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# Representations of Reductive and Parabolic Groups

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## §1. Introduction.

This article attempts to sketch the background for the theory of representations of linear groups, and to indicate by example of a new Fourier inversion formula that the representation theory of affine groups is very different.

The natural context of linear groups is that of reductive Lie groups, and the corresponding setting for affine groups is that of parabolic subgroups. Neither can be effectively studied without the other.

I'll recall the finite dimensional representation theory of reductive groups, to establish terminology, and then sketch some parts of general unitary representation theory. That includes $C^\infty$ and analytic vectors and infinitesimal characters. Then the role of large compact subgroups and distribution character is discussed, leading to the nondegenerate (tempered) series and the Plancherel formula for reductive groups. Finally, for contrast I show the Plancherel formula for the affine group. In the reductive case, most representations of physical interest are absent from the Plancherel formula. In the affine case, the Plancherel formula uses just one representation instead of a many-parameter family, and that is a physically interesting representation.

## §2. Finite Dimensional Representations of Reductive Groups

Recall that a Lie algebra $\mathfrak{g}$ has adjoint representation $\xi \mapsto \mathrm{ad}(\xi)$ given by $\mathrm{ad}(\xi)\eta = [\xi, \eta]$ and has Cartan-Killing form

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$\langle \zeta, \eta \rangle = \mathrm{trace}(\mathrm{ad}(\zeta)\mathrm{ad}(\eta))$ . The algebra $\mathfrak{g}$ is semisimple if $\langle\ ,\ \rangle$ is nonsingular. Equivalent: $\mathfrak{g}$ is direct sum of simple algebras. Here a Lie algebra is simple if it is noncommutative and has no proper ideal. Examples: the Lie algebra $\mathfrak{sl}(n;\mathbb{R})$ of the special linear group $SL(n,\mathbb{R})$ , which is all $n \times n$ real matrices of determinant 1, or the Lie algebra $\mathfrak{so}(2, 4)$ of the conformal group $SO(2, 4)$ .

A real or complex Lie algebra $\mathfrak{g}$ is reductive if it has a faithful (kernel = 0) completely reducible (every invariant subspace has an invariant complement) finite dimensional representation. Equivalent: $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ where $\mathfrak{z}$ is the center (all $\xi \in \mathfrak{g}$ with $[\xi, \mathfrak{g}] = 0$) and the $\mathfrak{g}_i$ are simple ideals. Examples: semi-simple Lie algebras (the case $\mathfrak{z} = 0$ ) , and the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$ of the general linear group $GL(n,\mathbb{R})$ .

A Lie group is called simple, semisimple or reductive if its Lie algebra has that property. Example: any compact Lie group is reductive.

A reductive subalgebra in a reductive Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{s}$ such that $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{s}}$ is completely reducible. A maximal commutative reductive subalgebra is a Cartan subalgebra (CSA). In the real case, $\mathfrak{h}$ is Cartan in $\mathfrak{g}$ iff $\mathfrak{h}_{\mathbb{C}}$ is Cartan in $\mathfrak{g}_{\mathbb{C}}$ . If $\mathfrak{g}$ is a semisimple Lie algebra of matrices, then its CSA are just the maxima among its subalgebras diagonalizable over $\mathbb{C}$ .

If $G$ is a reductive Lie group and $\mathfrak{h}$ is a CSA in its Lie algebra $\mathfrak{g}$, then the corresponding Cartan subgroup (CSG) is the centralizer,

$$H = \{g \in G : \mathrm{Ad}(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}\} .$$

It has Lie algebra $\mathfrak{h}$ . Example: the CSG in a compact connected Lie group are just the maximal tori.

Fix a reductive complex Lie algebra $\mathfrak{g}$ , a CSA $\mathfrak{h}$ in $\mathfrak{g}$ , and a completely reducible finite dimensional representation $\pi: \mathfrak{g} \to \mathfrak{gl}(V_\pi)$ . Then $V_\pi$ has a basis $\{v_1, \ldots, v_q\}$ of simultaneous eigenvectors,

$$\pi(\xi)\cdot v_j = \lambda_j(\xi)v_j \quad \text{where} \quad \lambda_j: \mathfrak{h} \to \mathbb{C} \quad \text{linear} .$$

These $\lambda_j$ are the *weights* of $\pi$ . Example: the nonzero weights of $ad_{\mathfrak{g}}$ are the *roots* or $\mathfrak{h}$ *-roots* of $\mathfrak{g}$ . Decompose $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the semisimple part. Then $\mathfrak{h} = \mathfrak{z} + \mathfrak{h}'$ where $\mathfrak{h}'$ is a CSA in $\mathfrak{g}'$ . The roots annihilate $\mathfrak{z}$ , span the linear dual $(\mathfrak{h}')^*$ , and sit in a real form $\mathfrak{h}^*_{\mathbb{R}}$ of $(\mathfrak{h}')^*$ .

Write $\Delta$ for the set of $\mathfrak{h}$ -roots of $\mathfrak{g}$ . If $\alpha \in \Delta$ then $(\alpha = 0)$ defines a hyperplane in $\mathfrak{h}^*_{\mathbb{R}}$ , the real span of $\Delta$ , and $\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta} (\alpha = 0)$ is a disjoint union of convex open cones cut out by these hyperplanes. These cones are called *Weyl chambers*. If $\mathcal{C}$ is one of them, then

$$\Delta^+ = \{\alpha \in \Delta: \ \alpha > 0 \ \text{ on } \ \mathcal{C}\}$$

is a *positive root system*. Its main properties are (i) $\Delta = \Delta^+ \cup \{-\alpha: \alpha \in \Delta^+\}$, disjoint, (ii) if $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta^+$ . Now the roots are partially ordered by

$$\alpha > \beta \ \text{ if } \ \alpha - \beta > 0 \ \text{ on } \ \mathcal{C} .$$

The minimal positive roots are called *simple roots*. Let $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ be the simple root system. Then $\Psi$ is a basis of $\mathfrak{h}^*_{\mathbb{R}}$ . More precisely, if $\alpha \in \Delta$ then $\alpha = \sum n_j \psi_j$ where the $n_j$ are integers, all $\geq 0$ if $\alpha \in \Delta^+$ , all $\leq 0$ if $-\alpha \in \Delta^+$ .

If $\pi$ is irreducible, then its weight system $\Delta_\pi$ also is partially ordered,

$$\lambda > \lambda' \ \text{ iff } \ \lambda - \lambda' > 0 \ \text{ on } \ \mathcal{C} \qquad \text{(definition)}$$

$$\text{iff } \ \lambda - \lambda' = \sum n_j \psi_j , \quad 0 \leq n_j \in \mathbb{Z} \quad \text{(theorem)} .$$

There is a unique maximal weight $\nu_\pi$ , and $\nu_\pi$ determines $\pi$ up to equivalence. For example, every weight sits in a chain of weights $\{\nu_\pi - \sum_{k=0}^{m} \psi_{i_k}\}$



where the $\psi_{i_k} \in \Psi$ . We carry the Cartan-Killing form over to $\mathfrak{h}'^*$ by duality. Then it is positive definite on $\mathfrak{h}^*_{\mathbb{R}}$ . Extend it in any way to $\mathfrak{h}^*$ so that $\mathfrak{z}^* \perp \mathfrak{h}'^*$ . Then a linear functional $\nu \in \mathfrak{h}^*$ is the highest weight of an irreducible representation if and only if the $2\langle \nu, \psi\rangle / \langle \psi, \psi\rangle$ are integers $\geq 0$ , for all $\psi$ in the simple root system $\Psi$ .

If $\mathfrak{g}$ is a reductive real Lie algebra we apply the above considerations to $\mathfrak{g}_{\mathbb{C}}$ .

If $G$ is a connected reductive Lie group, or at least is connected modulo its center, we apply the notions of weight and highest weight from its Lie algebra $\mathfrak{g}$ .

Let $\pi: G \to GL(V_\pi)$ be a finite dimensional representation. Its *character* is

$$\Theta_\pi: G \to \mathbb{C} \ \text{ by } \ \Theta_\pi(x) = \text{trace } \pi(x) .$$

If $f \in L_1(G)$ then $\pi(f) = \int_G f(x)\pi(x)dx$ is an operator on $V_\pi$ , and

$$\text{trace } \pi(f) = \int_G f(x) \text{ trace } \pi(x)dx = \Theta_\pi(f)$$

where $\Theta_\pi$ is viewed as a distribution. Later we will need this interpretation of "character."

When $G$ is a compact connected Lie group, $H$ the CSG for a CSA $\mathfrak{h} \subset \mathfrak{g}$, and $\pi$ irreducible with highest weight $\nu$ , the *Weyl character formula* gives $\Theta_\pi$ explicitly, as a distribution, as follows. First, it is a locally $L^1$ function, real-analytic on the *regular set* $G'$ . Here $G'$ consists of all $x \in G$ with minimal centralizer, i.e. such that $\{\xi \in \mathfrak{g}: Ad(x)\xi = \xi\}$ is a CSA. It is dense and open. Second, $\Theta_\pi(gxg^{-1}) = \Theta_\pi(x)$ , so $\Theta_\pi$ is specified on $H \cap G'$ . There it is invariant under the *Weyl group* $W = \{Ad(w)|_{\mathfrak{h}} : w \in G , \ Ad(w)\mathfrak{h} = \mathfrak{h}\}$ . Third, on $H \cap G'$ the character is

$$\Theta_\pi(\exp\,\xi) = \frac{\sum_{w\in W}\det(w)e^{(\nu+\rho)(w\xi)}}{\sum_{w\in W}\det(w)e^{\rho(w\xi)}}$$

$$= (\mathrm{const})\ \frac{\sum \det(w)e^{(\nu+\rho)(w\xi)}}{\prod_{\alpha\in\Delta^+}(e^{\alpha(\nu)/2}-e^{-\alpha(\nu)/2})}$$

Here $\rho$ denotes half the sum of the positive roots. If $f: G \to \mathbb{C}$ is a reasonable function, e.g. $C^\infty$, that leads to a Fourier Inversion formula

$$f(x) = \sum_{\hat G} \Theta_{\pi_\nu}(r_x f)\cdot\dim(\pi_\nu)\ .$$

Here $r_x f$ is the right translate $g \mapsto f(gx)$. $\hat G$ is the set of (equivalence classes of) irreducible representations of $G$; it is parameterized by highest weights $\nu$ as described above. The "Plancherel measure" here is

$$\dim(\pi_\nu) = \prod_{\alpha\in\Delta^+}\frac{\langle\alpha,\nu+\rho\rangle}{\langle\alpha,\rho\rangle}$$

times counting measure. The combination $\dim(\pi_\nu)\Theta_{\pi_\nu}$ is often called the "normalized character". We will see analogs for noncompact groups.

### §3. General Theory of Unitary Representations

Let $\pi$ be a unitary representation of a locally compact group $G$ on a Hilbert space $H$. This means that $\pi$ is a homomorphism from $G$ to the unitary group of $H$ that satisfies these equivalent continuity conditions

(i) the map $G \times H \to H$, $(g, v) \mapsto \pi(g)v$, is continuous;

(ii) if $v \in H$ then $G \to H$, by $g \mapsto \pi(g)v$, is continuous;

(iii) if $u, v \in H$ then the "matrix coefficient" $f_{u,v}: G \to \mathbb{C}$, by $f_{u,v}(g) = \langle u, \pi(g)v\rangle$, is continuous.

Here $\langle\ ,\ \rangle$ is the scalar product in $H$, taken linear in the first variable and conjugate linear in the second.

The representation $\pi$ is called irreducible or topologically irreducible if its representation space $H$ has no nontrivial closed $\pi(G)$-invariant subspace. In that case one has a version of Schur's Lemma: if a bounded operator on $H$ commutes with every $\pi(g)$, $g \in G$, then it is scalar. Consequence: if $Z$ is the center of $G$ then $\pi|_Z$ specifies a homomorphism

$$\zeta_\pi: Z \to \mathbb{C} = \{c \in \mathbb{C}: |c| = 1\}$$

called the central character of $\pi$.

The representation $\pi$ of $G$ defines a $*$-representation of $L_1(G)$ on $H$ by $\pi(f) = \int_G f(x)\pi(x)dx$. Evidently $\|\pi(f)\| \leqslant \|f\|_1$. So we have a family of continuous seminorms $\|f\|_\pi = \|\pi(f)\|$ on $L_1(G)$, as $\pi$ varies over the (equivalence classes of) unitary representations of $G$. The completion of $L_1(G)$ with respect to this family is called the $C^*$-algebra of $G$, denoted $C^*(G)$. Note that $\pi$ extends by continuity from $L_1(G)$ to a $*$-representation of $C^*(G)$. If a closed subspace of $H$ is invariant by one of $G$, $L_1(G)$ or $C^*(G)$, it is invariant under the others also.

When $G$ is a Lie group we also have associated representations $d\pi$ of the Lie algebra $\mathfrak{g}$ and the universal enveloping algebra $\mathfrak{G} = \mathcal{U}(\mathfrak{g}_\mathbb{C})$. They are more important than the representations of $L_1(G)$ and $C^*(G)$ when it comes to concrete matters. We look at that now.

Now let $G$ be a Lie group. We suppose that the topology is not bizarre: that it is countable at infinity. Retain $\pi$ and $H$.

A vector $v \in H$ is differentiable ($= C^\infty$) if $g \mapsto \pi(g)v$ is a $C^\infty$ map $G \to H$, i.e. if every coefficient $f_{u,v}: G \to \mathbb{C}$, $u \in H$, is a $C^\infty$ function. Write $H_\infty$ for the space of $C^\infty$ vectors in $H$. It is a dense subspace: if $v \in H$ and $f \in C_0^\infty(G) \subset L_1(G)$ then $\pi(f)v \in H_\infty$; so $\pi(C_0^\infty(G))\cdot H \subset H_\infty$. The differentiable representation $\pi_\infty$ of $G$, associated to $\pi$, is $\pi_\infty(g) = \pi(g)|_{H_\infty}$. Its virtue is that it lifts to $\mathfrak{g}$:

$$\pi_\infty(\xi)v = d\pi(\xi)v = \frac{d}{dt}\,\pi(\exp(t\xi)v)|_{t=0} \quad \text{for } v \in H_\infty\ .$$

$\mathfrak{G} = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is the associative algebra generated by $\mathfrak{g}_{\mathbb{C}}$. For this, view $\mathfrak{g}_{\mathbb{C}}$ as the left-invariant vector fields on $G$, i.e. the first order left-invariant differential operators that annihilate constants. Then $\mathfrak{G}$ consists of all left-invariant differential operators, and its center $\mathfrak{Z}$ $= \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ consists of those that are both left- and right-invariant, the "Casimir operators" of $G$. $\pi_\infty$ lifts to $\mathfrak{G}$ as an associative algebra homomorphism. Thus $\mathfrak{G}$ acts on $H$ by densely defined (on $H_\infty$) operators.

Suppose that every $\mathrm{Ad}(g)$, $g \in G$, gives an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$. Then every $\mathrm{Ad}(g)$ acts trivially on $\mathfrak{Z}$, and so every $\pi_\infty(g)$ commutes with every $\pi_\infty(D)$, $D \in \mathfrak{Z}$. A tricky variation on Schur's Lemma: if $\pi$ is irreducible then $\pi_\infty$ represents $\mathfrak{Z}$ by scalars on $H_\infty$. Consequence: then $\pi_\infty|_{\mathfrak{Z}}$ specifies an associative algebra homomorphism $\chi_\pi : \mathfrak{Z} \to \mathbb{C}$ called the infinitesimal character of $\pi$. It is very important because of an implicit role in geometric quantization, but more because of the differential equations $Df_{u,v} = \chi_\pi(D)f_{u,v}$ for $u, v \in H_\infty$, $D \in \mathfrak{Z}$.

The differentiable representation distorts the reducibility picture of the original representation. For example let $G = \mathbb{R}$, acting on $H = L_2(\mathbb{R})$ by translations

$$[\pi(x_0)v](x) = v(x + x_0) .$$

Then $H_\infty$ is the Schwartz space

$$S = \{v \in C^\infty(\mathbb{R}) : \text{every } \frac{d^n v}{dx^n} \in L_2(\mathbb{R})\} .$$

If $I$ is an open interval then $\{v \in S : v|_I = 0\}$ is $\pi_\infty(\mathfrak{g})$-invariant, but not $\pi_\infty(G)$-invariant, and its closure is not $\pi(G)$-invariant.

This difficulty is avoided by means of analytic vectors: $v \in H$ is analytic if $g \mapsto \pi(g)v$ is an analytic map $G \to H$, equivalently if $f_{u,v} : G \to H$ is real-analytic for all $u \in H$. The analytic vectors form a $\pi(G)$-stable subspace $H_\omega \subset H_\infty$ which is $\pi_\infty(\mathfrak{g})$-invariant. Write $\pi_\omega$ for the





representations of $G$, $\mathfrak{g}$ and $\mathfrak{G}$ on $H_\omega$. Given $v \in H_\omega$ there is a neighborhood $\mathcal{O}$ of $0$ in $\mathfrak{g}$ such that

$$\xi \in \mathcal{O} \Rightarrow \sum_{m=0}^{\infty} \frac{1}{m!}\pi_\omega(\xi)^m v \text{ converges to } \pi_\omega(\exp \xi)v$$

Consequence: every $\pi_\omega(\mathfrak{g})$-stable subspace has $\pi(G)$-stable closure; in particular, if $\pi$ is irreducible every nonzero $\pi_\omega(\mathfrak{g})$-stable subspace of $H_\omega$ is dense in $H$. To use this we need the rather deep existence theorem of Nelson: $H_\omega$ is dense in $H$.

Now let $K$ be a compact subgroup of $G$. As usual, $\hat{K}$ denotes the set of unitary equivalence classes of irreducible representations of $G$. If $\kappa \in \hat{K}$ has normalized character

$$\tau_\kappa(k) = (\dim \kappa)\operatorname{trace} \kappa(k)$$

then $\pi|_K(\overline{\tau_\kappa})$ is orthogonal projection of $H$ onto the maximal subspace $H(\kappa)$ on which $K$ acts by a multiple of $\kappa$. $H(\kappa)$ is called the $\kappa$-isotypic subspace of $H$, and $H$ is the Hilbert space direct sum of the $H(\kappa)$. More precisely, if $v \in H_\infty$ then $\sum_{\kappa\in K} \pi(\overline{\tau}_\kappa)v$ converges absolutely to $v$. Now denote

$$H_\omega(\kappa) = H_\omega \cap H(\kappa) \quad\text{and}\quad H_K = \sum_{\kappa\in\hat{K}} H_\omega(\kappa) \quad \text{(finite sums)}$$

then $H_K \subset H_\omega \subset H_\infty \subset H$. Fact: $H_K$ is dense in $H$. Important consequences: (i) if $\dim H(\kappa) < \infty$ then $H(\kappa) \subset H_\omega$, (ii) $H_K$ is $\pi_\omega(\mathfrak{G})$-invariant.

## §4. The Role of Large Compact Subgroups

Now let us specialize to the case where $G$ is a reductive group that satisfies some technical conditions whose importance will appear soon:

(i) $G/G^0$ is finite, where $G^0$ is the identity component

(ii) if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$,

(iii) the analytic subgroup for $[\mathfrak{g}, \mathfrak{g}]$ has finite center.

A group that satisfies these is said to be in "class $H$." Here (i) and (iii) say that the identity component $Z^0$ of the center sits nicely in $G$, with $G/Z^0$ almost a linear semisimple group, and (ii) ensures that every $\pi \in \hat{G}$ has an infinitesimal character.

Let $K$ be a maximal compact subgroup of $G$. A very deep result of Harish-Chandra says: if $\pi \in \hat{G}$ and $\kappa \in \hat{K}$ then $\mathrm{mult}(\kappa, \pi|_K) \leq \dim \kappa$. This result has two fundamental consequences.

The first consequence is the existence of the global character:

(i) if $f \in C_0^\infty(G)$ then $\pi(f)$ is of trace class,

(ii) $C_0^\infty(G) \ni f \mapsto \operatorname{trace} \pi(f)$ is a distribution on $G$.

The distribution $\Theta_\pi(f) = \operatorname{trace} \pi(f)$ is the global (distribution) character of $\pi$. A class $\pi \in \hat{G}$, in fact any K-finite unitary representation class $\pi$, is specified by $\Theta_\pi$. Furthermore, $\Theta_\pi$ can be analyzed through its properties of

invariance: $\Theta_\pi(f \bullet \mathrm{Ad}(g)) = \Theta_\pi(f)$, all $f, g$;

eigendistribution: $D\Theta_\pi = \chi_\pi(D)\Theta_\pi$, all $D \in \mathfrak{Z}$.

Here $\chi_\pi$ is the infinitesimal character. For example, it is not too difficult to see from this that $\Theta_\pi$ is real-analytic on the regular set $G' = \{g \in G$: the fixed point set of $\mathrm{Ad}(g)$ is a CSA in $\mathfrak{g}\}$. Much more delicate is the fact that $\Theta_\pi$ is a locally integrable function on $G$, so $\Theta_\pi(f)$ can be evaluated by integrating $f$ against the analytic function $\Theta_\pi$ on the regular set $G'$.

The second consequence of K-finiteness is the utility of $H_K$ as a $(\mathfrak{g}, K)$-module characterizing $\pi$. It is called the Harish-Chandra module and is the basic object in the relatively new infinitesimal approach to semisimple representation theory. The class $\pi$ is specified by any irreducible submodule of $H_K$ for the centralizer $\mathfrak{G}^K$, in particular by the Harish-Chandra module structure of $H_K$.





## §5. Unitary Representations of Reductive Groups.

$G$ is a reductive Lie group of the class $\mathcal{H}$ described above. Thus, any irreducible $\pi \in \hat{G}$ has infinitesimal and global characters $\chi_\pi$ and $\Theta_\pi$, and we have the corresponding Harish-Chandra module $H_K$ for $H = H_\pi$. Write $\theta$ for the Cartan involution of $G$ with fixed point set $K$.

If $\mathfrak{h}$ is a $\theta$-invariant CSA in $\mathfrak{g}$, its $\theta$-eigenspace decomposition

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \quad \text{with} \quad \mathfrak{t} = \mathfrak{h} \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{a} = \{\xi \in \mathfrak{h} : \theta\xi = -\xi\}$$

extends to a decomposition of the CSG

$$H = T \times A \quad \text{with} \quad T = H \cap K \quad \text{and} \quad A = \exp(\mathfrak{a}).$$

The centralizer of $T$ in $G$ splits as $M \times A$ where $M = \theta(M)$ and $M$ is a reductive Lie group of class $\mathcal{H}$. Also, $T$ is a compact CSG in $M$, and so a certain type of representation, of the

$$\text{discrete series } \hat{M}_{disc} \subset \hat{M}$$

is present. We will describe the discrete series and then indicate how every class in $\hat{G}$ comes out of these various $\hat{M}_{disc}$.

We describe $\hat{M}_{disc}$ in three stages: for the identity component $M^0$, for

$$M^\dagger = \{m \in M: \mathrm{Ad}(m) \text{ is inner on } \mathfrak{m}\},$$

and then for $M$ itself.

Choose a positive $\mathfrak{t}_{\mathbb{C}}$-root system $\Delta_{\mathfrak{t}}^+$ for $\mathfrak{m}_{\mathbb{C}}$ and denote

$$\Lambda_{\mathfrak{t}}^+ = \{\lambda \in \mathfrak{t}_{\mathbb{C}}^*: e^\lambda \in \widehat{T^0} \text{ and } \langle\lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta_{\mathfrak{t}}^+\}.$$

Harish-Chandra's famous result: if $\lambda \in \Lambda_{\mathfrak{t}}^+$ then $M^0$ has a unique discrete series representation, say $\eta_\lambda$, whose distribution character is given on $M' \cap T^0$ by

$$\Psi_{\eta_\lambda}(\exp \xi) = \prod_{\alpha \in \Delta_{\mathfrak{t}}^+} (e^{\alpha(\xi)/2} - e^{-\alpha(\xi)/2}) \cdot \sum_{w \in W} \mathrm{sgm}(w) e^{\lambda(w\xi)}$$

where $W$ is the Weyl group $W(M^0, T^0)$. Further, $\eta_\lambda = \eta_{\lambda'}$ in $\widehat{M^0}$ just when $\lambda' \in W(\lambda)$. This exhausts the discrete series of $M^0$. If $M^0$ is compact and $\rho_{\mathfrak{k}} = \frac{1}{2}\sum_{\Delta^+_{\mathfrak{k}}} \alpha$, then $\eta_\lambda$ is the representation of highest weight $\lambda - \rho_{\mathfrak{k}}$. In any case, it has the same infinitesimal character as that finite dimensional representation.

As defined above, $M^\dagger = Z_M(M^0)M^0$ where we write $Z_*(\cdot)$ for the centralizer of $\cdot$ inside $*$. Note $Z_M(M^0) \cap M^0 = Z_{M^0}$, center of $M^0$. Note also that the discrete series class $\eta_\lambda \in (M^0)^\wedge_{disc}$ has central character

$$\zeta = e^{\lambda-\rho_{\mathfrak{k}}}\big|_{Z_{M^0}} .$$

Now it is easy to check that

$$(M^\dagger)^\wedge_{disc} = \{\chi \otimes \eta_\lambda : \lambda \in \Lambda^+_{\mathfrak{k}} \text{ and } \chi \in Z_M(M^0)^\wedge \text{ with } \chi|_{Z_{M^0}} = \zeta_\lambda\} .$$

It is slightly less routine to verify

$$\hat{M}_{disc} = \{\eta_{\chi,\lambda} = \mathrm{Ind}_{M^\dagger\uparrow M}(\chi\otimes\eta_\lambda) : \chi\otimes\eta_\lambda \in (M^\dagger)^\wedge_{disc}\} .$$

That requires the class $H$ condition that every $\mathrm{Ad}(m)$, $m \in M$, is inner on $\mathfrak{m}_{\mathbb{C}}$.

Now decompose $\mathfrak{g} = (\mathfrak{m}+\mathfrak{a}) + \sum_{\Delta_{\mathfrak{a}}} \mathfrak{g}^\gamma$ where $\Delta_{\mathfrak{a}} \subset \mathfrak{a}^* \setminus \{0\}$ is the set of $\mathfrak{a}$-roots of $\mathfrak{g}$. In the case where $\mathfrak{h}$ is maximally split, $\Delta_{\mathfrak{a}}$ is the "real root system" ( = "restricted root system") of $\mathfrak{g}$. Choose a positive system $\Delta^+_{\mathfrak{a}}$ as usual, let $\mathfrak{n} = \sum_{\gamma\in\Delta^+_{\mathfrak{a}}} \mathfrak{g}^\gamma$ and $N = \exp(\mathfrak{n})$, and consider the cuspidal parabolic subalgebra of $\mathfrak{g}$ and subgroup of $G$ given by

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \quad \text{and} \quad P = MAN .$$

Different choices of $\Delta^+_{\mathfrak{a}}$ lead to associated parabolics that are not necessarily conjugate. Nevertheless, the representation classes

$$\pi_{\chi,\lambda,\nu} = \mathrm{Ind}_{P\uparrow G}(\eta_{\chi,\eta} \otimes e^{\sqrt{-1}\nu}) ,$$





with $\lambda \in \Lambda^+_{\mathfrak{k}}$ and $\chi \in Z_M(M^0)^\wedge$ restricting to $\zeta_\lambda$ and with $\nu \in \mathfrak{a}^*$, are independent of choice of $\Delta^+_{\mathfrak{k}}$. They thus depend only on the G-conjugacy class of H. Almost all are irreducible, and in any case their irreducible constituents form a nondegenerate series that I call the H-series, but which has no general standard name except in the cases

- H compact : the discrete series of G
- T a CSG in K : the fundamental series of G,
- $\mathfrak{a}$ maximal : the principal series of G.

In any case, the discrete series is the basic object here.

G has only finitely many conjugacy classes of CSG. Choose θ-stable representations $H_1, \ldots, H_p$ of these classes. If $1 \leq i \leq p$ consider the $H_i$-series

$$\hat{G}_i = \{\text{irred. constituents of the } \{\pi_{\chi,\lambda,\nu} : \lambda \in \Lambda^+_{\mathfrak{k}_i},\ \chi \in Z_{M_i}(M_i^0)^\wedge_{\zeta_\lambda},\ \nu \in \mathfrak{a}_i^*\}$$

and their union, which is the reduced dual

$$\hat{G}_{red} = \bigcup_{1\leq i\leq p} \hat{G}_i$$

in the sense that it is the support of Plancherel measure for a certain topology on $\hat{G}$. More precisely, there are meromorphic functions on the $(\mathfrak{a}_i)_{\mathbb{C}}$ regular on $\mathfrak{a}_i$ which combine with the $\deg(\chi)$ and the formal degree $|\prod_{\alpha\in\Delta^+_{\mathfrak{k}_i}} \langle\lambda, \alpha\rangle|$ of $\eta_\lambda$ to give us measures $d\mu_i(\chi, \lambda, \nu)$ on $\hat{G}_i$, such that G has a Plancherel formula

$$f(x) = \sum_{i=1}^{p} \int_{\hat{G}_i} \Theta_{\pi_{\chi,\eta,\nu}}(r_x f)\, d\mu_i(\chi, \lambda, \nu)$$

where $r_x f$ is the right translate $g \mapsto f(xg)$ as before. Here the distribution character $\Theta_{\pi_{\chi,\lambda,\nu}}$ comes out of $\Psi_{\eta_\lambda}$, trace $\chi$ and $e^{\sqrt{-1}\nu}$, and $f: G \to \mathbb{C}$ is any smooth rapidly decreasing (Schwartz class) function. As to the latter, it can be shown that the reduced dual $\hat{G}_{red}$ coincides with the set of

<u>tempered</u> <u>representations</u>: $\{\pi \in \hat{G}: \Theta_\pi$ is tempered$\}$

where tempered distribution is defined in terms of an appropriate Schwartz space on $G$ . This notion is, in fact, basic to Harish Chandra's construction of the discrete series representations $\eta_\lambda$ .

An <u>admissible</u> <u>representation</u> is a pre-Hilbert K-finite $(\mathfrak{g}, K)$-module. Harish-Chandra modules of K-finite, in particular irreducible, unitary representations of $G$ , appear to be the main case. But in fact one can discuss growth rate of coefficients of admissible representations, and this leads to a number of relatively recent developments. One of them is a classification up to infinitesimal equivalence of irreducible bounded representations of $G$ on a **Banach** space. Unfortunately there still is a unitarization problem there. See Duflo's complete description for $Sp(2; \mathbb{C})$ and Šijački's for $SL(3; \mathbb{R})$ and closely associated groups. Another is a better understanding of $L_p$ and Sobolev norm behavior of unitary representations, which in turn is instrumental in the Atiyah-Schmid construction of the discrete series.

## <u>§6. Fourier Inversion on the Affine Group</u>

We write $\underline{\underline{A}}(n)$ for the affine group $\mathbb{R}^n \cdot GL(n, \mathbb{R})$ on euclidean n-space. It has multiplication $(x, A)(x', A') = (x + Ax', AA')$ and matrix representation $(x, A) \leftrightarrow \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix}$ where $x$ is $n \times 1$ and $A$ is $n \times n$ .

$\underline{\underline{A}}(1)$ has right invariant measure $d\begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} = a^{-1}dxda$ . Choose a nontrivial unitary character on the subgroup $a = 1$ , say

$$\chi_1\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = e^{ix} ,$$

then $\underline{\underline{A}}(1)$ has just one "generic" representation,

$$\eta_1 = \mathrm{Ind}_{(\text{subgroup } a=1)\uparrow \underline{\underline{A}}(1)} \chi_1 .$$

A formal calculation: if $f: \underline{\underline{A}}(1) \to \mathbb{C}$ is $C^\infty$ and compactly supported then





$$f\begin{pmatrix} 1 & 0 \\ y & b \end{pmatrix} = c \cdot \mathrm{trace}\ \eta_1\left(\frac{\partial}{\partial x}\left(r\begin{pmatrix} 1 & 0 \\ y & b \end{pmatrix} f\right)\right)$$

for a certain constant $c$ . That is the Fourier inversion formula for $\underline{\underline{A}}(1)$. Note that $\frac{\overset{\rightharpoonup}{\partial}}{\partial x}$ compensated non-unimodularity.

**Explicit** inversion formulae are known for many parabolic subgroups of reductive groups, but not for the $\underline{\underline{A}}(n)$ , $n > 1$ .

$\underline{\underline{A}}(2)$ has right invariant measure $d\begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix}$ that is product of the two entries of $dx$ and the four entries of $dA \cdot A^{-1}$ . Choose a nontrivial unitary character, say $\chi_2$ , on the translation group $A = I$ in $\underline{\underline{A}}(2)$ . It has stabilizer in the linear group $x = 0$ ,

$$L = \{A \in GL(2; \mathbb{R}): \ A \ \text{fixes} \ \chi_2\} \cong \underline{\underline{A}}(1) .$$

Notice that $\chi_2$ extends to (translations)$\cdot$L by $\tilde{\chi}_2\begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} = \chi_2\begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix}$ . The Bargman-Mackey little group method says that $\underline{\underline{A}}(2)$ has just one generic representation,

$$\eta_2 = \mathrm{Ind}_{(\text{translations})\cdot L \uparrow \underline{\underline{A}}(2)} (\tilde{\chi}_2 \otimes \eta_1)$$

where $\eta_1$ acts through $L \cong \underline{\underline{A}}(1)$ .

Proceed recursively: choose a nontrivial unitary character $\chi_{n+1}$ on the translation subgroup of $\underline{\underline{A}}(n + 1)$ , note that

$$L = \{A \in GL(n + 1; \mathbb{R}): \ A \ \text{fixes} \ \chi_{n+1}\} \cong \underline{\underline{A}}(n) ,$$

extend $\chi_{n+1}$ to $\tilde{\chi}_{n+1} \in [(\text{translations})\cdot L]\hat{}$ by ignoring $L$ , and note that $\underline{\underline{A}}(n + 1)$ has just one generic representation,

$$\eta_{n+1} = \mathrm{Ind}_{(\text{translations})\cdot L \uparrow \underline{\underline{A}}(n+1)} (\tilde{\chi}_{n+1} \otimes \eta_n)$$

where $\eta_n$ acts through $L \cong \underline{\underline{A}}(n)$ .

The group $\underline{\underline{A}}(n)$ has modular function $\delta\begin{pmatrix} 1, & 0 \\ x' & A' \end{pmatrix} = \det A'$ . General theory provides an unbounded positive selfadjoint operator semi-invariant of type $\delta$ on the representation space of $\eta_n$ . Uniqueness of $\eta_n$ lets us carry that operator back to an invertible positive selfadjoint semi-invariant operator $D$ on $L^2(\underline{\underline{A}}(n))$ . Then we get a Fourier inversion formula

$$f(g) = \text{trace}\ \eta_n(D(r(g)f))$$

as in the case $n = 1$. In fact, for $n > 1$ the operator $D$ remains mysterious, as we now explain.

The Lie algebra of $\underline{\underline{A}}(n)$ is $\mathfrak{a}(n) = \{\begin{pmatrix} 0 & 0 \\ x & \alpha \end{pmatrix}\}$ where $x$ is $n \times 1$ and $\alpha$ is $n \times n$. Its real linear dual space is $\mathfrak{a}(n)^* = \{\begin{pmatrix} 0 & \xi \\ 0 & \alpha \end{pmatrix}\}$ where $\xi$ is $1 \times n$ and $\alpha$ is $n \times n$, and there the coadjoint action of $\underline{\underline{A}}(n)$ is

$$Ad^*\begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix}: \begin{pmatrix} 0 & \xi \\ 0 & \alpha \end{pmatrix} \mapsto \begin{bmatrix} 0 & \xi A^{-1} \\ 0 & x\xi A^{-1} + A\alpha A^{-1} \end{bmatrix}$$

From this, for $n > 1$ there is no nonconstant real analytic function on $\mathfrak{a}(n)^*$ semi-invariant of type a power of $\det(A)$. So we cannot hope to find $D$ as a pseudo-differential operator on $\underline{\underline{A}}(n)$.

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