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FOUNDATIONS OF REPRESENTATION THEORY FOR SEMISIMPLE LIE GROUPS

CONTENTS

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0. Introduction

In this paper I hope to communicate some of the basic facts on unitary representations of semisimple Lie groups, staring with the material on general harmonic analysis in R. Blattner's paper, and leading into the more advanced subjects in the papers of M. Atiyah and W. Schmid (the nature of distribution characters; analytic construction of the discrete series), P. Trombi (Plancherel formula for semisimple Lie groups) and V. S. Varadarajan (the infinitesimal approach, including algebraic construction of the discrete series).

The material to be covered is reasonably described by the Table of Contents and the three chapter introductions, so I will only add a word on references.

Chapter I. A good general reference is V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Prentice-Hall, 1974. Also good: J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag GTM #9, 1972, as well as the excellent introductions by Hochschild, Mostow, Samelson and others. This material is absolutely standard.

Chapter II. Besides the books of Varadarajan and Humphreys, one should add Chapter 7 of J. Dixmier, *Algèbres Enveloppantes*, Gauthier-Villars, 1974, and the last chapter of S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, 1972.

Chapter III. Here the most comprehensive reference is G. Warner, *Harmonic Analysis on Semisimple Lie Groups, I*, Springer-Verlag, 1972. Also, my 'Unitary Representations on Partially Holomorphic Cohomology Spaces,' AMS Memoir #138, 1974, may be useful. And of course there are various papers of Harish-Chandra, Schmid, and others.

Chapter I

STRUCTURE OF SEMISIMPLE LIE GROUPS

Chapter I presents a brief resume, with occasional indications of proofs, of the theory of semisimple Lie groups up to (but not including) Cartan's highest weight theory for finite-dimensional representations and the theory of parabolic subgroups. We start with some basic notions of linear algebra (Section 1) and do the representations of $\mathfrak{sl}(2)$ which have a highest or lowest weight vector (Section 2). We then discuss some basic facts about semisimple (Section 3) and reductive (Section 4) groups. After that, we look at root systems and the Weyl group (Section 5), Weyl bases and real forms (Section 6), Dynkin diagrams and classification (Section 7), and have a brief glimpse of the structure of real semisimple groups (Section 8). All this can be viewed as a sort of study guide to Varadarajan's book, through Chapter 4, Section 5.

1. Preliminaries

Let V be a vector space and S a set of linear transformations of V. Then S is called

irreducible if V has no proper S-invariant subspace,

semisimple if every S-invariant subspace has an S-invariant complement,

nilpotent if $\dim V < \infty$ and V has a basis in which every element of S has matrix of the form $\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$,

unipotent if $\{s-1 : s \in S\}$ is nilpotent.

A single linear transformation s of V is called semisimple, nilpotent, unipotent, if the set $\{s\}$ has that property. So s is nilpotent if and only if some power s^n is nilpotent, and s is semisimple if and only if it is diagonable over the algebraic closure of the base field.

Let G be a group (resp. $\mathfrak{g}$ a Lie algebra). A *representation* π of G (resp. $\mathfrak{g}$) on V is a homomorphism $\pi: G \to GL(V)$ of G to the group of invertible linear transformations of V (resp. Lie algebra homomorphism $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ to the set of linear transformations of V with Lie product $[a, b] = ab - ba$). The representation π is called irreducible, semisimple, unipotent (in the group case) or nilpotent (in the algebra case) if its image has that property.

From now on 'representation' means 'continuous representation' for Lie groups.

Now suppose that V is a real or complex vector space and that G is a Lie group with Lie algebra $\mathfrak{g}$, and for the moment suppose further that $\dim V < \infty$. If π is a representation of G on V, then it is real analytic as a map $G \to GL(V)$, and it induces a representation (denoted $d\pi$ or $\dot{\pi}$ or π) of $\mathfrak{g}$ on V by

$$d\pi(\xi)\colon v \mapsto \frac{d}{dt}\bigg|_{t=0} \pi(\exp_G(t\xi))v.$$

One recovers π from $d\pi$ – at least on the identity component G_0 of G – by $\pi(\exp_G(t\xi)) = e^{t\cdot d\pi(\xi)}$. On the other hand, if ψ is a representation of $\mathfrak{g}$ on V, and if $\tilde{G}$ is the connected simply connected Lie group with Lie algebra $\mathfrak{g}$, then $\tilde{\pi}(\exp_{\tilde{G}}(\xi)) = e^{\psi(\xi)}$ defines a representation of $\tilde{G}$ on V with $d\tilde{\pi} = \psi$. Here $\tilde{G}$ is the universal covering group of G_0, and $\tilde{\pi}$ may or may not factor through G_0, and even then it need not extend to G.

Now suppose that G is connected and that π is a representation of G on V. The relation between π and $d\pi$ shows: π is irreducible if and only if $d\pi$ is irreducible, π is semisimple if and only if $d\pi$ is semisimple, and π is unipotent if and only if $d\pi$ is nilpotent.

Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. It is the tensor algebra mod the ideal generated by the $\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]$. If we view $\mathfrak{g}_{\mathbb{C}}$ as the Lie algebra of left invariant vector fields on G then $\mathcal{U}(\mathfrak{g})$ is the associative algebra of all left invariant differential operators on G. If ψ is a representation of $\mathfrak{g}$ on V then it 'extends' uniquely to an associative algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathfrak{gl}(V)$, also denoted ψ, which is irreducible or semisimple if and only if $\psi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is. Further, $\psi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is semisimple or nilpotent if and only if the associative algebra $\psi(\mathcal{U}(\mathfrak{g}))$ has that property in the category of associative algebras. Finally, if ψ is irreducible and $0 \neq v \in V$ then $V = \psi(\mathcal{U}(\mathfrak{g})) \cdot v$.

2. Representations of $\mathfrak{sl}(2)$

Let $\mathfrak{g} = \mathfrak{sl}(2)$, linear Lie algebra with basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and multiplication table

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Let ψ be an *algebraically* irreducible representation of $\mathfrak{g}$ on some vector space V, and suppose that h has an eigenvector, i.e.

$$V_\lambda = \{v \in V : \psi(h)v = \lambda v\}, \qquad \lambda \in \mathbb{C},$$

is nonzero for some value $\lambda = \lambda_0$. Writing $\psi(h)\psi(e) = \psi[h, e] + \psi(e)\psi(h)$ and $\psi(h)\psi(f) = \psi[h, f] + \psi(f)\psi(h)$ we see

$$\psi(e)V_\lambda \subset V_{\lambda+2} \qquad \text{and} \qquad \psi(f)V_\lambda \subset V_{\lambda-2}.$$

By irreducibility, now,

$$V = \sum_{n=-\infty}^{\infty} V_{\lambda_0+2n} \qquad \text{algebraic direct sum.}$$

In particular, if $\dim V < \infty$ then $\psi(h)$ is semisimple and $\psi(e)$ and $\psi(f)$ are nilpotent.

Now suppose that V has a 'highest weight vector,' that is a nonzero vector v in some V_λ such that $\psi(e)v = 0$. Of course this is automatic if $\dim V < \infty$. Then we take $\lambda_0 = \lambda$, $v_0 = v$, and define $v_n = \psi(f)^n v_0 \in V_{\lambda_0 - 2n}$. Evidently the algebraic span of the v_n is stable under $\psi(f)$ and $\psi(h)$. I claim that it also is stable under $\psi(e)$. For one has

$$ef^n = f^n e + nf^{n-1}(h - n + 1) \qquad \text{in } \mathcal{U}(\mathfrak{g}) \text{ for } n = 1, 2, \ldots$$

(easy induction), so

$$\psi(e)v_n = \psi(f)^n\psi(e)v_0 + n(\lambda_0 - n + 1)v_{n-1} = n(\lambda_0 - n + 1)v_{n-1}.$$

Now, again by irreducibility,

$$\begin{aligned} &V \text{ has basis } \{v_0, v_1, \ldots\} \text{ and } \mathfrak{g} \text{ acts by } \psi(f)v_n = v_{n+1}, \\ &\psi(h)v_n = (\lambda_0 - 2n)v_n, \qquad \text{and } \psi(e)v_n = n(\lambda_0 - n + 1)v_{n-1}. \end{aligned}$$

Case 1. λ_0 is an integer $\geqslant 0$. Then $\psi(e)v_n = 0$ for $n = \lambda_0 + 1 > 0$. If $v_n \neq 0$ then $\{v_n, v_{n+1}, \ldots\}$ spans an invariant subspace, contradicting irreducibility. Thus V has finite dimension $\lambda_0 + 1$, basis $\{v_0, v_1, \ldots, v_{\lambda_0}\}$, with ψ as above. And for any integer $d \geqslant 0$ these formulae define an irreducible representation of degree $d + 1$ of $\mathfrak{g}$.

Case 2. λ_0 is not an integer $\geqslant 0$. Then $\psi(e)v_n \neq 0$ for all $n \geqslant 1$ (easy induction), so dim $V=\infty$. Later we will see that V is the space of $SO(2)$-finite vectors in an irreducible unitary representation π_{λ_0} of $SL(2,\mathbb{R})$ just when λ_0 is a negative integer; for $\lambda_0<-1$ it will be an 'antiholomorphic discrete series' representation, and for $\lambda_0=-1$ it will be a summand of the unique reducible 'principal series' representation of $SL(2;\mathbb{R})$.

If V has a 'lowest weight vector' $v_0\in V_{\lambda_0}$, $v_0\neq 0$ but $\psi(f)v_0=0$, then as above

V has basis $\{v_n=\psi(e)^n v_0\}_{n=0,1,\dots}$ and $\mathfrak{g}$ acts by

$$\psi(e)v_n=v_{n+1},\qquad \psi(h)v_n=(\lambda_0+2n)v_n,\qquad \psi(f)v_n=n(\lambda_0+n-1)v_{n-1}.$$

If λ_0 is an integer $\leqslant 0$ then, as above, $v_{-\lambda_0+1}=0$ and we have the irreducible $\mathfrak{g}$-module of dimension $|\lambda_0|+1$. Also,

Case 3. λ_0 is not an integer $\leqslant 0$. Then $\psi(f)v_n\neq 0$ for all $n\geqslant 1$ (easy induction), dim $V=\infty$, and it will turn out that V is the space of $SO(2)$-finite vectors in an irreducible unitary representation π_{λ_0} of $SL(2;\mathbb{R})$ just when λ_0 is a positive integer; for $\lambda_0>1$ it will be a 'holomorphic discrete series' representation, and for $\lambda_0=1$ it will be the other summand of the unique reducible 'principal series' representation of $SL(2;\mathbb{R})$.

If V has neither highest weight vector nor lowest weight vector, one needs some analytic tools to see dim $V_\lambda\leqslant 1$; when we do that, we will have the spaces of $SO(2)$-finite vectors in the unitary 'principal' and 'complementary' series representations of $SL(2;\mathbb{R})$.

Finally, we note that the irreducible finite-dimensional representations of $\mathfrak{sl}(2;\mathbb{C})$ and $SL(2;\mathbb{C})$ are given by the formulae in Case 1, and those of $\mathfrak{su}(2)$ and $SU(2)$ follow by restriction.

3. Cartan's criterion for semisimplicity

Fix a Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ and $\mathfrak{k}$ are subalgebras then $[\mathfrak{h},\mathfrak{k}]$ denotes the subalgebra spanned by the $[\xi,\eta]$, $\xi\in\mathfrak{h}$ and $\eta\in\mathfrak{k}$. $\mathfrak{g}$ is *solvable* if the derived series $\mathfrak{g}=\mathfrak{g}^{(0)}\supset\mathfrak{g}^{(1)}\supset\dots,\ \mathfrak{g}^{(i+1)}=[\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}]$, terminates at 0, *nilpotent* if the series $\mathfrak{g}=\mathfrak{g}^{[0]}\supset\mathfrak{g}^{[1]}\supset\dots,\ \mathfrak{g}^{[i+1]}=[\mathfrak{g},\mathfrak{g}^{[i]}]$, terminates at 0. Note that $\mathfrak{g}$ is nilpotent just when $\mathrm{ad}_{\mathfrak{g}}$ is nilpotent.

$\mathfrak{g}$ is *semisimple* if it has no nonzero solvable ideal.

A sum of solvable (resp. nilpotent) ideals in $\mathfrak{g}$ again is a solvable (resp. nilpotent) ideal. Thus $\mathfrak{g}$ has a *radical* (= solvable radical) and a *nilradical* (= nilpotent radical) defined by

rad $\mathfrak{g}$: maximal solvable ideal in $\mathfrak{g}$

nilrad $\mathfrak{g}$: maximal nilpotent ideal in $\mathfrak{g}$

and $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is semisimple.

THE LEVI–WHITEHEAD THEOREM. *$\mathfrak{g}$ has a (necessarily semisimple) subalgebra $\mathfrak{s}$ such that the projection $\mathfrak{g}\to\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ restricts to an isomorphism of $\mathfrak{s}$ onto $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$.*

In other words $\mathfrak{g}$ is the semidirect sum rad $\mathfrak{g}+\mathfrak{s}$. Any such algebra $\mathfrak{s}$ is called a *Levi subalgebra* or Levi factor of $\mathfrak{g}$.

THE MAL'CEV–HARISH-CHANDRA THEOREM. *Any two Levi subalgebras of $\mathfrak{g}$ are conjugate by an automorphism $e^{\mathrm{ad}(\xi)}$, $\xi\in[\mathrm{rad}\,\mathfrak{g},\mathfrak{g}]$.*

These results can be stated thus: Every semisimple subalgebra of $\mathfrak{g}$ is contained in a maximal one, any two maximal ones are $\exp(\mathrm{ad}[\mathrm{rad}\,\mathfrak{g},\mathfrak{g}])$-conjugate, and if $\mathfrak{s}$ is a maximal one then $\mathfrak{g}=\mathrm{rad}\,\mathfrak{g}+\mathfrak{s}$ semidirect sum. These theorems are not easy; they use cohomology of $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ for a certain representation.

ENGEL'S THEOREM. *If π is a finite-dimensional representation of $\mathfrak{g}$ such that each $\pi(\xi)$ is nilpotent, then π is nilpotent, so in particular $\mathfrak{g}/\ker(\pi)$ is nilpotent.*

If G has nilpotent Lie algebra $\mathfrak{g}$ and we apply Engel's Theorem to the adjoint representation then, from the Campbell–Hausdorff Theorem, there is a polynomial map $p\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that $\exp_G(\xi)\cdot\exp_G(\eta)=\exp_G(p(\xi,\eta))$ for $\xi,\eta\in\mathfrak{g}$. It follows that $\exp_G\colon\mathfrak{g}\to G$ is a covering space for $\mathfrak{g}$ nilpotent and G connected. Thus, in a simply connected nilpotent Lie group, every analytic subgroup is closed and exp(center of $\mathfrak{g}$) = (center of G).

LIE'S THEOREM. *Let π be a representation of a solvable Lie algebra $\mathfrak{g}$ on a finite-dimensional vector space V, both over the same algebraically closed field. Then there is a finite set $\{\lambda_1,\dots,\lambda_r\}$ of linear functionals on $\mathfrak{g}$ that vanish on $[\mathfrak{g},\mathfrak{g}]$, and there is a composition series $V=V_1\supsetneqq\cdots\supsetneqq V_r\supsetneqq V_{r+1}=0$ of V under $\pi(\mathfrak{g})$, such that the action π_i of $\mathfrak{g}$ on V_i/V_{i+1} is* *given, in an appropriate basis, by*

$$\pi_i(\xi)=\begin{pmatrix}\lambda_i(\xi) & & * \\ & \ddots & \\ 0 & & \lambda_i(\xi)\end{pmatrix},\qquad \xi\in\mathfrak{g}.$$

Some consequences: (1) *if* π *is irreducible then* $\dim V=1$, (2) *a Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g},\mathfrak{g}]$ *is nilpotent*, (3) *if* $\xi,\eta,\zeta\in\mathfrak{g}$ *then* $\operatorname{trace}(\pi([\xi,\eta])\cdot\pi(\zeta))=0$.

Further, the *Cartan–Killing form* on $\mathfrak{g}$ is the bilinear form $\langle\xi,\eta\rangle=\operatorname{trace}(\operatorname{ad}(\xi)\cdot\operatorname{ad}(\eta))$. It is symmetric, and is invariant in the sense $\langle[\zeta,\xi],\eta\rangle+\langle\xi,[\zeta,\eta]\rangle=0$.

CARTAN'S CRITERION. (i) $\mathfrak{g}$ *is solvable if and only if* $\langle\mathfrak{g},[\mathfrak{g},\mathfrak{g}]\rangle=0$, (ii) $\mathfrak{g}$ *is semisimple if and only if* $\langle\,,\rangle$ *is nondegenerate* ($\langle\xi,\mathfrak{g}\rangle=0\Rightarrow\xi=0$).

If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, so is $\mathfrak{h}^\perp=\{\xi\in\mathfrak{g}:\langle\xi,\mathfrak{h}\rangle=0\}$, and thus also $\mathfrak{h}\cap\mathfrak{h}^\perp$. Whenever $\mathfrak{k}$ is an ideal the Cartan–Killing forms satisfy $\langle,\rangle_{\mathfrak{k}}=\langle,\rangle_{\mathfrak{g}}|_{\mathfrak{k}\times\mathfrak{k}}$, so $\mathfrak{h}\cap\mathfrak{h}^\perp$ is solvable by Cartan's Criterion. If $\mathfrak{g}$ is semisimple this says $\mathfrak{h}\cap\mathfrak{h}^\perp=0$ so, again by Cartan's Criterion, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^\perp$. A Lie algebra is *simple* if it is nonAbelian and has no proper ideal, i.e. if the adjoint representation is nontrivial and irreducible. Now, if $\mathfrak{g}$ is semisimple then it is a direct sum of simple ideals, and the only ideals are the partial sums.

WEYL'S THEOREM. *Every finite-dimensional representation of a semisimple Lie group is semisimple.*

So, for example, every finite-dimensional representation of $\mathfrak{sl}(2)$ is a direct sum of irreducible representations listed in Section 2. Weyl's original proof: a finite-dimensional representation of a compact group K is semisimple (easy), so the same holds for the Lie algebra $\mathfrak{k}$ of a compact Lie group, thus also for $\mathfrak{k}_{\mathbf{C}}$ and then for any real Lie algebra $\mathfrak{g}$ with $\mathfrak{k}_{\mathbf{C}}\simeq\mathfrak{g}_{\mathbf{C}}$; and (this is the hard part) if $\mathfrak{g}$ is real semisimple then there is a compact Lie group K with $\mathfrak{k}_{\mathbf{C}}\simeq\mathfrak{g}_{\mathbf{C}}$. A more algebraic proof uses cohomology.

Now we transcribe these results over to Lie groups. If H is a normal (resp. Abelian, resp. nilpotent, resp. solvable) subgroup of a Lie group G, then its closure has the same property, so we have closed subgroups

rad G: maximal connected normal solvable subgroup of G,

nilrad G: maximal connected normal nilpotent subgroup of G,

whose respective Lie algebras are rad $\mathfrak{g}$ and nilrad $\mathfrak{g}$.

G is semisimple if rad G is trivial, i.e. if rad $\mathfrak{g}=0$. In any case $G/\text{rad } G$ is semisimple. If G is connected then the Levi decomposition $\mathfrak{g}=\text{rad }\mathfrak{g}+\mathfrak{s}$ goes over to $G=(\text{rad } G)\cdot S$, S analytic subgroup for $\mathfrak{s}$ and $S\cap\text{rad } G$ discrete; and here one has a semidirect product when G is simply connected.

Weyl's Theorem goes over directly to connected semisimple Lie groups and then to semisimple Lie groups G with G/G_0 finite: *for such groups, every finite-dimensional real or complex representation is semisimple.*

4. Reductive groups and trace forms

We say that a Lie group G (resp. Lie algebra $\mathfrak{g}$) is *reductive* if it has a finite-dimensional semisimple representation with discrete kernel (resp. kernel zero). For example let $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{g}'$ where $\mathfrak{z}$ is Abelian and $\mathfrak{g}'$ is semisimple, choose a basis $\{\zeta_1,\ldots,\zeta_s\}$ of $\mathfrak{z}$, and consider

$$\pi:\sum x_i\zeta_i+\eta\to\begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_s\end{pmatrix}\oplus\operatorname{ad}_{\mathfrak{g}'}(\eta),\qquad \eta\in\mathfrak{g}';$$

then π is faithful and semisimple, so $\mathfrak{g}$ is reductive. In fact we are about to see that this is the only example.

If π is a finite-dimensional representation of $\mathfrak{g}$, then the associated *trace form* is the symmetric bilinear form on $\mathfrak{g}$ given by

$$\langle\xi,\eta\rangle_\pi=\operatorname{trace}(\pi(\xi)\cdot\pi(\eta)).$$

It is invariant in the sense $\langle[\zeta,\xi],\eta\rangle_\pi+\langle\xi,[\zeta,\eta]\rangle_\pi=0$. The Cartan-Killing form of $\mathfrak{g}$ is the trace form of the adjoint representation.

THEOREM. *The following conditions are equivalent*: (i) $\mathfrak{g}$ *is reductive*, (ii) $\mathfrak{g}$ *has a nondegenerate* (qua bilinear form) *trace form*, (iii) $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{g}'$ *where* $\mathfrak{z}$ *is its center and* $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]$ *is a semisimple ideal* (the 'semisimple part'), (iv) *the adjoint representation* $\operatorname{ad}_{\mathfrak{g}}$ *is semisimple.*

Indication of proof. If $\operatorname{ad}_{\mathfrak{g}}$ is semisimple it is of the form $\pi_0\oplus\pi_1\oplus\ldots\oplus\pi_r$ where π_0 represents by zero and the other π_i are irreducible and nontrivial. Now $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_r$ where $\mathfrak{g}_i$ is the representation space of π_i, so $\mathfrak{g}_0$ is the center and the other $\mathfrak{g}_i$ are simple. Thus (iv) implies (iii) with $\mathfrak{g}_0=\mathfrak{z}$ and $\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_r=\mathfrak{g}'$. The converse (iii)$\Rightarrow$(iv) is obvious. Given (iii), the representation described just after the definition of 'reductive' is faithful, semisimple, and has nondegenerate trace form; so (iii) implies (i) and (ii). From Schur's Lemma and Engel's Theorem, an irreducible linear Lie algebra is either semisimple or of the form (some scalars)$\oplus$(semisimple) and thus (i) implies (iii). Given (ii), the method used to decompose a semisimple algebra as a direct sum of simple ideals, can be modified to prove (iii).

We also need the relative concept. A closed subgroup $H \subset G$ (resp. subalgebra $\mathfrak{h} \subset \mathfrak{g}$) is *reductive in* G (resp. $\mathfrak{g}$) if $\mathrm{Ad}_{G|H}$ (resp. $\mathrm{ad}_{\mathfrak{g}|\mathfrak{h}}$) is semisimple. In that case it is reductive.

THEOREM. *Let* $\mathfrak{g}$ *be reductive,* $\mathfrak{g}=\mathfrak{z}+\mathfrak{g}'$ *as in* (iii) *above, and let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$. *Then these are equivalent*: (i′) $\mathfrak{h}$ *is reductive in* $\mathfrak{g}$, (ii′) *some trace form on* $\mathfrak{g}$ *is nondegenerate on* $\mathfrak{h}$, (iii′) *the Cartan–Killing form of* $\mathfrak{g}'$ *is nondegenerate on the image of* $\mathfrak{h}$ *under the projection* $\mathfrak{g}\to\mathfrak{g}'$ *with kernel* $\mathfrak{z}$.

COROLLARY. *If* $\mathfrak{g}$ *is reductive, and* Γ *is a group of automorphisms whose action on* $\mathfrak{g}$ *is semisimple, then the fixed point set* $\mathfrak{g}^{\Gamma}$ *is reductive in* $\mathfrak{g}$.

5. Root systems – basic properties

By *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$, we mean a nilpotent subalgebra $\mathfrak{h}\subset\mathfrak{g}$ which is its own normalizer, i.e.

$$\mathfrak{h}=\{\xi\in\mathfrak{g}: [\xi,\mathfrak{h}]\subset\mathfrak{h}\}.$$

If the base field is algebraically closed, say $\mathbb{C}$, we apply Lie's Theorem to $\mathrm{ad}_{\mathfrak{g}|\mathfrak{h}}$, and nilpotence of $\mathfrak{h}$ gives a decomposition $\mathfrak{g}=\Sigma_{\lambda\in\mathfrak{h}^*}\mathfrak{g}_\lambda$ where $\mathfrak{h}^*$ is the linear dual space and

$$\mathfrak{g}_\lambda=\{\eta\in\mathfrak{g}: (\mathrm{ad}_{\mathfrak{g}}(\xi)-\lambda(\xi))^m\eta=0 \quad \text{for} \quad m\gg 0\},$$

Now the fact that $\mathfrak{h}$ is nilpotent and self-normalizing says: $\mathfrak{g}_0=\mathfrak{h}$. Let $\Delta=\Delta(\mathfrak{g},\mathfrak{h})=\{\lambda\in\mathfrak{h}^*: \lambda\neq 0 \text{ and } \mathfrak{g}_\lambda\neq 0\}$; it is a finite subset of $\mathfrak{h}^*$ called the *root system* of $\mathfrak{g}$ relative to $\mathfrak{h}$, and we have

$$\mathfrak{g}=\mathfrak{h}+\sum_{\lambda\in\Delta}\mathfrak{g}_\lambda, \qquad \textit{root space decomposition.}$$

If the base field is not algebraically closed, say $\mathbb{R}$, then we pass to the algebraic closure by scalar extension; $\mathfrak{h}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$ and we have $\mathfrak{g}^{\mathbf{C}}=\mathfrak{h}^{\mathbf{C}}+\Sigma_{\lambda\in\Delta}\mathfrak{g}_\lambda^{\mathbf{C}}$.

Define polynomial functions p_i on $\mathfrak{g}$ by

$$\det(\tau\cdot 1-\mathrm{ad}(\xi))=\sum(-1)^{m-i}p_i(\xi)\tau^i$$

where τ is an indeterminate and $m=\dim\mathfrak{g}$. The smallest integer $r\geqslant 0$ with p_r not identically zero is called the *rank* of $\mathfrak{g}$. It is not changed by scalar extension of the base field, and rank $\mathfrak{g}=\dim\mathfrak{g}$ only when $\mathfrak{g}$ is nilpotent. If $l=\text{rank}\,\mathfrak{g}$, then an element $\xi\in\mathfrak{g}$ is called *regular* if $p_l(\xi)\neq 0$, *singular* otherwise. The regular elements are dense.

THEOREM. *The Cartan subalgebras of* $\mathfrak{g}$ *are just the*

$$\mathfrak{h}_\xi=\{\eta\in\mathfrak{g}: \mathrm{ad}(\xi)^m\eta=0 \quad \textit{for} \quad m\gg 0\}$$

where ξ *is a regular element of* $\mathfrak{g}$. *In particular they all have the same dimension,* rank $\mathfrak{g}$.

CHEVALLEY'S THEOREM. *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$. *If* $\mathfrak{g}$ *is a complex Lie algebra then any two Cartan subalgebras are* $\mathrm{Ad}(G)$*-conjugate. If* $\mathfrak{g}$ *is a real Lie algebra then there are only finitely many* $\mathrm{Ad}(G)$*-conjugacy classes of Cartan subalgebras.*

Now we specialize to the case where $\mathfrak{g}$ is semisimple. An element $\xi\in\mathfrak{g}$ is called *semisimple* if $\mathrm{ad}(\xi)$ is semisimple, *nilpotent* if $\mathrm{ad}(\xi)$ is nilpotent. Looking at the Jordan normal form of $\mathrm{ad}(\xi)$ on $\mathfrak{g}$ or $\mathfrak{g}^{\mathbf{C}}$, we see $\xi=\xi_s+\xi_n$ with ξ_s semisimple, ξ_n nilpotent, and both $\mathrm{ad}(\xi_s)$ and $\mathrm{ad}(\xi_n)$ polynomials in $\mathrm{ad}(\xi)$. If ξ is regular it follows that ξ is semisimple. But if ξ is semisimple then the Cartan–Killing form is nondegenerate on the centralizer $\mathfrak{g}^\xi=\{\eta\in\mathfrak{g}: [\xi,\eta]=0\}$, which thus is reductive in $\mathfrak{g}$.

CONCLUSIONS. (i) *an element* $\xi\in\mathfrak{g}$ *is semisimple if and only if it is contained in a Cartan subalgebra,* (ii) *a subalgebra* $\mathfrak{h}\subset\mathfrak{g}$ *is Cartan if and only if it is a maximal Abelian subalgebra and consists of semisimple elements.*

Now fix a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$, and consider the root space decomposition $\mathfrak{g}=\mathfrak{h}+\Sigma_{\lambda\in\Delta}\mathfrak{g}_\lambda$. The Cartan–Killing form $\langle\,,\rangle$ is nondegenerate on $\mathfrak{h}$, so to each $\nu\in\mathfrak{h}^*$ we have $h_\nu\in\mathfrak{h}$ defined by $\nu(\xi)=\langle h_\nu,\xi\rangle$. We transfer $\langle\,,\rangle$ to $\mathfrak{h}^*$ by $\langle\nu,\mu\rangle=\langle h_\nu,h_\mu\rangle$. The elementary basic facts are

(i) if $\lambda\in\Delta$ then $\dim\mathfrak{g}_\lambda=1$, and $-\lambda\in\Delta$,

(ii) if $\lambda,\mu\in\Delta$ with $\lambda+\mu\neq 0$ then $[\mathfrak{g}_\lambda,\mathfrak{g}_\mu]=\mathfrak{g}_{\lambda+\mu}$ and $\mathfrak{g}_\lambda\perp\mathfrak{g}_\mu$,

(iii) if $\xi\in\mathfrak{g}_\lambda$ and $\eta\in\mathfrak{g}_{-\lambda}$ then $[\xi,\eta]=\langle\xi,\eta\rangle h_\lambda$,

(iv) $\mathfrak{g}$ is the $\langle\,,\rangle$-orthogonal sum of $\mathfrak{h}$ and the $\mathfrak{g}_\lambda+\mathfrak{g}_{-\lambda}(\lambda\in\Delta)$.

Looking at the representation of $\mathfrak{g}[\alpha]=\mathfrak{g}_{-\alpha}+h_\alpha\cdot\mathbf{C}+\mathfrak{g}_\alpha$ on $\mathfrak{g}$, described in Section 2, one sees

(v) Let $\alpha,\lambda\in\Delta$ with $\alpha\neq\pm\lambda$. Then there are integers $p,q\geqslant 0$ such that $2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle=q-p$ and $\lambda+k\alpha\in\Delta$ precisely when k is an integer with $-q\leqslant k\leqslant p$.

(vi) If $\lambda\in\Delta$, $c\in\mathbf{C}$, $c\lambda\in\Delta$ then $c=\pm 1$.

The h_λ, $\lambda\in\Delta$, span a real form $\mathfrak{h}_{\mathbf{R}}$ of $\mathfrak{h}$ on which the Cartan–Killing form is positive definite. Thus also Δ spans a real form $\mathfrak{h}^*_{\mathbf{R}}$ of $\mathfrak{h}^*$ which is positive definite. If $\alpha\in\Delta$ then the reflections in the hyperplanes $\alpha=0$ and $\alpha^\perp$ are

$$s_\alpha\colon \mathfrak{h}\to\mathfrak{h} \qquad \text{by} \qquad s_\alpha(\xi)=\xi-\left(\frac{2\alpha(\xi)}{\langle\alpha,\alpha\rangle}\right)h_\alpha,$$

and

$$s_\alpha\colon \mathfrak{h}^*\to\mathfrak{h}^* \qquad \text{by} \qquad s_\alpha(\lambda)=\lambda-\left(\frac{2\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\right)\alpha.$$

From (v), the s_α permute Δ, hence generate a finite group $W=W(\mathfrak{g},\mathfrak{h})$, the *Weyl group*. If $w\in W$ then $w(\Delta)=\Delta$.

A subset $\Delta^+\subset\Delta$ is a *positive root system* if (a) Δ is disjoint union of Δ^+ and $-\Delta^+$ and (b) whenever $\lambda,\mu\in\Delta^+$ with $\lambda+\mu\in\Delta$ one has $\lambda+\mu\in\Delta^+$.

Each root $\lambda\in\Delta$ defines a hyperplane $(\lambda=0)$ in $\mathfrak{h}_{\mathbf{R}}$, and $\mathfrak{h}_{\mathbf{R}}\setminus\bigcup_{\lambda\in\Delta}(\lambda=0)$ is a disjoint union of convex open cones which are its topological components. Each such cone is called a *Weyl chamber.*

A Weyl chamber $\mathscr{C}\subset\mathfrak{h}_{\mathbf{R}}$ defines a positive root system $\Delta^+=\{\lambda\in\Delta\colon\lambda>0$ on $\mathscr{C}\}$. Conversely a positive root system Δ^+ defines a Weyl chamber $\mathscr{C}=\{\xi\in\mathfrak{h}_{\mathbf{R}}\colon\lambda(\xi)>0$ for all $\lambda\in\Delta^+\}$.

A positive root is called *simple* if it is not the sum of two positive roots. Let $S=S(\Delta^+)$ denote the system of simple roots for Δ^+, and enumerate $S=\{\alpha_1,\dots,\alpha_l\}$. Then S is a basis of $\mathfrak{h}^*_{\mathbf{R}}$, so $l=\operatorname{rank}\mathfrak{g}$ and the α_i are linearly independent. Every $\alpha\in\Delta$ has expression $\alpha=\Sigma n_i\alpha_i$ where the n_i are integers, all $\geqslant 0$ if $\alpha\in\Delta^+$, all $\leqslant 0$ if $-\alpha\in\Delta^+$, so in particular S determines Δ^+. Also, $s_{\alpha_i}(\Delta^+)=(\Delta^+\setminus\{\alpha_i\})\cup\{-\alpha_i\}$. Now the Weyl chamber $\mathscr{C}=\mathscr{C}(\Delta^+)$ determines S directly as $\{\alpha\in\Delta\colon\alpha>0$ on $\mathscr{C}$ and $\overline{\mathscr{C}}\cap(\alpha=0)$ is open in $(\alpha=0)\}$. Conversely, of course, $\mathscr{C}=\{\xi\in\mathfrak{h}_{\mathbf{R}}\colon$ each $\alpha_i(\xi)>0\}$.

The Weyl group W acts in a simply transitive manner on the set of all Weyl chambers (resp. set of all positive root systems, resp. set of all simple root systems). If S is a simple root system then $\{s_\alpha\colon\alpha\in S\}$ is a minimal generating set for W. Also, if $\alpha\in\Delta$ then $\overline{\mathscr{C}}\cap(\alpha=0)$ is open in $(\alpha=0)$ for some chamber, so $\Delta=W(S)$.

6. Isomorphism; compact and split real forms

Select elements $e_\lambda\in\mathfrak{g}_\lambda$, $\lambda\in\Delta$, such that $\langle e_\lambda,e_{-\lambda}\rangle=-1$. So $[e_\lambda,e_{-\lambda}]=-h_\lambda$ by (iii). If $\lambda,\mu\in\Delta$ with $\lambda+\mu\neq 0$, define numbers $n_{\lambda,\mu}$ by

$$[e_\lambda,e_\mu]=n_{\lambda,\mu}e_{\lambda+\mu}\quad\text{if }\lambda+\mu\in\Delta,\qquad n_{\lambda,\mu}=0\quad\text{if }\lambda+\notin\Delta.$$

Then one has $n_{\lambda,\mu}+n_{\mu,\lambda}=0$, and one can prove

(vii) if $\lambda,\mu,\nu\in\Delta$ and $\lambda+\mu+\nu=0$ then $n_{\lambda,\mu}=n_{\mu,\nu}=n_{\nu,\lambda}$

(viii) if $\kappa,\lambda,\mu,\nu\in\Delta$, none the negative of any other, and if $\kappa+\lambda+\mu+\nu=0$, then $n_{\kappa,\lambda}n_{\mu,\nu}+n_{\lambda,\mu}n_{\kappa,\nu}+n_{\mu,\kappa}n_{\lambda,\nu}=0$

(ix) If $\alpha,\lambda\in\Delta$, $\alpha\neq\pm\lambda$, and $p,q\geqslant 0$ as in (v), then $n_{\alpha,\lambda}n_{-\alpha,-\lambda}=\frac{1}{2}\langle\alpha,\alpha\rangle p(q+1)$.

Suppose that $\mathfrak{g}_1$ is another complex semisimple Lie algebra, $\mathfrak{h}_1$ a Cartan subalgebra of $\mathfrak{g}_1$, and $\Delta_1=\Delta(\mathfrak{g}_1,\mathfrak{h}_1)$ the root system. Let $f\colon\mathfrak{h}\to\mathfrak{h}_1$ be a linear isomorphism such that $f^*\Delta_1=\Delta$.

THEOREM. *f extends to an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}_1$.*

First, f is an isometry relative to the Cartan–Killing form; for

$$\langle f\xi,f\xi'\rangle_1=\sum_{\Delta_1}\gamma(f\xi)\gamma(f\xi')=\sum_{\Delta_1}(f^*\gamma)(\xi)\cdot(f^*\gamma)(\xi')$$

$$=\sum_{\Delta}\lambda(\xi)\lambda(\xi')=\langle\xi,\xi'\rangle.$$

Choose a positive root system Δ_1^+ for $(\mathfrak{g}_1,\mathfrak{h}_1)$, let S_1 be the corresponding simple root system, and set $\Delta^+=f^*\Delta_1^+$ and $S=f^*S_1$. Then Δ^+ is a positive root system, and S the corresponding simple root system, for $(\mathfrak{g},\mathfrak{h})$. Enumerate $S_1=\{\beta_i\}$ and $S=\{\alpha_i\}$ where $\alpha_i=f^*\beta_i$. We say that a root $\lambda\in\Delta^+$ (resp. $\gamma\in\Delta_1^+$) has *level* Σn_i if $\lambda=\Sigma n_i\alpha_i$ (resp. $\gamma=\Sigma n_i\beta_i$). As above, select $e_\lambda\in\mathfrak{g}_\lambda(\lambda\in\Delta)$ with $\langle e_\lambda,e_{-\lambda}\rangle=-1$ and define $n_{\lambda,\mu}$ by $[e_\lambda,e_\mu]=n_{\lambda,\mu}e_{\lambda+\mu}$ if $\lambda+\mu\in\Delta$, $n_{\lambda,\mu}=0$ if $\lambda+\mu\notin\Delta$. We want elements $f_\gamma\in(\mathfrak{g}_1)_\gamma$ such that $\langle f_\gamma,f_{-\gamma}\rangle_1=-1$ and $[f_\gamma,f_\delta]=n_{f^*\gamma,f^*\delta}f_{\gamma+\delta}$ if $\gamma+\delta\in\Delta_1$; then $f(e_{f^*\gamma})=f_\gamma$ will give an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}_1$. Suppose that $k>0$ is an integer and we have such f_γ for $|\text{level }\gamma|<k$. Let $\varepsilon\in\Delta_1^+$ of level k. If there do not exist $\gamma,\delta\in\Delta_1$ with $|\text{level }\gamma|$, $|\text{level }\delta|<k$ then let $f_\varepsilon\in(\mathfrak{g}_1)_\varepsilon$ be any nonzero element. Otherwise choose $\gamma,\delta\in\Delta_1$, $|\text{level }\gamma|<k$, $|\text{level }\delta|<k$ with $\gamma+\delta=\varepsilon$ and define

f_ε by $[f_\gamma, f_\delta] = \eta_{\gamma,\delta} f_\varepsilon$, define $f_{-\varepsilon} \in (\mathfrak{g}_1)_{-\varepsilon}$ by $\langle f_\varepsilon, f_{-\varepsilon} \rangle_1 = -1$. Using (vii), (viii) and (ix) one shows that $f_{\pm\varepsilon}$ are independent of choice of γ, δ. Thus $f: \mathfrak{h} \to \mathfrak{h}_1$ extends to $f: \mathfrak{g} \cong \mathfrak{g}_1$.

Suppose, in the above theorem, that $\mathfrak{g} = \mathfrak{g}_1$, $\mathfrak{h} = \mathfrak{h}_1$ and $f(\xi) = -\xi$ for all $\xi \in \mathfrak{h}$. Then in the extension $f: \mathfrak{g} \cong \mathfrak{g}$, we have $f(e_\lambda) = c_\lambda e_{-\lambda}$, so $f^2(e_\lambda) = c_\lambda c_{-\lambda} e_\lambda$. But $\langle e_\lambda, e_{-\lambda} \rangle = \langle f(e_\lambda), f(e_{-\lambda}) \rangle$ shows $c_\lambda c_{-\lambda} = 1$. So we have

COROLLARY. *$\mathfrak{g}$ has an automorphism f which is -1 on $\mathfrak{h}$, and any such automorphism is involutive.*

A *Weyl basis* of $\mathfrak{g}$ is a basis of the form $\{h_i\} \cup \{e_\lambda\}$ where $\{h_i\}$ is a real basis of $\mathfrak{h}_{\mathbb{R}}$, $e_\lambda \in \mathfrak{g}_\lambda$, $\langle e_\lambda, e_{-\lambda} \rangle = -1$, and the $n_{\lambda,\mu} = n_{-\lambda,-\mu}$. To construct a Weyl basis: start with an automorphism f such that $f|_{\mathfrak{h}} = -1$, choose a positive root system Δ^+, and for each $\lambda \in \Delta^+$ select $e_\lambda \in \mathfrak{g}_\lambda$ such that $\langle e_\lambda, f(e_\lambda) \rangle = -1$. Set $e_{-\lambda} = f(e_\lambda)$. Then the $n_{\lambda,\mu} = n_{-\lambda,-\mu}$.

A real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is called a *real form* if $(\mathfrak{g}_0)_{\mathbb{C}} = \mathfrak{g}$. A real form $\mathfrak{g}_0$ is *split* if it has a Cartan subalgebra $\mathfrak{h}_0$ on which all the roots are real-valued. This means, setting $\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$, that the root space decomposition $\mathfrak{g} = \mathfrak{h} + \Sigma_\Delta \mathfrak{g}_\lambda$ intersects $\mathfrak{g}_0$ to give a real root space decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 + \Sigma_\Delta (\mathfrak{g}_0)_\lambda$. A split real form is also called a normal real form. If $\{h_i\} \cup \{e_\lambda\}$ is a Weyl basis of $\mathfrak{g}$, then (ix) gives us

$$n^2_{\alpha,\lambda} = \tfrac{1}{2}\langle \alpha, \alpha \rangle p(q+1) \geqslant 0,$$

so the $n_{\alpha,\lambda}$ are real. As the roots take real values on the real span $\mathfrak{h}_{\mathbb{R}}$ of $\{h_i\}$, now *the real span of that Weyl basis is a split real form of* $\mathfrak{g}$.

A real form $\mathfrak{g}_u \subset \mathfrak{g}$ is called *compact* if the Cartan–Killing form is negative definite on it. An equivalent formulation: *if $\xi \in \mathfrak{g}_u$ then every eigenvalue of* $\mathrm{ad}(\xi)$ *is pure imaginary.* If G has Lie algebra $\mathfrak{g}$, then the analytic subgroup for a real form $\mathfrak{g}_u$ is a compact group if and only if $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}$. Also, if $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}$ and $\mathfrak{h}_u$ is a Cartan subalgebra, then the roots of $\mathfrak{g}$ relative to $\mathfrak{h} = (\mathfrak{h}_u)_{\mathbb{C}}$ take pure imaginary values on $\mathfrak{h}_u$, so $\mathfrak{h}_u = i\mathfrak{h}_{\mathbb{R}}$.

THEOREM. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and G any Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has compact real forms, and any two are* Ad(G)*-conjugate.*

To see this, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{h_i\} \cup \{e_\lambda\}$ a Weyl basis, and f an automorphism of $\mathfrak{g}$ with $f|_{\mathfrak{h}} = -1$ and $f(e_\lambda) = e_{-\lambda}$. Let $\mathfrak{g}_0$ be the split real form of $\mathfrak{g}$ spanned over $\mathbb{R}$ by $\{h_i\} \cup \{e_\lambda\}$ and let σ_0 denote complex conjugation of $\mathfrak{g}$ over $\mathfrak{g}_0$. Then $\sigma_u = f \circ \sigma_0$ is a real algebra automorphism of $\mathfrak{g}$ which is complex conjugation over some vector space real form $\mathfrak{g}_u$. Now $\mathfrak{g}_u$ is a Lie algebra real form. If $\xi = h + \Sigma c_\lambda e_\lambda \in \mathfrak{g}$, $h \in \mathfrak{h}$, then $\sigma_u(\xi) = \sigma_u(h) + \Sigma \bar{c}_\lambda e_{-\lambda}$, so $\xi \in \mathfrak{g}_u$ if and only if $h \in i\mathfrak{h}_{\mathbb{R}}$ and $\bar{c}_\lambda = c_{-\lambda}$. In that case, since $i\mathfrak{h}_{\mathbb{R}}$ is negative definite, $\langle \xi, \xi \rangle = \langle h, h \rangle + \Sigma |c_\lambda|^2 \leqslant 0$ with equality just when $\xi = 0$. So $\mathfrak{g}_u$ is a compact real form. Let $\mathfrak{g}'_u$ be another compact real form, $\mathfrak{h}'_u$ a Cartan subalgebra. Since $\mathfrak{h}' = (\mathfrak{h}'_u)_{\mathbb{C}}$ is Ad(G)-conjugate to $\mathfrak{h}$ we may suppose that $\mathfrak{h}'_u = i\mathfrak{h}_{\mathbb{R}}$. Let σ, σ' be conjugations of $\mathfrak{g}$ over $\mathfrak{g}_u, \mathfrak{g}'_u$. Then $\tau = \sigma\sigma'$ is an automorphism of $\mathfrak{g}$ which is the identity on $\mathfrak{h}$, so τ preserves each $\mathfrak{g}_\lambda$. Now we have nonzero b_λ with $\sigma'(e_\lambda) = b_\lambda e_{-\lambda}$, and $\langle e_\lambda, \sigma' e_\lambda \rangle < 0$ so b_λ is real and positive. Conjugating by an element of $\exp(\mathfrak{h}_{\mathbb{R}})$ we may assume $b_\lambda = 1$ for λ in some simple root system, and then we have $\mathfrak{g}_u = \mathfrak{g}'_u$.

7. DYNKIN DIAGRAMS AND CLASSIFICATION

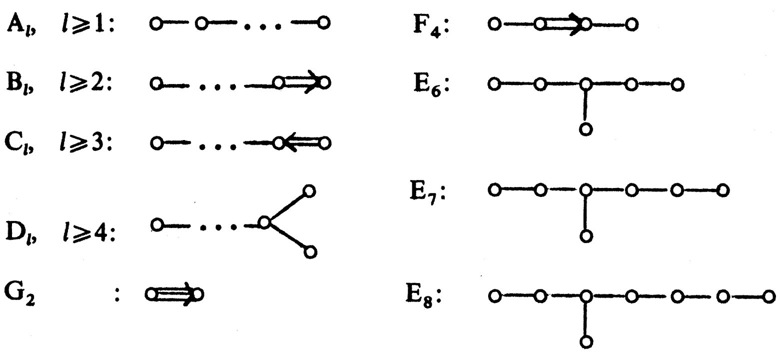
Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and S a simple $\mathfrak{h}$-root system for $\mathfrak{g}$. Enumerate $S = \{\alpha_1, \ldots, \alpha_l\}$. The *Cartan matrix* $A_S = (a_{ij})$ is defined by $a_{ij} = 2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$. Here $a_{ii} = 2$, and for $i \neq j$ we have $a_{ij} = 0, -1, -2$ or -3.

THEOREM. *$\mathfrak{g}$ determines the Cartan matrix A_S up to permutation-equivalence, i.e. conjugation by a permutation matrix. A_S determines $\mathfrak{g}$ up to isomorphism.*

First, if G has Lie algebra $\mathfrak{g}$, then any two Cartan subalgebras are Ad(G)-equivalent, and given $\mathfrak{h}$ any two simple root systems are equivalent under the Weyl group. Second, if S' is a simple root system for $(\mathfrak{g}', \mathfrak{h}')$ with $A_S = A_{S'}$, then we have an isometry of $\mathfrak{h}$ onto $\mathfrak{h}'$ which pulls S' back to S, hence pulls the root system $\Delta' = \Delta(\mathfrak{g}', \mathfrak{h}')$ back to Δ, and thus extends to an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.

The information carried by the Cartan matrix A_S can be summarized in a *Dynkin diagram* as follows. For each $\alpha \in S = \{\alpha_1, \ldots, \alpha_l\}$ we have a vertex, represented by a small circle. Vertices α_i, α_j are jointed by $a_{ij}a_{ji}$ lines if $i \neq j$. If $0 \neq a_{ij} \neq -1$, so a_{ij} is -2 or -3, then one has $a_{ji} = -1$, so the number of lines joining two vertices is 0, 1, 2 or 3. If the number is 2 or 3, say $a_{ij} = -2$ or -3 and $a_{ji} = -1$, then we insert an arrow toward the vertex representing the shorter root α_i. (Another convention: darken the shorter root. But this causes problems later with Satake diagrams.)

Every such diagram is a disjoint union of copies of the irreducible diagrams, which give the isomorphism classes of complex simple Lie algebras



A splitting of the Dynkin diagram of A_S as disjoint union of subdiagrams, corresponds to decomposition of S as disjoint union of mutually orthogonal sets, which in turn corresponds to splitting of as direct sum of ideals. So $\mathfrak{g}$ is simple just when its Dynkin diagram is connected.

Type A$_l$. Here $\mathfrak{g}=\mathfrak{sl}(l+1;\mathbb{C})$, complex $(l+1)\times(l+1)$ matrices of trace 0, and we can choose

$$\mathfrak{h}=\left\{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{l+1}\end{pmatrix} : \sum a_i=0\right\} \quad \text{with}$$

$$\alpha_i\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{l+1}\end{pmatrix}=a_i-a_{i+1}.$$

The simply connected group is the complex *special linear group*

$$G=SL(l+1;\mathbb{C})=\{g\in GL(l+1;\mathbb{C}): \det g=1\}.$$

We also have

$$\text{split:} \quad \mathfrak{g}_0=\mathfrak{sl}(l+1;\mathbb{R}) \quad \text{and} \quad G_0=SL(l+1;\mathbb{R})$$

$$\text{compact:} \quad \mathfrak{g}_u=\mathfrak{su}(l+1) \quad \text{and} \quad G_u=SU(l+1)$$

Type B$_l$. Here $\mathfrak{g}=\mathfrak{o}(2l+1;\mathbb{C})$, antisymmetric complex $(2l+1)\times(2l+1)$ matrices. The simply connected group is a 2-sheeted cover of the complex special orthogonal group $SO(2l+1;\mathbb{C})$. The split real form is the identity

component $SO(l, l+1)$ of the real orthogonal group of

$$(x,y)=\sum_1^l x_iy_i-\sum_{l+1}^{2l+1} x_jy_j,$$

and the compact real form is the ordinary special orthogonal group $SO(2l+1)$.

Type C$_l$. Here $\mathfrak{g}=\mathfrak{sp}(l;\mathbb{C})$, complex $2l\times 2l$ matrices that annihilate the antisymmetric form

$$A\colon \mathbb{C}^{2l}\times\mathbb{C}^{2l}\to\mathbb{C} \quad \text{by} \quad A((x,x'),(y,y'))={}^tx\cdot y'-{}^tx'\cdot y$$

The simply connected group is $G=Sp(l;\mathbb{C})$, complex symplectic group, and one has the real forms

$$\text{split: } \mathfrak{g}_0=\mathfrak{sp}(l;\mathbb{R}) \text{ and } G_0=Sp(l;\mathbb{R}), \text{ real matrices in } \mathfrak{g} \text{ and } G$$

$$\text{compact: } \mathfrak{g}_u=\mathfrak{sp}(l;\mathbb{C})\cap\mathfrak{u}(2l)=\mathfrak{sp}(l), \quad G_u=Sp(l;\mathbb{C})\cap U(2l)=Sp(l).$$

Type D$_l$. Here $\mathfrak{g}=\mathfrak{o}(2l;\mathbb{C})$, $G=SO(2l;\mathbb{C})$, $G_0=SO(l,l)$ (split), $G_u=SO(2l)$ (compact).

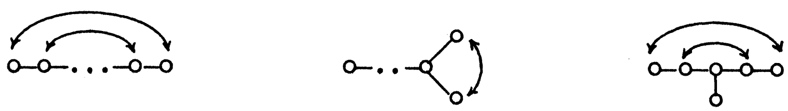
Type G$_2$. Here $\mathfrak{g}$ is the algebra of derivations of the complex Cayley algebra $\mathscr{C}$ and G is the automorphism group. The split real forms $\mathfrak{g}_0$, G_0 are the derivation algebra and automorphism group of the real form of $\mathscr{C}$ that is not a division algebra, and the compact real forms $\mathfrak{g}_u$ and G_u correspond to the real Cayley division algebra.

Type F$_4$ can be described in terms of the 27-dimensional exceptional Jordan algebra, but this sort of thing is not so useful for E$_6$, E$_7$, E$_8$.

Beside the classification, Dynkin diagrams allow an easy description of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$. First, let $\mathrm{Int}(\mathfrak{g})$ denote the group of ***inner automorphisms***, that is, automorphisms of the form $\mathrm{Ad}(g)$ where g runs through a connected Lie group with Lie algebra $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ is generated by the ***elementary automorphisms*** $\exp(\mathrm{ad}(\xi))$, ξ nilpotent in $\mathfrak{g}$. Second, note that $\mathrm{Int}(\mathfrak{g})$ is transitive on the pairs $(\mathfrak{h},\Delta^+)$ where $\mathfrak{h}$ is a Cartan subalgebra and Δ^+ is a positive $\mathfrak{h}$-root system for $\mathfrak{g}$, and if $\gamma\in\mathrm{Int}(\mathfrak{g})$ with $\gamma(\mathfrak{h})=\mathfrak{h}$ and $\gamma^*\Delta^+=\Delta^+$ then (*simple transitivity of the Weyl group*) $\gamma|_{\mathfrak{h}}=1$. Third, note that every symmetry of the Dynkin diagram extends to an automorphism of $\mathfrak{g}$. Assembling these facts,

$$\mathrm{Int}(\mathfrak{g}) \text{ is a normal subgroup of finite index in } \mathrm{Aut}(\mathfrak{g}),$$
$$\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \text{ is the symmetry group of the Dynkin diagram.}$$

If $\mathfrak{g}$ is simple of type A_1, B_l, C_l, G_2, F_4, E_7 or E_8, the Dynkin diagram has no nontrivial symmetries, so $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Int}(\mathfrak{g})$. If $\mathfrak{g}$ is simple of type A_l $(l > 1)$, D_l $(l > 4)$ or E_6, the Dynkin diagram has symmetry group $\mathbb{Z}_2$ given by



So $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \cong \mathbb{Z}_2$. And if $\mathfrak{g}$ is simple of type D_4 then the symmetry group of the diagram is the permutation group $\mathfrak{S}_3$ on the extremal vertices, so $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ has order 6. If $\mathfrak{g}$ is not simple, one also has permutations of isomorphic simple ideals.

8. Real semisimple groups

We say that a group G is of *Harish-Chandra class* if (i) G is a reductive Lie group, (ii) G has only finitely many topological components, (iii) the derived group $[G, G]$ has finite center, and (iv) if $g \in G$ then $\mathrm{Ad}(g) \in \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. This is the class that contains connected semisimple Lie groups with finite center and is closed under passage to certain reductive subgroups that we will encounter when we study parabolic subgroups. From now on, G is of Harish-Chandra class

Let G^0 denote the identity component. Simply because G is locally compact with G/G^0 finite, every compact subgroup of G is contained in a maximal compact subgroup, and any two maximal compact subgroups are $\mathrm{Ad}(G^0)$-conjugate. If K is a maximal compact subgroup of G, then $K \cap G^0 = K^0$, K meets every component of G, and G/K has the structure of riemannian symmetric space of curvature $\leqslant 0$. The symmetry gives an involutive automorphism θ of G, called a *Cartan involution*, with fixed point set $G^\theta = K$. Some examples:

G	θ	K
$GL(n;\mathbb{R})$	$\theta(g) = {}^t g^{-1}$	$O(n)$
$U(p, q)$	$\theta(g) = \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix} g \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix}$	$U(p) \times U(q)$
connected complex semisimple Lie group	conjugation over G_u	G_u

THEOREM. *A closed subgroup $H \subset G$ is reductive in G if, and only if, $\theta(H) = H$ for some Cartan involution θ of G.*

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ the corresponding *Cartan subgroup* is $H = \{g \in G\colon \mathrm{Ad}(g)\xi = \xi \text{ for all } \xi \in \mathfrak{h}\}$, closed subgroup with identity component $H^0 = \exp(\mathfrak{h})$. Since $\mathfrak{h}$ is reductive in $\mathfrak{g}$, it is stable under a Cartan involution θ, and evidently $\theta(H) = H$. So we have

COROLLARY. *Fix a Cartan involution θ of G; then every Cartan subgroup of G is conjugate to a θ-stable Cartan subgroup of G.*

Fix a Cartan involution θ, thus also a maximal compact subgroup $K = G^\theta$; this decomposes $\mathfrak{g}$ into (± 1)-eigenspaces,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}, \qquad \textit{Cartan decomposition,}$$

where $\mathfrak{k}$ is the Lie algebra of K. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$, and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$. The Killing form is negative on $\mathfrak{k}$, positive on $\mathfrak{s}$, and $\langle \mathfrak{k}, \mathfrak{s} \rangle = 0$. So a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is reductive in $\mathfrak{g}$ if and only if $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{l} \cap \mathfrak{s}$ for some Cartan involution θ.

A subspace $\mathfrak{a} \subset \mathfrak{g}$ is called a *Cartan subalgebra of* $(\mathfrak{g}, \mathfrak{k})$ if it is maximal among the subspaces such that

$$\mathfrak{a} \subset \mathfrak{s} \qquad \text{and} \qquad [\mathfrak{a}, \mathfrak{a}] = 0.$$

THEOREM. *Any two Cartan subalgebras of $(\mathfrak{g}, \mathfrak{k})$ are $\mathrm{Ad}_G(K)$-conjugate, so in particular they have the same dimension.*

This dimension is called the *rank* of the symmetric space G/K and the *real rank* of the group G. Note that G is a split real form of a complex group $G_{\mathbb{C}}$ if and only if $\operatorname{rank}_{\mathbb{R}} G = \operatorname{rank} G$. Also note that $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{s}$ is a compact real form $\mathfrak{g}_{\mathbb{C}}$.

Fix a Cartan subalgebra $\mathfrak{a}$ of $(\mathfrak{g}, \mathfrak{k})$. Every $\xi \in \mathfrak{s}$ belongs to some Cartan subalgebra of $(\mathfrak{g}, \mathfrak{k})$. It follows that

$$\mathfrak{s} = \bigcup_{k \in K} \operatorname{Ad}(k) \cdot \mathfrak{a} \text{ and so } \exp_G(\mathfrak{s}) = \bigcup_{k \in K} \operatorname{Ad}(k) \cdot \exp_G(\mathfrak{a}).$$

Writing $A = \exp_G(\mathfrak{a})$ and using $G = K \cdot \exp_G(\mathfrak{s})$, this gives a decomposition $G = KAK$. In a moment we will refine that decomposition.

The centralizer $Z_G(A)$ of A in G is θ-stable and intersects $\exp_G(\mathfrak{s})$ in A only. So we have

$$Z_G(A) = M \times A \qquad \text{where} \qquad M = Z_G(A) \cap K = Z_K(A).$$

Let $\mathfrak{t}$ (resp. T) be a Cartan subalgebra of $\mathfrak{m}$ (resp. Cartan subgroup of M). Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ (resp. $H = T \times A$) is a Cartan subalgebra of $\mathfrak{g}$ (resp. Cartan subgroup of G) stable under θ.

Every $\mathfrak{h}_{\mathbb{C}}$-root of $\mathfrak{g}_{\mathbb{C}}$ takes real values on $\mathfrak{a}$ and pure imaginary values on $\mathfrak{t}$. From the former, we have an $\mathfrak{a}$-root decomposition

$$\mathfrak{g} = (\mathfrak{m} + \mathfrak{a}) + \sum_{\nu \in \Delta_{\mathfrak{a}}} \mathfrak{g}_\nu$$

Here $\Delta_{\mathfrak{a}} \subset \mathfrak{a}^* \backslash \{0\}$ is the set of linear functionals ν on $\mathfrak{a}$ such that $\mathfrak{g}_\nu = \{\eta \in \mathfrak{g}\colon [\xi, \eta] = \nu(\xi)\eta \text{ for all } \xi \in \mathfrak{a}\} \neq 0$; $\Delta_{\mathfrak{a}}$ is the *$\mathfrak{a}$-root system* of $\mathfrak{g}$. The elements of $\Delta_{\mathfrak{a}}$ are called *$\mathfrak{a}$-roots* or *restricted roots*. They are the $\lambda|_{\mathfrak{a}}$ where $\lambda \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with $\lambda|_{\mathfrak{a}} \neq 0$. As in the case of ordinary roots, we have positive root systems $\Delta_{\mathfrak{a}}^+$ corresponding to Weyl chambers

$$\mathscr{C}\text{: topological component of } \mathfrak{a} \backslash \bigcup_{\nu \in \Delta_{\mathfrak{a}}} (\text{hyperplane } \nu = 0)$$

by $\Delta_{\mathfrak{a}}^+ = \{\nu \in \Delta_{\mathfrak{a}}\colon \nu > 0 \text{ on } \mathscr{C}\}$ and $\mathscr{C} = \{\xi \in \mathfrak{a}\colon \nu(\xi) > 0 \text{ for all } \xi \in \Delta_{\mathfrak{a}}^+\}$. There is a Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ as before, sometimes called the "baby Weyl group" of G. It is simply transitive on the set of positive $\mathfrak{a}$-root systems.

Fix a positive $\mathfrak{a}$-root system $\Delta_{\mathfrak{a}}^+$. Then we have nilpotent subalgebras of $\mathfrak{g}$

$$\mathfrak{n} = \sum_{\nu \in \Delta_{\mathfrak{a}}^+} \mathfrak{g}_\nu \qquad \text{and} \qquad \mathfrak{n}^- = \theta(\mathfrak{n}) = \sum_{\nu \in \Delta_{\mathfrak{a}}^+} \mathfrak{g}_{-\nu},$$

and connected simply connected nilpotent subgroups of G

$$N = \exp_G(\mathfrak{n}) \qquad \text{and} \qquad N^- = \theta(N) = \exp_G(\mathfrak{n}^-).$$

Their respective G-normalizers are the *minimal parabolic subgroups*

$$B = MAN \qquad \text{and} \qquad B^- = \theta(B) = MAN^-.$$

Note that B and B^- are conjugate – by any element of K that normalizes $\mathfrak{a}$ and gives the Weyl group element sending $\Delta_{\mathfrak{a}}^+$ to its negative. More generally, any two minimal parabolic subgroups of G are conjugate.

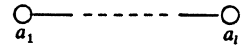
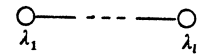
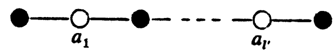
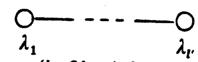
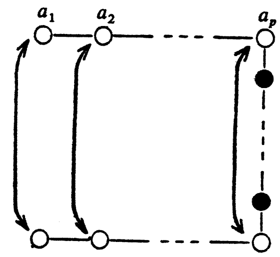
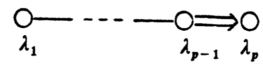
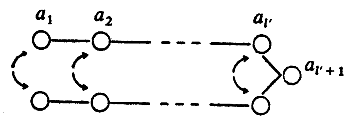
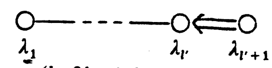
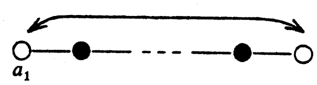
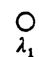
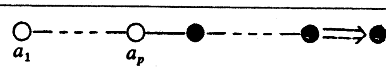
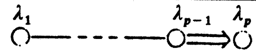
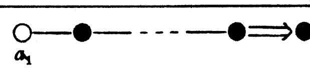
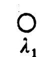
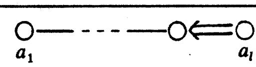
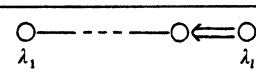
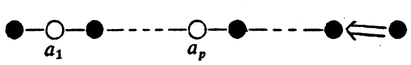
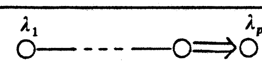
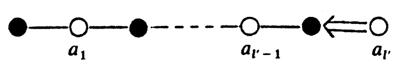
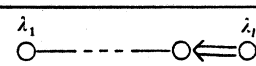
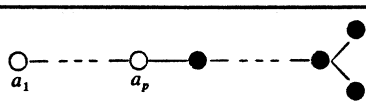
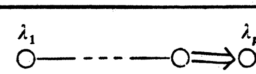
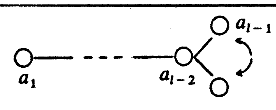
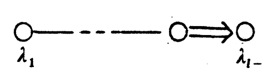
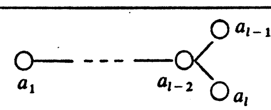
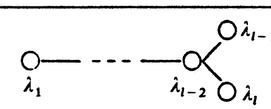
THE IWASAWA DECOMPOSITION. $G = KAN$; *more precisely, the map $K \times A \times N \to G$, by $(k, a, n) \mapsto kan$, is a diffeomorphism.*

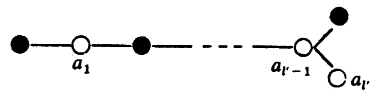
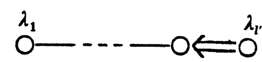
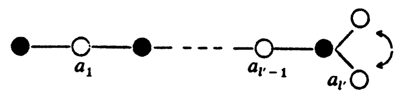
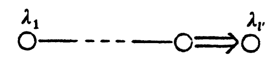
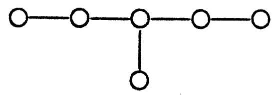
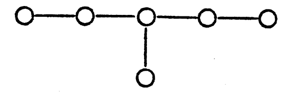
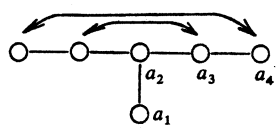
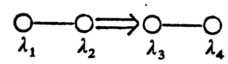
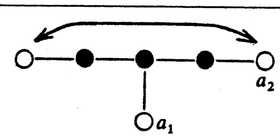
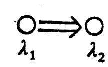
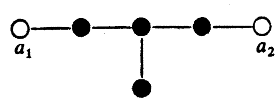
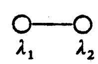
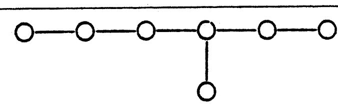
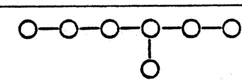
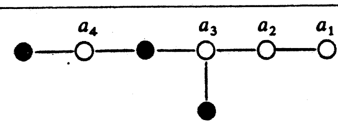
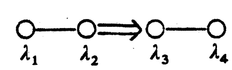
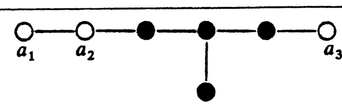
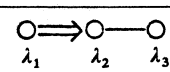
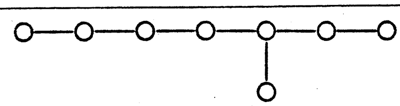
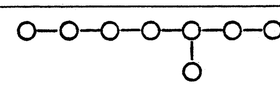
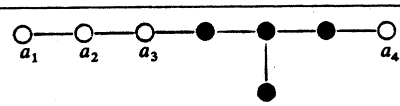
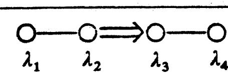
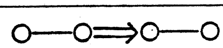
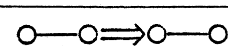
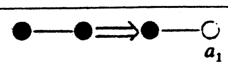
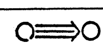
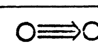
The space $G/B \cong K/M$ is called a *maximal boundary* of G/K. It appears both in the construction of 'principal series' respresentations and in the theory of bounded harmonic functions.

Let $\mathfrak{a}^+$ denote the positive Weyl chamber for $\Delta_{\mathfrak{a}}^+$, $A^+ = \exp_G(\mathfrak{a}^+)$, and $\overline{A^+}$ the closure of A^+ in G. The promised refinement of $G = KAK$ is: $G = K \cdot \overline{A^+} \cdot K$.

Let Δ be the $\mathfrak{h}_{\mathbb{C}}$-root system of $\mathfrak{g}_{\mathbb{C}}$, so $\Delta_{\mathfrak{a}} = \{\lambda|_{\mathfrak{a}}\colon \lambda \in \Delta,\ \lambda|_{\mathfrak{a}} \neq 0\}$. We say that choices Δ^+, $\Delta_{\mathfrak{a}}^+$ of positive root systems are *consistent* if $\lambda \in \Delta^+$, $\lambda|_{\mathfrak{a}} \neq 0$ implies $\lambda|_{\mathfrak{a}} \in \Delta_{\mathfrak{a}}^+$. Given $\Delta_{\mathfrak{a}}^+$, the consistent choices of Δ^+ are in one-one correspondence with the positive $\mathfrak{t}_{\mathbb{C}}$-root systems on $\mathfrak{m}_{\mathbb{C}}$; so consistent choices exist.

Now fix consistent choices Δ^+, $\Delta_{\mathfrak{a}}^+$ of positive $\mathfrak{h}_{\mathbb{C}}$-root and $\mathfrak{a}$-root systems. Let S and $S_{\mathfrak{a}}$ denote the corresponding simple systems. The *Satake diagram* of G is obtained as follows. In the Dynkin diagram $\mathscr{D}_S$ of S, we darken the vertices corresponding to simple roots $\alpha \in S$ with $\alpha|_{\mathfrak{a}} = 0$, and we join (by 2-headed arrows) vertices corresponding to simple roots $\alpha, \alpha' \in S$ with $0 \neq \alpha|_{\mathfrak{a}} = \alpha'|_{\mathfrak{a}}$. Following Araki, we list all real simple $\mathfrak{g}$ except those where $\mathfrak{g}$ is compact or complex. Here $\lambda_i = \alpha_i|_{\mathfrak{a}}$, $m(\lambda) = \dim \mathfrak{g}_\lambda$, $\mathbb{H}$ denotes the quaternions, $l = \operatorname{rank} \mathfrak{g}$, and we identify the real exceptional Lie algebras by listing the type of a maximal compact subalgebra $\mathfrak{k}$ as subscript with T_1 meaning 1-dimensional Abelian.

	$\mathfrak{g}$	$\mathscr{D}_S$	$\mathscr{D}_{S_a}$	$m(\lambda_i)$	$m(2\lambda_i)$
AI	$\mathfrak{sl}(l+1;\mathbb{R})$			1	0
AII	$\mathfrak{sl}(l+1;\mathbb{H})$		 $(l=2l'+1, l'\geq 1)$	4	0
AIII	$\mathfrak{su}(p, l+1-p)$		 $\left(2\leq p\leq \frac{l}{2}\right)$	2 (for $i<p$) $2(l-2p+1)$ (for $i=p$)	0 1
	$\mathfrak{su}(l'+1, l'+1)$		 $(l=2l'+1, l'\geq 1)$	2 (for $i\leq l'$) 1 ($i=l'+1$)	0 0
	$\mathfrak{su}(1, l)$			$2(l-1)$	1
BI	$\mathfrak{so}(p, 2l+1-p)$		 $(l\geq 2, 2\leq p\leq l)$	1 ($i<p$) $2(l-p)+1$ ($i=p$)	0 0
BII	$\mathfrak{so}(1, 2l)$			$2l-1$	0
CI	$\mathfrak{sp}(l;\mathbb{R})$			1	0
CII	$\mathfrak{sp}(p, l-p)$		 $\left(l\geqq 3, 1\leqq p\leqq \frac{l-1}{2}\right)$	4 ($i<p$) $4(l-2p)$ ($i=p$)	0 3
	$\mathfrak{sp}(l', l')$		 $(l=2l', l'\geqq 2)$	4 ($i<l'$) 3 ($i=l'$)	0 0
DI	$\mathfrak{so}(p, 2l-p)$		 $(l\geqq 4, 2\leqq p\leqq l-2)$	1 ($i<p$) $2(l-p)$ ($i=p$)	0
	$\mathfrak{so}(l-1, l+1)$			1 ($i<l-1$) 2 ($i=l-1$)	0
	$\mathfrak{so}(l, l)$			1	0

	$\mathfrak{so}(1, 2l-1)$			$2(l-1)$	0
DII	$\mathfrak{so}^*(4l')$		 $(l=2l', l'\geqq 2)$	4 $(i<l')$ 1 $(i=l')$	0 0
	$\mathfrak{so}^*(4l'+2)$		 $(l=2l'+1, l'\geqq 2)$	4 $(i<l')$ 4 $(i=l')$	0 1
EI	$\mathfrak{e}_{6,C_4}$			1	0
EII	$\mathfrak{e}_{6,A_5A_1}$			1 $(i=1,2)$ 2 $(i=3,4)$	0 0
EIII	$\mathfrak{e}_{6,D_5T_1}$			6 $(i=1)$ 8 $(i=2)$	0 1
EIV	$\mathfrak{e}_{6,F_4}$			8	0
EV	$\mathfrak{e}_{7,A_7}$			1	0
EVI	$\mathfrak{e}_{7,D_6A_1}$			1 $(i=1,2)$ 4 $(i=3,4)$	0 0
EVII	$\mathfrak{e}_{7,E_6T_1}$			1 $(i=1)$ 8 $(i=2,3)$	0 0
EVIII	$\mathfrak{e}_{8,D_8}$			1	0
EIX	$\mathfrak{e}_{8,E_7A_1}$			1 $(i=1,2)$ 8 $(i=3,4)$	0 0
FI	$\mathfrak{f}_{4,C_3A_1}$			1	0
FII	$\mathfrak{f}_{4,B_4}$			8	7
G	$\mathfrak{g}_{2,A_1A_1}$			1	0

Chapter II

FINITE-DIMENSIONAL REPRESENTATION THEORY

This chapter is intended to present a short tour of the finite-dimensional representation theory for reductive and semisimple Lie groups. We start by looking at Verma modules and representations with highest weights (Section 9) and specializing (in Section 10) to E. Cartan's highest weight theory for finite-dimensional irreducible representations. Then we recall some results on invariants (Section 11), define the Harish-Chandra homomorphism (Section 12), and prove the Weyl Character Formula (Section 13). In Section 14 we try to fix these ideas by specializing to $\mathfrak{sl}(2)$. We then illustrate to some extent how these results can be applied to homogeneous vector bundles, proving Frobenius' Reciprocity Theorem for compact groups and the Borel-Weil Theorem for compact semisimple groups in Section 15. Finally, in Section 16, we specialize to the decomposition of the L_2 space of a compact symmetric space and give Cartan's highest weight theory for class one representations.

9. Representations with highest weight

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$, let Δ denote the root system, choose a positive root system Δ^+, and let $S=\{\alpha_1,\dots,\alpha_l\}$ be the simple roots. Then we have

root lattice $\Lambda_{rt}=\mathbb{Z}[S]=$ integral linear combinations of roots,

weight lattice $\Lambda_{wt}=\{\lambda\in\mathfrak{h}^*: 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\in\mathbb{Z}$ for all $\alpha\in S\}$,

dominant weights $\Lambda^+_{wt}=\{\lambda\in\Lambda_{wt}: 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\geqslant 0$ for all $\alpha\in S\}$,

and, for convenience,

$\Lambda^+_{rt}=$ non-negative integral linear combinations from S.

The latter gives a partial order on $\mathfrak{h}^*$: $\lambda\geqslant\mu$ if $\lambda-\mu\in\Lambda^+_{rt}$. We will also have use for the notation $\rho=\frac{1}{2}\Sigma_{\Delta^+}\lambda$ and for

$$\mathfrak{n}=\sum_{\lambda\in\Delta^+}\mathfrak{g}_\lambda \quad\text{and}\quad \mathfrak{b}=\mathfrak{h}+\mathfrak{n},$$

$$\mathfrak{n}^-=\sum_{\lambda\in\Delta^+}\mathfrak{g}_{-\lambda} \quad\text{and}\quad \mathfrak{b}^-=\mathfrak{h}+\mathfrak{n}^-,$$

Finally, we will have use for the *partition function* $P=P_{\mathfrak{g}}$ on $\mathfrak{h}^*$ given by

$$P(\lambda)=\text{number of distinct expressions } \lambda=\sum n_i\alpha_i,$$

where the n_i are integers $\geqslant 0$ and the α_i run over Δ^+. Here note $P(\lambda)>0$ if and only if $\lambda\in\Lambda^+_{rt}$.

Of course Λ_{rt} is a lattice only in $(\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}])_{\mathbb{R}}$.

Let ψ represent $\mathfrak{g}$ on a vector space V. If $\lambda\in\mathfrak{h}^*$, the corresponding *weight space* $V_\lambda=\{v\in V: \psi(\xi)v=\lambda(\xi)v$ for $\xi\in\mathfrak{h}\}$. We call dim V_λ the *multiplicity* of the weight λ for ψ. If $V_\lambda\neq 0$ we say that λ is a *weight* of ψ and that any $v\in V_\lambda$ is a *λ-weight vector*. The algebraic sum $\Sigma_{\lambda\in\mathfrak{h}^*}V_\lambda$ is an invariant subspace because $\psi(\mathfrak{g}_\mu)V_\lambda\subset V_{\lambda+\mu}$. Here we will be concerned with the case where $V=\Sigma_{\lambda\in\mathfrak{h}^*}V_\lambda$; that will be automatic when dim $V<\infty$. And as in Section 2 it is automatic if ψ is irreducible and has a nonzero weight vector.

Fix $\lambda\in\mathfrak{h}^*$. We define the *Verma module* $M(\lambda)$ to be $\mathscr{U}(\mathfrak{g})\otimes_{\mathscr{U}(\mathfrak{b})}\mathbb{C}$ where $\mathscr{U}(\mathfrak{b})$ acts on $\mathbb{C}$ by

$$\tau_{\lambda-\rho}: h+n\mapsto(\lambda-\rho)(h) \qquad\text{for}\qquad h\in\mathfrak{h}, \qquad n\in\mathfrak{n}.$$

In other words, $M(\lambda)=\mathscr{U}(\mathfrak{g})/\{\mathscr{U}(\mathfrak{g})\mathfrak{n}+\Sigma_{\xi\in\mathfrak{h}}\mathscr{U}(\mathfrak{g})(\xi-(\lambda-\rho)(\xi))\}$. Note that $u\mapsto u\otimes 1$ is an $\mathfrak{n}^-$-module isomorphism of $\mathscr{U}(\mathfrak{n}^-)$ onto $M(\lambda)$. If we enumerate

$$\Delta^+=\{\alpha_1,\dots,\alpha_n\}$$

and choose $0\neq e_\alpha\in\mathfrak{g}_\alpha$ then the Poincaré–Birkhoff–Witt Theorem gives

$$M(\lambda)_\mu=\sum_{\substack{p_i\in\mathbb{Z},\,p_i\geqslant 0,\\ \lambda-\rho-\Sigma p_i\alpha_i=\mu}}(e^{p_1}_{-\alpha_1}\cdots e^{p_n}_{-\alpha_n})\otimes\mathbb{C}$$

In particular, $\dim M(\lambda)_\mu=P(\lambda-\rho-\mu)$ is finite, and is zero unless $\lambda-\rho-\mu\in\Lambda^+_{rt}$, that is, $\lambda-\rho\geqslant\mu$. In consequence, $M(\lambda)=\Sigma_{\nu\in\Lambda^+_{rt}}M(\lambda)_{\lambda-\rho-\nu}$, sum of weight spaces; and $M(\lambda)$ has highest weight space $M(\lambda)_{\lambda-\rho}=1\otimes\mathbb{C}$, which is annihilated by $\mathfrak{n}$ and which generates $M(\lambda)$ under $\mathscr{U}(\mathfrak{n}^-)$. Now denote $v_{\lambda-\rho}=1\otimes 1\in M(\lambda)_{\lambda-\rho}$.

THEOREM. *Let ψ represent $\mathfrak{g}$ on a vector space V and suppose there exists $v\in V_{\lambda-\rho}$ such that $\psi(\mathfrak{n})v=0$ and $\psi(\mathscr{U}(\mathfrak{g}))v=V$. Then* (i) *there is a unique $\mathfrak{g}$-module homomorphism $M(\lambda)\to V$ sending $v_{\lambda-\rho}$ to v; it is surjective, and is injective if and only if $\psi(u)$ is one-to-one for $0\neq u\in\mathscr{U}(\mathfrak{n}^-)$,* (ii) *$V$ is a sum of weight spaces, each weight is in $\lambda-\rho-\Lambda^+_{rt}$, each weight space has* *finite dimension, and the highest weight space $V_{\lambda-\rho}$ has dimension* 1. *Every* $\mathfrak{g}$*-endomorphism of V is scalar, so $\psi|_{\text{center of } \mathscr{U}(\mathfrak{g})}$ is an algebra homomorphism* $\mathscr{Z}(\mathfrak{g})\to\mathbb{C}$.

For existence and uniqueness of the $\mathfrak{g}$-map $M(\lambda)\to V$ sending $v_{\lambda-\rho}$ to v we note that relations in $M(\lambda)$ are generated by $\mathfrak{n}\cdot v_{\lambda-\rho}=0$ and $v_{\lambda-\rho}\in M(\lambda)_{\lambda-\rho}$ only. It is surjective because $\mathscr{U}(\mathfrak{n}^-)\cdot v_{\lambda-\rho}=M(\lambda)$ and $\mathscr{U}(\mathfrak{n}^-)\cdot v=V$. The assertions on weights of V follow from the facts on weights of $M(\lambda)$. If T is a $\mathfrak{g}$-endomorphism of V then $\xi\in\mathfrak{h}$ gives $\psi(\xi)Tv= =T\psi(\xi)v=(\lambda-\rho)(\xi)\cdot Tv$ so $Tv\in V_{\lambda-\rho}$, say $Tv=tv$ where $t\in\mathbb{C}$, and now if $u\in\mathscr{U}(\mathfrak{g})$ then $T\psi(u)v=\psi(u)tv=t\psi(u)v$ shows $T=c\cdot 1$. Finally, if $M(\lambda)\to V$ is injective then $\psi(u)$ is injective for $0\neq u\in\mathscr{U}(\mathfrak{n}^-)$ from the corresponding fact on $M(\lambda)$; and if $M(\lambda)\to V$ is not injective then some $u\cdot v_{\lambda-\rho}$, $0\neq u\in\mathscr{U}(\mathfrak{n}^-)$, goes to zero, so $\psi(u)$ is not injective.

Following Harish-Chandra's notation, we write χ_λ: $\mathscr{Z}(\mathfrak{g})\mapsto\mathbb{C}$ for the action of the center of $\mathscr{U}(\mathfrak{g})$ on $M(\lambda)$. Since we generally look from the viewpoint of a group representation, we call χ_λ the *infinitesimal character* of a $\mathfrak{g}$-module generated by a highest weight vector of weight $\lambda-\rho$.

Every proper $\mathfrak{g}$-submodule of $M(\lambda)$ is contained in $\Sigma_{\mu\neq\lambda-\rho}M(\lambda)_\mu$, so there is a maximal such submodule. Denote

$$L(\lambda)=M(\lambda)/\text{maximal proper } \mathfrak{g}\text{-submodule}.$$

Then the representation of $\mathfrak{g}$ on $L(\lambda)$ is an irreducible representation with highest weight $\lambda-\rho$.

THEOREM. *Let V be an irreducible $\mathfrak{g}$-module with highest weight $\lambda-\rho$; then $V\cong L(\lambda)$.*

For V is a quotient of $M(\lambda)$.

10. Finite-dimensional representations

Suppose that $\mathfrak{g}$ is semisimple and ψ is a finite dimensional representation, say on V. Then $V=\Sigma_{\lambda\in\mathfrak{h}^*}V_\lambda$ because ψ is sum of irreducibles, and, in each irreducible, Lie's Theorem applied to $\mathfrak{h}$ gives a nonzero weight vector. For each root $\alpha\in\Delta$ denote

$$\mathfrak{g}[\alpha]=\mathfrak{g}_{-\alpha}+h_\alpha\mathbb{C}+\mathfrak{g}_\alpha\cong\mathfrak{sl}(2;\mathbb{C})$$

and consider $\psi|_{\mathfrak{g}[\alpha]}$ on a weight vector v_λ. From Section 2,

$$2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle \text{ is an integer, and is } \geqslant 0 \text{ in case } \psi(\mathfrak{g}_\alpha)v_\lambda=0.$$

In particular *every weight λ of ψ satisfies $\lambda\in\Lambda_{\text{wt}}$, and if λ is a maximal weight in the sense $\psi(\mathfrak{n})V_\lambda=0$ then $\lambda\in\Lambda^+_{\text{wt}}$*. Further *if $w\in W$ and λ is a weight, then $w(\lambda)$ is a weight of the same multiplicity*. For if $\alpha\in\Delta$ and $n=2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$ then, looking at $\mathfrak{g}[\alpha]$, $\psi(e_{-\alpha})^n$ injects V_λ into $V_{\lambda-n\alpha}$, so $s_\alpha(\lambda)=\lambda-n\alpha$ is a weight of multiplicity$\geqslant$(mult of λ).

Now suppose further that ψ is irreducible. Then ψ *has a highest weight* $\lambda\in\Lambda^+_{\text{wt}}$ because dim $V<\infty$, $V\cong L(\lambda+\rho)$ in consequence, and so λ is unique and has multiplicity 1. If μ is any weight of ψ then $\mu\leqslant\lambda$, and since $W(\mu)$ meets Λ^+_{wt} one can show $\langle\mu,\mu\rangle\leqslant\langle\lambda,\lambda\rangle$.

These considerations also hold for reductive $\mathfrak{g}=\mathfrak{z}\oplus[\mathfrak{g},\mathfrak{g}]$ provided that $\psi|_{\mathfrak{z}}$ is semisimple, and that condition is automatic in case ψ is irreducible.

É. Cartan's 'highest weight theory' for finite-dimensional representations:

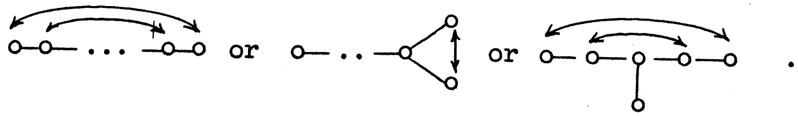
THEOREM. *If $\mathfrak{g}$ is a complex reductive Lie algebra, then the map $\lambda\mapsto L(\lambda+\rho)$ is a bijection of Λ^+_{wt} onto the set of equivalence classes of irreducible finite-dimensional $\mathfrak{g}$-modules.*

To see this, first consider a representation ψ of $\mathfrak{g}$ on V where V is generated by a highest weight vector $v\in V_\lambda$ and, for every simple root α, $\psi(e_{-\alpha})^n\cdot v=0$ for n sufficiently large; we claim that ψ is irreducible and finite dimensional. For if dim $V>1$ then $\psi(e_{-\alpha})\cdot v\neq 0$ for some simple root α, so v is in a finite-dimensional nontrivial irreducible $\mathfrak{g}[\alpha]$-submodule V^0 of V; then $\mathfrak{g}\otimes V^0\to V$ by $(\xi,w)\mapsto\psi(\xi)w$ is a surjective $\mathfrak{g}[\alpha]$-module map, and $\mathfrak{g}\otimes V^0$ is a finite sum of finite-dimensional $\mathfrak{g}[\alpha]$-modules, so V is a finite sum of finite dimensional $\mathfrak{g}[\alpha]$-modules; now dim $V<\infty$, and irreducibility comes from dim $V_\lambda=1$. Second, let $\lambda\in\Lambda^+_{\text{wt}}$, let J be the maximal proper $\mathfrak{g}$-submodule of $M(\lambda+\rho)$ and let $v_\lambda=1\otimes 1\in M(\lambda+\rho)_\lambda$; for every simple root $\alpha\in S$ let $m_\alpha=1+2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$; we claim $J=\Sigma_{\alpha\in S}\mathscr{U}(\mathfrak{g})e^{m_\alpha}_{-\alpha}\cdot v_\lambda$ and that J has finite codimension in $M(\lambda+\rho)$. For if $\alpha\in S$ then $m_\alpha=2\langle\lambda+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ is a positive integer and one can check that the $\mathfrak{g}$-submodule (say Y_α) of $M(\lambda+\rho)$ generated by $e^{m_\alpha}_{-\alpha}\cdot v_\lambda$ is

$$\mathscr{U}(\mathfrak{n}^-)e^{m_\alpha}_{-\alpha}\cdot v_\lambda\subsetneqq M(\lambda+\rho);$$

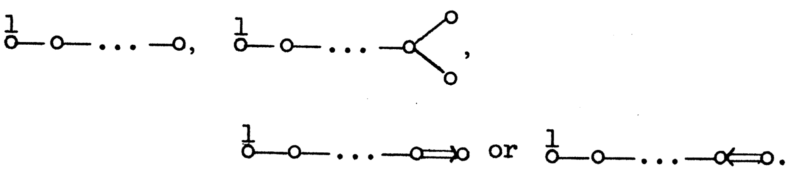
so $\Sigma_{\alpha\in S}Y_\alpha\subset J$; and our first consideration shows that $M(\lambda+\rho)/\Sigma_{\alpha\in S}Y_\alpha$ is a finite-dimensional simple $\mathfrak{g}$-module. The theorem is proved.

An addendum: Let $w_0 \in W$ be the element that sends Δ^+ to $-\Delta^+$. If $\mathfrak{g}$ is simple then $w_0 = -1$ except when $\mathfrak{g}$ is of type $A_l\,(l>1)$, D_{2n+1} or E_6, in which cases $-w_0$ acts on Δ^+ by

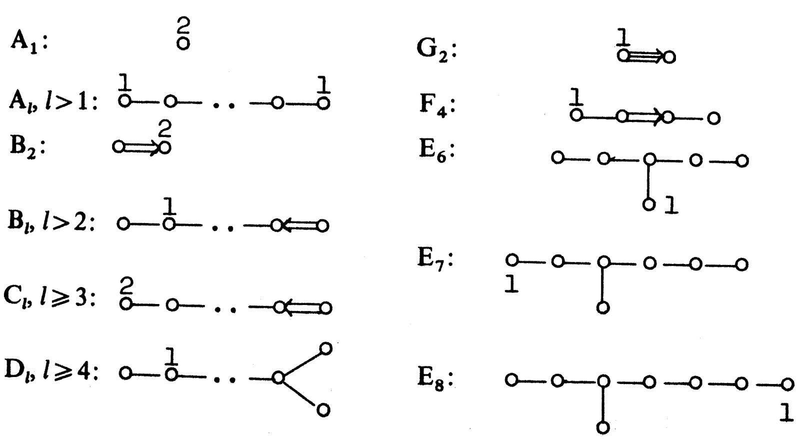


Now suppose that V is a finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Then V has lowest weight $w_0(\lambda)$, and V^* is the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight $-w_0(\lambda)$.

Let us write ψ_λ for the finite-dimensional irreducible representation of highest weight $\lambda \in \Lambda^+_{\mathrm{wt}}$. If $\mathfrak{g}$ is semisimple we can 'describe' ψ_λ on the Dynkin diagram as follows. Whenever the non-negative integer $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle$ is positive, we write it next to the vertex that corresponds to the simple root α. Thus, for the classical groups, the ordinary ('vector') representation is 'described' as

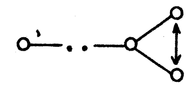


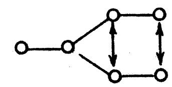
The adjoint representations are



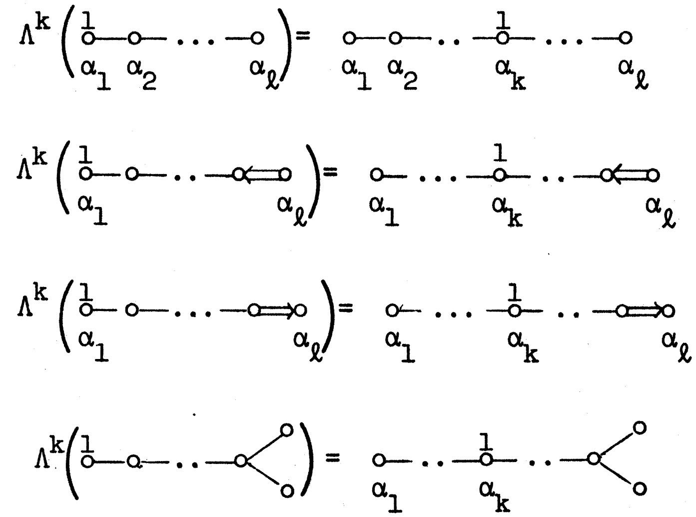
If $\mathfrak{g}$ is simple, then ψ_1 is equivalent to its dual $\psi_{-w_0(\lambda)}$ if and only if its Dynkin diagram is stable under

 for $A_l\,(l \geq 2)$,

 for D_{2n+1},

 for E_6.

One also has some handy computational tricks, such as



11. Invariant polynomials

Let f be a polynomial function on $\mathfrak{g}$, i.e. an element of the symmetric algebra $S(\mathfrak{g}^*)$. We say that f is *invariant* if it is annihilated by every $\xi \in \mathfrak{g}$, i.e. preserved by every $\gamma\in\mathrm{Int}(\mathfrak{g})$. Denote the space of all such polynomial invariants by $S(\mathfrak{g}^*)^{\mathfrak{g}}$. Similarly one has the space $S(\mathfrak{h}^*)^W$ of W-invariant polynomials on $\mathfrak{h}$. Evidently the restriction map

$$i: S(\mathfrak{g}^*)\to S(\mathfrak{h}^*) \quad \text{by} \quad i(f)=f|_{\mathfrak{h}}$$

is an algebra homomorphism that maps $S(\mathfrak{g}^*)^{\mathfrak{g}}$ into $S(\mathfrak{h}^*)^W$.

THEOREM. (i) *For each integer $n\geqslant 0$, the space $S^n(\mathfrak{g}^*)^{\mathfrak{g}}$ of invariants homogeneous of degree n is spanned by the $\xi\mapsto\mathrm{trace}(\psi(\xi)^n)$ as ψ runs over the semisimple finite-dimensional representations of* $\mathfrak{g}$, (ii) *similarly $S^n(\mathfrak{h}^*)^W$ is spanned by the $\xi\mapsto\mathrm{trace}(\psi)^n)$*, (iii) *$i$: $S(\mathfrak{g}^*)^{\mathfrak{g}}\to S(\mathfrak{h}^*)^W$ is an isomorphism.*

The Cartan–Killing form on $[\mathfrak{g},\mathfrak{g}]$, direct sum with any nondegenerate symmetric bilinear form on the center $\mathfrak{z}$ of $\mathfrak{g}$, defines isomorphisms a: $S(\mathfrak{g})\mapsto S(\mathfrak{g}^*)$ and b: $S(\mathfrak{h})\to S(\mathfrak{h}^*)$, the latter W-equivariant. If J denotes the ideal in $S(\mathfrak{g})$ generated by $\mathfrak{n}\cup\mathfrak{n}^-$ then $S(\mathfrak{g})=S(\mathfrak{h})\oplus J$. Now let

$$j: S(\mathfrak{g})\to S(\mathfrak{h}), \text{ projection with kernel } J.$$

Then j is a homomorphism and $i(a(f))=b(j(f))$ for $f\in S(\mathfrak{g})$. Further, *$j|_{S(\mathfrak{g})^{\mathfrak{g}}}$ is an isomorphism of $S(\mathfrak{g})^{\mathfrak{g}}$ onto $S(\mathfrak{h})^W$.*

CHEVALLEY'S THEOREM. *There are $l=\mathrm{rank}\,\mathfrak{g}$ homogeneous, algebraically independent, elements $f_1,\dots,f_l\in S(\mathfrak{g})^{\mathfrak{g}}$ which generate $S(\mathfrak{g})^{\mathfrak{g}}$. Their degrees $v_1,\dots,v_l$ are independent (except for a permutation) of choice of the f_i, and $\Sigma\, v_i=\frac{1}{2}(l+\dim\mathfrak{g})$.*

COROLLARY. *The center $\mathscr{Z}(\mathfrak{g})$ of $\mathscr{U}(\mathfrak{g})$ is a polynomial algebra in l indeterminates.*

For the Poincaré–Birkhoff–Witt Theorem gives a vector space isomorphism of $\mathscr{U}(\mathfrak{g})$ onto $S(\mathfrak{g})$ that is $\mathrm{Ad}(G)$-invariant, thus sends $\mathscr{Z}(\mathfrak{g})$ onto $S(\mathfrak{g})^{\mathfrak{g}}$.

12. THE HARISH-CHANDRA HOMOMORPHISM

Enumerate $\Delta^+=\{\alpha_1,\dots,\alpha_n\}$ and choose a basis $\{h_1,\dots,h_l\}$ of $\mathfrak{h}$. Then $\mathscr{U}(\mathfrak{g})$ has vector space basis consisting of the

$$u(\bar{q},\bar{m},\bar{p})=e_{-\alpha_1}^{q_1}\dots e_{-\alpha_n}^{q_n}h_1^{m_1}\dots h_l^{m_l}e_{\alpha_1}^{p_1}\dots e_{\alpha_n}^{p_n},$$

where $\bar{q}$, $\bar{m}$, $\bar{p}$ are multi-indices. If $\xi\in\mathfrak{h}$ then

$$[\xi, u(\bar{q},\bar{m},\bar{p})]=\left(\sum_1^n (p_i-q_i)\alpha_i\right)(h)u(\bar{q},\bar{m},\bar{p}),$$

and this gives the decomposition

$$\mathscr{U}(\mathfrak{g})=\sum_{\lambda\in\Lambda\mathfrak{n}}\mathscr{U}(\mathfrak{g})_\lambda \qquad \text{under } \mathfrak{h}.$$

As $\mathrm{ad}(\xi)$ is a derivation, $\mathscr{U}(\mathfrak{g})_\lambda\mathscr{U}(\mathfrak{g})_\mu\subset\mathscr{U}(\mathfrak{g})_{\lambda+\mu}$, and in particular $\mathscr{U}(\mathfrak{g})_0=$ $=\mathscr{U}(\mathfrak{g})^{\mathfrak{h}}$ is a subalgebra of $\mathscr{U}(\mathfrak{g})$. As $\mathscr{U}(\mathfrak{g})_0$ is spanned by the $u(\bar{q},\bar{m},\bar{p})$, $\Sigma\, p_i\alpha_i=\Sigma\, q_i\alpha_i$,

$$\mathscr{U}(\mathfrak{g})\mathfrak{n}\cap\mathscr{U}(\mathfrak{g})_0=\mathfrak{n}^-\mathscr{U}(\mathfrak{g})\cap\mathscr{U}(\mathfrak{g}_0): \qquad \text{call it } L.$$

So *L is a 2-sided ideal in $\mathscr{U}(\mathfrak{g})_0$, and* one checks that $\mathscr{U}(\mathfrak{g})_0=\mathscr{U}(\mathfrak{h})\oplus L$. Now consider the projection

$$\phi: \mathscr{U}(\mathfrak{g})_0\to\mathscr{U}(\mathfrak{h}), \textit{ Harish-Chandra homomorphism.}$$

THEOREM. *Let V be a $\mathfrak{g}$-module generated by a highest weight vector $v\in V_\lambda$, and let χ: $\mathscr{Z}(\mathfrak{g})\to\mathbf{C}$ its infinitesimal character. If $z\in\mathscr{Z}(\mathfrak{g})$ then $\chi(z)=\phi(z)(\lambda)$.*

For $z=\phi(z)+\Sigma\mu_i\eta_i$ with $\mu_i\in\mathscr{U}(\mathfrak{g})$ and $\eta_i\in\mathfrak{n}$, so

$$\chi(z)v=z\cdot v=\phi(z)\cdot v+\sum\mu_i\eta_i\cdot v=\phi(z)\cdot v=\phi(z)(\lambda)v.$$

Now let γ denote the 'shift by ρ' automorphism of $S(\mathfrak{h})$, $\gamma(p)(\lambda)=p(\lambda-\rho)$.

THEOREM. *$\gamma\circ\phi|_{\mathscr{Z}(\mathfrak{g})}$ is an isomorphism of $\mathscr{Z}(\mathfrak{g})$ onto $S(\mathfrak{h})^W$, independent of the choice Δ^+ of positive root system.* We call $\gamma\circ\phi|_{\mathscr{Z}(\mathfrak{g})}$ the *Harish-Chandra isomorphism* of $\mathscr{Z}(\mathfrak{g})$ onto $S(\mathfrak{h})^W$.

Toward the end of Section 9 we agreed to write χ_λ for the infinitesimal character of a $\mathfrak{g}$-module generated by a highest weight vector of weight $\lambda-\rho$. Now observe

$$\chi_\lambda(z)=\phi(z)(\lambda-\rho)=(\gamma\circ\phi)(z)(\lambda).$$

Furthermore, one can check (i) *if χ: $\mathscr{Z}(\mathfrak{g})\to\mathbf{C}$ is any homomorphism then there exists $\lambda\in\mathfrak{h}^*$ such that $\chi=\chi_\lambda$*, (ii) *if λ, $\lambda'\in\mathfrak{h}^*$ then $\chi_\lambda=\chi_{\lambda'}$ if and only if $\lambda'\in W(\lambda)$*, and (iii) *if $w_0\in W$ is the element that sends Δ^+ to $-\Delta^+$, and if $u\to u^T$ is the anti-automorphism of $\mathscr{U}(\mathfrak{g})$ generated by $\mathfrak{g}\ni\xi\mapsto-\xi$, then for all $\lambda\in\mathfrak{h}^*$ and all $z\in\mathscr{Z}(\mathfrak{g})$ one has $\chi_{\lambda+\rho}(z)=\chi_{-w_0(\lambda)+\rho}(z^T)$.*

13. Formal characters

Let $\mathbf{Z}\langle \mathfrak{h}^*\rangle$ denote the set of all measures f on $\mathfrak{h}^*$ such that (i) if $\lambda \in \mathfrak{h}^*$ then $f(\lambda) \in \mathbf{Z}$ and (ii) for some $\mu \in \mathfrak{h}^*$, f has support in $\mu - \Lambda_{\mathrm{rt}}^+$. Then $\mathbf{Z}\langle \mathfrak{h}^*\rangle$ is a ring, with ordinary addition and with convolution product. For convenience with the product, let e^λ denote the measure supported in $\{\lambda\}$ with value 1 at λ; then the multiplication is

$$\left(\sum_{\lambda\in\mathfrak{h}^*} c'_\lambda e^\lambda\right)\left(\sum_{\mu\in\mathfrak{h}^*} c''_\mu e^\mu\right) = \sum_{\nu\in\mathfrak{h}^*}\left(\sum_{\lambda+\mu=\nu} c'_\lambda c''_\mu\right)e^\nu,$$

in other words $e^\lambda e^\mu = e^{\lambda+\mu}$; the $\Sigma_{\lambda+\mu=\nu} c'_\lambda c''_\mu$ are finite sums because of the condition on supports, and if $\mathrm{supp}(f) \subset \alpha - \Lambda_{\mathrm{rt}}^+$ and $\mathrm{supp}(f') \subset \beta - \Lambda_{\mathrm{rt}}^+$ then $\mathrm{supp}(ff') \subset (\alpha+\beta) - \Lambda_{\mathrm{rt}}^+$.

The Weyl group W preserves the set of all finitely supported measures in $\mathbf{Z}\langle \mathfrak{h}^*\rangle$, but does not preserve $\mathbf{Z}\langle \mathfrak{h}^*\rangle$.

We say that a $\mathfrak{g}$-module V *has a formal character* if $V = \Sigma_{\lambda\in\mathfrak{h}^*} V_\lambda$ and each $\dim V_\lambda < \infty$. Then the *formal character* of V is $\mathrm{ch}(V) = \Sigma_{\lambda\in\mathfrak{h}^*}(\dim V_\lambda)e^\lambda$. If V' is submodule then also V' and V/V' have formal characters, and $\mathrm{ch}(V) = \mathrm{ch}(V') + \mathrm{ch}(V/V')$. If V_1, V_2 are $\mathfrak{g}$-modules that have formal characters, so does $V_1 \otimes V_2$, and $\mathrm{ch}(V_1\otimes V_2) = \mathrm{ch}(V_1)\mathrm{ch}(V_2)$.

Example: $\mathrm{ch}\, M(\lambda) = (\Sigma_{w\in W}\varepsilon(w)e^{w\rho})^{-1}e^\lambda \in \mathbf{Z}\langle\mathfrak{h}^*\rangle$, where $\varepsilon(w) = \det_{\mathfrak{h}}(w) = \pm 1$. This is seen as follows. Set

$$d = \left(\sum_{w\in W}\varepsilon(w)e^{w\rho}\right) = e^{-\rho}\prod_{\alpha\in\Delta^+}(e^\alpha - 1)$$

$$= e^\rho \prod_{\alpha\in\Delta^+}(1-e^{-\alpha}) = \prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2})$$

and

$$k = \sum_{\gamma\in\Lambda_{\mathrm{rt}}^+} P(\gamma)e^{-\gamma} = \prod_{\alpha\in\Delta^+}(1 + e^{-\alpha} + e^{-2\alpha} + \ldots).$$

Then d, $k \in \mathbf{Z}\langle\mathfrak{h}^*\rangle$ and $e^{-\rho}d = \Pi_{\alpha\in\Delta^+}(1-e^{-\alpha})$, so $ke^{-\rho}d = 1$, i.e. d has inverse $e^{-\rho}k$ in $\mathbf{Z}\langle\mathfrak{h}^*\rangle$. But $\mathrm{ch}\, M(\lambda) = e^{\lambda-\rho}\Sigma_{\gamma\in\Lambda_{\mathrm{rt}}^+}P(\gamma)e^{-\gamma} = e^{-\rho}ke^\lambda$.

We now express a fairly general formal character in terms of characters of Verma modules. More precisely, *let V be a $\mathfrak{g}$-module with infinitesimal character χ_{λ_0} and with formal character* $\mathrm{ch}(V) \in \mathbf{Z}\langle\mathfrak{h}^*\rangle$, *let* $D_V = \{\lambda \in W(\lambda_0)$: $\lambda - \rho + \Lambda_{\mathrm{rt}}^+$ *meets* supp $\mathrm{ch}(V)\}$; *then* $\mathrm{ch}(V)$ *is a* $\mathbf{Z}$*-linear combination of* $\{\mathrm{ch}M(\lambda)$: $\lambda \in D_V\}$. First, if $V \neq 0$ then $D_V \neq 0$. For if $\mu - \rho$ is a maximal element of supp $\mathrm{ch}(V)$ and $m = \dim V_{\mu-\rho}$ we have a $\mathfrak{g}$-homomorphism

$$\phi\colon M(\mu)\otimes\mathbf{C}^m \to V$$

that maps $M(\mu)_{\mu-\rho}\otimes\mathbf{C}^m$ isomorphically onto $V_{\mu-\rho}$. So $M(\mu)$ has infinitesimal character $\chi_\mu = \chi_{\lambda_0}$, i.e. $\mu \in W(\lambda_0)$, so $\mu \in D_V$. Second, let L and N be the kernel and cokernel of ϕ,

$$0 \to L \to M(\mu)\otimes\mathbf{C}^m \xrightarrow{\phi} V \to N \to 0.$$

They have infinitesimal character χ_{λ_0} and formal character in $\mathbf{Z}\langle\mathfrak{h}^*\rangle$. Evidently $D_N \subset D_V$; but $\mu \notin D_N$, so the cardinality $|D_N| < |D_V|$. By induction, $\mathrm{ch}(N)$ is a $\mathbf{Z}$-linear combination from $\{\mathrm{ch}(\lambda)$: $\lambda \in D_N\}$. Similarly $D_L \subsetneqq D_V$ so $\mathrm{ch}(L)$ is a $\mathbf{Z}$-linear combination from $\{\mathrm{ch}(\lambda)$: $\lambda\in D_L\}$. Now the assertion follows because $\mathrm{ch}(V) = -\mathrm{ch}(L) + m\cdot\mathrm{ch}\, M(\mu) + \mathrm{ch}(N)$.

WEYL CHARACTER FORMULA. *Let V be a finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Then*

$$\mathrm{ch}(V) = \left(\sum_{w\in W}\varepsilon(w)e^{w\rho}\right)^{-1}\left(\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}\right).$$

For $V \cong L(\lambda+\rho)$, hence has infinitesimal character $\chi_{\lambda+\rho}$, so $d\cdot\mathrm{ch}(V)$ is a $\mathbf{Z}$-linear combination of the $e^{w(\lambda+\rho)}$, $w\in W$. If $w \in W$ then $w(d) = \varepsilon(w)d$ and $w(\mathrm{ch}(V)) = \mathrm{ch}(V)$, so now $d\cdot\mathrm{ch}(V) = n\Sigma_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}$ for some integer n. But $\dim V_\lambda = 1$, so $n = 1$.

KOSTANT MULTIPLICITY FORMULA. *Let μ be a weight of a finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ. Then μ has multiplicity $\Sigma_{w\in W}\varepsilon(w)P(w(\lambda+\rho)-(\mu+\rho))$.*

For in our earlier notation,

$$\mathrm{ch}(V) = (ke^{-\rho})(d\cdot\mathrm{ch}(V)) = \left(\sum_{\gamma\in\Lambda_{\mathrm{rt}}^+}P(\gamma)e^{-\gamma-\rho}\right)\left(\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}\right),$$

so the coefficient of e^μ is

$$\sum_{w(\gamma+\rho)-(\lambda+\rho)=\mu} P(\gamma)\varepsilon(w) = \sum_{w\in W}\varepsilon(w)P(w(\lambda+\rho)-(\mu+\rho)).$$

WEYL DEGREE FORMULA. *The finite-dimensional irreducible* $\mathfrak{g}$-*module of highest weight* λ *has dimension*

$$\dim L(\lambda+\rho)=\prod_{\alpha\in\Delta^+}\frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

For this, consider the function $\deg(\Sigma c_\nu e^\nu)=\Sigma c_\nu$ on the sub-ring $\mathbb{Z}[\mathfrak{h}^*]$ of finitely supported elements in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Set $d_\nu=\Sigma_{w\in W}\varepsilon(w)e^{w(\nu)}$, so $\mathrm{ch}(V)=d_\rho^{-1}\cdot d_{\lambda+\rho}$. Given a root α, $e^\lambda\mapsto\langle\lambda,\alpha\rangle e^\lambda$ extends to a derivation ∂_α of $\mathbb{Z}[\mathfrak{h}^*]$. Now apply $\partial=\Pi_{\alpha\in\Delta^+}\partial_\alpha$ to $d_\rho\mathrm{ch}(V)=d_{\lambda+\rho}$ using Liebnitz' Rule for the ∂_α:

$$\deg\partial(d_\rho)\cdot\deg\mathrm{ch}(V)=\deg\partial(d_\rho\,\mathrm{ch}(V))=\deg\partial(d_{\lambda+\rho}).$$

As $\deg\partial(e^\nu)=\Pi_{\alpha\in\Delta^+}\langle\nu,\alpha\rangle$ now

$$\deg\partial(d_\nu)=\sum_{w\in W}\varepsilon(w)\prod_{\alpha\in\Delta^+}\langle w\nu,\alpha\rangle=|W|\cdot\prod_{\alpha\in\Delta^+}\langle\nu,\alpha\rangle.$$

So

$$\left\{|W|\cdot\prod_{\alpha\in\Delta^+}\langle\rho,\alpha\rangle\right\}\deg\mathrm{ch}(V)=|W|\cdot\prod_{\alpha\in\Delta^+}\langle\lambda+\rho,\alpha\rangle$$

which gives the formula for the dimension of $V=L(\lambda+\rho)$.

A Weyl character formula for the infinite-dimensional irreducible modules $L(\lambda+\rho)$ would have to rely either on a new idea or on a knowledge of the Jordan–Holder series (it exists) of $M(\lambda+\rho)$. Here one knows (i) every irreducible subquotient of $M(\lambda)$ is isomorphic to $L(\mu)$ for some $\mu\in W(\lambda)\cap(\lambda-\Lambda_{\mathrm{rt}}^+)$, and (ii) a certain root-chain condition says which μ actually occur. But the multiplicities cause problems.

14. Case of $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$

Let $\mathfrak{g}=\mathfrak{sl}(2;\mathbb{C})$ and $\mathfrak{h}=\left\{\begin{pmatrix}t&0\\0&-t\end{pmatrix}: t\in\mathbb{C}\right\}$. We use the root order $\Delta^+=\{\alpha\}$ where $\alpha\begin{pmatrix}t&0\\0&-t\end{pmatrix}=2t$, so $e_\alpha=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $e_{-\alpha}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Then $\Lambda_{\mathrm{rt}}=\{n\alpha: n\in\mathbb{Z}\}$ with Λ_{rt}^+ given by $n\geqslant 0$, $\Lambda_{\mathrm{wt}}=\{\frac{1}{2}n\alpha: n\in\mathbb{Z}\}$ with Λ_{wt}^+ given by $n\geqslant 0$. The irreducible $\mathfrak{g}$-module of highest weight $\frac{1}{2}(n-1)\alpha$ is the one that has dimension n, and its formal character is

$$\mathrm{ch}\,L(\tfrac{1}{2}n)=\frac{e^{n\alpha/2}-e^{-n\alpha/2}}{e^{\alpha/2}-e^{-\alpha/2}}.$$

Let's compare this with the representation on the group level: there, writing π_n for the irreducible of degree n, highest weight $\frac{1}{2}(n-1)\alpha$

$$\mathrm{trace}\,\pi_n\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}=\sum_{1\leqslant j\leqslant n}e^{(n+1-2j)t}=\frac{e^{nt}-e^{-nt}}{e^t-e^{-t}}$$

The same holds by restriction for $\mathfrak{g}_0=\mathfrak{sl}(2;\mathbb{R})$ and $SL(2;\mathbb{R})$, and $\mathfrak{g}_u=\mathfrak{su}(2)$ and $SU(2)$.

15. Homogeneous vector bundles

Let G be a compact group and K a closed subgroup, and consider the homogeneous space $X=G/K$. The 'left regular' representation of G on $L_2(X)$ comes out of the Peter–Weyl theorem as follows. Write $E(\pi)$ for the space of a representation $\pi\in\hat{G}$, so $L_2(G)=\bigoplus_{\pi\in\hat{G}}E(\pi)\otimes E(\pi^*)$. Then, if l and r denote the left and right regular representations of G,

$$\begin{aligned}L_2(X)=L_2(G)^{r(K)}&=\left\{\bigoplus_{\pi\in\hat{G}}E(\pi)\otimes E(\pi^*)\right\}^{r(K)}\\&=\bigoplus_{\pi\in\hat{G}}E(\pi)\otimes E(\pi^*)^{\pi^*(K)}\\&=\bigoplus_{\pi\in\hat{G}}m(1_K,\pi|_K)E(\pi)\end{aligned}$$

as unitary $l(G)$-module. This technique is due to Hermann Weyl.

More generally consider a homogeneous vector bundle $\mathscr{V}\to X$, say with typical fibre $V=V(\kappa)$ for some $\kappa\in\hat{K}$. Here $\mathscr{V}\to X$ is associated to the principal K-bundle $G\to X$ by the action κ of K on V. We recall the construction: $\mathscr{V}=G\times_K V$, set of equivalence classes from $G\times V$ under $(gk,v)\sim(g,\kappa(k)v)$. A section $\sigma: X\to\mathscr{V}$ is of the form $gK\mapsto[g,f_\sigma(g)]$ with $f_\sigma: G\to V$ such that $f_\sigma(gk)=\kappa(k)^{-1}f_\sigma(g)$, and every such function f determines a section σ_f. So now the space of L_2 sections of $\mathscr{V}\to X$, as unitary G-module, is

$$\begin{aligned}L_2(X;\mathscr{V})=\{L_2(G)\otimes V\}^{(r\otimes\kappa)(K)}&=\bigoplus_{\pi\in\hat{G}}E(\pi)\otimes\{E(\pi^*)\otimes V\}^{(\pi^*\otimes\kappa)(K)}\\&=\bigoplus_{\pi\in\hat{G}}m(1_K,\pi^*|_K\otimes\kappa)E(\pi)=\bigoplus_{\pi\in\hat{G}}m(\kappa,\pi)E(\pi).\end{aligned}$$

Glancing back to the definition of induced representation this gives us the following theorem.

FROBENIUS RECIPROCITY THEOREM. *If* $\pi \in \hat{G}$ *and* $\kappa \in \hat{K}$ *then* $m(\pi, \mathrm{Ind}_K^G(K)) = m(\kappa, \pi)$, *whenever G is a compact group and K is a closed subgroup.*

An application. Let G be a compact simply connected Lie group and T a Cartan subgroup (maximal torus). Relative to a positive $\mathfrak{t}_{\mathbb{C}}$-root system Δ^+ for $\mathfrak{g}_{\mathbb{C}}$ we have the Borel subalgebra

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} + \sum_{\Delta^+} (\mathfrak{g}_{\mathbb{C}})_{-\alpha} = \mathfrak{t}_{\mathbb{C}} + \mathfrak{n}^-$$

and the Borel subgroup $B = HN^-$, $H = \exp(\mathfrak{t}_{\mathbb{C}})$ and $N^- = \exp(\mathfrak{n}^-)$, in $\mathfrak{g}_{\mathbb{C}}$ and $G_{\mathbb{C}}$. Given $\lambda \in \Lambda^+_{\mathrm{wt}}$ we have well defined holomorphic homomorphism

$$e^{\lambda}: B \to \mathbb{C}\backslash\{0\} \quad \text{by} \quad e^{\lambda}(\exp_{G_{\mathbb{C}}}(\xi + \eta)) = e^{\lambda(\xi)} \quad \text{for} \quad \xi \in \mathfrak{t}_{\mathbb{C}}, \quad \eta \in \mathfrak{n}^-.$$

It is a representation of B on $\mathbb{C}$ so we have the associated

$$\mathscr{L}_{\lambda} \to G_{\mathbb{C}}/B \text{ holomorphic homogeneous line bundle}$$

A section $\sigma: G_{\mathbb{C}}/B \to \mathscr{L}_{\lambda}$ is holomorphic if and only if it is a continuously differentiable and satisfies

$$\eta(f_{\sigma}) = 0 \quad \text{for all} \quad \eta \in \mathfrak{n}^-, \text{ where } \eta(f)(x) = (d/dt)|_{t=0}\, f(\exp(t\eta)x).$$

We denote the space of holomorphic sections by, say, $\mathscr{H}^0(\mathscr{L}_{\lambda})$.

One more often is interested in $\mathscr{L}_{\lambda} \to G/T$. Here note $G \cap B = T$, so $G/T \subset G_{\mathbb{C}}/B$, where it is open by dimension and closed because G is compact. This gives the complex structure on G/T (one for each choice of Δ^+) and, from the viewpoint of G/T, the complex structure on $\mathscr{L}_{\lambda}$.

BOREL–WEIL THEOREM. $G_{\mathbb{C}}$ *acts on* $\mathscr{H}^0(\mathscr{L}_{\lambda})$ *by the irreducible representation of highest weight* λ. *In particular*

$$\dim \mathscr{H}^0(\mathscr{L}_{\lambda}) = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

This specializes to various formulae in algebraic geometry, such as the formulae for dimensions of linear systems of divisors on complex Grassmannians.

A short proof of Borel–Weil. Let V be the irreducible $G_{\mathbb{C}}$-module of highest weight λ, π the action of $G_{\mathbb{C}}$ on V and V^* and π^* their dual. Denote the pairing by (v, v^*). The lowest weight of V is $-w_0(\lambda)$; choose a weight vector $0 \neq v \in V_{-w_0(\lambda)}$. The lowest weight of V^* is $-\lambda$; choose a weight vector $0 \neq v^* \in V^*_{-\lambda}$. Now define

$$f: G_{\mathbb{C}} \to V \quad \text{by} \quad f(g) = (v, \pi^*(g)v^*).$$

If $\eta \in \mathfrak{n}^-$ then $d\pi(\eta)v = 0$ so

$$\eta(f)(g) = (d/dt)|_{t=0}(\pi(\exp(t\eta))^{-1}v, \pi^*(g)v^*)$$
$$= -(d\pi(\eta)v, \pi^*(g)v^*) = 0;$$

so $\eta(f) = 0$. And similarly $f(g \cdot \exp(\eta)) = f(g)$. If $\xi \in \mathfrak{t}_{\mathbb{C}}$ then

$$f(g \cdot \exp(\xi)) = (v, \pi^*(g)\pi^*(\exp(\xi))v^*)$$
$$= (v, \pi^*(g)e^{-\lambda(\xi)}v^*) = e^{-\lambda(\xi)}f(g)$$

because $v^* \in V^*_{-\lambda}$. Now

$$0 \neq f \in \mathscr{H}^0(\mathscr{L}_{\lambda}).$$

Let

$$W = \operatorname{span}\{g(f): g \in G_{\mathbb{C}}\} \subset \mathscr{H}^0(\mathscr{L}_{\lambda}).$$

Here note that

$$[g(f)](g') = f(g^{-1}g') = (\pi(g)v, \pi^*(g')v^*),$$

so W consists of

$$f_u: g \to (u, \pi^*(g)v^*), \qquad u \in V,$$

with $g(f_u) = f_{\pi(g)u}$. Now apply the Peter–Weyl Theorem: the L_2 sections of $\mathscr{L}_{\lambda} \to G/T$ constitute the space $\bigoplus_{\nu \in \Lambda^+_{\mathrm{wt}}} E(\nu) \otimes E(\nu)^*_{-\lambda}$. The condition that an L_2 section σ be holomorphic: decompose $\eta \in \mathfrak{n}^-$ as $\eta_1 + i\eta_2$ with $\eta_j \in \mathfrak{g}$, then $\eta_1(f_{\sigma}) + i\eta_2(f_{\sigma}) = 0$, which just says that the corresponding $E(\nu)^*_{-\lambda}$ is a lowest weight space. Now, if $E(\nu) \otimes E(\nu)^*_{-\lambda}$ contributes to $\mathscr{H}^0(\mathscr{L}_{\lambda})$ then $E(\nu)^*$ has lowest weight $-\lambda$, i.e. $\nu = \lambda$. Now $W = \mathscr{H}^0(\mathscr{L}_{\lambda}) = E(\lambda)$.

16. Functions on Symmetric Spaces

We consider an important case of the decomposition

$$L_2(G/K) = \bigoplus_{\pi \in \hat{G}} m(1_K, \pi|_K) E(\pi)$$

of the first paragraph of Section 15. Write π_ν for the class in $\hat{G}$ with highest weight $\nu \in \Lambda^+_{\mathrm{wt}}$ relative to a positive system Δ^+ of $\mathfrak{h}_{\mathbf{C}}$-roots on $\mathfrak{g}_{\mathbf{C}}$.

THEOREM. *If G/K is a symmetric space, i.e. if K has finite index in the fixed point set G^θ of an involutive automorphism θ, then $m(1_K, \pi|_K) \leqslant 1$ for every $\pi \in \hat{G}$, so $L_2(G/K) = \bigoplus_{m(1_K,\pi)\neq 0} E(\pi)$.*

Example: the 2-sphere $S^2 = SU(2)/U(1)$, and $m(1_{U(1)}, \pi_{\text{degree } n}) = 0$ for n even, $=1$ for n odd. So for each integer $k \geqslant 0$ one has the subspace $E(k\alpha)$ of dimension $2k+1$ in $L_2(S^2)$ generated by the $SU(2)$-translates of the spherical harmonics of degree k, and $L_2(S^2) = \bigoplus_{0 \leqslant k < \infty} E(k\alpha)$.

Fix a symmetric space G/K say K finite index in G^θ where θ is an involutive automorphism, and decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ into the (± 1)-eigenspaces of θ. Extending by complex linearity and restricting, θ is a Cartan involution of $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{s}_0$, $\mathfrak{s}_0 = i\mathfrak{s}$. Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be a CSA of $(\mathfrak{g}_0, \mathfrak{k})$, $\mathfrak{t}$ a CSA in the $\mathfrak{k}$-centralizer $\mathfrak{m}$ of $\mathfrak{a}_0$, and $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ where $\mathfrak{a} = i\mathfrak{a}_0$. Then $\mathfrak{h}$ is a CSA of $\mathfrak{g}$ and we use that choice of $\mathfrak{h}$. We also use a choice of Δ^+ compatible with some system $\Delta^+_{\mathfrak{a}_0}$ of positive $\mathfrak{a}_0$-roots on $\mathfrak{g}_0$.

THEOREM. *Let $\nu \in \Lambda^+_{\mathrm{wt}}$. Then $m(1_{K_0}, \pi_\nu|_{K_0}) = 1$ if and only if* (i) $\nu|_{\mathfrak{t}} = 0$, *so in effect $\nu \in \mathfrak{a}_0^*$, and* (ii) *if α is a simple $\mathfrak{a}_0$-root then $\langle \nu, \alpha\rangle / \langle \alpha, \alpha\rangle$ is an integer $\geqslant 0$.*

This is Cartan's highest weight theory for the *class* 1 *representations* of G relative to K, which thus describes L_2 of a compact symmetric space.

CHAPTER III

INFINITE-DIMENSIONAL REPRESENTATIONS

This third chapter presents a brief and somewhat sketchy introduction to the theory of unitary representations of reductive and semisimple Lie groups G. The basic fact for an irreducible unitary representation π of G on a Hilbert space $\mathscr{H}$, is that every irreducible representation κ of a maximal compact subgroup $K \subset G$ has multiplicity $m(\kappa, \pi|_K) \leqslant \dim \kappa$. This yields up the infinitesimal character $\chi_\pi : \mathscr{Z}(\mathfrak{g}) \to \mathbf{C}$ and the distribution character $\theta_\pi : C_c^\infty(G) \to \mathbf{C}$, and consequently the differential equations

$$z(\theta_\pi) = \chi_\pi(z)\theta_\pi \qquad \text{for} \qquad z \in \mathscr{Z}(\mathfrak{g})$$

which are the starting point for serious harmonic analysis on G.

Unfortunately the bound $m(\kappa, \pi|_K) \leqslant \dim \kappa$ has not been proved purely within the context of unitary representations, so we must consider representations of G on a Banach space. The basic facts on Banach representations are in Section 17. Then we look at C^∞ vectors (Section 18), analytic and K-finite vectors (Section 19), and finally the K-multiplicities (Section 20). We then establish the existence of the distribution character and a few of its properties in Section 21.

The representations that enter into harmonic analysis on a reductive or semisimple group G, come in several 'series,' one for each conjugacy class of Cartan subgroup H. The series for a compact CSG is fundamental in that it is the basic building block for the other series. More precisely, H defines a certain reductive subgroup $M \subset G$ such that $T = H \cap M$ is a compact CSG in M, and the 'H-series' for G is constructed from the 'T-series' for M by elementary methods. In Section 22 we state the basic facts on the 'discrete series,' which is the series for a compact Cartan subgroup. Then in Section 23 we define the 'cuspidal parabolic' subgroup $P = MAN$ associated to the Cartan $H = T \times A$, and in Section 24 we describe the H-series and state the Plancherel Theorem for G.

17. BANACH REPRESENTATIONS

Let G be a locally compact group countable at infinity. If $\mathscr{B}$ is a Banach space let $\mathrm{Aut}(\mathscr{B})$ denote the group of all topological (bounded, bounded inverse) automorphisms of $\mathcal{B}$. A homomorphism $\pi: G \to \mathrm{Aut}(\mathcal{B})$ is a *Banach representation* of G on $\mathcal{B}$ if it satisfies the following equivalent continuity conditions

(i) $G \times \mathcal{B} \to \mathcal{B}$, by $(g, v) \mapsto \pi(g)v$, is continuous

(ii) if $v \in \mathcal{B}$ then $G \to \mathcal{B}$, by $g \mapsto \pi(g)v$, is continuous

(iii) if $v \in \mathcal{B}$ and $v^* \in \mathcal{B}^*$ then $g \mapsto \langle \pi(g)v, v^* \rangle$ is continuous.

We say that π is *topologically irreducible* (TI) if $\mathcal{B}$ has no proper closed $\pi(G)$-invariant subspace. That notion usually is not the right one. If $f \in C_c(G)$ then $\|\pi(x)\|$ is bounded on $\mathrm{supp}(f)$ so we have a bounded operator $\pi(f) = \int_G f(x)\pi(x)\,dx$. Suppose, given $T: \mathcal{B} \to \mathcal{B}$ bounded, $n \geqslant 1$, $\{v_1, \ldots, v_n\} \subset \mathcal{B}$ and $\varepsilon > 0$ there exists $f \in C_c(G)$ with $\|(\pi(f) - T)v_i\| < \varepsilon$ for $i = 1, \ldots, n$. Then π is *topologically completely irreducible* (TCI). Equivalent formulation: use the algebra $M_c(G)$ of compactly supported Radon measures in place of $C_c(G)$; here $\langle \pi(\mu)v, v^* \rangle = \int_G \langle \pi(x)v, v^* \rangle \, d\mu(x)$ for $v \in \mathcal{B}$, $v^* \in \mathcal{B}^*$.

If π is TCI it is TI; for if $0 \neq v \in \mathcal{B}$ then $\pi(C_c(G))v$ is dense in $\mathcal{B}$. The converse holds if π is 'finite-dimensionally spanned' (FDS), which means: the linear span of the ranges of $\{\pi(\mu): \mu \in M_c(G)$ and $\pi(\mu)$ has finite rank$\}$ is dense in $\mathcal{B}$.

SCHUR'S LEMMA. *If π is TCI, then every bounded operator on $\mathcal{B}$ that commutes with all the $\pi(x)$, $x \in G$, is scalar.*

For this, let $0 \neq v \in \mathcal{B}$ and let T be the operator. If v and Tv are linearly independent we have a net $\{f_\alpha\} \subset C_c(G)$ with $\{\pi(f_\alpha)v\} \to v$ and $\{\pi(f_\alpha)Tv\} \to v$. But $\pi(f_\alpha)T = T\pi(f_\alpha)$ so $v = \lim \pi(f_\alpha)Tv = T \cdot \lim \pi(f_\alpha)v = Tv$, contradiction. Now $Tv = c(v)v$, $c(v) \in \mathbb{C}$, for all $v \in V$. If $u \neq 0 \neq v$ take a net $\{f_\alpha\} \subset C_c(G)$, $\{\pi(f_\alpha)u\} \to v$, and apply T to see $c(u) = c(v)$.

Let Z be the center of G. If π is TCI now $\pi|_Z$ is a homomorphism $\zeta_\pi: Z \to \mathbb{C} \backslash \{0\}$, called the *central character of* π.

If π is a unitary representation of G on a Hilbert space, then π is TI if and only if every bounded operator that commutes with all the $\pi(x)$, $x \in G$, is scalar. This comes right out of the spectral theorem.

THEOREM. *If π is unitary and TI then π is TCI.*

To see this, let $\mathfrak{A}$ denote the von Neumann algebra of all bounded T on the Hilbert space $\mathcal{H}$ such that if $\{v_1, \ldots, v_n\} \subset \mathcal{H}$ and $\varepsilon > 0$ then, for some $f \in M_c(G)$, each $\|(\pi(f) - T)v_i\| < \varepsilon$. As $\pi(G) \subset \pi(M_c(G)) \subset \mathfrak{A}$, the commutant $\mathfrak{A}' = \mathbb{C}$, so $\mathfrak{A} = (\mathfrak{A}')'$ consists of all bounded operators.

18. Smooth vectors

Now G is a Lie group countable at infinity. Fix a Banach representation π of G on $\mathcal{B}$. A vector $v \in \mathcal{B}$ is *differentiable* ($= C^\infty$) if $g \mapsto \pi(g)v$ is a C^∞ map $G \to \mathcal{B}$, that is, if $g \mapsto \langle \pi(g)v, v^* \rangle$ is a C^∞ function on G for every $v^* \in \mathcal{B}^*$. Write $\mathcal{B}_\infty$ for the space of C^∞ vectors in $\mathcal{B}$. If $v \in \mathcal{B}$ and $f \in C_c^\infty(G)$ then $\pi(f)v \in \mathcal{B}_\infty$; so $\pi(C_c^\infty(G))\mathcal{B}$ is a dense subspace of $\mathcal{B}$ contained in $\mathcal{B}_\infty$; in particular $\mathcal{B}_\infty$ is dense in $\mathcal{B}$. The *differentiable representation* π_∞ of G associated to π is the representation $g \mapsto \pi(g)|_{\mathcal{B}_\infty}$, where $\mathcal{B}_\infty$ carries the subspace topology from $\mathcal{B}$.

π_∞ lifts to a representation of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{B}_\infty$, by

$$\pi_\infty(\xi)v = (d/dt)\pi(\exp(t\xi))v|_{t=0}$$

for $\xi \in \mathfrak{g}$ and $v \in \mathcal{B}_\infty$. Here $\pi(C_c^\infty(G))\mathcal{B}$ is $\pi_\infty(\mathcal{U}(\mathfrak{g}))$-stable; in fact if $D \in \mathcal{U}(\mathfrak{g})$, $f \in C_c^\infty(G)$ and $v \in \mathcal{B}$ then $\pi_\infty(D)\pi(f)v = \pi(Df)v$.

THEOREM. *Suppose that every* $\mathrm{Ad}(g) \in \mathrm{Int}\,(\mathfrak{g}_{\mathbb{C}})$, *so that every* $\mathrm{Ad}(g)$ *is trivial on the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. If π is TCI then $\mathcal{Z}(\mathfrak{g})$ is represented by scalars on $\mathcal{B}_\infty$, that is, π_∞ has an infinitesimal character.*

This is a little bit tricky. One views $\mathcal{U}(\mathfrak{g})$ as the distributions on G supported at 1, so it sits in the convolution algebra $\mathcal{D}_c(G)$ of compactly supported distributions, and $\mathcal{Z}(\mathfrak{g})$ sits in the center of $\mathcal{D}_c(G)$. One lifts π_∞ to a representation of $\mathcal{D}_c(G)$ on $\mathcal{B}_{(\infty)} = \pi(C_c^\infty(G))\mathcal{B}$ and carries it over to a representation of $\mathcal{D}_c(G)$ on $\mathcal{B}^*_{(\infty)} = \pi^*(C_c^\infty(G))\mathcal{B}^*$, which is weakly dense in $\mathcal{B}^*$. Then one argues as in Schur's Lemma to show that $\pi_\infty(Z)$, Z central in $\mathcal{D}_c(G)$, is a scalar operator $\chi_\pi(Z) \cdot 1$ on $\mathcal{B}_{(\infty)}$, and a weak continuity argument shows $\pi_\infty(Z) = \chi_\pi(Z) \cdot 1$ on $\mathcal{B}_\infty$.

COROLLARY. *Unitary TI representations of connected Lie groups have infinitesimal characters.*

Let K be a compact Lie group, eventually a maximal compact subgroup of G. As usual, $\hat{K}$ denotes the set of equivalence classes of finite dimensional irreducible representations of K. (*Every TI Banach representation* *of a compact group is finite dimensional.*) Given $\kappa\in\hat{K}$ we have the *normalized character* $\tau_\kappa(k)=(\dim\kappa)\mathrm{trace}\,\kappa(k)$. In $L_2(K)=\bigoplus_{\kappa\in\hat{K}}V(\kappa)\otimes V(\kappa)^*$, left or right convolution by $\overline{\tau_\kappa}$ is orthogonal projection to $V(\kappa)\otimes V(\kappa)^*$.

Let π be a Banach representation of K on $\mathscr{B}$. If $\kappa\in\hat{K}$ then $\pi(\overline{\tau_\kappa})$ is a continuous projection of $\mathscr{B}$ onto the K-isotypic subspace $\mathscr{B}(\kappa)$ of type κ. In other words, $\mathscr{B}(\kappa)$ consists of all $v\in\mathscr{B}$ such that $\mathrm{span}(\pi(K)v)$ is finite dimensional and that K acts on it by a multiple of κ.

THEOREM. *If $v\in\mathscr{B}_\infty$ then the Fourier series $\Sigma_{\kappa\in\hat{K}}(\overline{\tau_\kappa})v$ converges absolutely ($\Sigma\|\pi(\overline{\tau_\kappa})v\|<\infty$) to v.*

The proof runs as follows. Fix a positive definite $\mathrm{Ad}(K)$-invariant bilinear form $\langle\,,\rangle$ on $\mathfrak{k}$, let $\{\xi_1,\ldots,\xi_n\}$ be an orthonormal basis, and consider the 'Casimir' $\Omega=-\Sigma\xi_i^2$. Then $\Omega\in\mathscr{Z}(\mathfrak{k})$, so by Schur's Lemma and skew adjointness of $\kappa(\xi_i)$, if $\kappa\in\hat{K}$ then $\kappa(1+\Omega)=c_\kappa\geqslant 1$ and $(1+\Omega)\tau_\kappa=c_\kappa\tau_\kappa$. If $v\in\mathscr{B}_\infty$ now $\pi(\overline{\tau_\kappa})\pi_\infty(1+\Omega)v=c_\kappa\pi(\overline{\tau_\kappa})v$. Since K is compact we have $a\geqslant 1$ such that $\|\pi(k)\|\leqslant a$ for all $k\in K$, so $\|\pi(\overline{\tau_\kappa})v\|\leqslant(\dim\kappa)^2a\|v\|$. For each integer $m\geqslant 0$ now $\pi(\overline{\tau_\kappa})v=c_\kappa^{-m}\pi(\overline{\tau_\kappa})\pi_\infty((1+\Omega)^m)v$ gives us

$$\|\pi(\overline{\tau_\kappa})v\|\leqslant c_\kappa^{-m}(\dim\kappa)^2a\|\pi_\infty((1+\Omega)^m)v\|.$$

To use this, we need

(*) if m is large enough then $\sum_{\kappa\in\hat{K}}(\dim\kappa)^2c_\kappa^{-m}<\infty$.

Frobenius Reciprocity reduces the proof of (*) to the case where K is connected. Now with K connected, choose a maximal torus T and consider a positive system Δ^+ of $\mathfrak{t}_{\mathbb{C}}$-roots of $\mathfrak{k}_{\mathbb{C}}$. Then $\hat{K}$ is parameterized by Λ_{wt}^+, intersection of the weight lattice Λ_{wt} with a cone in $\sqrt{-1}\mathfrak{t}^*$, say by $\kappa_\lambda\leftrightarrow\lambda$. Here $\dim\kappa_\lambda=\Pi_{\alpha\in\Delta^+}\langle\lambda+\rho,\alpha\rangle/\langle\rho,\alpha\rangle$, which is a polynomial $p(\lambda)$, and $c_{\kappa_\lambda}=1+\|\lambda+\rho\|^2-\|\rho\|^2$. Now, outside of a finite set $F\subset\hat{K}$ one has $\|\lambda+\rho\|^2-\|\rho\|^2\geqslant\frac{1}{2}\|\lambda\|^2$, so

$$\sum_{\kappa\in\hat{K}\setminus F}(\dim\kappa)^2c_\kappa^{-m}\leqslant 2^m\sum_{\lambda\in\Lambda_{wt}^+}(1+\|\lambda\|^2)^{-m}|p(\lambda)|^2,$$

which is finite for $m\gg 0$. That proves (*). Now

$$\sum_{\kappa\in\hat{K}}\|\pi(\overline{\tau_\kappa})v\|\leqslant\left\{\sum_{\kappa\in\hat{K}}(\dim\kappa)^2c_\kappa^{-m}\right\}a\|\pi_\infty((1+\Omega^m)v\|$$

shows absolute convergence of the Fourier series $\Sigma\pi(\overline{\tau_\kappa})v$ for $v\in\mathscr{B}_\infty$. To complete the proof, we must show that

$$v-v_0=0\qquad\text{where } v_0=\sum_{\kappa\in\hat{K}}\pi(\overline{\tau_\kappa})v.$$

First, every $\pi(\overline{\tau_\kappa})(v-v_0)=0$. Let $\{f_n\}$ be a C^∞ approximate identity in $L_1(K)$. For each n we have a finite $F_n\subset\hat{K}$ and a function

$$h_n\in\sum_{\kappa\in F_n}V(\kappa)\otimes V(\kappa)^*$$

with $\sup|h_n-f_n|<2^{-n}$. Now

$$v-v_0=\lim_{n\to\infty}\pi(f_n)(v-v_0)=\lim_{n\to\infty}\pi(h_n)(v-v_0),$$

but

$$\pi(h_n)(v-v_0)=\pi\left(h_n*\sum_{F_n}\overline{\tau_\kappa}\right)(v-v_0)=\sum_{\kappa\in F_n}\pi(h_n)\pi(\overline{\tau_\kappa})(v-v_0)=0.$$

Now again, G is a Lie group countable at infinity, K is a compact subgroup, and π is a Banach representation of G on $\mathscr{B}$.

THEOREM. *The space $\Sigma_{\kappa\in\hat{K}}\,\mathscr{B}_\infty\cap\mathscr{B}(\kappa)$ is dense in $\mathscr{B}$.*

First, as in the last theorem, if $f\in C^\infty(G)$ (resp. $f\in C_c^\infty(G)$) then the series $\Sigma_{\kappa\in\hat{K}}\overline{\tau_\kappa}*f$ and $\Sigma_{\kappa\in\hat{K}}f*\overline{\tau_\kappa}$ converge absolutely to f in $C^\infty(G)$ (resp. in $C_c^\infty(G)$). Second, let $v\in\mathscr{B}$ and $\varepsilon>0$, choose $f\in C_c^\infty(G)$ such that $\|\pi(f)v-v\|<\varepsilon$, and choose a compact set $F=KF$ in which f is supported. The L_1 norm is a continuous seminorm on $C_c^\infty(G)$. If $A\subset\hat{K}$ is finite set $\overline{\tau_A}=\Sigma_{\kappa\in A}\overline{\tau_\kappa}$, so $f-\overline{\tau_A}*f$ has support in F and $\|\pi(f-\overline{\tau_A}*f)v\|\leqslant c\|f-\overline{\tau_A}*f\|_1$ where $c=\sup_{x\in F}\|\pi(x)v\|$. Choose A so that $\|f-\overline{\tau_A}*f\|_1<\varepsilon/c$, then

$$\|\pi(\overline{\tau_A}*f)v-v\|\leqslant\|\pi(f-\overline{\tau_A}*f)v\|+\|\pi(f)v-v\|<2\varepsilon.$$

COROLLARY. *$\mathscr{B}_\infty\cap\mathscr{B}(\kappa)$ is dense in $\mathscr{B}(\kappa)$.*

19. Harish-Chandra module

G is a Lie group countable at infinity, K is a compact subgroup, and π is a Banach representation of G on $\mathscr{B}$. A vector $v\in\mathscr{B}$ is *analytic* if $x\mapsto\pi(x)v$ is an analytic map $G\to\mathscr{B}$. Equivalent: if $v^*\in\mathscr{B}$ then $x\mapsto\langle\pi(x)v,v^*\rangle$ is an analytic function on G. We write $\mathscr{B}_\omega$ for the space of all analytic vectors in

$\mathscr{B}$. It is $\pi(G)$-stable, so we have the *analytic representation* π_ω of G on $\mathscr{B}_\omega$. Note that $\mathscr{B}_\omega \subset \mathscr{B}_\infty$, and if $u \in \mathscr{U}(\mathfrak{g})$ then $\pi_\infty(u)\mathscr{B}_\omega \subset \mathscr{B}_\omega$. The representation of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{B}_\omega$ is also denoted π_ω. The term is justified by a glance at some Taylor series, showing the following proposition.

PROPOSITION. *If $v \in \mathscr{B}_\omega$ then there is a neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$ such that $\Sigma_{m=0}^{\infty}(1/m!)\pi_\omega(\xi)^m v$ converges to $\pi(\exp(\xi))v$ for all $\xi \in \mathcal{O}$.*

One studies analytic vectors because of the following corollary.

COROLLARY. *If $\mathscr{B}_0$ is a $\pi_\omega(\mathscr{U}(\mathfrak{g}))$-stable subspace of $\mathscr{B}_\omega$ then its closure is a $\pi(G)$-stable subspace of $\mathscr{B}$.*

We are going to get analogs of the results of Section 18 for analytic vectors, and in the process obtain information needed to define global characters of representations. Of course $\mathscr{B}_\omega$ is only useful because of the following theorem.

NELSON'S THEOREM. *$\mathscr{B}_\omega$ is dense in $\mathscr{B}$.*

If $\kappa \in \hat{K}$ denote $\mathscr{B}_\omega(\kappa) = \mathscr{B}_\omega \cap \mathscr{B}(\kappa)$ and $\mathscr{B}_K = \Sigma_{\kappa \in \hat{K}} \mathscr{B}_\omega(\kappa)$. Then $\mathscr{B}_K = \mathscr{B}_\omega \cap \Sigma_{\kappa \in \hat{K}} \mathscr{B}(\kappa)$ and we have $\mathscr{B}_K \subset \mathscr{B}_\omega \subset \mathscr{B}_\infty$.

THEOREM. *$\mathscr{B}_K$ is dense in $\mathscr{B}$.*

For let $u \in \mathscr{B}$ and $\varepsilon > 0$. As $\mathscr{B}_\omega$ is dense in $\mathscr{B}$ we have $v \in \mathscr{B}_\omega$ with $\|u - v\| < \varepsilon/2$. As $v \in \mathscr{B}_\infty$, $\Sigma_{\kappa \in \hat{K}} \pi(\overline{\tau_\kappa})v$ converges absolutely to v, so some partial sum $w = \Sigma_{\kappa \in F} \pi(\overline{\tau_\kappa})v$, $F \subset \hat{K}$ finite, has $\|v - w\| < \varepsilon/2$. If f: $G \to \mathscr{B}$ is analytic and h: $K \to \mathbb{C}$ is analytic then $x \mapsto \int_K f(xk)h(k)\,dk$ is an analytic map $G \to \mathscr{B}$; so $w \in \mathscr{B}_\omega$ and $\|u - w\| < \varepsilon$.

COROLLARY. *If $\kappa \in \hat{K}$ and* dim $\mathscr{B}(\kappa) < \infty$ *then $\mathscr{B}(\kappa) \subset \mathscr{B}_\omega$.*

For $\pi(\overline{\tau_\kappa})\mathscr{B}_\omega$ is a dense subspace of $\mathscr{B}(\kappa)$ contained in $\mathscr{B}_\omega$.

COROLLARY. *$\mathscr{B}_K$ is $\pi_\omega(\mathscr{U}(\mathfrak{g}))$-invariant.*

For if $D \in \mathscr{U}(\mathfrak{g})$ and $k \in K$ then $\pi(k)\pi_\omega(D)v = \pi_\omega(\mathrm{ad}(k)D)\pi(k)v$ for $v \in \mathscr{B}_K$. The $\mathrm{Ad}(k)D$, $k \in K$, lie in a finite-dimensional subspace of $\mathscr{U}(\mathfrak{g})$ and the $\pi(k)v$, $k \in K$, are in a finite-dimensional subspace of $\mathscr{B}$. So we have a representation π_K of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{B}_K$.

THEOREM. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let π be a TI Banach representation of G on $\mathscr{B}$. Then π has infinitesimal character if and only if it is TCI. In that case, π is K-finite (each $\mathscr{B}(\kappa)$ has finite dimension), so each $\mathscr{B}(\kappa) \subset \mathscr{B}_\omega$ and $\mathscr{B}_K = \Sigma_{\kappa \in \hat{K}} \mathscr{B}(\kappa)$.*

If π is TCI it has infinitesimal character.

Let π have infinitesimal character. Choose $0 \neq v \in \mathscr{B}_K$. Then $\mathscr{U}(\mathfrak{g})v = \pi_\omega(\mathscr{U}(\mathfrak{g}))v$ has $\pi(G)$-stable closure, hence is dense in $\mathscr{B}$. As $v \in \mathscr{B}_K$, $\mathscr{U}(\mathfrak{g})v \subset \mathscr{B}_K$, so $\mathscr{U}(\mathfrak{g})v = \Sigma \mathscr{U}(\mathfrak{g})v \cap \mathscr{B}_\omega(\kappa)$. Let $\mathfrak{J}$ be the annihilator of v in $\mathscr{U}(\mathfrak{k})$. It is a left ideal of finite codimension, $\mathscr{U}(\mathfrak{k})$ acts semisimply on $\mathscr{U}(\mathfrak{k})/\mathfrak{J}$, and a calculation shows that $\bar{\mathscr{U}} = \mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\mathfrak{J}$ is of the form $\Sigma_{\kappa \in \hat{K}} \bar{\mathscr{U}}(\kappa)$ where each $\bar{\mathscr{U}}(\kappa)$ has finite rank as a $\mathscr{Z}(\mathfrak{g})$-module. Since $\bar{\mathscr{U}} \to \mathscr{U}(\mathfrak{g})v$, by $D + \mathscr{U}(\mathfrak{g})\mathfrak{J} \to \pi_\omega(D)v$, is a $\mathscr{U}(\mathfrak{g})$-module map onto $\mathscr{U}(\mathfrak{g})v$, and since π has infinitesimal character, each $\mathscr{U}(\mathfrak{g})v \cap \mathscr{B}_\omega(\kappa)$ has finite dimension. By density of $\mathscr{U}(\mathfrak{g})v$ now each dim $\mathscr{B}_\omega(\kappa) < \infty$. By density of $\mathscr{B}_K$ now each $\mathscr{B}_\omega(\kappa) = \mathscr{B}(\kappa)$. So π is K-finite, hence FDS, and thus is TCI.

COROLLARY. *Let G be a connected semisimple Lie group with finite center. If π is an irreducible unitary representation of G and if $f \in L_1(G)$ then $\pi(f)$ is a completely continuous operator. In other words, G is* CCR *('liminaire'), and so G is a group of type* I *('postliminaire').*

For π is TCI, hence K-finite, so the $\pi(\overline{\tau_\kappa})\pi(f) = \pi(\overline{\tau_\kappa} * f)$ are operators of finite rank. $\Sigma_{\kappa \in \hat{K}} \overline{\tau_\kappa} * f$ converges L_1 to f so $\Sigma_{\kappa \in \hat{K}} \pi(\overline{\tau_\kappa})\pi(f)$ converges strongly to $\pi(f)$. Now $\pi(f)$ is a strong limit of finite rank operators.

Note that this theorem and its corollary hold whenever G is reductive such that (i) every $\mathrm{Ad}(g) \in \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ and (ii) Z has a closed normal Abelian subgroup $Z \subset Z_G(G_0)$ with ZG_0 of finite index in G and $Z \cap G_0$ co-compact in the center of G_0. Then $K = \mathrm{Ad}_G^{-1}$ (maximal compact subgroup of $\mathrm{Ad}(G)$) and $\hat{K}$ consists of finite-dimensional representations only.

A representation ψ of an associative algebra $\mathscr{A}$ on a vector space V is *algebraically irreducible* if V has no proper $\psi(\mathscr{A})$-invariant subspace. It is *algebraically completely irreducible* if, given $n \geqslant 0$ and $\{v_1, \ldots, v_N\}$, $\{u_1, \ldots, u_n\} \subset V$ with $\{u_i\}$ linearly independent, there is an $x \in \mathscr{A}$ with $\psi(x)u_i = v_i$ for $1 \leqslant i \leqslant n$.

THEOREM. *Let G be a connected unimodular Lie group, K a compact subgroup, and π a K-finite Banach representation of G on $\mathscr{B}$. Then the following are equivalent*: (i) *π is topologically irreducible*, (ii) *π is TCI*, (iii) *π_K (of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{B}_K$) is algebraically irreducible*, (iv) *π_K is algebraically completely irreducible.*

Now combine the last two theorems. So *G is a connected semisimple Lie group with finite center, K is a maximal compact subgroup, and π is a TCI Banach representation of G* on $\mathscr{B}$. Then *$\mathscr{U}(\mathfrak{g})$ acts on $\mathscr{B}_K = \Sigma_{\kappa\in\hat{K}}\mathscr{B}(\kappa)$* by *an algebraically completely irreducible representation π_K, K acts on $\mathscr{B}_K$ by the finite-multiplicity representation $\pi|_K$, and these are compatible in the sense $\pi|_K(k)\circ\pi_K(D)\circ\pi|_K(k)^{-1} = \pi_K(\mathrm{Ad}_G(k)D)$ and $d(\pi|_K) = \pi_K|_{\mathfrak{k}(\mathfrak{k})}$*. The space $\mathscr{B}_K$, with the 'compatible representation' $(\pi_K, \pi|_K)$ of the pair $(\mathscr{U}(\mathfrak{g}), K)$, is the *Harish-Chandra module* associated to π. These modules will be the subject of Varadarajan's paper.

20. THE K-MULTIPLICITIES

THEOREM. *Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let π be a TCI Banach representa-ation of G and κ an irreducible representation of K. Then the multiplicity $m(\kappa, \pi|_K) \leqslant \dim\kappa$.*

The proof depends on the following algebraic theorem.

THEOREM. *Let $\mathscr{A}$ be an associative algebra and $\mathscr{R}$ a set of representations of $\mathscr{A}$ such that* (i) *$\mathscr{R}$ is complete*, i.e. *if $0\neq x\in\mathscr{A}$ then $\psi(x)\neq 0$ for some $\psi\in\mathscr{R}$, and* (ii) dim *$\psi\leqslant n$ for every $\psi\in\mathscr{R}$ and some fixed integer n. Then every TCI Banach representation of $\mathscr{A}$ has dimension $\leqslant n$.*

To see this, let $r(n)$ be the least integer r such that $\Sigma(\mathrm{sign}\,\sigma)\xi_{\sigma(1)}\cdots\xi_{\sigma(r)}=0$ for all $\{\xi_1,\ldots,\xi_r\}\subset\mathfrak{gl}(n;\mathbb{C})$, where the sum runs over the permutations of $\{1,\ldots,r\}$. Then $r(n)\leqslant n^2+1$ because $\Lambda^r(\mathfrak{gl}(n;\mathbb{C}))=0$ for $r>n^2=\dim\mathfrak{gl}(n;\mathbb{C})$. A combinatorial argument shows $r(n)\geqslant r(n-1)+2$. Now let ϕ be a TCI Banach representation of $\mathscr{A}$ on $\mathscr{B}$ with $\dim\mathscr{B}>n$, and let $\mathscr{B}_0$ be a subspace of dimension $n+1$ in $\mathscr{B}$. As $r(n+1)>r(n)$ we have $T_1,\ldots,T_{r(n)}\in\mathrm{Hom}(\mathscr{B}_0,\mathscr{B}_0)$ such that $[T_1,\ldots,T_{r(n)}]=\Sigma(\mathrm{sign}\,\sigma)T_{\sigma(1)}T_{\sigma(2)}\cdots T_{\sigma(r(n))}\neq 0$. Extend T_i to a bounded linear operator $\tilde{T}_i$ on $\mathscr{B}$. As ϕ is TCI. There is a net $\{x_\alpha\}\subset\mathscr{A}$ with $\{\phi(x_\alpha)\}\to\tilde{T}_1$ in the strong topology. So $\lim\,[\phi(x_\alpha), \tilde{T}_2,\ldots,\tilde{T}_{r(n)}] = [\tilde{T}_1,\ldots,\tilde{T}_{r(n)}]\neq 0$, and thus for some $y_1\in\mathscr{A}$ we have $[\phi(y_1),\tilde{T}_2,\ldots,\tilde{T}_{r(n)}]\neq\neq 0$. Iterating, we have $\{y_1,\ldots,y_{r(n)}\}\subset\mathscr{A}$ with $[\phi(y_1),\ldots,\phi(y_{r(n)})]==\phi[y_1,\ldots,y_{r(n)}]\neq 0$. So $[y_1,\ldots,y_{r(n)}]\neq 0$, contradicting $\psi[y_1,\ldots,y_{r(n)}]=0$ for all $\psi\in\mathscr{R}$.

Given $\kappa\in\hat{K}$ set $C_{c,\kappa}(G)=\bar{\tau}_\kappa * C_c(G) * \bar{\tau}_\kappa$. It is an associative algebra under convolution, and its π-image preserves $\mathscr{B}(\kappa)$, defining a representation π_κ of $C_{c,\kappa}(G)$ on $\mathscr{B}(\kappa)$. As π is TCI, *each* π_κ *is* TCI. For if T is a bounded linear operator on $\mathscr{B}(\kappa)$ we extend it to $\tilde{T}=T\cdot\pi(\bar{\tau}_\kappa)$ on $\mathscr{B}$, and if $\{f_\alpha\}\subset C_c(G)$ with $\{\pi(f_\alpha)\}\to\tilde{T}$ then $\{\pi_\kappa(\bar{\tau}_\kappa * f_\alpha * \bar{\tau}_\kappa)\}\to T$. Now the algebraic theorem just proved, applies to G as follows.

THEOREM. *Suppose that G has a complete* (for $C_c(G)$) *set $\mathscr{R}$ of Banach representations such that, for a fixed $\kappa\in\hat{K}$ and a fixed integer n, every $\psi\in\mathscr{R}$ satisfies $m(\kappa,\psi|_K)\leqslant n$. Then, if π is a TCI Banach representation of G on $\mathscr{B}$, $m(\kappa,\pi|_K)\leqslant n$.*

For the ψ_κ, $\psi\in\mathscr{R}$, form a complete set of representations of $C_{c,\kappa}(G)$, each of dimension $\leqslant n\dim\kappa$, so $m(\kappa,\pi|_K)\leqslant n$.

Now the theorem $m(\kappa,\pi|_K)\leqslant\dim\kappa$ is reduced to the finding of a complete set $\mathscr{R}$ of Banach representations of G that satisfy $m(\kappa,\psi|_K)\leqslant\dim\kappa$ for all $\kappa\in\hat{K}$.

This is easy for linear groups; there the finite-dimensional irreducible representations form a complete set $\mathscr{R}$ of Banach representations.

THEOREM. *Every finite-dimensional irreducible representation ψ of G is equivalent to a subrepresentation of some $\pi_\nu=\mathrm{Ind}_{AN}^G(e^\nu)$, $\nu\in(\mathfrak{a}_\mathbb{C})^*$.*

Here $G=KAN$ is the Iwasawa decomposition, and e^ν is a quasi-character on A extended to AN by $e^\nu(an)=e^\nu(a)$. Frobenius Reciprocity gives $m(\kappa,\pi_\nu|_K)=m(\kappa,\mathrm{Ind}_{\{1\}}^K\{1\})=\dim\kappa$, so the theorem forces $m(\kappa,\psi|_K)\leqslant\dim\kappa$. The proof: ψ represents G on V and we apply Lie's Theorem to ψ^* and obtain

$$0\neq v^*\in V^* \quad\text{and}\quad \nu\in(\mathfrak{a}_\mathbb{C})^* \quad\text{with}\quad \psi^*(an)v^*=e^{-(\nu+\rho)}(a)v^*.$$

If $v\in V$ then $\phi_v\colon x\mapsto\langle\psi(x)^{-1}v, v^*\rangle$ satisfies $\phi_v(xan)=e^{-(\nu+\rho)}(a)\phi_v(x)$, so $v\mapsto\phi_v$ intertwines V with a subrepresentation of π_ν. That subrepresentation is equivalent to ψ because ψ is irreducible and $\phi_v(1_G)\neq 0$ for $\langle v, v^*\rangle\neq 0$.

For non-linear groups it is much more difficult to find the set $\mathscr{R}$. But the general idea is the same. The set $\hat{G}$ of irreducible unitary representations of G is a complete set of Banach representations. If $B=MAN$ is a minimal parabolic subgroup of G, then we have representations $\pi_{\mu,\nu}$ of G defined by $\mu\in\hat{M}$ and $\nu\in(\mathfrak{a}_C)^*$ as follows. Let $E(\mu)$ be the space of μ, so $\mathrm{Ind}_M^K(\mu)$ represents K on

$$V(\mu)=\{f\in L_2(K)\otimes E(\mu)\colon (r\otimes\mu)(m)f=f \quad \text{for all} \quad m\in M\}$$

where r is the right regular representation. Then $\pi_{\mu,\nu}=\mathrm{Ind}_B^G(\mu\otimes e^\nu)$ is a Banach representation of G on $V(\mu)$, and Frobenius Reciprocity gives

$$m(\kappa, \pi_{\mu,\nu}|_K)=m(\kappa, \mathrm{Ind}_M^K(\mu))=m(\mu, \kappa|_M)\leqslant \dim \kappa.$$

So the problem is to put every $\pi\in\hat{G}$ into some $\pi_{\mu,\nu}$. Of course this follows from the following theorem of Harish-Chandra.

SUBQUOTIENT THEOREM. *Every $\pi\in\hat{G}$ is equivalent on the K-finite level* (i.e. *as concerns the representation π_K of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{H}_K$ – 'infinitesimal equivalence'*) *to a subquotient of some $\pi_{\mu,\nu}$.*

Originally that was a deep analytic theorem based on a good knowledge of $\mathscr{U}(\mathfrak{g})$ and its representation π_K. A few years ago, Lepowski gave an enveloping algebra proof of the subquotient theorem, from which the multiplicity theorem comes out fairly early in the argument. Recently Jacquet* (in a conversation with Casselman) gave a short argument based on asymptotics of matrix coefficients, showing more – that every smooth irreducible K-finite ('irreducible admissible') representation of G which has an infinitesimal character is infinitesimally equivalent to a subrepresentation of some $\pi_{\mu,\nu}$.

21. The global character

THEOREM. *Let G be a connected semisimple Lie group with finite center and π a TCI Banach representation of G on a Hilbert space $\mathscr{H}$. If $f\in C_c^\infty(G)$, then $\pi(f)$ is a trace class operator on $\mathscr{H}$. Furthermore $\theta_\pi\colon C_c^\infty(G)\to\mathbb{C}$, by $\theta_\pi(f)=\mathrm{trace}\,\pi(f)$, is a distribution on G, and if χ_π is the infinitesimal character of π then $z(\theta_\pi)=\chi_\pi(z)\theta_\pi$ for all $z\in\mathscr{Z}(\mathfrak{g})$. Finally, if π and π' are K-finite (in particular if they are irreducible) unitary representations of G, then they are unitarily equivalent if and only if $\theta_\pi=\theta_{\pi'}$.*

*I am indebted to W. Schmid for this information.

θ_π is called the *global character* or *distribution character* of π.

This theorem holds for the larger class of reductive groups mentioned in Section 19.

Note that if π is a finite-dimensional representation of G, then formally

$$\mathrm{trace}\,\pi(f)=\int_G f(x)\,\mathrm{trace}\,\pi(x)\,dx$$

so the global character θ_π exists and, as distribution, is just integration against the classical character trace $\pi(x)$.

We start with a separable locally compact unimodular group G and a compact subgroup K. Let π be a Banach representation of G on a Hilbert space $\mathscr{H}$ such that, for some fixed m_π, each $\dim\mathscr{H}(\kappa)\leqslant m_\pi(\dim\kappa)^2$.

THEOREM. *If $f\in L_2(G)$ and f is compactly supported then $\pi(f)$ is a Hilbert-Schmidt operator on $\mathscr{H}$.*

First, if T is a bounded linear operator on $\mathscr{H}$ with bounded inverse then

$$\|\pi(f)\|_{HS}=\|T^{-1}\cdot\pi(f)\cdot T\|_{HS}\leqslant\|T^{-1}\|\cdot\|\pi(f)\|_{HS}\cdot\|T\|.$$

We use this with a T such that $k\to T\cdot\pi(k)\cdot T^{-1}$ is unitary. So we can assume $\pi|_K$ unitary thus the various $\mathscr{H}(\kappa)$ mutually orthogonal. Second, let F be a compact set in G whose interior contains supp(f), let $\phi\in C_c^+(G)$ with $\phi=1$ on KF, and choose $\{f_n\}$ continuous on G vanishing outside F with $\{f_n\}\to f$ in $L_2(G)$. Then $\{f_n\}\to f$ in $L_1(G)$ so $\|\pi(f_n)-\pi(f)\|\to0$. In a moment we will see that $\{\pi(f_n)\}$ is Cauchy HS. Then we will have an HS operator T with $\|\pi(f_n)-T\|_{HS}\to0$. But $\|\pi(f_n)-T\|\leqslant\|\pi(f_n)-T\|_{HS}$. So then $\pi(f)=T$ is HS. To see $\{\pi(f_n)\}$ Cauchy HS we need an estimate: *if $h\in C_c(G)$ vanishes outside F there exists $N=N(F,\pi)>0$ such that $\|\pi(h)\|_{HS}\leqslant N\|h\|_{L_2(G)}$.* For that, use $\int_G h(x)\pi(x)\,dx=\int_K(\int_G h(kx)\pi(kx)\,dx)\,dk$ to bound

$$\|\pi(h)\|_{HS}\leqslant\int_G\left\|\int_K h(kx)\pi(kx)\,dk\right\|_{HS}dx$$

$$\leqslant\int_G\left\|\int_K h(kx)\pi(k)\,dk\right\|_{HS}\|\pi(x)\|\,dx.$$

As $\kappa\in K$ appears at most $m_\pi\dim\kappa$ times in $\pi|_K$ and appears exactly $\dim\kappa$

times in the left regular representation L_K of K, and as $\pi|_K$ is unitary,

$$\left\| \int_K h(kx)\pi(k)\,\mathrm{d}k \right\|_{HS}^2 \; m_\pi \left\| \int_K h(kx)L_K(k)\,\mathrm{d}k \right\|_{HS}^2$$

$$= \text{(Peter–Weyl)} \quad m_\pi \int_K |h(kx)|^2\,\mathrm{d}k.$$

If $M=\sup_{KF}\|\pi(x)\|$ now

$$\|\pi(k)\|_{HS}\, Mm_\pi^{1/2} \int_G \left(\int_K |h(kx)|^2\,\mathrm{d}k \right)^{1/2} \mathrm{d}x$$

$$= Mm_\pi^{1/2} \int_G \phi(x) \left(\int_K |h(kx)|^2\,\mathrm{d}k \right)^{1/2} \mathrm{d}x$$

$$\leqslant Mm_\pi^{1/2} \|\phi\|_{L_2(G)} \|h\|_{L_2(G)}.$$

That proves the estimate, and the Cauchy sequence assertion follows directly.

Now suppose further that G is a Lie group with K and π as above.

THEOREM. *If $f \in C_c^\infty(G)$ then $\pi(f)$ is a trace class operator on $\mathscr{H}$.*

As with the Hilbert-Schmidt assertion we may assume $\pi|_K$ unitary. Let $\mathscr{A}$ be the closure of $\{\pi(f)\colon f \in C_c^\infty(G)\}$ in the Banach space of bounded linear operators on $\mathscr{H}$. If L_G and R_G are the left and right regular representations of G then $\pi(x)\pi(f)=\pi(L_G(x)f)$ and $\pi(f)\pi(x)=\pi(R_G(x)f)$. This defines Banach representations

$$l(x)\colon A \mapsto \pi(x)A \qquad \text{and} \qquad r(x)\colon A \to A\pi(x)^{-1}$$

of G on $\mathscr{A}$. If $f \in C_c^\infty(G)$ then $\pi(f)$ is a C^∞ vector for l and r. Now let $\psi = l \otimes r$, representation of $G \times G$ on $\mathscr{A}$ by $\psi(x,y)(A)=\pi(x)A\pi(y)^{-1}$. We may assume K connected so, as in Section 18, there exists $D_0 \in \mathscr{Z}([\mathfrak{k},\mathfrak{k}])$ with $\kappa(D_0)=(\dim\kappa)^2\kappa(1)$ for all $\kappa \in \hat{K}$. If $f \in C_c^\infty(G)$ then $\pi(f)$ is a C^∞ vector for ψ. Now $\psi(L_G(D_0)R_G(D_0)f)$ also is C^∞ for ψ, and as in Section 18

$$\sum_{\kappa_1,\kappa_2 \in \hat{K}} \|\psi(\overline{\tau_{\kappa_1 \otimes \kappa_2}})\pi(L_G(D_0)R_G(D_0)f)\| < \infty.$$

But

$$\begin{aligned} \psi(\overline{\tau_{\kappa_1 \otimes \kappa_2}})\pi(L_G(D_0)R_G(D_0)f) &= l(\overline{\tau_{\kappa_1}})r(\overline{\tau_{\kappa_2}})l_\infty(D_0)r_\infty(D_0)\pi(f) \\ &= l_\infty(D_0)l(\overline{\tau_{\kappa_1}})r_\infty(D_0)r(\overline{\tau_{\kappa_2}})\pi(f) \\ &= (\dim\kappa_1)^2(\dim\kappa_2)^2 l(\overline{\tau_{\kappa_1}})r(\overline{\tau_{\kappa_2}})\pi(f) \end{aligned}$$

so

$$(*) \qquad \sum_{\kappa_1,\kappa_2 \in \hat{K}} (\dim\kappa_1)^2(\dim\kappa_2)^2 \|\pi(\overline{\tau_{\kappa_1}})\pi(f)\pi(\overline{\tau_{\kappa_2}})\| < \infty.$$

Now let $\{v_i\colon i \in b_\kappa\}$ be an orthonormal basis of $\mathscr{H}(\kappa)$, $\kappa \in \hat{K}$. The $\mathscr{H}(\kappa)$ are mutually orthogonal because $\pi|_\kappa$ is unitary, so $\{v_i\colon i \in b = \bigcup b_\kappa\}$ is an orthonormal basis of $\mathscr{H}(\kappa)$. Calculate

$$\begin{aligned} \sum_{i,j \in b} |\langle \pi(f)v_i, v_j\rangle| &= \sum_{\kappa_1,\kappa_2 \in \hat{K}, i \in b_{\kappa_1}, j \in b_{\kappa_2}} |\langle \pi(f)v_i, v_j\rangle| \\ &\leqslant \sum_{\kappa_1,\kappa_2 \in \hat{K}} (\dim\mathscr{H}(\kappa_1))(\dim\mathscr{H}(\kappa_2)) \|\pi(\overline{\tau_{\kappa_1}})\pi(f)\pi(\overline{\tau_{\kappa_2}})\| \\ &\leqslant m_\pi^2 \sum_{\kappa_1,\kappa_1 \in \hat{K}} (\dim\kappa_1)^2(\dim\kappa_2)^2 \|\pi(\overline{\tau_{\kappa_1}})\pi(f)\pi(\overline{\tau_{\kappa_2}})\| < \infty. \end{aligned}$$

With G, K and π as above, we now prove the following theorem.

THEOREM. *The map $C_c^\infty(G) \ni f \to \theta_\pi(f) = \text{trace}\ \pi(f)$ is a distribution on G.*

As before we may assume $\pi|_K$ unitary. We have D_0 as above. We start by showing the existence of $D \in \mathscr{U}(\mathfrak{k})$ such that $|\theta_\pi(f)| \leqslant m_\pi \|\pi(L_G(DD_0)f)\|$ for all $f \in C_c^\infty(G)$. First,

$$\begin{aligned} |\theta_\pi(f)| &\leqslant \sum_{\kappa \in \hat{K}} |\text{trace}(\pi(\overline{\tau_\kappa})\pi(f)\pi(\overline{\tau_\kappa}))| \\ &\leqslant \sum_{\psi \in \hat{K}} (\dim\mathscr{H}(\kappa)) \|\pi(\overline{\tau_\kappa})\pi(f)\| \\ &\leqslant m_\pi \sum_{\kappa \in \hat{K}} (\dim\kappa)^2 \|\pi(\overline{\tau_\kappa})\pi(f)\| \\ &= m_\pi \sum_{\kappa \in \hat{K}} \|\pi(\overline{\tau_\kappa})\pi(L_G(D_0)f)\|. \end{aligned}$$

As $\pi(L_G(D_0)f)$ is a C^∞ vector for l we have, as in Section 18, some $D \in \mathscr{U}(\mathfrak{k})$

independent of f with

$$\sum_{\kappa\in\hat{K}} \|\pi(\overline{\tau_\kappa})\pi(L_G(D_0)f)\| = \sum_{\kappa\in\hat{K}} \|l(\overline{\tau_\kappa})\pi(L_G(D_0))f\|$$
$$\leqslant \|l(D)\pi(L_G(D_0)f)\| = \|\pi(L_G(DD_0)f)\|$$

yielding D as required. Now let $F\subset G$ be compact and $\{f_n\}\subset C_c^\infty(G)$ such that each $\mathrm{supp}(f_n)\subset F$ and $L_G(D)f_n\to 0$ uniformly on F. Then

$$\|\pi(L_G(D')f_n)\| \leqslant \int_G |L_G(D')f_n(x)|\,\|\pi(x)\|\,dx\to 0 \quad \text{for all} \quad D'\in\mathscr{U}(\mathfrak{g}).$$

So our estimate with D gives

$$|\theta_\pi(f_n)|\leqslant m_\pi\|\pi(L_G(DD_0)f_n\|\to 0$$

which shows that θ_π is a distribution.

For the theorem on semisimple G it remains only to show that equality of K-finite unitary characters θ_π, $\theta_{\pi'}$ implies unitary equivalence $\pi\cong\pi'$. That involves a certain amount of machinery comparing Naimark equivalence, infinitesimal equivalence. Banach equivalence and unitary equivalence, and we will not go into it.

Let π be a TCI representation of G and θ_π its distribution character. The distribution θ_π is *invariant* (under conjugation by elements of G) because

$$\theta_\pi(f\cdot\mathrm{Ad}(x)) = \mathrm{trace}\int_G f(xgx^{-1})\pi(g)\,dg \qquad \text{(definition)}$$
$$= \mathrm{trace}\int_G f(g)\pi(x^{-1}gx)\,dg \qquad (G \text{ unimodular})$$
$$= \mathrm{trace}\,(\pi(x^{-1})\pi(f)\pi(x)) = \mathrm{trace}\,\pi(f) = \theta_\pi(f)$$

θ_π is an *eigendistribution* (for the center $\mathscr{Z}(\mathfrak{g})$ of $\mathscr{U}(\mathfrak{g})$) because

$$(z\theta_\pi)(f) = \theta_\pi(zf) = \mathrm{trace}(\pi(zf)) = \mathrm{trace}(\pi_\infty(z)\cdot\pi(f))$$
$$= \mathrm{trace}(\chi_\pi(z)\pi(f)) = \chi_\pi(z)\ \mathrm{trace}\ \pi(f) = \chi_\pi(z)\theta_\pi(f)$$

where χ_π is the infinitesimal character of π. This system of differential equations

$$z\theta_\pi = \chi_\pi(z)\theta_\pi, \qquad z\in\mathscr{Z}(\mathfrak{g}),$$

together with invariance, has a strong influence on θ_π. To describe it, we

define the *regular set*

$$G' = \{x\in G: \mathfrak{g}^{\mathrm{Ad}(x)} \text{ is a Cartan subalgebra of } \mathfrak{g}\}$$

THEOREM. *Let θ be an invariant eigendistribution on G. Then θ is a locally L_1 function analytic on G'.*

This marvellous result of Harish-Chandra has recently been considerably simplified by Schmid and Atiyah, and is the subject of their paper.

22. Discrete series

G is a unimodular locally compact group countable at infinity. We write $\hat{G}$ for the set of unitary equivalence classes of irreducible unitary representations. It has a distinguished subset,

$$\hat{G}_{\mathrm{disc}} = \{[\pi]\in\hat{G}: [\pi]\subset L_G, \text{ the left regular representation}\}.$$

$\hat{G}_{\mathrm{disc}}$ is called the *discrete series* of G. Roughly speaking, it is the part of $\hat{G}$ that occurs discretely in the decomposition $L_2(G) = \int_{\hat{G}} \mathscr{H}_\pi\otimes\mathscr{H}_\pi^*\,d\pi$ relative to 'Plancherel measure' on $\hat{G}$. The Peter–Weyl Theorem $L_2(K) = \bigoplus_{\kappa\in\hat{K}} V(\kappa)\otimes V(\kappa)^*$ for a compact group K, says $\hat{K}=\hat{K}_{\mathrm{disc}}$.

The discrete series plays a fundamental role in harmonic analysis on semisimple and reductive groups. There, the Plancherel measure is concentrated on several 'series' of representations, which are constructed from the discrete series of certain subgroups.

Let G be a reductive Lie group with only finite many components, with compact center, such that every $\mathrm{Ad}(g)\in\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$, and such that the derived group $[G, G]$ has finite center. Then Harish-Chandra has given an explicit description of $\hat{G}$, which we now recall.

First, *$\hat{G}_{\mathrm{disc}}$ is non-empty if and only if G has a compact Cartan subgroup.*

Now suppose that G has a compact Cartan subgroup T. One can interpose a maximal compact subgroup, say $T\subset K\subset G$. Choose a positive $\mathfrak{t}_{\mathbb{C}}$-root system Δ^+ for $\mathfrak{g}_{\mathbb{C}}$ and define

$$\rho = \tfrac{1}{2}\sum_{\Delta^+}\alpha, \qquad \tilde{\omega}(\cdot) = \prod_{\Delta^+}\langle\cdot,\alpha\rangle \quad \text{and} \quad \Delta_{G,T} = \prod_{\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$$

Consider the shifted lattice

$$L = \{\lambda\in i\mathfrak{t}^*: e^{\lambda-\rho} \text{ is well defined in } T_0\}$$

which parameterizes $\hat{T}_0$ under $\lambda \leftrightarrow e^{\lambda-\rho}$. It has 'regular set' $L' = \{\lambda \in L: \tilde{\omega}(\lambda) \neq 0\}$. Given $\lambda \in L'$ set

$$q(\lambda) = |\{\alpha \in \Delta^+ : \mathfrak{g}_{\mathbf{C}}^{\alpha} \subset \mathfrak{k}_{\mathbf{C}} \quad \text{and} \quad \langle \lambda, \rangle < 0\}| + \\ + |\{\alpha \in \Delta^+ : \mathfrak{g}_{\mathbf{C}}^{\alpha} \not\subset \mathfrak{k}_{\mathbf{C}} \quad \text{and} \quad \langle \lambda, \alpha \rangle > 0\}|$$

so $(-1)^{q(\lambda)} = (-1)^q \operatorname{sign} \tilde{\omega}(\lambda)$ where $q = \frac{1}{2}\dim G/K$.

THEOREM. *If $\lambda \in L'$ there is a unique class $[\pi_\lambda] \in (G_0)\hat{}_{\text{disc}}$ whose distribution character satisfies*

$$\theta_{\pi\lambda}|_{T_0 \cap G'} = (-1)^{q(\lambda)} \frac{1}{\Delta_{G,T}} \sum_{w \in W(G_0, T_0)} \varepsilon(w) e^{w\lambda}$$

where $W(G_0, T_0)$ is the Weyl group (normalizer mod centralizer) *of T_0 in G_0. Every class in $(G_0)\hat{}_{\text{disc}}$ is one of these $[\pi_\lambda]$. Classes $[\pi_\lambda] = [\pi_{\lambda'}]$ if and only if $\lambda' \in W(G_0, T_0)(\lambda)$. In the appropriate normalization of Haar measure, $[\pi_\lambda]$ has formal degree $|\tilde{\omega}(\lambda)|$. Finally, $[\pi_\lambda]$ has central character $e^{\lambda-\rho}|_{ZG_0}$ and infinitesimal character χ_λ.*

Set $G\dagger = Z_G(G_0) \cdot G_0$. Then one can check that

$$(G\dagger)\hat{} = \{[\psi \otimes \pi]: [\psi] \in Z_G(G_0)\hat{}, [\pi] \in \hat{G}_0, \psi|_{ZG_0} \otimes \pi(1) = \psi(1) \otimes \pi|_{ZG_0}\}$$

Further, $[\psi \otimes \pi]$ has the same infinitesimal character χ_π as $[\pi]$ and has L_1^{loc} distribution character

$$\theta_{\psi\otimes\pi}(zg) = (\text{trace } \psi(z))\theta_\pi(g) \quad \text{for} \quad z \in Z_G(G_0), g \in G_0.$$

Finally, $[\psi \otimes \pi] \in (G\dagger)\hat{}_{\text{disc}}$ if and only if $[\pi] \in (G_0)\hat{}_{\text{disc}}$.

Let $\gamma = \operatorname{Ind}_{G\dagger}^G(\psi \otimes \pi)$ where $[\psi \otimes \pi] \in (G\dagger)\hat{}$. Its infinitesimal character $\chi_\gamma = \chi_\pi$, its distribution character $\theta_\gamma \in L_1^{\text{loc}}(G)$ with support in $G\dagger$ and formula

$$\theta_\gamma(zg) = \sum_{G/G\dagger} \text{trace } \psi(x^{-1}zx)\theta_\pi(x^{-1}gx),$$

and $[\gamma] \in \hat{G}_{\text{disc}}$ just when $[\pi] \in (G_0)\hat{}_{\text{disc}}$. Now set

$$\pi_{\psi,\lambda} = \operatorname{Ind}_{G\dagger}^G(\psi \otimes \pi_\lambda) \quad \text{for} \quad \lambda \in L', \psi \in Z_G(G_0)\hat{}, \quad \psi|_{ZG_0} = e^{\lambda-\rho}.$$

THEOREM. *$\hat{G}_{\text{disc}}$ consists precisely of the classes $[\pi_{\psi,\lambda}]$. Classes $[\pi_{\psi,\lambda}] = [\pi_{\psi',\lambda'}]$ just when $([\psi'], [\lambda']) \in W_{G,T}([\psi], [\lambda])$. The class $[\pi_{\psi,\lambda}]$ is the unique one in $\hat{G}$ with distribution character such that*

$$\theta_{\pi_{\psi,\lambda}}(zh) = \sum_{1 \leqslant i \leqslant r} (-1)^{q(w_j\lambda)} \operatorname{tr}\psi(x_j^{-1}zx_j) \frac{1}{\Delta_{G,T}} \sum_{w \in W_{G_0}} \det(ww_j) e^{ww_j(\lambda)}(h)$$

where $G = \bigcup_{1 \leqslant i \leqslant r} x_j G\dagger$, x_j normalizing T_0, and $w_j \in W_{G,T}$ is the element represented by x_j. Further, $\chi_{\pi_{\psi,\lambda}} = \chi_\lambda$ and $[\pi_{\psi,\lambda}{}^*] = [\pi_{\psi^*,-\lambda}]$, and in the appropriate normalization of Haar measures $\deg(\pi_{\psi,\lambda}) = r \cdot \dim(\psi) \cdot |\tilde{\omega}(\lambda)|$.

These results hold for the larger class of reductive groups described in Section 19, as do the various analogs of the Borel–Weil Theorem: As in the compact case, G/T is open in $G_{\mathbf{C}}/B^-$ and has complex structure. If $e^\nu \in \hat{T}_0$ and $[\psi] \in Z_G(G_0)\hat{}$ with $\psi|_{ZG_0} = e^\nu|_{ZG_0}$ then $[\psi \otimes e^\nu] \in (Z_G(G_0) \cdot T_0)\hat{} = \hat{T}$ defines a homogeneous holomorphic line bundle $\mathscr{L}_{\psi,\nu} \to G/T$ with G-invariant Hermitian metric. One can form the spaces $\mathscr{H}^{0,q}(G/T; \mathscr{L}_{\psi,\nu})$ of $\mathscr{L}_{\psi,\nu}$-valued square integrable harmonic $(0, q)$-forms on G/T.

THEOREM. *If $\nu + \rho \notin L'$ then every $\mathscr{H}^{0,q}(G/T; \mathscr{L}_{\psi,\nu}) = 0$. If $\nu + \rho \in L'$ then (i) $\mathscr{H}^{0,q}(G/T; \mathscr{L}_{\psi,\nu}) = 0$ for $q \neq q(\nu+\rho)$, and G acts irreducibly on $\mathscr{H}^{0,q(\nu+\rho)}(G/T, \mathscr{L}_{\psi,\nu})$ by the discrete series representation $[\pi_{\psi,\nu+\rho}]$.*

But this does not 'construct' the discrete series; just as in the compact case one needs the decomposition $L_2(G) = \int_{\hat{G}} \mathscr{H}(\pi) \otimes \mathscr{H}(\pi^*)\, d\pi$ to prove it, and in fact one needs a lot of information about the $\pi_{\psi,\lambda}|_K$.

The case of $SL(2; \mathbb{R})$ goes as follows. $G_{\mathbf{C}}/B^-$ is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with $G_{\mathbf{C}} = SL(2; \mathbb{C})$ acting by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: z \mapsto \frac{az+b}{cz+d}$$

and $B^- = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\right\}$ the subgroup fixing 0. For convenience we replace $SL(2; \mathbb{R})$ by its conjugate

$$G = \phi^{-1} \cdot SL(2; \mathbb{R}) \cdot \phi = \left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}: |a|^2 - |b|^2 = 1\right\},$$

which is $SU(1, 1)$, where

$$\phi = \sqrt{\frac{i}{2}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}.$$

Then

$$T=K=G\cap B^-=\left\{k_\theta=\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}: \theta \text{ real}\right\}$$

and so

$$G/T\cong G(0)=\{z\in\mathbb{C}: |z|<1\}.$$

The positive $\mathfrak{t}_{\mathbb{C}}$-root α is given by $\alpha\begin{pmatrix}t & 0\\ 0 & -t\end{pmatrix}=2t$, and $\rho=\frac{1}{2}\alpha$, so $L=\{h\alpha: 2h\in\mathbb{Z}\}$ with L' given by $h\neq 0$, and $\hat{T}$ consists of the $e^{h\alpha}$: $k_\theta\mapsto e^{2ih\theta}$ with $2h\in\mathbb{Z}$. Let $\mathcal{L}_h\to G/T$ denote the holomorphic Hermitian line bundle corresponding to $e^{h\alpha}\in\hat{T}$. It has a holomorphic trivialization under which the L_2 holomorphic sections are identified with the holomorphic functions f on the disc $D=\{z\in\mathbb{C}: |z|<1\}$ such that $\int_D |f(z)|^2(1-|z|^2)^{-2h-2}\,dx\,dy<\infty$. This Hilbert space is nonzero just when $h<-\frac{1}{2}$, and then it has orthonormal basis consisting of the functions

$$z\mapsto\left\{\frac{\pi}{-2h-1}\,\frac{\Gamma(-2h+n)}{\Gamma(-2h)\Gamma(n+1)}\right\}^{1/2} z^n, \qquad n \text{ integer} \geqslant 0.$$

Thus, for $h=-1, -3/2, -2, -5/2, \ldots$ we have $\mathcal{H}^{0,0}(G/T;\mathcal{L}_h)\neq 0$, and G acts on that space by the discrete series representation $\pi_{(-h+1/2)\alpha}$. Those representations form the 'holomorphic discrete series' of G. For $h>-\frac{1}{2}$, one can see that $\mathcal{H}^{0,1}(G/T;\mathcal{L}_h)\neq 0$, and obtain the 'antiholomorphic discrete series' representations $\pi_{(-h+1/2)\alpha}=\pi^*_{(h-1/2)\alpha}$. These representation exhaust the discrete series of $G\cong SL(2;\mathbb{R})$.

23. Cuspidal parabolic subgroups

G is a reductive Lie group of the class studied in Section 22. We fix a Cartan involution θ or, equivalently, the maximal compact subgroup $K=G^\theta$. Every Cartan subgroup of G is conjugate to a θ-stable one; let H be a θ-stable CSG in G. One checks $H=T\times A$ where $T=H\cap K$ and $A=\exp_G(\mathfrak{a})$, $\mathfrak{a}=\{\xi\in\mathfrak{h}: \theta(\xi)=-\xi\}$. Further, the G-centralizer of A is θ-stable and decomposes

$$Z_G(A)=M\times A, \qquad M=\theta(M), \qquad T \text{ a CSG in } M.$$

Here M is in general noncompact; it is compact if and only if $\mathfrak{a}$ is a CSA of $(\mathfrak{g},\mathfrak{k})$ – in general $\mathfrak{a}$ is smaller. But M is a reductive Lie group of the

class studied in Section 22, and $\hat{M}_{\text{disc}}\neq\emptyset$ because T is a compact CSG of M.

Just as in the case of CSA's of $(\mathfrak{g},\mathfrak{k})$, we have an $\mathfrak{a}$-root space decomposition $\mathfrak{g}=(\mathfrak{m}+\mathfrak{a})+\Sigma_{\nu\in\Delta_{\mathfrak{a}}}\mathfrak{g}_\nu$ where $\mathfrak{g}_\nu=\{\eta\in\mathfrak{g}: [\xi,\eta]=\nu(\xi)\eta \text{ for all } \xi\in\mathfrak{a}\}$ and $\Delta_{\mathfrak{a}}=\{\nu\in\mathfrak{a}^*: \nu\neq 0 \text{ and } \mathfrak{g}_\nu\neq\{0\}\}$. Any choice $\Delta_{\mathfrak{a}}^+$ of positive $\mathfrak{a}$-root system on $\mathfrak{g}$ comes from some choice Δ^+ of positive $\mathfrak{h}_{\mathbb{C}}$-root system on $\mathfrak{g}_{\mathbb{C}}$ by $\Delta_{\mathfrak{a}}^+=\{\alpha|_{\mathfrak{a}}: \alpha\in\Delta^+ \text{ and } \alpha|_{\mathfrak{a}}\neq 0\}$. Denote

$$\mathfrak{n}=\sum_{\nu\in\Delta_{\mathfrak{a}}^+}\mathfrak{g}_{-\nu} \qquad \text{and} \qquad N=\exp_G(\mathfrak{n})$$

and

$$\mathfrak{p}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n} \qquad \text{and} \qquad P=MAN.$$

Then P is a 'parabolic subgroup' of G, and it is called *cuspidal* because $\hat{M}_{\text{disc}}\neq\emptyset$. Every cuspidal parabolic subgroup of G is conjugate to one of the $P=P(\mathfrak{a},\Delta_{\mathfrak{a}}^+)$ just described.

In the case of a minimal parabolic subgroup, i.e. the case where $\mathfrak{a}$ is a CSA of $(\mathfrak{g},\mathfrak{k})$, the Weyl group $W(G,A)=$(normalizer of A)/(centralizer of A) is transitive on the set of all positive $\mathfrak{a}$-root systems, so $\mathfrak{h}$, or equivalently $\mathfrak{a}$, determines the conjugacy class of P. But that is not the case in general. So we say that two cuspidal parabolic subgroups $P_1, P_2\subset G$ are *associated* if their maximally split CSA are G-conjugate, in other words if they are respectively conjugate to groups $P(\mathfrak{a},\Delta_{\mathfrak{a}}^+)$ for the same $\mathfrak{a}$ but possibly different choices of $\Delta_{\mathfrak{a}}^+$.

A few words on parabolic subgroups in general. Fix a CSA of $(\mathfrak{g},\mathfrak{k})$, say $\mathfrak{a}_0$, and extend it to a θ-stable CSA in $\mathfrak{g}$, say $\mathfrak{h}_0=\mathfrak{t}_0+\mathfrak{a}_0$ where $\mathfrak{t}_0$ is any CSA in the $\mathfrak{k}$-centralizer, $\mathfrak{m}_0$, of $\mathfrak{a}_0$. Now fix a positive system $\Delta_{\mathfrak{a}_0}^+$ of $\mathfrak{a}_0$-roots and let $S_{\mathfrak{a}_0}$ be its simple subsystem. To every subset $T\subset S_{\mathfrak{a}_0}$ we associate an algebra $\mathfrak{p}_T=\mathfrak{m}_T+\mathfrak{a}_T+\mathfrak{n}_T$ as follows. Let $\langle T\rangle=\{\nu\in\Delta_{\mathfrak{a}_0}: \nu \text{ is a linear combination of the roots in } T\}$ and set

$$\mathfrak{a}_T=\{\xi\in\mathfrak{a}: \nu(\xi)=0 \quad \text{for all } \nu\in T\}$$
$$\mathfrak{m}_T=\{\xi\in\mathfrak{a}: \langle\xi,\mathfrak{a}_T\rangle=0\}+\sum_{\nu\in\langle T\rangle}\mathfrak{g}_\nu$$
$$\mathfrak{n}_T=\sum_{\nu\in\Delta_{\mathfrak{a}_0}^+,\,\nu\notin\langle T\rangle}\mathfrak{g}_{-\nu}.$$

For T empty we get the minimal parabolic $\mathfrak{b}^-$, and for $T=S_{\mathfrak{a}_0}$ we have $\mathfrak{p}_T=\mathfrak{g}$. Every subalgebra of $\mathfrak{g}$ that contains a minimal parabolic subalgebra is G_0-conjugate to exactly one of the $\mathfrak{p}_T$ and is called a *parabolic subalgebra* of $\mathfrak{g}$: the $\mathfrak{p}_T$ are called *standard* (rel $\mathfrak{a}_0$) parabolic subalgebras. On the group level

$$N_T=\exp_G(\mathfrak{n}_T),\quad A_T=\exp_G(\mathfrak{a}_T)\quad\text{and}\quad M_T\times A_T=Z_G(A_T),$$

and $P_T=M_TA_TN_T$ is the corresponding (standard) parabolic subgroup of G. Note that P_T is cuspidal just when $\mathfrak{a}_T$ is the $(\theta=-1)$-intersection of some CSA of $\mathfrak{g}$. This can be expressed in terms of $\Delta_{\mathfrak{a}_0}$.

24. Series of representations and the Plancherel formula

Let $P=MAN$, cuspidal parabolic subgroup of G constructed from $H=T\times A$ and $\Delta_{\mathfrak{a}}^+$. If $[\eta]\in\hat{M}$ and $\sigma\in\mathfrak{a}^*$, we extend $[\eta\otimes e^{i\sigma}]\in(MA)^\wedge$ to P by triviality on N and get $\pi_{\eta,\sigma}=\mathrm{Ind}_P^G(\eta\otimes e^{i\sigma})$, unitary representation of G. If η has infinitesimal character χ_ν relative to $\mathfrak{t}$ and distribution character Ψ_η, then $\pi_{\eta,\sigma}$ has infinitesimal character $\chi_{\nu+i\sigma}$ relative to $\mathfrak{h}$ and has distribution character $\theta_{\pi_{\eta,\sigma}}$ that is supported in the closure of $\bigcup_{g\in G}\mathrm{Ad}(g)(MA)$ and that is independent of choice of P within its association class. Thus $\pi_{\eta,\sigma}$ depends only on (H,η,σ).

Now the conjugacy class of the Cartan subgroup H specifies a series of unitary representations of G,

$$\{[\pi_{\eta,\sigma}]:\quad \eta\in\hat{M}_{\mathrm{disc}}\quad\text{and}\quad \sigma\in\mathfrak{a}^*\},$$

which we will call the *H-series.* Other people use other names, such as 'principal P-series' or 'cuspidal principal series' or, and this is misleading, 'degenerate principal series.' Anyway, at the two extremes the names are standard; for H maximally split (i.e. $\mathfrak{a}$ a CSA of $(\mathfrak{g},\mathfrak{k})$), principal series, and for H compact, discrete series.

Now we can characterize and parameterize the H-series as follows. Denote the positive $\mathfrak{t}_C$-root system on $\mathfrak{m}_C$ by $\Delta_{\mathfrak{t}}^+$, so $\Delta_{\mathfrak{t}}^+=\{\alpha|_{\mathfrak{t}}:\ \alpha\in\Delta^+$ and $\alpha|_{\mathfrak{a}}=0\}$. Set

$$\rho_{\mathfrak{t}}=\tfrac{1}{2}\sum_{\Delta^+}\nu,\quad \tilde{\omega}_{\mathfrak{t}}(\cdot)=\prod_{\Delta\mathfrak{t}^+}\langle\cdot,\nu\rangle\quad\text{and}\quad \Delta_{M,T}=\prod_{\Delta\mathfrak{t}^+}(e^{\nu/2}-e^{-\nu/2})$$

and consider

$$L_{\mathfrak{t}}''=\{\nu\in i\mathfrak{t}^*: e^{\nu-\rho_{\mathfrak{t}}}\in\hat{T}_0\quad\text{and}\quad \tilde{\omega}_{\mathfrak{t}}(\nu)\neq 0\}.$$

Set $M\dagger=Z_M(M_0)M_0$ as was done for G in Section 22. Thus $\hat{M}_{\mathrm{disc}}$ consists

of the classes of the

$$\eta_{\psi,\nu}=\mathrm{Ind}_{M\dagger}^M(\psi\otimes\eta_\nu)\quad\text{for}\quad \nu\in L_{\mathfrak{t}}''\quad\text{and}\quad \psi\in Z_M(M_0)^\wedge$$

where

$$\eta_\nu\in(M_0)^\wedge_{\mathrm{disc}}\text{ for parameter }\nu\text{ and }\psi|_{ZM_0}=e^{\nu-\rho_{\mathfrak{t}}}|_{ZM_0}.$$

So the H-series $\hat{G}_H$ of G consists of the classes of the

$$\pi_{\psi,\nu,\sigma}=\mathrm{Ind}_P^G(\eta_{\psi,\nu}\otimes e^{i\sigma});\qquad \psi\in Z_M(M_0),\qquad \nu\in L_{\mathfrak{t}}'',\qquad \sigma\in\mathfrak{a}^*$$

with the consistency condition between ψ and ν. Specializing the calculation of induced characters to which we alluded, $\pi_{\psi,\nu,\sigma}$ has infinitesimal character $\chi_{\nu+i\sigma}$ and it has distribution character $\theta_{\psi,\nu,\sigma}$ that satisfies

$$\theta_{\psi,\nu,\sigma}(ta)=\left|\frac{\Delta_{M,T}(t)}{\Delta_{G,H}(ta)}\right|\sum_{w(ta)\in NG(H)(ta)}\frac{1}{|N_M(T)(wt)|}\Psi_{\eta_{\chi,\nu}}(wt)\,e^{i\sigma}(a)$$

where $t\in T$, $a\in A$ and $ta\in G'$. The discrete series character $\Psi_{\eta_{\psi,\nu}}$ is specified in Section 22.

Every H-series representation is a finite sum of irreducibles, and if σ is regular relative to $(\mathfrak{g},\mathfrak{a})$ then $\pi_{\psi,\nu,\sigma}$ is irreducible. Also, if H_1 and H_2 are non-conjugate CSG then every H_1-series representation of G is disjoint from every H_2-series representation.

Harish-Chandra's Plancherel Formula goes roughly as follows. Choose a complete system $\{H_1,\dots,H_r\}$ of conjugacy classes of θ-stable CSG in G, split $H_j=T_j\times A_j$, and consider cuspidal parabolic subgroups $P_j==M_jA_jN_j$. Set $L_j''=L_{\mathfrak{t}_j}''$. Given $\nu\in L_j''$ the set

$$S(\nu)=\{\psi\in Z_{M_j}((M_j)_0)^\wedge:\psi=e^{\nu-\rho_{\mathfrak{t}_j}}\text{ on }Z_{(M_j)_0}\}$$

is finite, so one has finite sums

$$\pi_{j,\nu+i\sigma}=\sum_{S(\nu)}(\dim\psi)\pi_{\psi,\nu,\sigma}\quad\text{and}\quad \theta_{j,\nu+i\sigma}=\sum_{S(\nu)}(\dim\psi)\theta_{\psi,\nu,\sigma}.$$

THEOREM. *There are analytic $W(G,H_j)$-invariant functions $m_{j,\nu}$ on $\mathfrak{a}_j^*$, $1\leqslant j\leqslant r$, with the following property. Let $f\in C_c^\infty(G)$, and for $x\in G$ let r_xf denote right translate $g\mapsto f(gx)$. Then*

$$\sum_{1\leqslant j\leqslant r}\sum_{\nu\in L_j''}|\tilde{\omega}_{\mathfrak{t}_j}(\nu)|\int_{\mathfrak{a}_j^*}|\theta_{j,\nu+i\sigma}(r_xf)m_{j,\nu}(\sigma)|\,d\sigma<\infty$$

and

$$f(x)=\sum_{1\leqslant j\leqslant r}\sum_{v\in L_j^{\prime}}|\tilde{\omega}_{t_j}(v)|\int_{\mathfrak{a}_j^*}\theta_{j,v+i\sigma}(r_x f)m_{j,v}(\sigma)\,\mathrm{d}\sigma$$

This is the Plancherel formula for G, Fourier inversion for the Fourier transform $f\mapsto\hat{f}$ where $\hat{f}:\hat{G}\to\mathbb{C}$ by $\hat{f}(\pi)=\theta_\pi(f)$. It expresses the Plancherel measure

$$\mathrm{d}\pi_{\psi,v,\sigma}=(\dim\psi)|\tilde{\omega}_{t_j}(v)|m_{j,v}(\sigma)\,\mathrm{d}\sigma$$

as a very smooth measure on the union of the various H-series. Peter Trombi's paper will be concerned with the proof and the precise nature of the Plancherel measure.

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