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Completeness of Poincaré series for automorphic cohomology

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1. Introduction

Nearly a century ago, Poincaré introduced a construction for automorphic forms by summing over a discontinuous group. Poincaré studied the unit disc case of what usually now is formulated as

D: a bounded symmetric domain in $\mathbf{C}^n$;

$\mathbf{K}$: the canonical line bundle (of $(n, 0)$-forms) over D; and

Γ: a discontinuous group of analytic automorphisms of D.

In this formulation, he considered holomorphic sections φ of powers $\mathbf{K}^m \to D$, such as $(dz^1 \wedge \cdots \wedge dz^n)^m$, and formed the *Poincaré theta series*

$$\theta(\varphi) = \sum_{\gamma \in \Gamma} \gamma^* \varphi = \sum_{\gamma \in \Gamma} \varphi \cdot \gamma^{-1} .$$

$\mathbf{K}^m$ carries a natural Γ-invariant hermitian metric. If $m \geqq 2$ then $\mathbf{K}^m \to D$ has L_1 holomorphic sections, for example $(dz^1 \wedge \cdots \wedge dz^n)^m$. If φ is L_1, the series $\theta(\varphi)$ is absolutely convergent, uniformly on compact sets, to a Γ-invariant holomorphic section of $\mathbf{K}^m \to D$. The Γ-invariant holomorphic sections of $\mathbf{K}^m \to D$ are the Γ-*automorphic forms of weight m on D*. Their role is pervasive. See Borel [5] for a systematic discussion.

Consider $D = \{Z \in \mathbf{C}^{p \times p}\colon Z = {}^tZ \text{ and } I - ZZ^* \gg 0\}$, the bounded symmetric domain of $p \times p$ matrices equivalent to the Siegel half space of degree p. The latter is the space of normalized Riemann matrices of degree p. Thus, for appropriate choice of Γ, the equivalence classes of period matrices of Riemann surfaces of genus p sit in Γ/D.

In Griffiths' study ([6], [7]) of periods of integrals on algebraic manifolds, the period matrix domains D belong to a well-understood [16] class of homogeneous complex manifolds, of which the bounded symmetric domains are a small part. We refer to these more general domains as *flag domains*; see Section 2 for the definition. Except, essentially, in the symmetric case, one cannot expect a holomorphic vector bundle $\mathbf{E} \to D$ over a flag domain to have

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nontrivial holomorphic sections ([16], [11], [12]). In particular there are no automorphic forms in the classical sense. Instead, one must look to cohomology of degree $s = \dim_{\mathbf{C}} Y$ where Y is a maximal compact subvariety of D. Thus the substitute for automorphic forms is the *automorphic cohomology*, either in sheaf form:

$$H^s_\Gamma(D; \mathcal{O}(\mathbf{E})) = \{\Gamma\text{-invariant classes in } H^s(D; \mathcal{O}(\mathbf{E}))\} ,$$

in the Dolbeault form:

$$H^{0,s}_\Gamma(D; \mathbf{E}) = \{\Gamma\text{-invariant classes in } H^{0,s}(D; \mathbf{E})\} ,$$

or in the Kodaira-Hodge sense of harmonic forms:

$$\mathcal{H}^{0,s}(D/\Gamma; \mathbf{E}) = \{\Gamma\text{-invariant forms in } \mathcal{H}^{0,s}(D; \mathbf{E})\} .$$

Here one is quickly forced to assume that the bundle $\mathbf{E} \to D$ is *nondegenerate* as defined in Section 2 below. A holomorphic line bundle usually is degenerate.

Suppose $1 \leqq p \leqq \infty$, and let $H^s_p(D; \mathcal{O}(\mathbf{E}))$ (resp. $H^{0,s}_p(D; \mathbf{E})$) denote the subspace of $H^s(D; \mathcal{O}(\mathbf{E}))$ (resp. of $H^{0,s}(D; \mathbf{E})$) consisting of the classes with a Dolbeault representative φ such that $z \mapsto ||\varphi(z)||$ is in $L_p(D)$. Wells and I proved [15] that if $\mathbf{E} \to D$ is nondegenerate then the Poincaré series

$$\theta[\varphi] = \sum_{\gamma\in\Gamma} \gamma^*[\varphi] , \quad [\varphi] \in H^{0,s}_1(D; \mathbf{E}) ,$$

converge in the Fréchet topology of $H^{0,s}(D; \mathbf{E})$. We also showed that if, further, $\mathbf{E} \to D$ is L_1-*nonsingular* as defined in Section 2 below, then $H^{0,s}_2(D; \mathbf{E})$ is an infinite dimensional Hilbert space in which $H^{0,s}_1(D; \mathbf{E}) \cap H^{0,s}_2(D; \mathbf{E})$ is dense, so there are lots of these convergent Poincaré series. But we had no result on the kernel nor on the image of the Poincaré series operator θ: $H^{0,s}_1(D; \mathbf{E}) \to H^{0,s}_\Gamma(D; \mathbf{E})$, and in fact we did not exhibit a nonclassical Poincaré series $\theta[\varphi] \neq 0$. In this paper I show that the image of θ is as large as could be expected.

We start by studying harmonic forms. Let $\mathcal{H}^{0,s}_p(D; \mathbf{E})$ (resp. $\mathcal{H}^{0,s}_p(D/\Gamma; \mathbf{E})$) denote the space of harmonic $\mathbf{E}$-valued $(0, s)$-forms on D that are $L_p(D)$ (resp. Γ-invariant and $L_p(D/\Gamma)$). The main results, found in Section 7, are

PROPOSITIONS 7.2 AND 7.4. *Let $\mathbf{E} \to D$ be nondegenerate and L_1-nonsingular. If $\varphi \in \mathcal{H}^{0,s}_1(D; \mathbf{E})$, then the Poincaré series $\theta(\varphi) = \sum_{\gamma\in\Gamma} \gamma^*\varphi$ converges absolutely, uniformly on compacta, to an element of $\mathcal{H}^{0,s}_1(D/\Gamma; \mathbf{E})$. The resulting linear map*

$$\theta: \mathcal{H}^{0,s}_1(D; \mathbf{E}) \longrightarrow \mathcal{H}^{0,s}_1(D/\Gamma; \mathbf{E})$$

has $||\theta|| \leqq 1$, is surjective, and has for adjoint the inclusion $\mathcal{H}^{0,s}_\infty(D/\Gamma; \mathbf{E}) \hookrightarrow$

$\mathcal{H}^{0,s}(D; \mathbf{E})$.

THEOREM 7.9. *Let* $\mathbf{E} \to D$ *be nondegenerate and* L_1*-nonsingular, and let* $1 \leqq p \leqq \infty$. *Then the Poincaré series operator* θ *is defined on a certain subset of* $\mathcal{H}_p^{0,s}(D; \mathbf{E})$ *and maps that set onto* $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E})$.

The reader will now have guessed that we follow the rough outline of the Banach space approach, originated by Bers ([3], [4]) and Ahlfors ([1], [2]), described in Kra's book [10], for the case of the unit disc in $\mathbf{C}$. Our main problem is that of defining an appropriate harmonic projector. In the unit disc case there is an explicit formula for the reproducing kernel, and we just compute. Here, we use information on integrable discrete series representations to obtain L_p a priori estimates on a reproducing kernel form, and then use Banach space methods such as the Riesz-Thorin theorem to define projections from various spaces of L_p forms to the corresponding spaces of harmonic L_p forms. Once we have the projections, we establish the appropriate extension (Theorem 6.2) of Bers' result on the Petersson scalar product, and then our L_1 results (Propositions 7.2 and 7.4) are straightforward. The general L_p result (Theorem 7.9) requires some caution because θ does not converge on all of $\mathcal{H}_p^{0,s}(D; \mathbf{E})$ when $p > 1$ and Γ is infinite.

Having established that any harmonic form $\psi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E})$ is represented by a Poincaré series, we turn to the corresponding question for cohomolgy. This depends on Theorem 4.5, where we show that certain complete orthonormal sets $\{\varphi_i\} \subset \mathcal{H}_2^{0,s}(D; \mathbf{E})$ have the property that every $\mathcal{H}_p^{0,s}(D{:}\ \mathbf{E})$ is their L_p-closed span, and we use that to show that the natural map of a form to its Dolbeault class gives $\mathcal{H}_p^{0,s}(D; \mathbf{E}) \cong H_p^{0,s}(D; \mathbf{E})$. We define Poincaré series operators θ from (an appropriate subset of) $H_p^{0,s}(D; \mathbf{E})$ to

$$H_{p,\Gamma}^{0,s}(D; \mathbf{E})\text{: } \Gamma\text{-invariant } L_p \text{ cohomology on } D$$

and to

$$H_p^{0,s}(D/\Gamma; \mathbf{E})\text{: } L_p \text{ cohomology on } D/\Gamma$$

by $\theta(c) = [\theta(\psi)]$ where $\psi \in \mathcal{H}_p^{0,s}(D; \mathbf{E})$ is the harmonic representative of c, *provided that* $\theta(\psi)$ *is defined* as in Section 7. The main results, found in Section 8, are

THEOREM 8.6. *Let* $\mathbf{E} \to D$ *be nondegenerate and* L_1*-nonsingular, and let* $1 \leqq p \leqq \infty$. *Then the Poincaré series operator maps a certain subset of* $H_p^{0,s}(D; \mathbf{E})$ *onto* $H_{p,\Gamma}^{0,s}(D; \mathbf{E})$.

THEOREM 8.8. *Let* $\mathbf{E} \to D$ *be nondegenerate and* L_1*-nonsingular, let* $1 \leqq p < \infty$, *and suppose that* 0 *is not contained in the continuous spectrum* *of the laplacian on the space of* Γ*-invariant* $L_2(D/\Gamma)$ $\mathbf{E}$*-valued* $(0, s)$*-forms. Then the Poincaré series operator maps a certain subset of* $H_p^{0,s}(D; \mathbf{E})$ *onto* $H_p^{0,s}(D/\Gamma; \mathbf{E})$.

These theorems come down to the question of whether a class $[\psi] \in H_{p,\Gamma}^{0,s}(D; \mathbf{E})$ $\big($resp. $[\psi] \in H_p^{0,s}(D/\Gamma; \mathbf{E})\big)$ has a harmonic representative, i.e., whether $\psi - H\psi$ is cohomologous to zero on D (resp. on D/Γ). Here we use Fréchet space methods, based on a sharpening (8.3) of the L_p estimates for the reproducing kernel form. Those improved estimates depend on facts about integrable discrete series representations.

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2. Homogeneous vector bundles over flag domains

We recall the basic facts on the complex manifolds D and the holomorphic vector bundles $\mathbf{E} \to D$ which form the setting for automorphic cohomology.

A *complex flag manifold* is a compact complex homogeneous space $X = G_{\mathbf{C}}/P$ where $G_{\mathbf{C}}$ is a connected complex semisimple Lie group and P is a parabolic subgroup. Examples: the hermitian symmetric spaces of compact type.

A *flag domain* is an open orbit $D = G(x) \subset X = G_{\mathbf{C}}/P$ where X is a complex flag manifold, G is the identity component $G_{\mathbf{R}}^0$ of a real form of $G_{\mathbf{C}}$, and the isotropy subgroup of G at x is compact. Then that isotropy subgroup V [16] is the identity component of a compact real form of the reductive part of the conjugate $\{g \in G_{\mathbf{C}}: gx = x\}$ of P, V contains a compact Cartan subgroup H of G, and V is the centralizer in G of the torus $Z(V)^0$. Examples: the bounded symmetric domains and the period domains for compact Kähler manifolds.

Fix a flag domain $G/V = G(x_0) = D \subset X = G_{\mathbf{C}}/P$ and an irreducible unitary representation μ of V, say with representation space E_μ. Then we have the associated homogeneous hermitian C^∞ complex vector bundle $\mathbf{E}_\mu = G \times_\mu E_\mu \to D$. It is defined by the equivalence relation $(gv, z) \sim \big(g, \mu(v)z\big)$ on $G \times E_\mu$, and the sections over an open set $U \subset D$ are represented by the functions

$$f\colon \tilde{U} = \{g \in G\colon gx_0 \in U\} \longrightarrow E_\mu \quad \text{with} \quad f(gv) = \mu(v)^{-1} f(g) .$$

Furthermore [17], $\mathbf{E}_\mu \to D$ has a unique structure of a holomorphic vector bundle. Identify the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ of $G_{\mathbf{C}}$ with the corresponding algebra of complex vector fields on X, so the isotropy subalgebra

$$\mathfrak{p}_{x_0}\text{: Lie algebra of } P_{x_0} = \{g \in G_{\mathbf{C}}\colon gx_0 = x_0\}$$

consists of all $\xi \in g_{\mathbf{C}}$ whose value at x_0 is an antiholomorphic tangent vector there. Then a section over $U \subset D$, represented as above by $f\colon \widetilde{U} \to E_\mu$, is holomorphic just when $\xi(f) = 0$ for every ξ in the nilradical of $\mathfrak{p}_{x_0}$.

Let $\mathbf{T} \to D$ denote the holomorphic tangent bundle with a G-invariant hermitian metric. Then we have the Fréchet spaces

$$A^{p,q}(D;\, \mathbf{E})\colon C^\infty \quad \text{sections of } \mathbf{E} \otimes \Lambda^p(\mathbf{T}^*) \otimes \Lambda^q(\bar{\mathbf{T}})^* \longrightarrow D$$

of smooth $\mathbf{E}$-valued (p, q)-forms on D. Similarly, using pointwise norms from the hermitian metrics on $\mathbf{E}$ and $\mathbf{T}$ and the G-invariant measure on D derived from the metric on $\mathbf{T}$, we have the Banach spaces

$$L_r^{p,q}(D;\, \mathbf{E})\colon\ L_r \text{ sections of } \mathbf{E} \otimes \Lambda^p(\mathbf{T}^*) \otimes \Lambda^q(\bar{\mathbf{T}}^*) \longrightarrow D$$

for $1 \leqq r \leqq \infty$.

As usual, the $(0, 1)$-component of exterior differentiation is a well-defined Fréchet-continuous operator $\bar\partial\colon A^{p,q}(D;\, \mathbf{E}) \to A^{p,q+1}(D;\, \mathbf{E})$, and we have the *Dolbeault cohomology spaces*

$$H^{p,q}(D;\, \mathbf{E}) = \{\omega \in A^{p,q}(D;\, \mathbf{E})\colon \bar\partial\omega = 0\}/\bar\partial A^{p,q-1}(D;\, \mathbf{E}) \,.$$

In the cases studied in this paper, $\bar\partial$ has closed range, so $H^{p,q}(D;\, \mathbf{E})$ inherits the structure of Fréchet space from $A^{p,q}(D;\, \mathbf{E})$. In any case, for $1 \leqq r \leqq \infty$ we have

$$H_r^{p,q}(D;\, \mathbf{E}) = \{[\omega] \in H^{p,q}(D;\, \mathbf{E})\colon \omega \in L_r^{p,q}(D;\, \mathbf{E})\} \,,$$

the Dolbeault classes represented by L_r-forms.

Let

$$\#\colon \mathbf{E} \otimes \Lambda^p(\mathbf{T}^*) \otimes \Lambda^q(\bar{\mathbf{T}}^*) \longrightarrow \mathbf{E}^* \otimes \Lambda^{n-p}(\mathbf{T}^*) \otimes \Lambda^{n-q}(\bar{\mathbf{T}}^*)$$

denote the Hodge-Kodaira orthocomplementation operator. Here $\mathbf{E}^* \to D$ is the bundle dual to $\mathbf{E} \to D$, $n = \dim_{\mathbf{C}} D$, and $\#$ is conjugate linear on fibres. The global pairing between $L_r^{p,q}(D;\, \mathbf{E})$ and $L_{r'}^{p,q}(D;\, \mathbf{E})$, $1/r + 1/r' = 1$, is

$$\langle\varphi, \psi\rangle_D = \int_D \langle\varphi, \psi\rangle_x dx = \int_D \varphi \wedge \#\psi \tag{2.1}$$

where $\wedge$ is exterior product followed by contraction of $\mathbf{E}$ with $\mathbf{E}^*$. In particular this defines the Hilbert space structure on $L_2^{p,q}(D;\, \mathbf{E})$.

$\bar\partial$ has formal adjoint $\bar\partial^* = -\#\,\bar\partial\,\#$, and this defines the *Kodaira-Hodge-Laplace operator*

$$\Box = (\bar\partial + \bar\partial^*)^2 = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

on each $A^{p,q}(D; \mathbf{E})$. It is a second order, elliptic, G-invariant operator, and we also view it as a densely defined operator on each $L_r^{p,q}(D; \mathbf{E})$. We will need the spaces of L_r $\mathbf{E}$-valued *harmonic* (p, q)-forms on D, given by

$$\mathcal{H}_r^{p,q}(D; \mathbf{E}) = \{\omega \in L_r^{p,q}(D; \mathbf{E}) \colon \Box\omega = 0\} \tag{2.2}$$

where $\Box\omega = 0$ is understood in the sense of distributions,

$$\langle\omega, \Box\psi\rangle_D = 0 \quad \text{for all compactly supported} \quad \psi \in A^{p,q}(D; \mathbf{E}) .$$

Ellipticity gives $\mathcal{H}_r^{p,q}(D; \mathbf{E}) \subset \{\omega \in A^{p,q}(D; \mathbf{E}); \Box\varphi = 0\}$ where $\Box\omega = 0$ is understood with $\Box$ as differential operator. Since $\Box$ is elliptic and formally self-adjoint, it is not difficult to see that $\mathcal{H}_r^{p,q}(D; \mathbf{E})$ is a closed subspace of $L_r^{p,q}(D; \mathbf{E})$.

$\mathcal{H}_2^{p,q}(D; \mathbf{E})$ inherits a Hilbert space structure from $L_2^{p,q}(D; \mathbf{E})$. The natural action of G on $\mathcal{H}_2^{p,q}(D;\mathbf{E})$ is a unitary representation. We will need some detailed information about those unitary representations. For this, we must be specific about the bundles $\mathbf{E}$ and the corresponding representations.

Replace P by its conjugate $\{g \in G \colon gx_0 = x_0\}$, so the isotropy subgroup of G at $x_0 \in D$ is $V = G \cap P$. We have a compact Cartan subgroup H of G, and a maximal compact subgroup K, such that $H \subset V \subset K$. Further, we have a positive $\mathfrak{h}_{\mathbf{C}}$-root system Δ^+ on $\mathfrak{g}_{\mathbf{C}}$ and a subset Φ of the simple roots, such that $\mathfrak{p} = \mathfrak{p}_r + \mathfrak{p}_n$ where

$$\mathfrak{p}_n = \textstyle\sum_{\Delta^+\setminus\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{-\alpha} \quad \text{is the nilradical of } \mathfrak{p}$$

and

$$\mathfrak{p}_r = \mathfrak{v}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} + \textstyle\sum_{\langle\Phi\rangle} \mathfrak{g}_{\mathbf{C}}^{\beta} + \mathfrak{g}_{\mathbf{C}}^{-\beta} \quad \text{is a reductive complement} .$$

Here $\langle\Phi\rangle$ consists of all positive roots that are linear combinations of elements of Φ.

Let θ denote the Cartan involution of G with fixed point set K, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the Cartan decomposition into eigenspaces of θ. So

$$\mathfrak{k}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} + \textstyle\sum_{\Delta_k} \mathfrak{g}_{\mathbf{C}}^{\alpha} \quad \text{and} \quad \mathfrak{s}_{\mathbf{C}} = \textstyle\sum_{\Delta_s} \mathfrak{g}_{\mathbf{C}}^{\gamma}$$

where Δ_k consists of the "compact roots" and Δ_s of the "noncompact roots." Write Δ_k^+ for $\Delta_k \cap \Delta^+$ and Δ_s^+ for $\Delta_s \cap \Delta^+$, and Δ_v^+ for the positive root system $\langle\Phi\rangle$ of $\mathfrak{p}_r = \mathfrak{v}_{\mathbf{C}}$. Further define

$$\rho = \rho_G = \frac{1}{2}\textstyle\sum_{\Delta^+} \beta \ , \ \rho_k = \frac{1}{2}\textstyle\sum_{\Delta_k^+} \beta \quad \text{and} \quad \rho_V = \frac{1}{2}\textstyle\sum_{\Delta_v^+} \beta .$$

For convenience we replace $G_{\mathbf{C}}$ by its simply connected covering group. Then ρ_G exponentiates to a character on H.

Let Ψ be the simple root system of $(\mathfrak{g}_C, \Delta^+)$. Then $\lambda \in \mathfrak{h}_C^*$ is *integral* if the $2\langle\lambda, \psi\rangle/\langle\psi, \psi\rangle$, $\psi \in \Psi$, are integers, where $\langle , \rangle$ comes from the Killing form. As G_C is simply connected, λ is integral just when e^λ: $\exp(\xi) \mapsto e^{\lambda(\xi)}$, $\xi \in \mathfrak{h}$, is a well-defined character on the torus group H. Example: $2\langle\rho_G, \psi\rangle/\langle\psi, \psi\rangle = 1$ for all $\psi \in \Psi$.

If μ is an irreducible representation of V then $\mu|_H$ is a finite sum of characters e^λ; the $\lambda \in i\mathfrak{h}^*$ are integral and are called the *weights* of μ. There is a unique highest weight relative to a lexicographic order on $i\mathfrak{h}^*$ for which Δ^+ consists of positive elements. That highest weight does not depend on the order, and it determines μ up to equivalence. Denote

$$(2.3)\quad \begin{cases} \mu_\lambda: & \text{irreducible representation of } V \text{ with highest weight } \lambda \\ E_\lambda: & \text{representation space of } \mu_\lambda \\ \mathbf{E}_\lambda \to D: & \text{associated hermitian holomorphic bundle.} \end{cases}$$

Thus, for example, the canonical line bundle over D is $\Lambda^n(\mathbf{T}^*) = \mathbf{E}_{2(\rho_V - \rho_G)}$.

A bundle $\mathbf{E}_\lambda \to D$ is called *nondegenerate* if, whenever $\beta_1, \cdots, \beta_l$ are distinct noncompact positive $\mathfrak{h}_C$-roots of $\mathfrak{g}_C$,

$$(2.4)\qquad \begin{cases} \langle\lambda + \rho_V + \beta_1 + \cdots + \beta_l, \alpha\rangle > 0 & \text{for all} \quad \alpha \in \langle\Phi\rangle = \Delta_v^+ , \\ \langle\lambda + \rho_K + \beta_1 + \cdots + \beta_l, \gamma\rangle < 0 & \text{for all} \quad \gamma \in \Delta_k^+ \backslash \langle\Phi\rangle . \end{cases}$$

In that case, we have [15, § 3.2] the *Schmid Identity Theorem*:

$$(2.5)\qquad \begin{cases} \textit{If } c \in H^{0,s}(D; \mathbf{E}_\lambda),\ s = \dim_C Y = \dim_C K/V, \textit{ and} \\ \textit{if } c \textit{ restricts to the zero cohomology class on every} \\ \textit{fiber } gY \textit{ of } D \to G/K, \textit{ then } c = 0. \end{cases}$$

Among its consequences:

$$(2.6)\qquad \begin{cases} H^{0,q}(D; \mathbf{E}_\lambda) = 0 \quad \text{for} \quad q \neq s, \text{ and } H^{0,s}(D; \mathbf{E}_\lambda) \text{ is } \infty\text{-dimensional} \\ \text{Fréchet space on which the representation of } G \text{ is continuous.} \end{cases}$$

A homogeneous line bundle over D can be nondegenerate only under rather special conditions [15, Prop. 3.2.7].

Let Λ denote the set of integral linear forms on $\mathfrak{h}_C$ and Λ' its regular set,

$$\Lambda' = \{\lambda \in \Lambda: \langle\lambda, \alpha\rangle \neq 0 \quad \text{for all} \quad \alpha \in \Delta^+\} .$$

If $\lambda \in \Lambda'$ we denote

$$q(\lambda) = |\{\alpha \in \Delta_k^+: \langle\lambda, \alpha\rangle < 0\}| + |\{\gamma \in \Delta_s^+: \langle\lambda, \gamma\rangle > 0\}| .$$

From [17, Theorem 7.2.3] and the work of Schmid [13] on the Langlands Conjecture,

$$(2.7)\quad \begin{cases} \text{(i)} & \text{if } \lambda+\rho \notin \Lambda' \text{ then every } \mathcal{H}_2^{0,q}(D;\mathbf{E}_\lambda)=0; \\ \text{(ii)} & \text{if } \lambda+\rho \in \Lambda' \text{ and } q \neq q(\lambda+\rho) \text{ then } \mathcal{H}_2^{0,q}(D;\mathbf{E}_\lambda)=0; \\ \text{(iii)} & \text{if } \lambda+\rho \in \Lambda' \text{ then } G \text{ acts irreducibly on } \mathcal{H}_2^{0,q(\lambda+\rho)}(D;\mathbf{E}_\lambda) \text{ by the} \\ & \text{discrete series representation class } [\pi_{\lambda+\rho}]. \end{cases}$$

We will say that $\mathbf{E}_\lambda \to D$ is *L_1-nonsingular* if

$$(2.8)\quad \lambda+\rho \in \Lambda' \quad \text{and} \quad |\langle \lambda+\rho, \gamma\rangle| > \frac{1}{2}\sum_{\alpha\in\Delta^+} |\langle\alpha,\gamma\rangle| \quad \text{for all} \quad \gamma\in\Delta_s^+ .$$

According to Trombi-Varadarajan [14], and Hecht and Schmid ([8], [9]), given that $\lambda+\rho\in\Lambda'$ the other condition for L_1-nonsingularity is necessary and sufficient for $[\pi_{\lambda+\rho}]$ to have all K-finite matrix coefficients in $L_1(G)$.

In this paper, we are concerned with nondegenerate (2.4) L_1-nonsingular (2.8) homogeneous holomorphic vector bundles $\mathbf{E}_\lambda \to D$ such that the dimension $q(\lambda+\rho)$, in which square integrable cohomology occurs, is the complex dimension $s=\dim_{\mathbf{C}} Y$ of the maximal compact subvariety $Y=K(x_0)\cong K/V$.

Finally we recall the main results of [15]. The first [15, Theorem 4.1.6] says: *Let $\mathbf{E}_\lambda \to D$ be nondegenerate* (2.4), *let Γ be a discrete subgroup of G, and let $c\in H_1^{0,s}(D,\mathbf{E}_\lambda)$. Then the Poincaré series $\theta(c)=\sum_\Gamma \gamma^*(c)$ converges, in the Fréchet topology of $H^{0,s}(D;\mathbf{E}_\lambda)$, to a Γ-invariant class.* The second [15, Theorem 4.3.9] says: *If $\mathbf{E}_\lambda\to D$ is nondegenerate* (2.4), *then the natural map $\mathcal{H}_2^{0,s}(D;\mathbf{E}_\lambda)\to H^{0,s}(D;\mathbf{E}_\lambda)$ is a topological injection with image $H_2^{0,s}(D;\mathbf{E}_\lambda)$.* And the third [15, Theorem 4.3.8] tells us: *If $\mathbf{E}_\lambda\to D$ is L_1-nonsingular* (2.8) *and $q=q(\lambda+\rho)$, then $\mathcal{H}_2^{0,q}(D;\mathbf{E}_\lambda)\to H^{0,q}(D;\mathbf{E}_\lambda)$ maps every K-finite element of $\mathcal{H}_2^{0,q}(D;\mathbf{E}_\lambda)$ into $H_1^{0,q}(D;\mathbf{E}_\lambda)$.*

Thus, for $\mathbf{E}_\lambda\to D$ nondegenerate (2.4) and L_1-nonsingular (2.8) with $s=q(\lambda+\rho)$, $H_1^{0,s}(D;\mathbf{E}_\lambda)$ is very large—dense in the infinite dimensional Hilbert space $H_2^{0,s}(D;\mathbf{E}_\lambda)$ on which G acts by $[\pi_{\lambda+\rho}]$—and the Poincaré series operator θ maps it to the space $H_\Gamma^{0,s}(D;\mathbf{E}_\lambda)$ of Γ-invariant classes in $H^{0,s}(D;\mathbf{E}_\lambda)$. As described in the introduction, we are going to show that, in suitable senses, θ maps onto all spaces $\mathcal{H}_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ of $L_p(D/\Gamma)$ harmonic forms, onto all spaces $H_{p,\Gamma}^{0,s}(D;\mathbf{E}_\lambda)$ of Γ-invariant $L_p(D/\Gamma)$ classes in $H^{0,s}(D;\mathbf{E}_\lambda)$, and onto certain spaces $H_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ of L_p cohomology classes on D/Γ.

3. The reproducing kernel form

Surjectivity of the Poincaré series operator will depend on properties of a harmonic projector that derives from a reproducing kernel for $\mathcal{H}_2^{0,s}(D;\mathbf{E}_\lambda)$. In this section we study that kernel and its L_p properties.

Consider a homogeneous holomorphic vector bundle $\mathbf{E}\to D$. Given φ,

$\psi \in L_2^{p,q}(D; \mathbf{E})$ we have the exterior tensor product $(\varphi \otimes \# \psi)(z, \zeta) = \varphi(z) \otimes \# \psi(\zeta)$ and the corresponding integral operator

$$(T_{\varphi,\psi}\eta)(z) = \int_{\zeta \in D} \varphi(z) \otimes \# \psi(\zeta) \wedge \eta(\zeta) = \langle \eta, \psi \rangle_D \varphi(z)$$

on $L_2^{p,q}(D; \mathbf{E})$. If $\{\varphi_i\}$ is a complete orthonormal set in $\mathcal{H}_2^{p,q}(D; \mathbf{E})$, now $\mathcal{H}_2^{p,q}(D; \mathbf{E})$ has reproducing kernel

$$K_D(z, \zeta) = K_{D,\mathbf{E},p,q}(z, \zeta) = \sum \varphi_i(z) \otimes \# \varphi_i(\zeta) \, , \tag{3.1}$$

which converges absolutely because point evaluation norms $\varphi \mapsto ||\varphi(x)||$ are continuous on $\mathcal{H}_2^{p,q}(D; \mathbf{E})$. The kernel K_D is independent of choice of $\{\varphi_i\}$, hence G-invariant in the sense $K_D(gz, g\zeta) = K_D(z, \zeta)$. It is hermitian in that $\# K_D(z, \zeta) = K_D(\zeta, z)$. And since $\square$ is a self-adjoint elliptic operator, $K_D(z, \zeta)$ is weakly harmonic and thus harmonic in each variable.

THEOREM 3.2. *Let $\mathbf{E}_\lambda \to D$ be nondegenerate* (2.4) *and L_1-nonsingular* (2.8) *with $q(\lambda + \rho) = s$. Then the kernel form*

$$K_D(z, \zeta) = K_{D,\mathbf{E}_\lambda,0,s}(z, \zeta)$$

is L_p in each variable for $1 \leqq p \leqq \infty$, and the norms

$$||K_D(z, \cdot)||_p = ||K_D(\cdot, \zeta)||_p$$

independent of (z, ζ) for $1 \leqq p \leqq \infty$.

Proof. Fix $z \in D$. We are first going to show that $K_D(z, \zeta)$ is L_1 in ζ. For that, we may translate by an element of G and assume $z = x_0$, base point at which V is the isotropy subgroup of G. So our maximal compact subvariety $Y = K(z)$.

$U_\lambda = H^{0,s}(Y; \mathbf{E}_\lambda|_Y)$ is a finite dimensional K-module. It specifies a homogeneous complex vector bundle $\mathbf{U}_\lambda \to G/K$, whose sections are the functions $f \colon G \to U_\lambda$ with $f(gk) = k^* f(g)$. Thus we have the "direct image map"

$$\zeta \colon H^{0,s}(D; \mathbf{E}_\lambda) \longrightarrow \Gamma(\mathbf{U}_\lambda)$$

from cohomology to sections of $\mathbf{U}_\lambda$, given by $\xi(c)(g) = (g^* c)|_Y$. The Identity Theorem (2.5) says that ξ is injective. We know [15, Theorem 4.3.9] that the map $\mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda) \to \mathcal{H}^{0,s}(D; \mathbf{E}_\lambda)$, which sends a harmonic form to its Dolbeault class, is injective. Conclusion: the space

$$Q = \{\varphi \in \mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda) \colon \varphi|_Y = 0\}$$

is a closed K-invariant subspace of finite codimension in $\mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda)$.

We now have complete orthonormal sets $\{\varphi_1, \cdots, \varphi_m\}$ in $Q^\perp$ and $\{\varphi_{m+1}, \varphi_{m+2}, \cdots\}$ in Q, consisting of K-finite forms, and we use those to expand

$$K_D(z, \zeta) = \sum_1^\infty \varphi_i(z) \otimes \#\varphi_i(\zeta) = \sum_1^m \varphi_i(z) \otimes \#\varphi_i(\zeta) .$$

Each $\varphi_i \in L_1^{0,s}(D; \mathbf{E}_\lambda)$ by [15, Theorem 4.3.8]. So, for our fixed z, $K_D(z, \cdot)$ is a finite sum of L_1 forms, thus is L_1.

We need the fact that the forms φ_i are L_p for $1 \leqq p \leqq \infty$. Since they are L_1 and continuous, it suffices to prove that they are bounded. For that, consider the direct integral decomposition

$$L_2(G) = \int_{\hat{G}} \mathcal{H}_\pi \otimes \overline{\mathcal{H}_\pi} d\pi$$

where $\hat{G}$ consists of the equivalence classes of irreducible unitary representations of G, $\mathcal{H}_\pi$ is the representation space for $\pi \in [\pi] \in \hat{G}$, $\mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi$ is the Hilbert space completion of the span of the coefficients

$$f_{u,v}\colon g \longmapsto \langle u, \pi(g)v \rangle , \qquad u, v \in \mathcal{H}_\pi$$

with $\langle f_{u,v}, f_{w,x} \rangle = \langle u, w \rangle \langle v, x \rangle$, and $d\pi$ is Plancherel measure. Write $\mathcal{H}_\pi^\omega$ for the space of vectors $u \in \mathcal{H}_\pi$ whose K-type decomposition $u = \sum_{\tau \in \hat{K}} u_\tau$ satisfies $\sum c_\tau^{2n} ||u_\tau||^2 < \infty$, for all integers $n \geqq 0$, where c_τ is the value of τ on the K-component of the Casimir operator of G. Also write $\mathcal{H}_\pi^\infty$ for the space of K-finite vectors in $\mathcal{H}_\pi$. In the course of his proof [13] of the Langlands Conjecture, Schmid shows

$$\mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda) \subset \mathcal{H}_{\pi_{\lambda+\rho}} \otimes \overline{(\mathcal{H}_{\pi_{\lambda+\rho}})}^\omega \otimes \Lambda^s(\bar{T}^*) \otimes E_\lambda$$

where forms are viewed as $\Lambda^s(\bar{T}^*) \otimes E_\lambda$-valued functions on G. See [15, p. 443]. Now each

$$\varphi_i \in \mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda)^\infty \subset \mathcal{H}_{\pi_{\lambda+\rho}}^\infty \otimes \overline{(\mathcal{H}_{\pi_{\lambda+\rho}})}^\omega \otimes \Lambda^s(\bar{T}^*) \otimes E_\lambda .$$

Thus φ_i is a finite sum of terms $f_{u,v} \otimes w$ where $u \in \mathcal{H}_{\pi_{\lambda+\rho}}^\infty$, $v \in \mathcal{H}_{\pi_{\lambda+\rho}}^\omega$, $f_{u,v}$ is the coefficient function on G, and $w \in \Lambda^s(\bar{T}^*) \otimes E_\lambda$. Express $v = \sum_{\tau \in \hat{K}} v_\tau$. Then

$$||(f_{v,v} \otimes w)(g)|| = ||\textstyle\sum_{\hat{K}} \langle u, \pi_{\lambda+\rho}(g)v_\tau \rangle w|| \leqq ||u|| \cdot ||w|| \cdot \sum ||v_\tau|| < \infty .$$

Here we have the last inequality because [13, p. 379] $||v_\tau|| \leqq c(n)(1 + c_\tau)^{-n}$ for all integers $n \geqq 0$ and because c_τ is polynomial in the highest weight of τ. This completes the proof that φ_i is bounded, and thus completes the proof that each φ_i is L_p for $1 \leqq p \leqq \infty$.

Now the function $\zeta \mapsto ||K_D(z, \zeta)||$ is bounded by a finite linear combination $\sum_1^m ||\varphi_i(z)|| \cdot ||\varphi_i(\cdot)||$ of L_p functions, $1 \leqq p \leqq \infty$. So $K_D(z, \cdot)$ is L_p.

The kernel K_D, all the bundle metrics, and integration on D, are G-invariant. Thus the norm $||K_D(z, \cdot)||_p$ is finite and independent of $z \in D$, for $1 \leqq p \leqq \infty$.

Finally, $\# K_D(z, \zeta) = K_D(\zeta, z)$ says $\|K(z, \zeta)\| = \|K(\zeta, z)\|$, and this completes the proof of the theorem. q.e.d.

4. The harmonic projector on D

Retain the notation and setup of Section 3. In particular $\mathbf{E}_\lambda \to D$ is nondegenerate (2.4) and L_1-nonsingular (2.8) with $q(\lambda + \rho) = s$. Define a a constant $b = b(D, \lambda) > 0$ by

$$b = \|K_D(z, \cdot)\|_1 = \|K_D(\cdot, \zeta)\|_1, \qquad z, \zeta \in D. \tag{4.1}$$

THEOREM 4.2. *Let* $1 \leqq p \leqq \infty$. *If* $\psi \in L_p^{0,s}(D; \mathbf{E}_\lambda)$, *then its "harmonic projection"*

$$H\psi(z) = \int_{\zeta \in D} K_D(z, \zeta) \wedge \psi(\zeta) \tag{4.3a}$$

converges absolutely to an L_p *harmonic* $\mathbf{E}_\lambda$*-valued* $(0, s)$ *-form on* D. *Furthermore,*

$$H\colon L_p^{0,s}(D; \mathbf{E}_\lambda) \longrightarrow \mathcal{H}_p^{0,s}(D; \mathbf{E}_\lambda) \tag{4.3b}$$

has norm $\|H\| \leqq b$ *and if* $\psi \in \mathcal{H}_p^{0,s}(D; \mathbf{E}_\lambda)$ *then* $H\psi = \psi$.

Proof. Convergence and the bound on H are clear for $p = \infty$; there,

$$\begin{aligned}\left\|\int_{\zeta \in D} K_D(z, \zeta) \wedge \psi(\zeta)\right\| &\leqq \int_D \|K_D(z, \zeta) \wedge \psi(\zeta)\|\, d\zeta \\ &\leqq \int_D \|K_D(z, \zeta)\| \cdot \|\psi(\zeta)\|\, d\zeta \leqq b\, \|\psi\|_\infty .\end{aligned}$$

If ψ is continuous and compactly supported, it is L_∞, so $H\psi$ converges absolutely as just seen, and

$$\begin{aligned}\|H\psi\|_1 &= \int_D \left\|\int_{\zeta \in D} K_D(z, \zeta) \wedge \psi(\zeta)\right\| dz \\ &\leqq \int_D \int_D \|K_D(z, \zeta)\| \cdot \|\psi(\zeta)\|\, d\zeta dz \\ &= b \int_D \|\psi(\zeta)\|\, d\zeta = b\, \|\psi\|_1 .\end{aligned}$$

Extending H to $L_p^{0,s}(D; \mathbf{E}_\lambda)$ by continuity for $1 \leqq p < \infty$, we see that Riesz-Thorin gives us convergence of (4.3a) and the bound $\|H\| \leqq b$ on

$$H\colon L_p^{0,s}(D; \mathbf{E}_\lambda) \longrightarrow L_p^{0,s}(D; \mathbf{E}_\lambda) .$$

We check that $H\psi$ is harmonic. If φ is a C_c^∞ $\mathbf{E}_\lambda$-valued $(0, s)$-form on D, then

$$\langle H\psi, \square \varphi\rangle_D = \int_{z \in D} \left\{\int_{\zeta \in D} K_D(z, \zeta) \wedge \psi(\zeta)\right\} \wedge \# \square \varphi(z)$$

$$
\begin{aligned}
&= \int_{D\times D}\int K_D(z,\zeta)\wedge\psi(\zeta)\wedge\Box\,\#\varphi(z) \qquad (\Box\,\#=\#\,\Box)\\
&= \int_{D\times D}\int \Box_z\, K_D(z,\zeta)\wedge\psi(\zeta)\wedge\#\varphi(z) \qquad \text{(parts)}\\
&= 0 \quad \text{because } K_D(z,\zeta) \text{ is harmonic in } z\,.
\end{aligned}
$$

So $H\psi$ is weakly harmonic, and thus harmonic.

Next, we verify that the harmonic projector satisfies

$$
(4.4)\qquad \begin{cases}\text{If } \psi\in L_p^{0,s}(D;\mathbf{E}_\lambda) \text{ and } \varphi\in L_q^{0,s}(D;\mathbf{E}_\lambda) \text{ with } 1/p+1/q=1\,,\\ \text{then } \langle H\psi,\varphi\rangle_D=\langle\psi,H\varphi\rangle_D.\end{cases}
$$

If one of φ, ψ is continuous and compactly supported, and the other either is continuous and compactly supported or is essentially bounded, then $H\varphi$ and $H\psi$ converge absolutely and we calculate

$$
\begin{aligned}
\langle H\psi,\varphi\rangle_D &= \int_{z\in D}\left\{\int_{\zeta\in D}K_D(z,\zeta)\wedge\psi(\zeta)\right\}\wedge\#\varphi(z)\\
&= \int_{z\in D}\int_{\zeta\in D}\sum\nolimits_{i=1}^{\infty}\varphi_i(z)\otimes\#\varphi_i(\zeta)\wedge\psi(\zeta)\wedge\#\varphi(z)\\
&= \sum\nolimits_{i=1}^{\infty}\langle\varphi_i,\varphi\rangle_D\langle\psi,\varphi_i\rangle_D=\sum\nolimits_{i=1}^{\infty}\overline{\langle\varphi_i,\psi\rangle_D\langle\varphi,\varphi_i\rangle_D}\\
&= \overline{\langle H\varphi,\psi\rangle_D}=\langle\psi,H\varphi\rangle_D\,.
\end{aligned}
$$

Here we use the fact that the K-finite φ_i in K_D are L_r for $1\leqq r\leqq\infty$. As H is L_p and L_q bounded, now (4.4) follows whenever ψ and φ both are limits of C_c forms, or one is such a limit and the other is essentially bounded. The first case proves (4.4) for $1<p<\infty$, and the second case proves it for $p=1$ and for $p=\infty$.

Finally let $\psi\in L_p^{0,s}(D;\mathbf{E}_\lambda)$ be harmonic. Then it is weakly harmonic: $\langle\psi,\Box\varphi\rangle_D=0$ for every C_c^∞ $\mathbf{E}_\lambda$-valued $(0,s)$-form φ on D. That space of C_c^∞ forms is of course inside $L_2^{0,s}(D;\mathbf{E}_\lambda)$, and the Kodaira-Hodge decomposition expresses the latter as the orthogonal direct sum of its subspaces

$$\mathcal{H}_2^{0,s}(D;\mathbf{E}_\lambda)\text{: kernel of } \Box \text{ and image of } H$$

$$\mathrm{cl}\{\bar\partial L_2^{0,s-1}(D;\mathbf{E}_\lambda)+\bar\partial^* L_2^{0,s+1}(D;\mathbf{E}_\lambda)\}\text{: kernel of } H \text{ and closure of } \{\Box\varphi\colon \varphi \text{ is } C_c^\infty\}\,.$$

Conclusion: $\langle\psi,(1-H)\varphi\rangle_D=0$ for every C_c^∞ form φ. Here φ is L_q where $1/p+1/q=1$, so $(1-H)\varphi$ also is L_q, and (4.4) gives us $\langle\psi-H\psi,\varphi\rangle_D=0$. That proves $H\psi=\psi$. q.e.d.

We record a rather interesting consequence of Theorem 4.2.

THEOREM 4.5. *Let $E_\lambda\to D$ be nondegenerate* (2.4) *and L_1-nonsingular* (2.8) *with $q(\lambda+\rho)=s$ $(=\dim_{\mathbf{C}} Y)$. Let $\{\varphi_i\}$ be a complete orthonormal set in $\mathcal{H}_2^{0,s}(D;\mathbf{E}_\lambda)$ with each φ_i K-finite. Let $1\leqq p\leqq\infty$. Then*

(1) *$\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)$ is the L_p-closed span of $\{\varphi_i\}$ and*

(2) *the natural map of a form to its Dolbeault class is an injection* $\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)\hookrightarrow H^{0,s}(D;\mathbf{E}_\lambda)$ *with image* $H_p^{0,s}(D;\mathbf{E}_\lambda)$.

Proof. The φ_i were proved in Section 3 to be L_p, so each $\varphi_i\in\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)$. If $\psi\in\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)$ then Theorem 4.2 expresses it as the L_p limit of finite linear combinations of the φ_i. That proves (1), and it also shows that the restriction $\pi_p|_K$ of the Banach space representation of G on $\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)$ is independent of p. We know [15, Theorem 4.3.9] that $\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)\to H^{0,s}(D;\mathbf{E}_\lambda)$ is injective for $p=2$. Now for general p it is injective on K-finite vectors, and thus by G-equivariance is injective.

$\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)\hookrightarrow H^{0,s}(D;\mathbf{E}_\lambda)$ evidently has image in $H_p^{0,s}(D;\mathbf{E}_\lambda)$. Conversely, suppose that a class $c\in H_p^{0,s}(D;\mathbf{E}_\lambda)$ is represented by an L_p form ψ. By Theorem 4.2, $H\psi$ is another $\bar\partial$-closed L_p form; and of course $[H\psi]$ is in the image of $\mathcal{H}_p^{0,s}(D;\mathbf{E}_\lambda)\to H^{0,s}(D;\mathbf{E}_\lambda)$. To prove c is in that image, we will show $c=[H\psi]$, i.e., $[\psi-H\psi]=0$. In view of the Identity Theorem (2.5), it suffices to show that if $g\in G$ then $(\psi-H\psi)|_{gY}$ is cohomologous to zero on the compact subvariety gY. To do that we expand $K_D(z,\zeta)=\sum\varphi_i(z)\otimes\#\varphi_i(\zeta)$ where $\{\varphi_1,\cdots,\varphi_m\}$ span the orthocomplement of

$$\{\varphi\in\mathcal{H}_2^{0,s}(D;\mathbf{E}_\lambda):\ \varphi|_{gY}=0\}\,.$$

Then

$$\textstyle\sum_{i=1}^m\langle\psi,\varphi_i\rangle_D\varphi_i\,|_{gY}=(H\psi)\,|_{gY}$$

is the harmonic $\mathbf{E}_\lambda|_{gY}$-valued $(0,s)$-form on gY in the Dolbeault class of $\psi|_{gY}$. That shows $(\psi-H\psi)|_{gY}$ to be cohomologous to zero. q.e.d.

5. The harmonic projector on D/Γ

Retain the notation and setup of Sections 3 and 4. So $\mathbf{E}_\lambda\to D$ is nondegenerate and L_1-nonsingular with $q(\lambda+\rho)=s$. We are going to adapt the harmonic projection of Theorem 4.2 to forms invariant by the action of a discrete subgroup $\Gamma\subset G$.

Fix a discrete subgroup $\Gamma\subset G$ and let Ω be a fundamental domain for the action of Γ on D. Then we have Lebesgue spaces and harmonic subspaces

$$L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)\quad\text{and}\quad\mathcal{H}_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)\,.$$

Here the L_p consist of all measurable Γ-invariant $\mathbf{E}_\lambda$-valued $(0,s)$-forms ψ on D such that $||\psi|_\Omega(\cdot)||$ is in $L_p(\Omega)$, and the $\mathcal{H}_p$ consist of the harmonic ones. As before, $L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ is a Banach space, $\mathcal{H}_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ is a closed subspace, and for $1/p+1/q=1$ the pairing between $L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ and $L_q^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ is

(5.1) $$\langle \varphi, \psi \rangle_{D/\Gamma} = \int_{\Omega} \varphi \wedge \# \psi \ .$$

Now $L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ consists of the Γ-invariant elements of $L_\infty^{0,s}(D; \mathbf{E}_\lambda)$. If $\psi \in L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, we have $H\psi \in \mathcal{H}_\infty^{0,s}(D; \mathbf{E}_\lambda)$ as in Theorem 4.2, and for $\gamma \in \Gamma$ and $z \in D$,

$$\begin{aligned} H\psi(\gamma z) &= \int_{\zeta \in D} K_D(\gamma z, \zeta) \wedge \psi(\zeta) = \int_{\zeta \in D} K_D(z, \gamma^{-1}\zeta) \wedge \psi(\zeta) \\ &= \int_{\zeta \in D} K_D(z, \gamma^{-1}\zeta) \wedge \psi(\gamma^{-1}\zeta) = \int_{\zeta \in D} K_D(z, \zeta) \wedge \psi(\zeta) = H\psi(z) \ . \end{aligned}$$

Now Theorem 4.2 gives us

LEMMA 5.2. *Defined by* (4.3a), *the harmonic projection H sends $L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ to $\mathcal{H}^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ with norm $||H|| \leqq b$ and with $H\psi = \psi$ on $\mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$.*

The corresponding L_1 statement is

LEMMA 5.3. *If $\psi \in L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, then $H\psi$ is well-defined in the distribution sense,*

$$H\colon L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \longrightarrow \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \text{ with norm } ||H|| \leqq b \ ,$$

and if $\psi \in \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then $H\psi = \psi$.

Proof. Let A denote the space of all C^∞ Γ-invariant $\mathbf{E}_\lambda$-valued $(0, s)$-forms on D with support compact modulo Γ, and let $B = \{\varphi \in A; \max ||\varphi(z)|| = 1\}$. If ψ is a measurable Γ-invariant $\mathbf{E}_\lambda$-valued $(0, s)$-form on D, then $\psi \in L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ just when $\sup_{\varphi \in B} |\langle \psi, \varphi \rangle|_{D/\Gamma}$ is finite, and in that case $||\psi||_{D/\Gamma,1} = \sup_{\varphi \in B} |\langle \psi, \varphi \rangle_{D/\Gamma}|$.

We have H well-defined and of norm $\leqq b$ on $A \subset L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. Thus H is well-defined on $L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ in the distribution sense,

$$\langle H\psi, \varphi \rangle_{D/\Gamma} = \langle \psi, H\varphi \rangle_{D/\Gamma} \quad \text{for all} \quad \varphi \in A \ .$$

Further, $H\psi$ is Γ-invariant, and H has norm $\leqq b$, by duality from L_∞. The $H\psi$ are weakly harmonic by construction and thus harmonic. Finally, if $\psi \in \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then $H\psi = \psi$ as in the last paragraph of the proof of Theorem 4.2. q.e.d.

If ψ is a C^∞ $\mathbf{E}_\lambda$-valued $(0, s)$-form on D that is Γ-invariant and has support compact modulo Γ, then $H\psi$ is defined both by integration (4.3a) and in the sense of distributions as in Lemma 5.3, using L_p for any $p < \infty$. Integration by parts shows that the results are the same. Now we can combine Lemmas 5.2 and 5.3 with the Riesz-Thorin Theorem as follows.

THEOREM 5.4. *Let $1 \leqq p \leqq \infty$. If $\psi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, then its harmonic* *projection* $H\psi$ *is well-defined, by integration* (4.3a) *against the kernel form in case* $p = \infty$, *by* L_p *limits from* $C_c^\infty(D/\Gamma)$ *forms in case* $p < \infty$. *Furthermore*:

$$H: L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \longrightarrow \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \quad \textit{with norm } \|H\| \leqq b \,. \tag{5.5}$$

$$\textit{If } \psi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \quad \textit{then} \quad H\psi = \psi \,. \tag{5.6}$$

$$\begin{cases} \textit{If } \psi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \textit{ and } \varphi \in L_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \textit{ with} \\ 1/p + 1/q = 1, \textit{ then } \langle H\psi, \varphi\rangle_{D/\Gamma} = \langle \psi, H\varphi\rangle_{D/\Gamma}. \end{cases} \tag{5.7}$$

Here (5.6) follows from Lemma 4.4; and (5.7) is clear from integration by parts or the distribution definition of H if one of ψ, φ is $C_c^\infty(D/\Gamma)$, and then follows by the Riesz-Thorin limit procedure.

6. Analogue of the Petersson scalar product

Retain the notation and setup of Sections 3, 4, and 5. If $1/p + 1/q = 1$ then the pairing (5.1) restricts to a pairing

$$\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \times \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \longrightarrow \mathbf{C} \text{ by } \langle \varphi, \psi\rangle_{D/\Gamma} = \int_\Omega \varphi \wedge \#\psi \,. \tag{6.1}$$

The classical Petersson scalar product is the case where D is the unit disc and $\mathbf{E}_\lambda \to D$ is a power $\mathbf{K}^m \to D$ ($m \geqq 2$) of the canonical line bundle. We are going to apply Theorem 5.4 to obtain the following result, which is due to L. Bers ([3]; or see [10, p. 89]) in the classical case.

THEOREM 6.2. *For* $1 \leqq p < \infty$ *and* $1/p + 1/q = 1$, *the pairing* (6.1) *establishes a conjugate-linear isomorphism between* $\mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ *and the dual space of* $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. *If* $\psi \in \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ *corresponds to the linear functional* l, *then*

$$b^{-1}\cdot \|\psi\|_{D/\Gamma, q} \leqq \|l\| \leqq \|\psi\|_{D/\Gamma, q} \tag{6.3}$$

where b *is given by* (4.1).

Proof. Evidently (6.1) establishes a conjugate-linear map of $\mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ into the dual space of $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. We first prove it surjective.

Let l be a continuous linear functional on $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. By the Hahn-Banach Theorem it extends to $L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, and there the L_p, L_q version of the Riesz Representation Theorem represents it as integration against a form $\psi = \psi_l \in L_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$:

$$l(\varphi) = \langle \varphi, \psi\rangle_{D/\Gamma} = \int_\Omega \varphi \wedge \#\psi \,.$$

Using Theorem 5.4, we have

$$l(\varphi) = \langle \varphi, \psi\rangle_{D/\Gamma} = \langle H\varphi, \psi\rangle_{D/\Gamma} = \langle \varphi, H\psi\rangle_{D/\Gamma} \,,$$

so l corresponds to $H\psi \in \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ under (6.1).

We now check that our map of $\mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ onto the dual of $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ is injective. If $\psi \in \mathcal{H}_q^{0,s}(D; \mathbf{E}_\lambda)$ maps to the zero functional, then

$$\langle \varphi, \psi \rangle_{D/\Gamma} = 0 \quad \text{for} \quad \varphi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) .$$

On the other hand, if $\varphi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ with $H\varphi = 0$, then

$$\langle \varphi, \psi \rangle_{D/\Gamma} = \langle \varphi, H\psi \rangle_{D/\Gamma} = \langle H\varphi, \psi \rangle_{D/\Gamma} = 0 .$$

Combining these, we have $\langle \varphi, \psi \rangle_{D/\Gamma} = 0$ for all $\varphi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, which forces $\psi = 0$. The isomorphism is established.

Let $\psi \in \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ correspond to the functional l. Then $||l|| \leqq ||\psi||_{D/\Gamma,q}$ by the Hölder Inequality. On the other hand, using the full strength of Theorem 5.4, we have

$$\begin{aligned} ||\psi||_{D/\Gamma,q} &= \sup_{||\varphi||_p=1} |\langle \varphi, \psi \rangle_{D/\Gamma}| \\ &= \sup_{||\varphi||_p=1} |\langle \varphi, H\psi \rangle_{D/\Gamma}| && (5.6) \\ &= \sup_{||\varphi||_p=1} |\langle H\varphi, \psi \rangle_{D/\Gamma}| && (5.7) \\ &\leqq \sup_{||H\varphi||_p=1} |\langle H\varphi, \psi \rangle_{D/\Gamma}| && (5.5) \\ &= b \cdot \sup_{||H\varphi||_p=1} |\langle H\varphi, \psi \rangle_{D/\Gamma}| = b\,||l|| . \end{aligned}$$

That completes the proof of (6.3). q.e.d.

We record some consequences of Theorem 6.2.

COROLLARY 6.4. *Let* $\varphi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, $1 \leqq p < \infty$ *and* $1/p + 1/q = 1$. *Then* $H\varphi = 0$ *if and only if* $\langle \varphi, \psi \rangle_{D/\Gamma} = 0$ *for all* $\psi \in \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$.

COROLLARY 6.5. *Let* $\psi \in L_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, $1 < q \leqq \infty$ *and* $1/p + 1/q = 1$. *Then* $\psi \in \mathcal{H}_q^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ *if and only if* $\langle \varphi, \psi \rangle_{D/\Gamma} = \langle H\varphi, \psi \rangle_{D/\Gamma}$ *for all* $\varphi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$.

7. The Poincaré series operator on harmonic forms

Retain the notation and setup of Sections 3 through 6. Thus $\mathbf{E}_\lambda \to D$ is nondegenerate and L_1-nonsingular with $q(\lambda + \rho) = s$. If φ is a harmonic $\mathbf{E}_\lambda$-valued $(0, s)$-form on D, then we define the *Poincaré series* relative to Γ by

$$\theta(\varphi)(z) = \sum_{\gamma \in \Gamma} (\gamma^* \varphi)(z) \tag{7.1}$$

whenever the right side converges absolutely to a harmonic form. For example, if $\varphi \in \mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda)$ then

$$\begin{aligned} \int_\Omega ||\theta(\varphi)(z)||\,dz &= \int_\Omega ||\textstyle\sum_\Gamma (\gamma^*\varphi)(z)||\,dz \\ &\leqq \int_\Omega \textstyle\sum_\Gamma ||\varphi(\gamma z)||\,dz = ||\varphi||_1 , \end{aligned}$$

which says, more formally, the following:

PROPOSITION 7.2. *If $\varphi \in \mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda)$, then its Poincaré series $\theta(\varphi) = \sum_\Gamma \gamma^*\varphi$ converges absolutely, and uniformly on compact subsets of D, to an element of $\mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. The resulting linear map*

$$\theta\colon \mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda) \longrightarrow \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \quad \textit{has norm} \quad \|\theta\| \leqq 1\,. \tag{7.3}$$

In fact, the above calculation shows that $\theta\colon \varphi \to \sum_\Gamma \gamma^*\varphi$ converges, for $\varphi \in L_1^{0,s}(D; \mathbf{E}_\lambda)$, to an element $\theta(\varphi) \in L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, and that $\theta\colon L_1^{0,s}(D; \mathbf{E}_\lambda) \to L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ has norm $\leqq 1$. Further, if $\psi \in L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then

$$\begin{aligned}\langle \theta(\varphi), \psi\rangle_{D/\Gamma} &= \textstyle\sum_\Gamma \int_\Omega \gamma^*\varphi \wedge \#\psi = \sum_\Gamma \int_{\gamma\Omega} \varphi \wedge \gamma^*\#\psi \\ &= \textstyle\sum_\Gamma \int_{\gamma\Omega} \varphi \wedge \#\psi = \int_D \varphi \wedge \#\psi = \langle \varphi, \psi\rangle_D\,.\end{aligned}$$

Thus $\theta\colon L_1^{0,s}(D; \mathbf{E}_\lambda) \to L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ has adjoint $\theta^*\colon L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \hookrightarrow L_\infty^{0,s}(D; \mathbf{E}_\lambda)$, which is a continuous injection to a closed subspace, so here θ is surjective. The case $p = 1$ of Theorem 6.2 shows that the same considerations hold for the map (7.3) on L_1 harmonic forms; that says

PROPOSITION 7.4. *The Poincaré series map $\theta\colon \mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda) \to \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ is a continuous surjective linear map, and its adjoint is the inclusion $\theta^*\colon \mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \hookrightarrow \mathcal{H}_\infty^{0,s}(D; \mathbf{E}_\lambda)$.*

In fact one can do better. While θ need not converge on all of $\mathcal{H}_p^{0,s}(D; \mathbf{E}_\lambda)$, it certainly converges on the subspace $\mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda) \cap \mathcal{H}_p^{0,s}(D; \mathbf{E}_\lambda)$. So $\theta \cdot H$ converges on

$$\mathcal{J}_\Omega = \left\{\chi\varphi\colon \begin{array}{l}\varphi \text{ is an } \mathbf{E}_\lambda\text{-valued } \Gamma\text{-invariant } C^\infty \\ (0, s)\text{-form on } D \text{ with support compact mod } \Gamma\end{array}\right\}, \tag{7.5}$$

where χ is the indicator function of the fundamental domain Ω.

PROPOSITION 7.6. *Let $1 \leqq p < \infty$. If $\eta \in \mathcal{J}_\Omega$ then $\theta H(\eta)$ converges absolutely, and uniformly on compact subsets of D, to an element of $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, and*

$$\|\theta H(\eta)\|_{D/\Gamma, p} \leqq b \cdot \|\eta\|_p\,.$$

So θH extends to a continuous linear map

$$\theta H\colon\ (L_p\text{-}\textit{closure of } \mathcal{J}_\Omega) \longrightarrow \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \textit{ of norm} \leqq b\,.$$

This extension is surjective: if $\varphi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then $\chi\varphi$ is in the L_p-closure of $\mathcal{J}_\Omega$ and $\theta H(\chi\varphi) = \varphi$.

Proof. First suppose that $\varphi \in L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. Then $\chi\varphi \in L_1^{0,s}(D; \mathbf{E}_\lambda)$, so $H(\chi\varphi) \in \mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda)$ as in Lemma 5.3, and thus $\theta\big(H(\chi\varphi)\big) \in \mathcal{H}_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ as in

Proposition 7.2. In particular, if $\eta \in \mathcal{J}_\Omega$ then $\theta(H(\eta))$ converges absolutely, uniformly on compacta, to an L_1 harmonic form. Now let $\psi \in \mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ and calculate

$$\langle H\varphi, \psi\rangle_{D/\Gamma} = \langle \varphi, \psi\rangle_{D/\Gamma} \quad \text{by (5.6) and (5.7) ,}$$

and

$$\langle \theta H(\chi\varphi), \psi\rangle_{D/\Gamma} = \langle H(\chi\varphi), \psi\rangle_D = \langle \chi\varphi, \psi\rangle_D = \langle \varphi, \psi\rangle_{D/\Gamma} .$$

Using Theorem 6.2 with $p = 1$, we conclude

$$\theta H(\chi\varphi) = H\varphi \text{ for all } \varphi \in L_1^{0,s}(D/\Gamma; \mathbf{E}_\lambda) . \tag{7.7}$$

In particular, if $\eta = \chi\varphi \in \mathcal{J}_\Omega$ then

$$||\theta H(\eta)||_{D/\Gamma, p} = ||\theta H(\chi\varphi)||_{D/\Gamma, p} = ||H\varphi||_{D/\Gamma, p} \leqq b\,||\varphi||_{D/\Gamma, p} = b\,||\eta||_p .$$

That completes the proof of the first equation.

$L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ is isometric to $L_p^{0,s}(\Omega; \mathbf{E}_\lambda | \Omega)$ under $\varphi \to \chi\varphi$. Now suppose $p < \infty$. Then $\mathcal{J}_\Omega$ is dense in $L_p^{0,s}(\Omega; \mathbf{E}_\lambda|_\Omega)$, which of course contains $\chi \cdot \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. If $\varphi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$, now $\chi\varphi$ is in the L_p-closure of $\mathcal{J}_\Omega$, and $\theta H(\chi\varphi) = H\varphi = \varphi$ by continuity and (7.7). q.e.d.

The case $p = \infty$ is slightly different:

PROPOSITION 7.8. *If $\eta \in H(\chi \cdot L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda))$, then $\theta(\eta)$ converges absolutely to an element of $\mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. The map*

$$\theta \colon H(\chi \cdot L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)) \longrightarrow \mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$$

is surjective: if $\varphi \in \mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then $\theta(H(\chi\varphi)) = \varphi$.

Proof. Let $\psi = \chi\varphi \in \chi \cdot L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ and glance back at the proof of the $p = \infty$ case of Theorem 4.2. It gives

$$\int_{\zeta \in D} \sum\nolimits_{\gamma \in \Gamma} ||K(z, \zeta)|| \cdot ||(\gamma^* \psi)(\zeta)||\, d\zeta \leqq b\,||\psi||_\infty ,$$

so $H(\theta(\psi))$ is absolutely convergent. Since $\theta(\psi) = \theta(\chi\varphi) = \varphi$, now $\theta(H(\psi)) = H(\theta(\psi))$ is absolutely convergent to $H(\varphi)$. q.e.d.

We summarize for $1 \leqq p \leqq \infty$:

THEOREM 7.9. *Let $1 \leqq p \leqq \infty$. Then the Poincaré series operator is defined on*

$p = 1$:	*all of* $\mathcal{H}_1^{0,s}(D; \mathbf{E}_\lambda)$	*as in Proposition* 7.2;
$1 < p < \infty$:	$H(\chi \cdot L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda))$	*as in Proposition* 7.6;
$p = \infty$:	$H(\chi \cdot L_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda))$	*as in Proposition* 7.8;

and maps that space onto $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. In fact, if $\varphi \in \mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ then $||H(\chi\varphi)||_p \leqq b \cdot ||\varphi||_p$ and $\theta H(\chi\varphi) = \varphi$.

We record some consequences, the second of which uses Theorem 4.5.

COROLLARY 7.10. *$\psi \mapsto \theta H(\chi\psi)$ is a bounded linear projection of $\mathcal{H}_\infty^{0,s}(D; \mathbf{E}_\lambda)$ onto $\mathcal{H}_\infty^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$.*

COROLLARY 7.11. *Let $\{\varphi_i\}$ be a complete orthonormal set in $\mathcal{H}_2^{0,s}(D; \mathbf{E}_\lambda)$ with each φ_i K-finite. If $1 \leqq p \leqq \infty$ then $\mathcal{H}_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ is the L_p-closed span of $\{\theta H(\chi\varphi_i)\}$.*

8. The Poincaré series operator on cohomology

Retain the notation and setup of Sections 3–7. We are going to carry the surjectivity result of Theorem 7.9 over to Dolbeault and sheaf cohomology. Here there is an initial problem as to how to define θ, and there is the question of whether θ should map to cohomology of D/Γ or to Γ-invariant cohomology on D. So we first discuss these matters.

We first must decide just how to define the Poincaré series $\theta(c)$ of a Dolbeault class $c \in H^{0,s}(D; \mathbf{E}_\lambda)$. If $c \in H_1^{0,s}(D; \mathbf{E}_\lambda)$ then [15, Theorem 4.1.6] $\theta(c) = \sum_\Gamma \gamma^*(c)$ converges in the Fréchet topology of $H^{0,s}(D; \mathbf{E}_\lambda)$ to a Γ-invariant class. Further, from the proof, if $\psi \in L_1^{0,s}(D; \mathbf{E}_\lambda)$ represents c, then $\theta(\psi) = \sum_\Gamma \gamma^*\psi$ represents $\theta(c)$. If $c \in H_p^{0,s}(D; \mathbf{E}_\lambda)$, $1 < p \leqq \infty$, this argument breaks down because θ is defined from continuity considerations on $\theta \cdot H$. In view of this, we use Theorem 4.5 to obtain a harmonic representative $\psi \in \mathcal{H}_p^{0,s}(D; \mathbf{E}_\lambda)$ for c, and we define:

$$\theta(c) \text{ is the Dolbeault class of } \theta(\psi)\,, \tag{8.1}$$

whenever $\theta(\psi)$ is defined as in Section 7.

We can view θ as mapping either to Γ-invariant cohomology on D,

$$H_\Gamma^{0,s}(D; \mathbf{E}_\lambda) = \{c \in H^{0,s}(D; \mathbf{E}_\lambda) \colon \gamma^*(c) = c \text{ for } \gamma \in \Gamma\}\,,$$

or to cohomology on D/Γ,

$$H^{0,s}(D/\Gamma; \mathbf{E}_\lambda) = \frac{\{\bar\partial\text{-closed } \Gamma\text{-invariant forms in } A^{0,s}(D; \mathbf{E}_\lambda)\}}{\{\bar\partial\beta \colon \beta \in A^{0,s-1}(D; \mathbf{E}_\lambda) \text{ is } \Gamma\text{-invariant}\}}\,.$$

If $1 \leqq p \leqq \infty$, those cohomologies have L_p subspaces

$$H_{p,\Gamma}^{0,s}(D; \mathbf{E}_\lambda) = \{[\psi] \in H^{0,s}(D; \mathbf{E}_\lambda) \colon \psi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)\}$$

and

$$H_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda) = \{[\psi] \in H^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \colon \psi \in L_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)\}\,.$$

Evidently $H_{p,\Gamma}^{0,s}(D; \mathbf{E}_\lambda)$ is a quotient of $H_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. From (8.1), the image of θ will consist of classes with harmonic representatives. In general this means that we will take θ as mapping to the $H_{p,\Gamma}^{0,s}(D; \mathbf{E}_\lambda)$. But there are a few cases, detailed at the end of this section, where one can prove surjectivity to $H_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$.

PROPOSITION 8.2. *Every class* $[\psi] \in H_{p,\Gamma}^{0,s}(D;\mathbf{E}_\lambda)$ *has harmonic representative. In other words, the natural map* $\mathcal{H}_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda) \to H_{p,\Gamma}^{0,s}(D;\mathbf{E}_\lambda)$ *is surjective.*

Proof. $\psi \in L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ is $\bar\partial$-closed, so the same holds for $H\psi$, and it suffices to prove $\psi - H\psi$ cohomologous to zero over D.

If $p = \infty$ then $H\psi(z) = \int_{\zeta\in D} K_D(z,\zeta) \wedge \psi(\zeta)$, so the Identity Theorem argument at the end of the proof of Theorem 4.5 shows $\psi - H\psi$ cohomologous to zero.

Now let $p < \infty$. Then $H\psi$ is defined by continuity and Riesz-Thorin: if $\{\psi_i\}$ is a sequence of C^∞ Γ-invariant $\mathbf{E}_\lambda$-valued $(0,s)$-forms on D, each with support compact mod Γ, and if $\{\psi_i\} \to \psi$ in $L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$, then $H\psi$ is the limit in $L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ of the $H\psi_i = \int_{\zeta\in D} K_D(\cdot,\zeta) \wedge \psi_i(\zeta)$.

Exhaust D/Γ by an increasing sequence of compact sets and smooth the corresponding truncations of ψ. That gives a sequence $\{\psi_i\} \subset A^{0,s}(D;\mathbf{E}_\lambda)$ of Γ-invariant forms with supports compact mod Γ, such that if F is compact mod Γ then $\psi_i|_F = \psi|_F$ for i sufficiently large. Now $\{\psi_i\} \to \psi$ both in $L_p^{0,s}(D/\Gamma;\mathbf{E}_\lambda)$ and in the Fréchet space $A^{0,s}(D;\mathbf{E}_\lambda)$.

In order to prove that $\{H\psi_i\} \to H\psi$ in the Fréchet topology of $A^{0,s}(D;\mathbf{E}_\lambda)$, we will need some estimates that can be summarized as follows. Let Ξ belong to the universal enveloping algebra of $\mathfrak{g}_C$. Then

$$(8.3)\qquad \begin{cases} \zeta \to \|\Xi_z K_D(z,\zeta)\| \text{ is an } L_1 \text{ function on } D \text{ and} \\ \|\Xi_z K_D(z,\cdot)\|_1 \text{ is continuous in } z. \end{cases}$$

Once (8.3) is proved,

$$\begin{aligned} \|\Xi(H\psi_j - H\psi_i)(z)\| &= \left\|\int_{\zeta\in D} \Xi_z K_D(z,\zeta) \wedge \bigl(\psi_j(\zeta) - \psi_i(\zeta)\bigr)\right\| \\ &\leqq \|\Xi_z K_D(z,\cdot)\|_1 \, \|\psi_j - \psi_i\|_p \end{aligned}$$

which converges uniformly to zero on compact sets as $i, j \to \infty$. It then follows, if E is a C^∞ differential operator on $\mathbf{E}_\lambda \otimes \Lambda^s(\bar{\mathbf{T}}^*) \to D$, that $E(H\psi_i - H\psi)_j$ converges uniformly to zero on compact sets, so $\{H\psi_i\}$ converges in the Fréchet space $A^{0,s}(D;\mathbf{E}_\lambda)$. As $\{H\psi_i\} \to H\psi$ in L_p norm, now $\{H\psi_i\} \to H\psi$ in the Fréchet topology.

If $g \in G$ then $\psi_i|_{gY} \in A^{0,s}(gY;\mathbf{E}_\lambda)$ has harmonic component $H\psi_i|_{gY}$, as in the argument of Theorem 4.5. Taking limits, $\psi|_{gY} \in A^{0,s}(gY;\mathbf{E}_\lambda)$ has harmonic component $H\psi|_{gY}$, so $(\psi - H\psi)|_{gY}$ is cohomologous to zero on gY. Now as in the argument of Theorem 4.5, the Identity Theorem says that $\psi - H\psi$ is cohomologous to zero on D. That is the assertion of Proposition 8.2, which thus is proved pending verification of (8.3).

We turn to the proof of (8.3). Let U_d denote the set of all elements of degee $\leqq d$ in the universal enveloping algebra of $\mathfrak{g}_C$. Let K be a maximal compact subgroup of G and decompose

$$\mathcal{H}_2^{0,s}(D;\, \mathbf{E}_\lambda) = \sum_{\kappa \in \hat{K}} \mathcal{H}(\kappa)$$

into K-isotypic subspaces. This is just the K-decomposition of the discrete series class $[\pi_{\lambda+\rho}]$. Thus, if S is a finite subset of $\hat{K}$, there is another finite subset $F = F(S, d, \lambda + \rho)$ such that

$$\text{(8.4)} \qquad \begin{cases} \text{if } \varphi \in \mathcal{H}(\kappa),\ \Xi \in U_d, \ \text{ and } \ \Xi(\varphi) \text{ has nonzero} \\ \text{projection on } \sum_{\sigma \in S} \mathcal{H}(\sigma), \text{ then } \kappa \in F\,. \end{cases}$$

Now fix $z \in D$ and let K be the maximal compact subgroup of G that contains the isotropy subgroup at z. As in the proof of Theorem 3.2, we have a finite subset $S \subset \hat{K}$ such that

$$\text{if } \kappa \in \hat{K},\ \varphi \in \mathcal{H}(\kappa) \quad \text{and} \quad \varphi(z) \neq 0 \text{ then } \kappa \in S\,.$$

Fix an integer $d \geqq 0$, let F be a finite subset of $\hat{K}$ that satisfies (8.4), and choose a complete orthonormal set $\{\varphi_1, \varphi_2, \cdots\}$ in $\mathcal{H}_2^{0,s}(D;\, \mathbf{E}_\lambda)$ such that

(i) if $j \leqq m = \sum_{\kappa \in F} \dim \mathcal{H}(\kappa)$, then $\varphi_j \in \mathcal{H}(\kappa)$ for some $\kappa \in F$ and

(ii) if $j > m$ then $\varphi_j \in \mathcal{H}(\kappa)$ for some $\kappa \in \hat{K} - F$.

If $\Xi \in U_d$ now

$$\text{(8.5)} \qquad \Xi_z K_D(z, \zeta) = \sum_1^\infty \Xi(\varphi_j)(z) \otimes \# \varphi_j(\zeta) = \sum_{j=1}^m \Xi(\varphi_j)(z) \otimes \# \varphi_j(\zeta)\,.$$

First, this shows that $\zeta \mapsto \Xi_z K_D(z, \zeta)$ is an L_1 function on D, as required for (8.3). Second, (8.5) shows that $||\Xi_z K_D(z\cdot,)||_1$ is continuous in the coefficients of Ξ relative to a basis of U_d. If $g \in G$ then

$$||(\mathrm{Ad}(g)\Xi)_z K_D(z, \cdot)||_1 = ||\Xi_{gz} K_D(gz, \cdot)||_1\,.$$

So now $||\Xi_{gz} K_D(gz, \cdot)||_1$ is continuous in g. That shows $||\Xi_z K_D(z, \cdot)||_1$ to be continuous in z, and thus completes the proof of (8.3). q.e.d.

Proposition 8.2 combines with Theorem 7.9 to give us

THEOREM 8.6. *Let $1 \leqq p \leqq \infty$. Then the Poincaré series operator θ of* (8.1) *is defined on*

$$p = 1\colon \ \textit{all of } H_1^{0,s}(D;\, \mathbf{E}_\lambda)\,,$$
$$p > 1\colon \ \{[\psi] \in H_p^{0,s}(D;\, \mathbf{E}_\lambda)\colon \psi \in H(\chi \cdot L_p^{0,s}(D/\Gamma;\, \mathbf{E}_\lambda))\}\,,$$

and maps that space onto $H_{p,\Gamma}^{0,s}(D;\, \mathbf{E}_\lambda)$.

Thus every Γ-invariant $L_p(D/\Gamma)$ cohomology class for $\mathbf{E}_\lambda \to D$ is represented as an $L_p(D)$ Poincaré series, with no restriction on Γ.

Representation of cohomology on D/Γ by Poincaré series is less certain. The problem is that we use the Identity Theorem to show $\psi - H\psi = \bar{\partial}\eta$, starting with Γ-invariant ψ, but not necessarily obtaining η invariant under Γ. However, if D/Γ is compact, then the Green's operator $\mathcal{G}\colon (1 - H)L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda) \to L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ sends C^∞ forms to C^∞ forms, giving us $\psi - H\psi = \bar{\partial}(\bar{\partial}^*\mathcal{G}\psi)$ when $\bar{\partial}\psi = 0$. This argument extends a little bit past the compact case:

THEOREM 8.8. *Let $1 \leqq p < \infty$ and suppose that 0 is not contained in the continuous spectrum of $\square$ on $L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. Then every class $[\psi] \in H_p^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ has a harmonic representative, and so is of the form $\theta(c)$ for some $c \in H_p^{0,s}(D; \mathbf{E}_\lambda)$.*

Proof. The argument of Proposition 8.2 gives a sequence $\{\psi_i\} \subset A^{0,s}(D; \mathbf{E}_\lambda)$ of Γ-invariant forms with supports compact modulo Γ, such that $\{\psi_i\} \to \psi$ and $\{H\psi_i\} \to H\psi$ in the Fréchet topology.

Since $\square$ is uniformly elliptic and its continuous spectrum on $L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ omits 0, the Green's operator

$$\mathcal{G}(\square\varphi) = (1 - H)\varphi$$

is defined on all of $(1 - H)L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$ and there sends C^∞ forms to C^∞ forms. Thus the constituents of the Kodaira-Hodge decompositions

$$\psi_i = \bar{\partial}(\bar{\partial}^*\mathcal{G}\psi_i) + \bar{\partial}^*(\bar{\partial}\mathcal{G}\psi_i) + H\psi_i$$

all are C^∞ forms in $L_2^{0,s}(D/\Gamma; \mathbf{E}_\lambda)$. Since $\{\bar{\partial}\psi_i\} \to \bar{\partial}\psi = 0$, Fréchet, the

$$\zeta_i = \bar{\partial}\eta_i + H\psi_i\,, \quad \eta_i = \bar{\partial}^*\mathcal{G}\psi_i$$

satisfy

$$\{\zeta_i\} \longrightarrow \psi \quad \text{and} \quad \{H\zeta_i\} = \{H\psi_i\} \longrightarrow H\psi\ .$$

By its usual construction, $\mathcal{G}$ is continuous from the Hilbert space Γ-invariant forms, with derivatives of order $\leqq m$ square integrable modulo Γ, to the corresponding space of $m + 2$. It follows that $\{\mathcal{G}\psi_i\}$ is Fréchet convergent. That gives Fréchet convergence $\{\eta_i\} \to \eta \in A^{0,s-1}(D; \mathbf{E}_\lambda)$. Now

$$\psi - H\psi = \lim\{\zeta_i - H\zeta_i\} = \lim\{\bar{\partial}\eta_i\} = \bar{\partial}\eta\ .$$

As η is Γ-invariant by construction, we conclude that $\psi - H\psi$ is cohomologous to zero on D/Γ.

The representation $[\psi] = \theta\big([H(\chi\psi)]\big)$ now follows from Theorem 7.9.

q.e.d.

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