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FOURIER INVERSION PROBLEMS ON LIE GROUPS AND A CLASS OF PSEUDODIFFERENTIAL OPERATORS

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The Classical Plancherel Formula

Let G be a unimodular separable locally compact groups and, to avoid technicalities, suppose that G is of type I. The unitary dual $\hat{G}$, the set of all equivalence classes $[\pi]$ of irreducible unitary representations π of G, has a standard Borel structure. Given $\pi \in [\pi] \in \hat{G}$, let $\mathcal{H}_\pi$ denote the representation space, and also let π denote the corresponding $*$-representation of $L^1(G)$,

$$\langle \pi(f)u,v\rangle_{\mathcal{H}_\pi} = \int_G f(x)\langle \pi(x)u,v\rangle_{\mathcal{H}_\pi} dx \quad \text{for } f \in L^1(G), \quad u,v \in \mathcal{H}_\pi$$

Segal's Plancherel Theorem (1950), which extends earlier results on compact groups and on abelian groups, says: there is a unique Borel measure μ on $\hat{G}$ such that

(i) if $f \in L^1(G) \cap L^2(G)$, then $\pi(f)$ is a Hilbert-Schmidt operator on $\mathcal{H}_\pi$ for μ-almost-all $[\pi] \in \hat{G}$ and

(ii)

$$\|f\|^2_{L^2(G)} = \int_{\hat{G}} \|\pi(f)\|^2_{HS} d\mu[\pi] \tag{1.1}$$

Here μ is the "Plancherel measure" and (1.1) is the "Plancherel Formula."

If f is of the form $h^* * h$, $h^*(x) = \overline{h(x^{-1})}$ so that $\pi(h^*) = \pi(h)^*$, then

$$\|h\|^2_{L^2(G)} = f(1_G)$$

and $\|\pi(h)\|^2_{HS} = \text{trace } \pi(f)$. Then, (1.1) becomes a Fourier inversion formula

$$f(1_G) = \int_{\hat{G}} \text{trace } \pi(f) d\mu[\pi] \tag{1.2}$$

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When G is not Unimodular

Let us now drop the unimodularity condition of G. Let dx denote right Haar measure,

$$\int_G f(xg)dx = \int_G f(x)dx \quad \text{for} \quad f \in C_0(G)$$

and let $\delta = \delta_G$ denote the modular function

$$\delta(g)\int_G f(gx)dx = \int_G f(x)dx \quad \text{for} \quad f \in C_0(G)$$

Set $f^g(x) = f(g^{-1}xg)$. Then, for $[\pi] \in \hat{G}$,

$$\text{trace } \pi(f^g) = \text{trace} \int_G f(g^{-1}xg)\pi(x)dx = \delta(g)\text{trace } \pi(f)$$

while, of course, $f^g(1_G) = g(1_G)$. So, the Plancherel Formula (1.2) does not make sense.

The solution to this problem is to insert an operator that is semiinvariant of type δ_G, either the <u>infinitesimal operators</u> D_π, $[\pi] \in \hat{G}$, specified up to scalar multiple by

$$\pi(g)D_\pi\pi(g)^{-1} = \delta(g)D \quad \text{on} \quad \mathcal{H}_\pi$$

or a <u>global operator</u> D on $L^2(G)$ such that

$$D^g(f) = \delta(g)D(f) \quad \text{where} \quad D^g(f) = D(f^{g^{-1}})$$

The relation is that, in some suitable sense, $D_\pi = \pi(D)$.

We illustrate this with the case of the Heisenberg group with scale. The <u>Heisenberg group of dimension</u> $2n + 1$ is $N = \mathbb{R} + \mathbb{R}^n + \mathbb{R}^n$ with

$$(z,y,x)(z',y',x') = (z + z' + x\cdot y' - y\cdot x', y + y', x + x')$$

and the <u>Heisenberg group with scale</u> is $G = N\cdot A$, $A \cong \mathbb{R}^+$, with

$$(z,y,x,a)(z',y',x',a') = (z + a^2a' + ax\cdot y' - ay\cdot x', y + ay', x + ax', aa')$$

Write Z for the center $\{(z,0,0): z \in \mathbb{R}\}$ of N. Recall that $\hat{N}$ consists of (i) the unitary characters

$$\chi_f(z,y,x) \to e^{if(y,x)}$$

where f is a linear functional on $\mathbb{R}^n + \mathbb{R}^n$ and (ii) the infinite dimensional representation classes for the

$$\gamma_\lambda = \text{Ind}_Q^N((z,y,0) \mapsto e^{i\lambda z}), \quad \lambda \neq 0, \quad Q = \{(z,y,x) \in N; x = 0\}$$

So Mackey's "little group method" tells us that $\hat{G}$ consists of the equivalence classes of (i) the trivial representation, (ii) the $\text{Ind}_N^G(\chi_f)$, $\|f\| = 1$, and (iii) the two representations $\pi_\lambda = \text{Ind}_N^G(\gamma_\lambda)$, $\lambda = \pm 1$. The Plancherel formula will use only these $\pi_{\pm 1} = \text{Ind}_Q^G((z,y,0,1) \mapsto e^{\pm iz})$.

G has right Haar measure $d(z,y,x,a) = a^{-1}dzdydxda$ where dy, dx are Lebesgue measure on $\mathbb{R}^n$. The Hilbert space for π_λ is the space of all measureable $f: G \to \mathbb{C}$ such that $f((z,y,0,1)g) = e^{i\lambda z}f(g)$ and $\int|f(0,0,x,a)|^2a^{-1}dxda < \infty$, with π_λ given by $[\pi_\lambda(g')f](g) = f(gg')$. We view π_λ as a representation on $L^2(\mathbb{R}^n \times \mathbb{R}^+, a^{-1}dxda)$ by setting $\phi_f(x,a) = f(0,0,x,a)$, so

$$[\pi_\lambda(z',y',x',a')\phi_f](x,a) = f((0,0,x,a)(z',y',x',a'))$$

$$= f(a^2z' + ax\cdot y', ay', x + ax', aa')$$

$$= f((a^2z' + a^2x'\cdot y' + 2ax\cdot y', ay', 0, 1)(0m0mx + ax', aa'))$$

$$= e^{i\lambda(a^2z'+a^2x'\cdot y'+2ax\cdot y')}\phi_f(x + ax', aa')$$

If $\psi \in C_0^\infty(G)$, now

$$[\pi_\lambda(\psi)\phi_f](x.a) = \int\psi(z',y',x',a')[\pi_\lambda(z',y',x',a')\phi_f](x,a)a'^{-1}dz'dy'dx'da'$$

$$= \int K_\psi(x,a;x'',a'')\phi_f(x'',a'')dx''da''$$

where $x'' = x + ax'$, $a'' = aa'$, and

$$K_\psi(x,a;x'',a'') = \int\psi(z',y',x',a')e^{i\lambda(a^2z'+z^2x'\cdot y'+2ax\cdot y')}\,\frac{a^{-n}}{a''}\,dz'dy'$$

$$= (2\pi)^{n+1/2}\psi(\cdot,\cdot,\frac{x''-x}{a}, \frac{a''}{a})^\wedge(\lambda a^2, \lambda a(x''+x))\,\frac{a^{-n}}{a''}$$

so

$$\text{trace } \pi_\lambda(\psi) = \int K_\psi(x,a;x,a)dxda$$

$$= (2\pi)^{n+1/2}\int\psi(\cdot,\cdot,0,1)^\wedge(\lambda a^2, 2ax)a^{-(n+1)}dxda$$

$$= (2\pi)^{n+1/2}2^{-(n+1)}\int\psi(\cdot,\cdot,0,1)^\wedge(\lambda b, x)b^{-(n+1)}dxdb$$

Here a and $b = a^2$ run from 0 to ∞, so for the appropriate real constant c, we have the Plancherel formula

$$\mathrm{trace}\ \pi_+(D\psi) + \mathrm{trace}\ \pi_-(D\psi) = \psi(1_G) \quad D = (ic \frac{\partial}{\partial z})^{n+1} \tag{2.1}$$

In the case $n = 0$, where G is the "ax + b group," the Plancherel formula (2.1) is due independently to A. Hohari (1961) and C. C. Moore (1971 and 1973). The general (2.1) was known to Moore, and probably also to Tatsuuma in 1972 and to Pukánszky in 1971.

The idea of using semiinvariants D_π or D was first suggested by Dixmier's work (1952) on quasi-Hilbert algebras and the Tomita-Takesaki theory (1967 and 1970) of modular Hilbert algebras. It then developed along several lines. Kohari's method (1961) for the ax + b group was perfected by Tatsuuma (1972), who defined the D_π as corresponding to multiplication by the modular function δ_G, and showed that the Plancherel formula for G need only involve representations induced from the kernel of δ_G. Somewhat more direct approaches were taken by Kleppner and Lipsman (1972 and 1973) for the global operator and by Duflo and Moore (1976) for the infinitesimal operators. In the case where G is of type I, the Duflo-Moore result says: There exist a positive standard Borel measure μ on $\hat{G}$, and measurable fields $\{(\pi, D_\pi):[\pi] \in \hat{G} \setminus (\mu\text{-null set})\}$ where D_π is a nonzero selfadjoint operator on $\mathcal{H}_\pi$ semiinvariant of type δ_G, such that

if $f \in L^1(G) \cap L^2(G)$, then $D_\pi^{1/2}\pi(f)$ is Hilbert-Schmidt a.e. $(\hat{G},\mu)$; if $f \in C_0^\infty(G)$, then $D_\pi^{1/2}\pi(f)D_\pi^{1/2}$ is of trace class a.e. $(\hat{G},\mu)$ (2.2)

$f \to D_\pi^{1/2}\pi(f)$ extends to an isometry of $L^2(G)$ onto $\int_{\hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* d\mu[\pi]$ intertwining the left (resp. right) regular representation with $\int_{\hat{G}} \pi \otimes 1\, d\mu[\pi]$ (resp. $\int_{\hat{G}} 1 \otimes \pi^* d\mu[\pi]$) (2.3)

the D_π are unique up to scalars depending on π and the quantity $D_\pi^{1/2} d\mu[\pi]$ is unique up to a scalar multiple that depends only on normalization of Haar measure (2.4)

The infinitesimal operators D_π have been computed (Duflo and Rais, to appear, and Charbonnel, 1975) for simply connected solvable Lie groups of type I. Also, for simply connected solvable Lie groups, Pukánszky, (1971) showed that the global operator may be realized as the quotient of of two elements in the center of the universal enveloping algebra of the nilradical. But the only cases in which the global operator is known explicitly, besides (2.1) above, are the cases studied by Keene (to appear), by Keene, Lipsman and Wolf (to appear) and by Lipsman and Wolf (to appear). There, there remain some interesting analytic problems, and that is what I want to discuss in the remainder of this paper.

Maximal Parabolic Subgroups of Unitary Groups

Let $\mathbb{F}$ be one of the fields $\mathbb{R}$ (real), $\mathbb{C}$ (complex) or $\mathbb{Q}$ (quaternion). Given integers $p,q \geq 0$, we denote

$$\mathbb{F}^{p,q}: (p+q)\text{-tuples over } \mathbb{F} \text{ with } \langle x,y\rangle = \sum_1^p x_a\bar{y}_a - \sum_{p+1}^{p+q} x_b\bar{y}_b$$
$$U(p,q;\mathbb{F}): \text{all } \mathbb{F}\text{-linear } \langle,\rangle\text{-isometries of } \mathbb{F}^{p,q} \tag{3.1}$$

Here, $U(p,q;\mathbb{R})$ is the indefinite orthogonal group $O(p,q)$; $U(p,q;\mathbb{C})$ is the usual indefinite unitary group $U(p,q)$; and $U(p,q;\mathbb{Q})$ is the indefinite symplectic group $Sp(p,q)$.

A subspace $E \subset \mathbb{F}^{p,q}$ is <u>totally isotropic</u> if $\langle E,E\rangle = 0$. The $U(p,q;\mathbb{F})$ have an important class of subgroups, the <u>parabolic subgroups</u>, which are the normalizers of nested sequences $0 \neq E_1 \subsetneq \cdots \subsetneq E_k$ of totally isotropic subspaces,

$$P_{E_1,\ldots,E_k} = \{g \in U(p,q;\mathbb{F}); gE_\ell = E_\ell \text{ for } 1 \leq \ell \leq k\}$$

The conjugacy class of

$$P_{E_1,\ldots,E_k}$$

is determined by the dimension sequence $\dim_{\mathbb{F}} E_1 < \cdots < \dim_{\mathbb{F}} E_k$. Thus, $U(p,q;\mathbb{F})$ has $\min(p,q)$ conjugacy classes of maximal parabolic subgroups,

$$P_E = \{g \in U(p,q;\mathbb{F}): gE = E\} \quad E \text{ nonzero totally isotropic in } \mathbb{F}^{p,q} \tag{3.2}$$

We know (Wolf, 1976) the structure of P_E and $\hat{P}_E$. The group P_E is a semidirect product $N\cdot(M \times A)$ - its Langlands decomposition is MAN - as follows. To describe the nilradical N, we let $\mathbb{F}^{r\times s}$ denote the $r\times s$ matrices over $\mathbb{F}$, we let $\mathrm{Im}:\mathbb{F}^{s\times s} \to \mathbb{F}^{s\times s}$ and $\mathrm{Re}:\mathbb{F}^{s\times s} \to \mathbb{F}^{s\times s}$ denote the projections $z \mapsto \frac{1}{2}(z - z^*)$ and $z \mapsto \frac{1}{2}(z + z^*)$ where $z^* = {}^t\bar{z}$, and we let $\mathbb{F}^{s\times(t,u)}$ denote $\mathbb{F}^{s\times(t+u)}$ with the "hermitian" map

$$\mathcal{H}: \mathbb{F}^{s\times(t,u)} \times \mathbb{F}^{s\times(t,u)} \to \mathbb{F}^{s\times s} \quad \text{by} \quad \mathcal{H}((v_0,w_0),(v,w)) = v_0v^* - w_0w^*$$

To describe $M \times A$, we need

$$GL'(s;\mathbb{F}) = \{g \in GL(s;\mathbb{F}): \ g \text{ preserves Lebesgue measure on } \mathbb{F}^s\}$$

Here note $GL(s:\mathbb{F}) = GL'(s;\mathbb{F}) \times \mathbb{R}^+$ where $\mathbb{R}^+$ is the multiplicative group of positive real numbers. Now, if $s = \dim_{\mathbb{F}} E$,

$$\begin{aligned}
&\text{(i)} \quad N = \mathrm{Im}\mathbb{F}^{s\times s} + \mathbb{F}^{s\times(p-s,q-s)} \text{ with } (z,v)(z',v') \\
&\qquad = (z + z' + \mathrm{Im}\mathcal{H}(v,v'), v + v') \\
&\text{(ii)} \quad A = \mathbb{R}^+ \text{ and } M = GL'(s;\mathbb{F}) \times U(p-s,q-s;\mathbb{F}) \\
&\text{(iii)} \quad A \times M \text{ acts on } N \text{ by } (a,\gamma,g):(z,v) \to (a^2\gamma z\gamma^*, a\gamma vg^*)
\end{aligned} \tag{3.3}$$

For example, if $\mathbb{F} = \mathbb{C}$ and $s = 1$, then N is the Heisenberg groups of real dimension $2p + 2q - 3$; if $\mathbb{F} = \mathbb{R}$, $s = 1$, $p = 2$, and $q = 4$, then $U(p,q;\mathbb{F})$ is the conformal group $O(2,4)$, $N \cdot M$ is its Poincaré sub-group and P_E is the Poincaré group with scale.

In any case, N really is just a fancy sort of Heisenberg group. N <u>has center</u> $Z = [N,N] = \mathrm{Im}\,\mathbb{F}^{s\times s}$ <u>except in the cases</u>

$$\begin{aligned}
&p = q = s: \quad \text{here } N = \mathrm{Im}\mathbb{F}^{s\times s}, \text{ commutative} \\
&s = 1 \text{ and } \mathbb{F} = \mathbb{R}: \quad \text{here } N = \mathbb{R}^{p-1,q-1}, \text{ commutative}
\end{aligned} \tag{3.4}$$

In any case, N is 2-step nilpotent and $\hat{N}$ comes directly out of the Kirillov theory.

A class, $[\pi] \in \hat{N}$, is called <u>square integrable</u> (mod Z) if its matrix coefficients $f_{\xi,\eta}(n) = \langle\xi,\eta(n)\eta\rangle$ satisfy $|f_{\xi,\eta}| \in L^2(N/Z)$. If $\hat{N}$ has a square integrable representation, one knows (see Moore and Wolf, 1973) that Plancherel-almost-all classes in $\hat{N}$ are square integrable and that those classes correspond, using Kirillov theory, to the coadjoint orbits $Ad^*(N)\cdot\phi$ of the form $Ad^*(N)\cdot\phi = \{\psi \in n^*: \psi|_z = \phi|_z\}$, where n and z are the Lie algebras of N and Z, and $*$ denotes real linear dual space. In our case, one can see (Wolf, 1976) that N has square integrable representations in all cases except

$$\mathbb{F} = \mathbb{R} \quad s \text{ odd} \quad s > 1 \quad (p - s) + (q - s) > 0 \tag{3.5}$$

Leaving (3.5) aside, the coadjoint orbits giving square integrable representations are the coadjoint orbits of the linear functionals

$$\begin{aligned}
&s = 1 \text{ and } \mathbb{F} = \mathbb{R}: \phi_w(v) = \mathrm{Re}\langle v,w\rangle \quad w \in \mathbb{R}^{p-1,q-1} \\
&p = q = s: \phi_z(z_0) = \text{trace } \mathrm{Re}(z_0z^*) \quad z \in \mathrm{Im}\mathbb{F}^{s\times s} \\
&\text{otherwise:} \quad \phi_z(z_0,v_0) = \text{trace } \mathrm{Re}(z_0z^*) \quad z \in \mathrm{Im}\mathbb{F}^{s\times s} \text{ nonsingular}
\end{aligned} \tag{3.6}$$

According to the Mackey machine, the corresponding representations of P_E obtained by the little-group method are sufficient for harmonic analysis in $L^2(P_E)$. Here we may take $w \neq 0$ in the $s = 1$, $\mathbb{F} = \mathbb{R}$ case, and may take z of maximal possible rank in the $p = q = s$ case.

The Case of Real Rank One

Keene (to appear) and Keene, Lipsman and Wolf (to appear) studied the case of parabolic subgroups of the real rank one unitary groups $U(p,1;\mathbb{F})$. There, $s = 1$ and $P_E = NAM$ is given by

$$\begin{aligned}
&N = \mathrm{Im}\mathbb{F} + \mathbb{F}^{p-1} \text{ with } (z_0,v_0)(z,v) = (z_0 + z + \mathrm{Im}\langle v_0,v\rangle, v_0 + v) \\
&A = \mathbb{R}^+ \text{ and } M = \{\gamma \in \mathbb{F}: |\gamma| = 1\} \times U(p-1;\mathbb{F}) \\
&A \times M \text{ acts on } N \text{ by } (a,\gamma,g):(z,v) \to (a^2\gamma z\bar{\gamma}, a\gamma vg^*)
\end{aligned} \tag{4.1}$$

Square integrable representations of N are associated to the linear functionals $\lambda \in z^* - \{0\}$, the corresponding class $[\gamma_\lambda] \in \hat{N}$ being characterized by its central character

$$\gamma_\lambda(zn) = e^{i\lambda(\log z)}\gamma_\lambda(n) \quad \text{for } z \in Z, \ n \in N$$

If $\mathbb{F} = \mathbb{R}$, γ_λ is just the unitary character $z \to e^{i\lambda(z)}$ on $\mathbb{R}^{p-1}$. If $\mathbb{F} = \mathbb{C}$, $[\gamma_\lambda]$ is the infinite dimensional representation class of the Heisenberg group N with central character $e^{i\lambda}$, and one may view

$$\gamma_\lambda = \mathrm{Ind}_Q^N((z,v) \to e^{i\lambda(z)}) \qquad Q = \mathrm{Im}\mathbb{C} + \mathbb{R}^{p-1} \subset N \tag{4.2}$$

as in Section 1. If $\mathbb{F} = \mathbb{Q}$, the situation is similar.

A acts on $\{[\gamma_\lambda]: \lambda \in z^* - \{0\}\}$ by $a\cdot[\gamma_\lambda] = [\gamma_{a^{-1}\lambda}]$ for $\mathbb{F} = \mathbb{R}$ and $a\cdot[\gamma_\lambda] = [\gamma_{a^{-2}\lambda}]$ for $\mathbb{F} \neq \mathbb{R}$, so the

$$\eta_\nu = \mathrm{Ind}_N^{NA}(\gamma_\lambda) \qquad \lambda \in z^* - \{0\} \tag{4.3}$$

are irreducible, and $[\eta_\lambda] = [\eta_{\lambda'}]$ exactly when $\lambda = r\lambda'$ for some $r > 0$. If we denote

$$S: \text{ unit sphere in } z^* \tag{4.4}$$

then, the generic representations of NA are $\{[\eta_\lambda]:\lambda \in S\}$.

Conjugation by $m \in M$ commutes with induction from N to NA, so it sends $[\gamma_\lambda]$ to $[\gamma_{\mu(m)*\lambda}]$ and $[\eta_\lambda]$ to $[\eta_{\mu(m)*\lambda}]$ where μ is the representation of M on z. The latter is given by

$\mathbb{R}: M = \{\pm 1\} \times O(p-1)$ μ is nontrivial scalar on $\{\pm 1\}$ and μ is the usual representation of $O(p-1)$ on $\mathbb{R}^{p-1}$

$\mathbb{C}: M = \{t \in \mathbb{C}: |t| = 1\} \times U(p-1)$ and μ is trivial (4.5)

$\mathbb{Q}: M = \{t \in \mathbb{Q}: |t| = 1\} \times Sp(p-1)$ μ is the 3-dimensional representation of $\{t \in \mathbb{Q}: |t| = 1\} \cong SU(2)$ and μ is trivial on $Sp(p-1)$

Now, the action of M on z^* and the $[\gamma_\lambda]$, $\lambda \in S$, satisfies

$\mathbb{R}: M$ is transitive with isotropy $M_\lambda \cong \{\pm 1\} \times O(p-2)$

$\mathbb{C}: M$ fixes the 2 points of S, so isotropy $M_\lambda = M$ (4.6)

$\mathbb{Q}: M$ is transitive with $M_\lambda \cong \{t \in \mathbb{C}: |t| = 1\} \times Sp(p-1)$

One can check that η_λ extends to a linear representation $\tilde{\eta}_\lambda$ of NAM. Now, apply Mackey's little-group method. If $\mathbb{F} \neq \mathbb{C}$, fix $\lambda_1 \in S$ and set $M_1 = M_{\lambda_1}$; then, the generic representations of $P_E = NAM$ are the

$$\pi_\tau = \mathrm{Ind}_{NAM_1}^{NAM}(\widetilde{\eta_{\lambda_1}} \otimes \tau) \qquad [\tau] \in \hat{M}_1 \tag{4.7}$$

If $\mathbb{F} = \mathbb{C}$, then $S = \{\lambda_1, \lambda_{-1}\}$ and the generic representations of NAM are the

$$\pi_\tau^+ = \widetilde{\eta_{\lambda_1}} \otimes \tau \quad \text{and} \quad \pi_\tau^- = \widetilde{\eta_{\lambda_{-1}}} \otimes \tau \qquad [\tau] \in \hat{M} \tag{4.8}$$

The final ingredient of the Plancherel formula on P_E is the Laplacian $\Delta = -\sum \partial^2/\partial z_i^2$ on the Euclidean vector group Z. View Δ as an operator on P_E using the diffeomorphic splittings $\mathrm{Im}\mathbb{F} \times \mathbb{F}^{p-1} \times A \times M$. Set

$$k = \dim_{\mathbb{R}} Z \qquad \ell = \dim_{\mathbb{R}} N/Z \qquad r = k + \tfrac{1}{2}\ell \tag{4.9}$$

Then, for certain constants c_i the pseudodifferential operators

$$D_{NA} = c_1 \Delta^{r/2} \text{ on } NA \qquad D_{NAM} = c_2 \Delta^{r/2} \text{ on } NAM \tag{4.10}$$

come into the Plancherel formula as follows (see Keene, Lipsman and Wolf, to appear.)

For NA: if $f \in C_c^\infty(NA)$, then

(i) $D_{NA}f \in L_1(NA)$, so the $\eta_\lambda(D_{NA}f)$ are defined

(ii) each $\eta_\lambda(D_{NA}f)$ is of trace class

(iii) trace $\eta_\lambda(D_{NA}f)$ is a C^∞ function of $\lambda \in S$, and

(iv) we have

$$f(1_{NA}) = \int_S \mathrm{trace}\, \eta_\lambda(D_{NA}f)\, d\sigma(\lambda) \tag{4.11}$$

where σ is the standard volume element on S.

For NAM: if $f \in C_c^\infty(NAM)$, then

(i) $D_{NAM}(f) \in L_1(NAM)$, so the $\pi_\tau^{(\pm)}(f)$ are defined

(ii) each $\pi^{(\pm)}(f)$ is of trace class, and

(iii)

$$f(1_p) = \sum_{\tau \in \hat{M}_1} (\dim \tau)\, \mathrm{trace}\, \pi_\tau(D_{NAM}f) \qquad \mathbb{F} \neq \mathbb{C}$$
$$f(1_p) = \frac{1}{2} \sum_{\tau \in \hat{M}} (\dim \tau)\, \mathrm{trace}\, \{\pi_\tau^+(D_{NAM}f) + \pi_\tau^-(D_{NAM}f)\} \qquad \mathbb{F} = \mathbb{C} \tag{4.12}$$

The group theoretic and measure theoretic aspects of the proof use methods of Duflo (1972) and Kleppner and Lipsman (1972 and 1973), but they are not our concern here. Rather, we are concerned with the first conclusion

$$D: C_0^\infty(G) \to L_1(G) \qquad G = NA \text{ or } NAM \tag{4.13}$$

which is needed simply in order to make sense of the transformations

$$f \mapsto \eta(D_{NA}f) \qquad f \mapsto \pi(D_{NAM}f)$$

that appear in the Plancherel formulae (4.11) and (4.12).

There is nothing to (4.13) when D is differential. But Keene's discovery (to appear) that D may be strictly pseudodifferential (e.g., when $\mathbb{F} = \mathbb{Q}$, so $r/2 = \frac{3}{2} + (p - 1)$) raises the question of L_1 estimates of the form (4.13). In the real rank one case here, where D is a multiple of a positive power of Δ, this is not so serious (see Keene, Lipsman and Wolf, to appear). But the next examples will show that (4.13) can be very tricky and, in fact, still open.

The Estimate $DC_0(G) \subset L^1(G)$

If we compare the Plancherel formula (2.1) with the case $\mathbb{F} = \mathbb{C}$ of (4.11), we see that the global operator D is not quite unique. Indeed, D can be replaced by any densely defined invertible operator on $L^2(G)$, say D', that also is semiinvariant of type δ_G ($G = NA$ or NAM). Then the corresponding local operators $\pi(D') = a_\pi \cdot \pi(D)$, a_π nonzero scalars, for Plancherel almost all $[\pi] \in \hat{G}$. This leaves a lot of freedom with the solvable groups NA. For example, if $z:Z \to \mathbb{R}^k$ is a linear coordinate on the vector group Z, then (4.11) can be recast as

$$f(1_{NA}) = \int_S \text{trace } \eta_\lambda(D'f)d\sigma'(\lambda)$$

for an appropriate measure σ' on S, where $D' = (\partial/\partial z_1)^r$, $r = \dim_{\mathbb{R}} Z + \frac{1}{2}\dim_{\mathbb{R}} N/Z$ as in Section 4.

The matter is somewhat different for the full parabolic $P_E = NAM$. In order to manage the group theoretic aspects of the proof of the Plancherel formula, and also for esthetic reasons, one wants that D be M-invariant. Let us see what this means in the case $s = 1$. There,

$$N = \text{Im}\mathbb{F} + \mathbb{F}^{p-1,q-1} \qquad M = \{\gamma \in \mathbb{F} : |\gamma| = 1\} \times U(p-1,q-1;\mathbb{F})$$

If $\mathbb{F} \neq \mathbb{R}$, then $Z = \text{Im}\mathbb{F}$, $U(p-1,q-1;\mathbb{F})$ acts trivially on Z, and $\{\gamma \in \mathbb{F} : |\gamma| = 1\}$ is $\mathbb{R}$-irreducible on Z. If we require that $D = D_{NAM}$ be an operator on Z, it follows that D must be the Z-Fourier-transform of a radial function homogeneous of degree $\dim_{\mathbb{R}} Z + \frac{1}{2}\dim_{\mathbb{R}} N/Z$, and if we further require positivity, then D must essentially be the

$$\tfrac{1}{2}(\dim_{\mathbb{R}} Z + \tfrac{1}{2}\dim_{\mathbb{R}} N/Z)$$

power of the Laplacian on Z. In this case, things go more or less as for (4.12) with no serious problem in the L_1 estimate (4.13)

If $\mathbb{F} = \mathbb{R}$, the situation changes. There, $Z = \mathbb{R}^{p-1,q-1}$ and $M = \{\pm 1\} \times O(p-1,q-1)$ acts on it in the vector representation. With the Z-Fourier-transform and the positivity requirement as above, $D = D_{NAM}$ must be a positive multiple of the $\frac{1}{2}\dim Z$ power of a wave operator,

$$D = c|\Box|^{k/2} \qquad \Box = \sum_1^k \varepsilon(j)\partial^2/\partial x_j^2 \qquad k = (p-1) + (q-1) \tag{5.1}$$

where $(x_1,\ldots,x_k)$ is a linear coordinate on $Z = \mathbb{R}^{p-1,q-1}$ in which $\langle x,x'\rangle = \sum\varepsilon(j)x_jx_j'$, $\varepsilon = \pm 1$. There, one can show[+] that $|\Box|^\sigma C_0^\infty(Z)$ is contained in $L^1(Z)$ for $\sigma > 2[\frac{k}{4}] + 2$, but is not contained in $L^1(Z)$ for $\sigma < \frac{1}{2}k - 1$ and $\min(p,q) > 1$. There, the question of (4.13), where $\sigma = k/2$, remains open.

More generally, assuming $s = \dim_{\mathbb{F}} E$ even in case $\mathbb{F} = \mathbb{R}$, that is, leaving aside the case (3.5), where N does not have square integrable representations, and also the case $s = 1$, $\mathbb{F} = \mathbb{R}$ just discussed, one takes (see Lipsman and Wolf, to appear) the operators $D = D_{NA}$ and D_{NAM} in the form

$$Df = cF^{-1}(|p(\zeta)|^t \cdot F(f)) \qquad c > 0 \tag{5.2}$$

as follows. ζ is the coordinate on the real dual vector space Z^*, F is Fourier transform on the real vector space Z, p is the real polynomial function on $Z = \text{Im}\mathbb{F}^{s\times s}$ such that $z \in Z$ sends Lebesgue measure $d\nu$ on $\mathbb{F}^s$ to $|p(z)|d\nu$, and t is the positive real number such that $|p(z)|^t$ is homogeneous of degree $r = \dim_{\mathbb{R}} Z + \frac{1}{2}\dim_{\mathbb{R}} N/Z$. If $s > 1$, then $p(z)$ is considerably more complicated than the quadratic polynomial $\sum\varepsilon(j)z_j^2$ implicit in (5.1), and the problem of proving an L_1 a priori estimate (4.13) seems even stickier.

[+] Information received from R. Prosser and C. Fefferman.

Avoiding the Estimate-Possibility and Disadvantages

Lipsman and I recently (to appear) managed, without an estimate of the type (4.13), to prove a Plancherel formula for all the cases in which N has square integrable representations. The formula is of the form

$$f(1_P) = \sum_i \int_{\hat{M}_i} \text{trace}\,\pi_\tau^{(i)}(Df)d\nu_i(\tau) \tag{6.1}$$

where the sum runs over the (finite) set of MA-orbits on the space of square integrable classes in $\hat{N}$, where M_i is the M-stabilizer of an element $\lambda_i \in Z^*$ corresponding to the i^{th} orbit, where $\pi_\tau^{(i)}$ is defined as in (4.7) for $[\tau] \in \hat{M}_i$, and where $D = D_{NAM}$ is given by (5.1) or (5.2). The formula applies to functions $f \in C^\infty(P_E)$ whose Z-Fourier-transform $F(f) \in C_c^\infty((Z^*\backslash(\text{zeros of } p)) \times N/Z \times A \times M)$. Of course, one can enlarge the domain of D in the obvious way, using Schwartz functions f such that $F(f)$ vanishes, in the Z^* variable, to sufficiently high order at the zeros of $p(\zeta)$. Still, this is not completely satisfactory for harmonic analysis on P_E, and it would be very good to know whether the estimate (4.13) holds for the pseudodifferential operators D of (5.1) and (5.2)

In summary, we have the Fourier inversion formula (6.1) for a moderately large class of nonunimodular groups, but its utility is limited by our lack of exact knowledge of the class of functions to which it applies. This lack would be remedied by some special cases of a problem in psuedo-differential equations. The problem: Let $p(z)$ be a polynomial on $\mathbb{R}^k$, homogeneous of degree d, and let $D_{p,t}$ denote the pseudodifferential operator on $\mathbb{R}^k$ given by

$$D_{p,t}(f)^\wedge(\zeta) = |p(\zeta)|^t \hat{f}(\zeta) \qquad t > 0$$

For which values of t does $D_{p,t}$ keep $C_0^\infty(\mathbb{R}^k)$ inside $L^1(\mathbb{R}^k)$?

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