PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

EXPLICIT QUANTIZATION OF THE KEPLER MANIFOLD

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ABSTRACT. Any representation π of SO(2, 4) quantizing the Kepler manifold has the same lowest highest weight as the representation ν_0 in the Sternberg-Wolf description of the $U(2, 2)$-restriction of the metaplectic representation of Sp(4; $\mathbf{R}$). Hence, modulo covering groups, π is unitarily equivalent to ν_0.

0. Introduction. The Kepler manifold $T^+(S^3)$ is the cotangent bundle of the 3-sphere, minus the zero section, with the symplectic structure induced by that of the cotangent bundle. It is a Hamiltonian symplectic homogeneous space of the conformal group SO(2, 4) (cf. **[11]**). The action of SO(2, 4) on $T^+(S^3)$ has been quantized by various authors (**[3]**, **[6]**, **[7]**, **[10]**, **[11]**) to give an irreducible unitary representation π of SO(2, 4) on $L^2(S^3)$. However, these constructions of π suffer from being either ad hoc or else arrived at by a limiting procedure. Here, we give an explicit identification of π within the framework of the metaplectic representation.

In this note we show that π is essentially unitarily equivalent to a certain representation ν_0 of the 2-sheeted cover SU(2, 2) of SO(2, 4), described in **[9]**. It was noted in **[12]** that the coadjoint orbit of SU(2, 2) which corresponds to ν_0 under the moment map **[4]** is the coadjoint orbit of SO(2, 4) symplectomorphic to $T^+(S^3)$, which of course suggests, but does not prove, that $\pi = \nu_0$. Here we prove this equivalence by examining restrictions to a maximal compact subgroup of SU(2, 2). The interpretation of this result in terms of geometric quantization will be the subject of another paper.

1. The representations π and π_1. Rawnsley **[8]** and Blattner **[1]** have discussed two positive polarizations of the symplectic manifold $T^+(S^3)$ and their corresponding Hilbert spaces. The first polarization F is just the cotangent fibration, and the associated Hilbert space $\mathcal{H}_F$ is naturally isomorphic to $L^2(S^3)$. The second polarization, G, is obtained as follows: Identify $T^+(S^3)$ with

$$\{(e, x) \in \mathbf{R}^4 \times \mathbf{R}^4: e \cdot e = 1, e \cdot x = 0, x \neq 0\}. \tag{1}$$

Send (e, x) to $|x|e + ix \in \mathbf{C}^4$. The image is

$$X = \{z \in \mathbf{C}^4: z \cdot z = 0, z \neq 0\}, \tag{2}$$

Received by the editors September 27, 1978.
AMS (MOS) subject classifications (1970). Primary 22E45, 22E70; Secondary 53C15, 81A54.
[1]Research partially supported by NSF Grant MCS 78-01332 and MCS 75-17621.
[2]Research partially supported by NSF Grant MCS 76-01692.

X is then a Kähler manifold with Kähler form

$$-2^{-5/2} i d\left(|z|^{-1}\{z \cdot d\bar{z} - \bar{z} \cdot dz\}\right).$$

This structure defines G. The associated Hilbert space $\mathcal{H}_G$ consists of holomorphic functions on X square integrable with respect to the measure

$$\exp(-4\pi|x|) 2^{5/2} |x|^{1/2} \gamma, \tag{3}$$

where γ is the Liouville measure on X.

There is no positive polarization of $T^+(S^3)$ stable under SO(2, 4) **[11]**. However F is SO(1, 4) stable and G is SO(2) $\times$ SO(4) stable, and hence geometric quantization **[2]** provides unitary representations π_F of SO(1, 4) on $\mathcal{H}_F$ and π_G of SO(2) $\times$ SO(4) on $\mathcal{H}_G$. Moreover, the half form pairing of $\mathcal{H}_F$ with $\mathcal{H}_G$ (see **[8]**) gives a bounded nonunitary operator T of $\mathcal{H}_F$ onto $\mathcal{H}_G$, with bounded inverse, which intertwines $\pi_F|_{SO(4)}$ and $\pi_G|_{SO(4)}$.

For our purposes π will be any irreducible unitary representation of SO(2, 4) such that $\pi|_{SO(1,4)} \cong \pi_F$ and $\pi|_{SO(2)\times SO(4)} \cong \pi_G$.

As usual, the indefinite unitary group $U(2, 2) = \{g \in \mathbf{C}^{4\times 4}: ghg^* = h\}$, where

$$h = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

and $SU(2, 2) = \{g \in U(2, 2): \det g = 1\}$. Now $\bigwedge^2(\mathbf{C}^4) = \mathbf{C}^6$ has a real form $\mathbf{R}^6$ invariant under $\{\bigwedge^2(g): g \in SU(2, 2)\}$. This action of SU(2, 2) on $\mathbf{R}^6$ preserves a nondegenerate quadratic form of signature (2, 4). In this way we get a homomorphism $\alpha: SU(2, 2) \to SO(2, 4)$, where $\alpha(g) = \bigwedge^2(g)|_{\mathbf{R}^6}$, and this α is in fact a double covering. Letting $\pi_1 = \pi \circ \alpha$, we obtain an irreducible unitary representation of SU(2, 2).

2. The representation ν_0. Fix a nondegenerate antisymmetric bilinear form $\{u, v\}$ on $\mathbf{R}^8$. The *symplectic group* Sp(4; **R**) is the automorphism group of $(\mathbf{R}^8, \{\cdot, \cdot\})$. If $u, v \in \mathbf{R}^8$, then $\xi_{u,v}: x \mapsto \frac{1}{2}(\{u, x\}v + \{v, x\}u)$ belongs to the Lie algebra $\mathfrak{sp}(4; \mathbf{R})$. Fix a basis $p_1, \ldots, p_4, q_1, \ldots, q_4$ of $\mathbf{R}^8$ with $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_j, q_k\} = \delta_{jk}$. Then $\mathfrak{sp}(4; \mathbf{R})$ has basis

$$\xi_{a,b} = \xi_{p_a, p_b}, \qquad \xi'_{a,b} = \xi_{p_a, q_b}, \qquad \xi''_{a,b} = \xi_{q_a, q_b}. \tag{4}$$

Let λ be Lebesgue measure on $\mathbf{C}^4$. We have a Hilbert space

$$\mathcal{H} = \left\{ f: \mathbf{C}^4 \to \mathbf{C} \text{ holomorphic: } \int |f(z)|^2 \exp(-|z|^2)\, d\lambda < \infty \right\} \tag{5}$$

with inner product

$$\langle f_1, f_2 \rangle = \pi^{-4} \int f_1(z) \overline{f_2(z)} \exp(-|z|^2)\, d\lambda.$$

In multi-index notation $z^n = z_1^{n_1} \cdots z_4^{n_4}$, $n! = n_1! \cdots n_4!$, where $n = (n_1, \ldots, n_4)$, the $\varphi_n(z) = z^n/\sqrt{n!}$ form an orthonormal basis of $\mathcal{H}$.

The *metaplectic group* Mp(4; **R**) is the two-sheeted covering group of

$\mathrm{Sp}(4; \mathbf{R})$. It has a unitary representation μ on $\mathcal{H}$, called the *metaplectic representation*, specified by

$$d\mu(\xi_{a,b}) = -\frac{i}{2}(\partial_a\partial_b - z_a\partial_a - z_b\partial_a + z_a z_b - \delta_{a,b}),$$

$$d\mu(\xi'_{a,b}) = \frac{1}{2}(\partial_a\partial_b - z_a\partial_b + z_b\partial_a - z_a z_b),$$

$$d\mu(\xi''_{a,b}) = \frac{i}{2}(\partial_a\partial_b + z_a\partial_b + z_b\partial_a + \delta_{a,b}), \tag{6}$$

where $\partial_a = \partial/\partial z_a$ (see [**9**]).

Now $\mathrm{U}(2, 2)$ is naturally isomorphic to the subgroup of $\mathrm{Sp}(4; \mathbf{R})$ with Lie algebra spanned by the

$$\begin{aligned}
&\xi_{a,b} + \xi''_{a,b}, && 1 \leqslant a \leqslant b \leqslant 2 \quad \text{or} \quad 3 \leqslant a \leqslant b \leqslant 4,\\
&\xi'_{a,b} - \xi'_{b,a}, && (a, b) = (1, 2) \quad \text{or} \quad (3, 4),\\
&\xi_{a,b} - \xi''_{a,b}, && 1 \leqslant a \leqslant 2 \quad \text{and} \quad 3 \leqslant b \leqslant 4,\\
&\xi'_{a,b} + \xi'_{b,a}, && 1 \leqslant a \leqslant 2 \quad \text{and} \quad 3 \leqslant b \leqslant 4.
\end{aligned} \tag{7}$$

Moreover

$$\begin{aligned}
d\mu(\xi_{a,b} + \xi''_{a,b}) &= i(z_a\partial_b + z_b\partial_a + \delta_{a,b}),\\
d\mu(\xi'_{a,b} - \xi'_{b,a}) &= z_a\partial_b - z_b\partial_a,\\
d\mu(\xi_{a,b} - \xi''_{a,b}) &= -i(\partial_a\partial_b + z_a z_b),\\
d\mu(\xi'_{a,b} + \xi'_{b,a}) &= \partial_a\partial_b - z_a z_b.
\end{aligned} \tag{8}$$

Let $\mathrm{MU}(2, 2)$ denote the inverse image of $\mathrm{U}(2, 2)$ in $\mathrm{Mp}(4; \mathbf{R})$. Then

$$\nu = \det{}^{1/2} \otimes \mu|_{\mathrm{MU}(2,2)} \tag{9}$$

is a well-defined unitary representation of $\mathrm{U}(2, 2)$. There, it agrees with μ as given in (8), except for the cases $a = b$ in the first line, which become

$$\begin{aligned}
d\nu\begin{bmatrix} ix_1 & & \\ & \ddots & \\ & & ix_4 \end{bmatrix} &= \frac{1}{2}d\nu\left\{\sum_1^2 x_a(\xi_{a,a} + \xi''_{a,a}) - \sum_3^4 x_b(\xi_{b,b} + \xi''_{b,b})\right\}\\
&= i\left\{\sum_1^2 x_a(z_a\partial_a + 1) - \sum_3^4 x_b(z_b\partial_b)\right\}.
\end{aligned} \tag{10}$$

We know [**4**, Theorem 4.23] that $\mathcal{H}$ is the direct sum of subspaces

$$\mathcal{H}_d = \text{closed linear span of } \{\varphi_n\colon n_1 + n_2 - n_3 - n_4 = d\} \tag{11}$$

and that

$$\nu = \bigoplus_{d=-\infty}^{\infty} \nu_d, \tag{12}$$

where ν_d represents $\mathrm{U}(2, 2)$ irreducibly on $\mathcal{H}_d$.

In this note, we are concerned with $\nu_0|_{\mathrm{SU}(2,2)}$.

3. Restriction to S(U(2) $\times$ U(2)). S(U(2) $\times$ U(2)) is a maximal compact subgroup of SU(2, 2), and consists of matrices $g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A, B \in$ U(2) and $(\det A)(\det B) = 1$. Its center Z consists of those g with $A = e^{i\theta}I_2$ and $B = e^{-i\theta}I_2$, while its derived group consists of those g with $A, B \in$ SU(2). The kernel of α: SU(2, 2) $\to$ SO(2, 4) is just $\{\pm I_4\}$. Moreover, α maps S(U(2) $\times$ U(2)) onto SO(2) $\times$ SO(4), Z onto SO(2), and SU(2) $\times$ SU(2) onto SO(4).

Now the natural action of SO(4) on S^3 lifts to $T^+(S^3)$ and so to X. By **[8]** and **[1]**, the action of $g \in$ SO(4) on $\mathcal{H}_G$ sends f to $z \mapsto f(g^{-1} \circ z)$. Hence $\mathcal{H}_G$ is a direct sum of subspaces

$$(\mathcal{H}_G)_k = \text{span of } \{f|_X : f \text{ homogeneous polynomial of degree } k\}, \tag{13}$$

and

$$\begin{gathered} \pi_G|_{SO(4)} \text{ preserves } (\mathcal{H}_G)_k \text{ and acts irreducibly on} \\ \text{it by the } (k+1)^2\text{-dimensional representation } \overset{k}{\circ} \otimes \overset{k}{\circ}. \end{gathered} \tag{14}$$

Moreover, results of **[8]** and **[1]** prove that

$$(\mathcal{H}_G)_k \text{ is the } i(k+1)\text{-eigenspace of } d\pi_G \left[\begin{array}{cc|c} 0 & 1 & \\ & & 0_{2,4} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0_{4,2}} & 0_{4,4} \end{array}\right]. \tag{15}$$

This describes $\pi|_{SO(2)\times SO(4)}$ and hence $\pi_1|_{S(U(2)\times U(2))}$.

On the other hand, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{u}(2, 2)$ consisting of the diagonal matrices seen in (10), and let ε_j ($j = 1, 2, 3, 4$) be the linear functional

$$\begin{bmatrix} ix_1 & & \\ & \ddots & \\ & & ix_4 \end{bmatrix} \mapsto ix_j \quad \text{on } \mathfrak{h}.$$

We use the simple root system $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ}$ for $\mathfrak{u}(2, 2)$ and $\overset{\alpha_1}{\circ} \oplus \overset{\alpha_3}{\circ}$ for $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$, where $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Then **[9**, Lemma 5.3**]** $\mathcal{H}_0$ is a direct sum of subspaces

$$\mathcal{H}_{r,r} = \text{span of } \{\varphi_n : n_1 + n_2 = r = n_3 + n_4\} \tag{16}$$

and

$$\begin{gathered} \nu_0|_{U(2)\times U(2)} \text{ preserves } \mathcal{H}_{r,r} \text{ and acts irreducibly on it} \\ \text{by the representation with highest weight } (\varepsilon_1 + \varepsilon_2) + r(\varepsilon_1 - \varepsilon_4). \end{gathered} \tag{17}$$

Hence

$$\begin{gathered} \nu_0|_{SU(2)\times SU(2)} \text{ acts irreducibly on } \mathcal{H}_{r,r} \text{ by the} \\ (r+1)^2\text{-dimensional representation } \overset{r}{\circ} \otimes \overset{r}{\circ}, \end{gathered} \tag{18}$$

and

$$\mathcal{H}_{r,r} \text{ is the } 2i(r+1)\text{-eigenspace of } d\nu_0\begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}. \tag{19}$$

Comparing (14) with (18) and (15) with (19), and remembering that

$$d\alpha\begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} = \left[\begin{array}{cc|c} 0 & 2 & \\ & & 0_{2,4} \\ -2 & 0 & \\ \hline \multicolumn{2}{c|}{0_{4,2}} & 0_{4,4} \end{array}\right],$$

we have

LEMMA. $\pi_1|_{S(U(2)\times U(2))}$ *and* $\nu_0|_{S(U(2)\times U(2))}$ *are unitarily equivalent.*

4. Equivalence of π_1 and ν_0. In the simple root system $\{\alpha_1, \alpha_2, \alpha_3\}$ of §3, π_1 and ν_0 have the same lowest highest weight $\varepsilon_1 + \varepsilon_2$. Thus [**9**, Theorem 5.8] gives our result:

THEOREM. *The representations* π_1 *and* ν_0 *of* SU(2, 2) *are unitarily equivalent.*

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