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REPRESENTATIONS THAT REMAIN IRREDUCIBLE ON PARABOLIC SUBGROUPS

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§1. Introduction

In 1969 Mack and Todorov [6] showed that ladder representations of the conformal group remain irreducible on the Poincaré group. Then in 1975 (see [7, Theorem 4.27]) Sternberg and I extended that irreducibility result to ladder representations of any $U(k,\ell)$ on a maximal parabolic subgroup or its maximal unimodular subgroup. We noted that this seemed to be a combination of near transitivity and totally complex polarizations, as implicit in the reproducing kernel arguments ([3], [4]) of Kobayashi.

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Here I explicitly combine a notion of near transitivity with the uniqueness principle for reproducing kernels and obtain a very strong irreducibility theorem in §2. After a digression (§3) to explain dual reductive pairs and their classification over the real and complex numbers, I apply the result of §2 to

the group G of an irreducible dual reductive pair,

a summand π of the G-restriction of the metaplectic representation,

a subgroup $L \subset G$ constructed from parabolic subgroups,

in §4, and to certain other types of subgroups L in §5. The case $G = U(k, \ell)$ and L a maximal parabolic or the maximal unimodular subgroup of one, is a very special case. But this does seem to be the natural geometric setting for the Mack-Todorov irreducibility result.

I spoke on Theorem 2.1 and some of the examples in §4B at Oberwolfach in the summer of 1977. The example of §5A appeared in [10], in another setting. Now the general pattern for irreducible dual reductive pairs of type I seems clear, so publication is appropriate.

Finally let us note that there is a connection between representations that remain irreducible on parabolics and construction of L-functions in number theory. See ([3], §§3 and 4).

§2. An Irreducibility Criterion

We formalize an argument that I used in [7 , Theorem 4.27] to show that certain unitary representations remain irreducible on parabolic subgroups. The argument is a variation on Kobayashi's use of reproducing kernels to prove irreducibility theorems.

2.1. Theorem. Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on a complex manifold M such that point evaluations $f \mapsto f(m)$ are continuous functionals on $\mathcal{H}$. Let G be a Lie group acting on M such that the action lifts to a unitary representation π of G on $\mathcal{H}$.

Let L be a subgroup of G that satisfies the following near-transitivity condition on G-orbits. There is a G-invariant open set $U \subset M$ that meets every topological component of M such that: if $\mathcal{O} \subset U$ is a G-orbit then there is a point $m \in \mathcal{O}$ such that $L(m)$ is open in $\mathcal{O}$ and meets every component of $\mathcal{O}$.

Then every closed $\pi(L)$-invariant subspace of $\mathcal{H}$ is $\pi(G)$-invariant. In particular if π is irreducible then $\pi|_L$ is irreducible.

Remarks. G need not be a Lie group so long as its orbits are real analytic submanifolds of M. Also, as in [5], the theorem holds for spaces of holomorphic sections of vector bundles.

Proof. As point evaluations are continuous $\mathcal{H}$ has a reproducing Bergman kernel $K(z,\zeta)$, holomorphic in z, antiholomorphic in ζ, and hermitian in the sense $K(\zeta,z) = \overline{K(z,\zeta)}$. If $\{\varphi_i\}$ is any complete orthonormal set in $\mathcal{H}$ then $K(z,\zeta) = \Sigma\, \varphi_i(z)\overline{\varphi_i(\zeta)}$, uniformly and absolutely convergent on compact subsets of M.

Let $\mathcal{H}_1$ be a closed subspace of $\mathcal{H}$. As above, it has a reproducing Bergman kernel $K_1(z,\zeta) = \Sigma\, \psi_\alpha(z)\overline{\psi_\alpha(\zeta)}$ where $\{\psi_\alpha\} \subset \mathcal{H}_1$ is any complete orthonormal set. If $f \in \mathcal{H}$ then $f_1(z) = \langle K_1(z,\zeta), f(\zeta)\rangle_\zeta$ is its orthogonal projection to $\mathcal{H}_1$.

Now suppose that $\mathcal{H}_1$ is $\pi(L)$-invariant. Then $K_1(z,\zeta)$ is L-invariant,

$$K_1(gz, g\zeta) = K_1(z,\zeta) \quad \text{for} \quad g \in L \quad \text{and} \quad z,\zeta \in M ,$$

by uniqueness. In particular the function $K_1(z,z)$ is constant on L-orbits. Given $L(m) \subset \mathcal{O} \subset U$ as in the statement of the theorem, $z \mapsto K_1(z,z)$ is a real analytic function on the real analytic manifold $\mathfrak{G}$ that is constant on the open subset $L(m)$ which meets every component; so $K_1(z,z)$ is constant on the G-orbit $\mathfrak{G}$. Now the function $K_1(z,z)$ is G-invariant on U, hence also on M. But $K_1(z,\zeta)$ is determined by its restriction to the diagonal of $M \times M$, so it must be G-invariant,

$$K_1(gz, g\zeta) = K_1(z,\zeta) \quad \text{for} \quad g \in G \quad \text{and} \quad z,\zeta \in M \quad .$$

Now $\mathcal{K}_1 = \{\langle K_1(z,\zeta), f(\zeta)\rangle_\zeta = f \in \mathcal{K}\}$ is a $\pi(G)$-invariant subspace of $\mathcal{K}$.

q. e. d.

<u>2. 2. Corollary.</u> <u>Assume the conditions of Theorem 2. 1 and decompose</u>

$$\pi = \int \pi_\alpha \, d\nu(\alpha)\,, \quad \text{direct intergral of irreducibles .}$$

<u>Then</u> $\pi_\alpha|_L$ <u>is irreducible for</u> ν-<u>almost-every</u> α.

<u>Proof</u> (C. C. Moore). The result of Theorem 2. 1 can be phrased: every projection in $\mathcal{K}$ that commutes with $\pi(L)$ also commutes with $\pi(G)$. As these projections generate the commuting rings $A(\pi)$ and $A(\pi|_L)$ now $A(\pi) = A(\pi|_L)$. A direct integral decomposition into irreducibles corresponds to a choice of maximal abelian subalgebra of the commuting ring. Let $B \subset A(\pi)$ be the maximal abelian subalgebra whose Boolean algebra of projections gives $\pi = \int \pi_\alpha \, d\nu(\alpha)$. It also gives $\pi|_L = \int (\pi_\alpha|_L)\, d\nu(\alpha)$, and ν-almost-every $\pi_\alpha|_L$ is irreducible because B is maximal abelian in $A(\pi|_L)$.

§3. Dual Reductive Pairs

W is a symplectic vector space over a field $\mathbb{F}$ of characteristic $\neq 2$. Thus W has finite dimension over $\mathbb{F}$ and is equipped with a nondegenerate antisymmetric bilinear form $\langle\ ,\ \rangle_W$. The automorphism group of W is the symplectic group

$Sp(W) \cong Sp(\frac{1}{2}\dim W; \mathbb{F})$.

A pair of reductive subgroups $G_1, G_2 \subset Sp(W)$ forms a <u>dual reductive pair</u> if each is the centralizer of the other inside $Sp(W)$. If W is direct sum $W' \oplus W''$ of symplectic subspaces (i. e. if $\langle\ ,\ \rangle_W$ is nondegenerate on each and if $\langle W', W''\rangle_W = 0$) that are invariant under G_1 and G_2, then the dual reductive pair (G_1, G_2) is <u>reducible.</u> Otherwise it is <u>irreducible.</u>

In §§ 4 and 5 we will apply the result of §2 to the groups $G = G_1G_2$ where (G_1, G_2) is an irreducible dual reductive pair of type I in a real or complex symplectic group. There, π will be a summand of the G-restriction of the metaplectic representation. Here we are going to describe those pairs explicitly.

First let us recall Howe's general classification [1] of irreducible dual reductive pairs in $Sp(W)$.

Type I. $G = G_1G_2$ acts irreducibly on W. Then there exist

(3. 1) $\begin{cases} \text{a division algebra } \mathbb{D} \text{ over } \mathbb{F} \text{ with involution } x \mapsto \bar{x} \\ \text{a right vector space } W_1 \text{ over } \mathbb{D} \text{ and a left vector space } W_2 \\ \text{forms } (\ ,\ )_1 \text{ on } W_1 \text{ and } (\ ,\ )_2 \text{ on } W_2\text{, one hermitian and the other} \\ \qquad \text{skew hermitian} \end{cases}$

such that

(3. 2) $\qquad W \cong W_1 \otimes_{\mathbb{D}} W_2$ and $\langle\ ,\ \rangle_W = \mathrm{trace}_{\mathbb{D}/\mathbb{F}}(\ ,\ )_1 \otimes (\ ,\ )_2$

in such a way that

(3. 3) $\qquad G_1, G_2$ go to the isometry groups of $(W_1, (\ ,\ )_1)$ and $(W_2, (\ ,\ )_2)$.

Type II. $G = G_1G_2$ acts reducibly on W. Then W is the vector space direct sum of G-invariant maximal totally isotropic subspaces V, V' and there exist

(3.4) $\begin{cases} \text{a division algebra } \mathbb{D} \text{ over } \mathbb{F} \\ \text{a right vector space } V_1 \text{ over } \mathbb{D} \text{ and a left vector space } V_2 \end{cases}$

such that

(3.5) $$V \cong V_1 \otimes_{\mathbb{D}} V_2 \quad \text{with } G_i \text{ identified to } GL(V_i;\mathbb{D}) \quad .$$

Here V' is identified to the $\mathbb{F}$-linear dual space of V under $\langle\ ,\ \rangle_W$. The $\mathbb{D}$-linear dual spaces of V_1 and V_2 are

(3.6) a left vector space V_1' over $\mathbb{D}$ and a right vector space V_2' such that

(3.7) $$V' \cong V_2' \otimes V_1' \quad \text{with } G_i \text{ identified to } GL(V_i';\mathbb{D})$$

where

(3.8) the action of G_i on V_i' is the $\mathbb{D}$-dual to its action on V_i.

Next we specialize to the case $\mathbb{F} = \mathbb{C}$. Then $\mathbb{D} = \mathbb{C}$, and $\bar{x} = x$ in (3.1), so up to interchange of G_1, G_2 the only irreducible dual reductive pairs are

(3.9) $$O(u;\mathbb{C}),\ Sp(v;\mathbb{C}) \quad \text{in} \quad Sp(uv;\mathbb{C}) \quad ,$$

(3.10) $$GL(u;\mathbb{C}),\ GL(v;\mathbb{C}) \quad \text{in} \quad Sp(uv;\mathbb{C}) \quad .$$

In (3.9), $W \cong \mathbb{C}^u \otimes \mathbb{C}^{2v} = \mathbb{C}^{2uv}$ where $\langle\ ,\ \rangle_W$ is the tensor product of the symmetric form on $\mathbb{C}^u$ with the antisymmetric form on $\mathbb{C}^{2v}$. In (3.10), $W \cong V \oplus V'$ where $V = \mathbb{C}^{u \times v}$ space of $u \times v$ complex matrices, $V' = \mathbb{C}^{v \times u}$ with pairing $x'(x) = \operatorname{trace}(xx')$, and

$$(a,b) \in GL(u;\mathbb{C}) \cdot GL(v;\mathbb{C})$$

acts on $(x,x') \in V \oplus V' = W$ by

(3.11) $$(a,b) : (x,x') \mapsto (axb^{-1},\ bx'a^{-1}) \quad .$$

Next consider type I pairs with $\mathbb{F} = \mathbb{R}$. Then $\mathbb{D}$ is $\mathbb{R}$, $\mathbb{C}$ or the

quaternion algebra $\mathbb{H}$, and $x \to \bar{x}$ has its usual meaning in (3.1). Suppose $(\ ,\ )_1$ hermitian and $(\ ,\ )_2$ skew hermitian. Then we can take W_1 to be

(3.12) $\begin{cases} \mathbb{D}^{k,\ell} : \text{right vector space of } (k+\ell)\text{-tuples over } \mathbb{D} \text{ with form} \\ \qquad (x,y)_1 = \sum_1^k x_j \bar{y}_j - \sum_{k+1}^{k+\ell} x_j \bar{y}_j \quad . \end{cases}$

Its isometry group G_1 is the (indefinite)

(3.13)

orthogonal group	$O(k,\ell)$	$\mathbb{D} = \mathbb{R}$
unitary group	$U(k,\ell)$	$\mathbb{D} = \mathbb{C}$
unitary symplectic group	$Sp(k,\ell)$	$\mathbb{D} = \mathbb{H}$

A skew-hermitian form over $\mathbb{R}$ is just an antisymmetric bilinear form. If $\mathbb{D} = \mathbb{R}$ now W_2 and G_2 are $\mathbb{R}^v$ and $Sp(\frac{1}{2}v;\mathbb{R})$. A skew hermitian form over $\mathbb{C}$ is just i times a hermitian one, and they have the same isometry group. If $\mathbb{D} = \mathbb{C}$ now W_2 and G_2 are $\mathbb{C}^v$ and $U(p,q)$, $p + q = v$. A skew hermitian form on a left vector space $\mathbb{H}^v$ over $\mathbb{H}$ is equivalent to $(x,y)_2 = \Sigma\, \bar{x}_j\, i\, y_j$. Its isometry group is the real form $SO^*(2v)$ of $SO(2v;\mathbb{C})$ whose maximal compact subgroup is $U(v)$. See [9] and [10] for details on this. Now, G_2 and W_2 are

(3.14)

$Sp(\frac{1}{2}v;\mathbb{R})$	and	$\mathbb{R}^v$	$\mathbb{D} = \mathbb{R}$
$U(p,q)$	and	$\mathbb{C}^v$, $v = p+q$	$\mathbb{D} = \mathbb{C}$
$SO^*(2v)$	and	$\mathbb{H}^v$	$\mathbb{D} = \mathbb{H}$

Thus, the irreducible dual reductive pairs of type I with $\mathbb{F} = \mathbb{R}$ are

$$(3.15)\qquad \begin{cases} O(k,\ell)\ ,\ Sp(\tfrac{1}{2}v\,;\mathbb{R}) & \text{in} \quad Sp(\tfrac{1}{2}(k+\ell)v\,;\mathbb{R}) \\ U(k,\ell)\ ,\ U(p,q)\ ,\ v = p+q & \text{in} \quad Sp((k+\ell)v\,;\mathbb{R}) \\ Sp(k,\ell)\ ,\ SO^*(2v) & \text{in} \quad Sp(2(k+\ell)v\,;\mathbb{R}) \end{cases}$$

In each case, the action on $W \cong \mathbb{D}^{k,\ell} \otimes_{\mathbb{D}} \mathbb{D}^{v}$, viewed as the space $\mathbb{D}^{(k+\ell)\times v}$ of $(k+\ell) \times v$ matrices over $\mathbb{D}$, is $(a,b) : x \mapsto axb^{-1}$.

Finally consider type II pairs with $\mathbb{F} = \mathbb{R}$. As in the case $\mathbb{F} = \mathbb{C}$, the pairs are

$$(3.16)\qquad \begin{cases} GL(u\,;\mathbb{R})\ ,\ GL(v\,;\mathbb{R}) & \text{in} \quad Sp(uv\,;\mathbb{R}) \\ GL(u\,;\mathbb{C})\ ,\ GL(v\,;\mathbb{C}) & \text{in} \quad Sp(2uv\,;\mathbb{R}) \\ GL(u\,;\mathbb{H})\ ,\ GL(v\,;\mathbb{H}) & \text{in} \quad Sp(4uv\,;\mathbb{R}) \end{cases}$$

They act on $W = \mathbb{D}^{u\times v} \oplus \mathbb{D}^{v\times u}$ as in (3.11). Here $\mathbb{D}^{u\times v} = \mathbb{D}^{u} \otimes_{\mathbb{D}} \mathbb{D}^{v} = V_1 \otimes V_2 = V$, $V' = V_2' \otimes V_1' = \mathbb{D}^{v\times u}$ under $x'(x) = \operatorname{Re}\operatorname{trace}(xx')$.

§4. Patterns of Near Transitivity in Type I Pairs

Let (G_1, G_2) be an irreducible dual reductive pair of type I in a complex or real symplectic group $Sp(W)$. It is given by (3.9) or (3.15). W is a space $W_1 \otimes W_2$ of matrices and $G = G_1 \cdot G_2$ acts on it by $(a,b) : x \mapsto axb^{-1}$. In the setting of the metaplectic representation, W will be viewed as a complex vector space.

4A. $G = Sp(W)$. This is (3.9) with $u = 1$ or (3.15) with $k + \ell = 1$. The parabolic subgroups of G are the normalizers of "flags" $0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_t$ where the E_i are $\mathbb{F}$-subspaces of W that are totally isotropic, i.e. $\langle E_i, E_i\rangle_W = 0$.

The parabolic is

$$(4.1)\qquad P = P_{E_1,\dots E_t} = \{g \in G : gE_i = E_i \ \text{ for } \ 1 \le i \le t\}\ .$$

The set $L(m)$ in Theorem 2.1 will be

$$(4.2)\qquad Q = \{x \in W : x \notin E_t \ \text{ and } \ \langle x, E_1\rangle_W \ne 0\}\ .$$

Note that Q is a single P-orbit on W. For if $x, y \in Q$ then we have bases $\{v_1, \dots v_q, x\}$ of $E_t + x\mathbb{F}$ and $\{w_1, \dots, w_q, y\}$ of $E_t + y\mathbb{F}$ such that $\{v_1, \dots, v_{\dim E_i}\}$ and $\{w_1, \dots w_{\dim E_i}\}$ are bases of E_i $(1 \le i \le t)$ and

$$\langle v_1, x\rangle = 1 = \langle w_1, y\rangle\ , \ \text{ and } \ \langle v_j, x\rangle = 0 = \langle w_j, y\rangle \ \text{ for } \ j > 1\ .$$

Witt's Theorem provides $g \in G$ with $gv_i = w_i$ $(1 \le i \le q)$ and $gx = y$. So $g \in P$ sends x to y.

We just showed that the parabolic P is transitive on Q. By rescaling the w_i, $i > 1$, we may assume that the element $g \in G$ there, belongs to

$$(4.3)\qquad \begin{cases} \text{if } \dim E_1 > 1 : L = \{g \in P : \det(g|_{E_i/E_{i-1}}) = 1 \ \text{ for } \ 1 \le i \le t\} \\ \text{if } \dim E_1 = 1 : L = \{g \in P : \det(g|_{E_i/E_{i-1}}) = 1 \ \text{ for } \ 2 \le i \le t\} \end{cases}$$

This group L now satisfies the near transitivity condition of Theorem 2.1 with $M = W$, $U = W\setminus\{0\}$ a single G-orbit, and $L(m) = Q$ there.

4B. $G = O(k,\ell) \cdot Sp(\tfrac{1}{2}v\,;\mathbb{R})$ with $k + \ell > 1$, or $U(k,\ell) \cdot U(p,q)$ with $p + q = v$, or $Sp(k,\ell) \cdot SO^*(2v)$; and $\operatorname{rank}_{\mathbb{R}} G_1 = \min(k,\ell) \ge v$. The parabolic subgroups of $G_1 = U(k,\ell\,;\mathbb{D})$ are the normalizers

$$(4.4)\qquad P = P_{E_1,\dots E_t} = \{g \in G_1 : gE_i = E_i \ \text{ for } \ 1 \le i \le t\}$$

of $(\ ,\ )_1$-isotropic flags $0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_t$. Thus these E_i are $\mathbb{D}$-subspaces of $W_1 = \mathbb{D}^{k,\ell}$ with $(E_i, E_i)_1 = 0$. Similarly the parabolics in G_2 are the normalizers

(4.5) $$P' = P'_{F_1,\ldots F_s} = \{g \in G : gF_i = F_i \quad \text{for} \quad 1 \leqq i \leqq s\}$$

of $(\ ,\ )_2$-isotropic flags $0 \neq F_1 \subsetneqq \cdots \subsetneqq F_s$ in $W_2 = \mathbb{D}^v$.

We suppose $\min(k,\ell) \geqq v$, i.e. that W_1 has a totally isotropic subspace of dimension $\geqq \dim W_2$. In [2] this is the defining condition for (G_1, G_2) to be "stable." It says that G_1 has a parabolic subgroup $P = P_{E_1,\ldots,E_t}$ as in (4.4) with $\dim E_1 \geqq \dim W_2$. Define

(4.6) $$\begin{cases} \text{if } \dim E_1 > \dim W_2 : L_1 = \{g \in P : \det(g|_{E_i/E_{i-1}}) = 1 \quad \text{for} \quad 1 \leqq i \leqq t\} \\ \text{if } \dim E_1 = \dim W_2 : L_1 = \{g \in P : \det(g|_{E_i/E_{i-1}}) = 1 \quad \text{for} \quad 2 \leqq i \leqq t\} \end{cases}$$

where we note that the condition $\det = r$, r real, is well defined over $\mathbb{H}$.

Since $\min(k,\ell) \geqq \dim E_1 \geqq v$, $W_1 = \mathbb{D}^{k,\ell}$ has subspaces $V \cong \mathbb{D}^{k',\ell'}$ of every signature (k',ℓ') with $k' + \ell' = v$, such that $V \cap E_1^\perp = 0$. Fix one such (k',ℓ'); that gives us

$$U(k',\ell';\mathbb{D}) \subset GL'(v;\mathbb{D}) = GL'(W_2)$$

where $GL' = \{g \in GL : |\det g| = 1\} = \{g \in GL : g \text{ preserves Lebesgue measure}\}$. Let

(4.7) $$\begin{cases} L_2 : \text{any subgroup of } G_2 \text{ such that } L_2 \text{ and} \\ \quad U(k',\ell';\mathbb{D}) \text{ generate } GL'(W_2) \end{cases},$$

(4.8) $$L = L_1 \cdot L_2 \subset G_1 \cdot G_2 = G \quad .$$

There are many groups L_2 because $U(k',\ell';\mathbb{D})$ is a maximal subgroup of $GL'(W_2)$.

Our G-invariant open set $U \subset W$ will consist of all $x \in \mathbb{D}^{(k+\ell)\times v} = W$ whose columns span a subspace $V \cong \mathbb{D}^{k',\ell'}$ with $V \cap E_1^\perp = 0$. We will verify that every G-orbit on V is an L-orbit.

Fix $y \in U$. We first check that every L-orbit on U contains a positive real multiple ry of y. For that, let y_i be the columns of y, $y = (y_1, \ldots, y_v)$, and let T be their span. Given $x \in U$ we have $x = (x_1, \ldots, x_v)$; let S be its column span. Since $S \cong \mathbb{D}^{k',\ell'} \cong T$ we have an isometry $\varphi : S \cong T$. As $S \cap E_1^\perp = 0$ we have $\{f_j\}$ with $\{f_1, \ldots, f_{\dim E_i}\}$ basis of E_i for $1 \leqq i \leqq t$, such that $(f_j, x_i)_1 = \delta_{ij}$. Similarly $T \cap E_1^\perp = 0$ gives $\{e_j\}$ such that $\{e_1, \ldots, e_{\dim E_i}\}$ is a basis of E_i and $(e_j, \varphi x_i)_1 = \delta_{ij}$. Now $f_j \mapsto e_j$, $x_i \mapsto \varphi x_i$ is an isometry of $E_t + S$ onto $E_t + T$ that sends each E_i to itself. Witt's Theorem extends this to an isometry g_1 of W_1. Thus we have $g_1 \in P_{E_1,\ldots,E_t}$ with $g_1 S = T$. We can freely replace the f_j, $j > v$, by nonzero elements of $f_j\mathbb{D}$, so we may assume $g_1 \in L_1$.

Let $L_1^T = \{g|_T : g \in L_1 \text{ and } gT = T\}$. If we repeat the above argument with $x = y$ and any isometry of T we see $L_1^T \cong U(k',\ell';\mathbb{D})$. The action $z \mapsto z \cdot g_2^{-1}$ of G_2 does not change column span. Let L^T denote the group of all transformations of T generated by L_1^T and the $z \mapsto zg_2^{-1}$, $g_2 \in L_2$. By hypothesis on L_2 we have $L^T = GL'(T)$. Now, for a unique $r > 0$ some $\gamma \in L^T$ carries φx_i to ry_i for $1 \leqq i \leqq v$. So $ry \in L(x)$.

We next check that the positive multiple of y in a G-orbit on U is unique. For let $a \cdot b \in G$ and $r, r' > 0$ with $a \cdot ry \cdot b^{-1} = r'y$. As $ry \cdot b^{-1}$ and ry

have the same column span, our old T, now a preserves T acting there as an element of $U(k',\ell';\mathbb{D})$, so $a_{|T} \in GL'(T)$. Also $b \in G_2 \subset GL'(W_2)$ acts on T as an element of $GL'(T)$. It follows that $r = r'$.

We have shown that the group L of (4.10) is transitive on every G-orbit in U, thus proving the near-transitivity condition of Theorem 2.1 with $M = W$ and U as above.

4C. $G = O(k,\ell)\cdot Sp(\tfrac{1}{2}v;\mathbb{R})$ with $k+\ell > 1$, or $U(k,\ell)\cdot U(p,q)$ with $p+q = v$, or $Sp(k,\ell)\cdot SO^*(2v)$; and $\operatorname{rank}_{\mathbb{R}} G_2$ $(= \tfrac{1}{2}v$, $\min(p,q)$ or $[v/2]$, resp.$) \geqq k+\ell$. Thus W_2 has a totally isotropic subspace of dimension $\geqq \dim W_1$, i.e. G_2 has a parabolic subgroup $P' = P_{F_1,\dots,F_s}$ with $\dim F_1 \geqq \dim W_1$. Define

$$(4.9)\quad \begin{cases} \text{if } \dim F_1 > \dim W_1 : L_2 = \{g \in P' : \det(g_{|F_i/F_{i-1}}) = 1 \text{ for } 1 \leqq i \leqq s\} \\ \text{if } \dim F_1 = \dim W_1 : L_2 = \{g \in P' : \det(g_{|F_i/F_{i-1}}) = 1 \text{ for } 2 \leqq i \leqq s\} \end{cases}$$

As $\dim F_1 \geqq \dim W_1$, so $W_2 = \mathbb{D}^v$ has subspaces V of dimension $k+\ell$ with $V \cap F_1^{\perp} = 0$ and $(\ ,\ )_{2|V\times V}$ of maximal possible rank. If

$$(4.10)\qquad J = \{g_{2|V} : g_2 \in G_2 \text{ and } g_2 V = V\}$$

then the possibilities are

(i) $\mathbb{D} = \mathbb{R}$ and $k+\ell$ odd : $(\ ,\ )_{2|V\times V}$ has rank $k+\ell-1$ and

$$J \cong \left\{\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} : 0 \neq \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^{k+\ell-1},\ \gamma \in Sp(\tfrac{k+\ell-1}{2};\mathbb{R})\right\}$$

(ii) $\mathbb{D} = \mathbb{R}$ and $k+\ell$ even : $(\ ,\ )_{2|V\times V}$ has rank $k+\ell$ and

$$J \cong Sp(\tfrac{k+\ell}{2};\mathbb{R})$$

(iii) $\mathbb{D} = \mathbb{C}$: $(\ ,\ )_{|V\times V}$ has rank $k+\ell$, in fact $\sqrt{-1}\,(\ ,\ )_{|V\times V}$ has any specified signature (p',q') with $p'+q' = k+\ell$, and $J \cong U(p',q')$

(iv) $\mathbb{D} = \mathbb{H}$: $(\ ,\ )_{V\times V}$ has rank $k+\ell$ and $J \cong SO^*(2(k+\ell))$.

Identify J to a subgroup of $GL(W_1)$ and let

$$(4.11)\quad \begin{cases} L_1 : \text{any subgroup of } G_1 \text{ such that } L_1 \text{ and } J \text{ generate} \\ \quad GL'(W_1) \text{ or } GL(W_1), \end{cases}$$

$$(4.12)\qquad L = L_1 \cdot L_2 \subset G_1 \cdot G_2 = G .$$

J is maximal in $GL'(W_1)$ or nearly maximal in $GL(W_1)$, so there are many groups L_1.

Our G-invariant open set $U \subset W$ will consist of all $x \in \mathbb{D}^{(k+\ell)\times v} = W$ whose rows span a subspace V of dimension $k+\ell$ in W_2 with $V \cap F_1^{\perp} = 0$ and $(\ ,\ )_{2|V\times V}$ of maximal possible rank. If $\mathbb{D} = \mathbb{C}$ we specify the signature (p',q') of $\sqrt{-1}\,(\ ,\ )_{2|V\times V}$.

Fix $y \in U$. As in the second and third paragraph following (4.10), if $x \in U$ then $ry \in L(x)$ for some $r > 0$. As in the paragraph after that, when $(\ ,\ )_{2|V\times V}$ is nondegenerate and L_1, J generate $GL'(W_1)$, ry is the only positive real multiple of y in $G(x)$, and when $(\ ,\ )_{2|V\times V}$ degenerates and L_1, J generate $GL(W_1)$, $L(x)$ contains every real multiple of y. Thus L is transitive on every G-orbit in U, and we have the near-transitivity condition of Theorem 2.1.

§5. Isolated Cases of Near Transitivity in Type I Pairs

As in §4, (G_1, G_2) is an irreducible dual reductive pair of type I in a symplectic group $Sp(W)$. Here we describe certain subgroups $L \subset G = G_1 G_2$ that satisfy the near transitivity condition of Theorem 2.1 but, in contrast to those of §4, are not modeled on parabolic subgroups.

5A. $G = U(2k, 2\ell)$. This is the second case of (3.15) with $v = 1$. The G-orbits on $W = \mathbb{C}^{2k, 2\ell}$ are $\{0\}$, the light cone and the mass shells. The group

(5.1) $$L = Sp(k, \ell) \subset U(2k, 2\ell) = G$$

is transitive on them. We mentioned this example in [10, (4.20)].

5B. $G = O(7) \cdot Sp(1; \mathbb{R})$. This is the first case of (3.15) with $(k, \ell) = (7, 0)$ and $v = 2$. Here $W = \mathbb{R}^{7 \times 2}$. Let U be the subspace of rank 2 matrices; the G-orbits are the sets

$$U_r = \{x = (x_1 x_2) \in W : \|x_1 \wedge x_2\| = r\} \quad , \quad 0 < r < \infty \quad .$$

Every $x \in U_r$ is in the $Sp(1, \mathbb{R})$-orbit of some $x' = (x'_1, x'_2)$ where $\|x'_1\| = 1$, $\|x'_2\| = r$ and $(x'_1, x'_2)_1 = 0$. The subgroup $G_2 \subset O(7)$ is transitive on orthonormal 2-frames, hence for each r is transitive on pairs x'_1, x'_2 of vectors with $\|x'_1\| = 1$, $\|x'_2\| = r$, $(x'_1, x'_2) = 0$. See [8] for the transitivity. Now

(5.2) $$L = G_2 \cdot Sp(1; \mathbb{R}) \subset O(7) \cdot Sp(1; \mathbb{R}) = G$$

is transitive on every G-orbit in U.

5B'. $G = O(3,4) \cdot Sp(1; \mathbb{R})$. This is the first case of (3.15) with $(k, \ell) = (7, 0)$ and $v = 2$. Let U be all $x \in W = \mathbb{R}^{7 \times 2}$ whose column span is a positive definite

2-plane in $W_1 = \mathbb{R}^{3,4}$. The subgroup $G_2^* \subset O(3,4)$, the noncompact real group of type G_2, is transitive on (positive) orthonormal 2-frames. So, as above,

(5.3) $$L = G_2^* \cdot Sp(1; \mathbb{R}) \subset O(3,4) \cdot Sp(1; \mathbb{R}) = G$$

is transitive on every G-orbit in U.

5C. $G = O(8) \cdot Sp(1; \mathbb{R})$, the first case of (3.15) with $(k, \ell) = (8, 0)$ and $v = 2$. Let U = all rank 2 matrices in $W = \mathbb{R}^{8 \times 2}$. Just as in §5B.

(5.4) $$L = Spin(7) \cdot Sp(1; \mathbb{R}) \subset O(8) \cdot Sp(1; \mathbb{R}) = G$$

is transitive on every G-orbit in U.

5C'. $G = O(4,4) \cdot Sp(1; \mathbb{R})$, the first case of (3.15) with $(k, \ell) = (4, 4)$ and $v = 2$. Just as in §5B',

(5.5) $$L = Spin(3,4) \cdot Sp(1; \mathbb{R}) \subset O(4,4) \cdot Sp(1; \mathbb{R}) = G$$

is transitive on every G-orbit in the space of x in $W = \mathbb{R}^{8 \times 2}$ whose column span is a positive definite 2-plane.

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Along the lines of [2, Theorem 1] we note that, for the subgroups $L \subset G$ considered in this paper, equality $A(\pi) = A(\pi|_L)$ of commuting algebras shows, in the notation of Corollary 2.2, that $\pi|_L$ is multiplicity-free. So, for ν-almost-every α in the decomposition $\pi = \int_{\hat{G}} \pi_\alpha \, d\nu(\alpha)$ into irreducibles, the restriction $\pi_\alpha|_L$ determines π_α.