_{w_0(\chi - \tilde{n}\tilde{\gamma} + \rho) - \rho}]_L \cong U_{-\chi + \tilde{n}\gamma - 2\rho_{K/L}}$.

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